A study of 1/R expansions in cylindrical coordinates PhL 5.22.10

I have just finished doing this for spherical coordinates, so let's try to replicate things here for the cylindrical case. From "cylindrical atoms.doc" there are two atomic systems often used,

(a) $e^{\pm kz}$ [$J_m(k\rho)$, $N_m(k\rho)$] $e^{\pm im\varphi}$ // expo in z

(b) [$K_m(k\rho)$, $I_m(k\rho)$] $e^{\pm ikz}$ $e^{\pm im\varphi}$ // expo in ρ

As usual, I assume oscillatory in φ. So I am perhaps going to have to do everything twice.

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Overview (4 pages, added 9.9.10).

In **Part A** I seek the **J-J Green's Function for an infinite cylinder** of radius b using the J_m atomic form. This problem is not really the 1/R expansion, but is interesting in its own right. In Part **Pre 0** I state the Smythian form as follows, where m is all integers and $J_m(k_n b) = 0$ so the "k" sum is over k_n :

$$V(\mathbf{r} \mid \mathbf{r}') = \Sigma_{mk}\, A_{mk} \exp(-k|z-z'|)\, J_m(k\rho)\, e^{im\varphi}$$

I write $\mathbf{r} = (\rho\cos\varphi, \rho\sin\varphi, z)$ for the observation point, and $\mathbf{r}' = (\rho'\cos\varphi', \rho\sin\varphi', z')$ for the Green's charge location, and I derive the famous results that

$$R^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi') \qquad \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = [\,\rho\rho'\cos(\varphi-\varphi') + zz']/[\,\sqrt{\rho^2+z^2}\,\sqrt{\rho'^2+z'^2}\,] = \cos(\gamma)$$

Then in part **Pre 1** I do the usual pillbox method (perp to z, since ρ and φ are oscillatory) to find A_{mk}. This involves the usual φ orthogonality and the Fourier-Bessel ρ orthogonality, and the result is

$$A_{mn} = q\, e^{-im\varphi'}\, J_m(k_n\rho')\, [b\, J_{m+1}(bk_n)/\sqrt{2}\,]^{-2} / k_n \qquad \text{where } J_m(k_n b) = 0$$

so that our Green's Function is then [I call this a J-J form since the product of two J's appears]

$$V(\mathbf{r} \mid \mathbf{r}') = q\,[b/\sqrt{2}\,]^{-2} \Sigma_{mn}\, [J_{m+1}(bk_n)]^{-2}\, (1/k_n) \exp(-k_n|z-z'|)\, J_m(k_n\rho)\, J_m(k_n\rho')\, e^{im\varphi}\, e^{-im\varphi'}$$

which is manifestly symmetric. At this point, I review Stak's general proof that a Green's Function must in fact *be* symmetric, and this also implies what is called the reciprocity principle of electrostatics. I was not able to find verification of my result for V above but will keep my eye out for it.

It should be noted that this cylinder Green's problem can be solved using the I-K atomic form rather than the J-J form. I have not done the problem this way, but based on the 1/R analysis in Part C below, we will get the sum over k_n replaced by an integral over positive k. I have seen this result in the literature, and I think this is the way people usually do such a problem since the Fourier-Bessel is a bit messy. See Part C below. [One could say you use k = imaginary in the Mellin-Barnes sense for your J-J solution, and this is analogous to using the conical/toroidal/Mehler functions in spherical atomic forms. Someday I hope to carry out this little program and show that you pick up the series by closing the Barnes contour.]

In **Part B** I seek the **(J-J) 1/R expansion** using J_m functions. In **Part 0** I repeat work from Part A, but now the spectrum of k is the continuous range $(0,\infty)$, that is the key difference. I wanted to make Part A and Part B "look similar" which is why I used the Σ_k notation in Part A ($k = k_n$ etc) and $\int dk$ in Part B. The Smythian form for 1/R this time is claimed to be:

$$f(\mathbf{r} \mid \mathbf{r}') = \Sigma_m \int dk\, A_{mk} \exp(-k|z-z'|)\, J_m(k\rho)\, e^{im\varphi}$$

In **Part 1** I redo the pillbox analysis (again perp to z) to find the A_{mk}. This time of course we use the Hankel Transform for our second orthogonality, and the result is found to be simply this:

$$A_{mk} = q\, e^{-im\varphi'}\, J_m(k\rho') \qquad \text{// } A_{mk} \text{ is really } A_{mk}(\mathbf{r}') \text{ but } \mathbf{r}' \text{ is suppressed}$$

so our Green's function 1/R is then given by (Σ_m over all integers in first form, and set q=1)

$$V(\mathbf{r} \mid \mathbf{r}') = 1/R = \Sigma_m \int_0^\infty dk \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')\, e^{im\varphi}\, e^{-im\varphi'}$$

$$= \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')] \int_0^{\infty} dk \exp(-k|z-z'|) J_m(k\rho) J_m(k\rho') \qquad \text{// J form (*)}$$

which is again manifestly symmetric and which may be compared to the cylinder result above from Part A. I found **verification** of the above 1/R expansion in MF. In **Part 2** I derive the **addition theorem**. I first let **r'** = (0,0 ,z') to obtain the special case result of (*) above,

$$1/R^{\sim} = 1/\sqrt{\rho^2 + (z-z')^2} = \int_0^{\infty} dk \exp(-k|z-z'|) J_0(k\rho)$$

which I verify in GR. Since the above is true for any value of ρ^2, I replace ρ^2 by $\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')$ to obtain the result fully general result [since $R^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$]

$$1/R = \int_0^{\infty} dk \exp(-k|z-z'|) J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}) \qquad (**)$$

This, then, is our 1/R expansion *analogous* to the famous sphericals result (note m=0 everywhere)

$$1/R = \Sigma_n (r_<)^n (r_>)^{-n-1} P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))$$

If we now equate the 1/R form (**) with the J-form for 1/R shown above (*), we find that

$$J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}) = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')] J_m(k\rho) J_m(k\rho')$$

which is the "addition theorem" [verified GR7 page 940] analogous to the spherical addition theorem

$$P_n[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')] f(n,-m) P_n^m(\cos\theta) P_n^m(\cos\theta')]$$

I am sure the cylindrical addition theorems have group theoretic underpinnings as the sphericals do, but I have never pursued this subject. The group is of course continuous but different, not SO(3).

In **Part C** I repeat all of the above 1/R expansion work using the I-K form instead of the J-J form. We start with this Smythian form in **Part 0**,

$$f(\mathbf{r} \mid \mathbf{r}') = q/R = \Sigma_m \int_0^{\infty} dk\, A_{mk} \cos[k(z-z')] I_m(k\rho_<) K_m(k\rho_>) e^{im\varphi}$$

Then in **Part 1** we do the pillbox to get the A_{mk} . But now z and φ are oscillatory, so we do the pillbox perp to ρ instead of z. We use the usual φ orthogonality, but this time we use the Cosine Integral Transform for our z orthogonality, and we find this simple result:

$$A_{mk} = (2/\pi) q\, e^{-im\varphi'}$$

which then tells us our resulting 1/R expansion

$$1/R = (2/\pi)\,\Sigma_m \int_0^\infty dk\, \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)\, e^{im\varphi}\, e^{-im\varphi'}$$

$$= (2/\pi)\,\Sigma_{m=0}^{\infty}\, \varepsilon_m \cos[m(\varphi-\varphi')] \int_0^\infty dk\, \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)$$

which is verified Jackson page 86 (3.148). In **Part 2** I develop the addition theorem for this case. As before, I start with **r'** = (0,0 ,z') to get

$$1/R\tilde{} = 1/\sqrt{\rho^2 + (z-z')^2} = (2/\pi) \int_0^\infty dk\, \cos[k(z-z')]\, K_0(k\rho)$$

or:
$$1/\sqrt{\rho^2 + z^2} = (2/\pi) \int_0^\infty dk\, \cos(kz)\, K_0(k\rho)$$

which I then verify in Jackson and GR7. Doing the same addition theorem trick as above, we find the 1/R expansion and the addition theorem, both of which appear in Jackson:

$$1/R = (2/\pi) \int_0^\infty dk\, \cos[k(z-z')]\, K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) \qquad \text{// 1/R expansion}$$

$$K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) = \Sigma_{m=0}^{\infty}\, \varepsilon_m\, I_m(k\rho_<)\, K_m(k\rho_>) \cos[m(\varphi-\varphi')] \qquad \text{// addition theorem}$$

In **Part D** I summarize all the results of Parts A,B,C, but I also *add* some new results concerning the $Q_{n-1/2}$ function. First, we have the 1/R expansion which appears in the "Selvaggi" paper

$$1/R = 1/(\pi\sqrt{\rho\rho'}\,)\, \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')]\, Q_{m-1/2}(\beta) \qquad \beta = [\,\rho^2 + \rho'^2 + (z-z')^2\,]/(2\rho\rho')$$

and which I derive in another of my docs called "integral doc" (see also toroidal 1/R doc). If I then equate this to the J-J expansion for 1/R, I conclude that

$$Q_{m-1/2}(\beta) / (\pi\sqrt{\rho\rho'}\,) = \int_0^\infty dk\, \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')$$

which I then verify in GR. I now have no doubt but that the above 1/R Selvaggi expansion is really just the 1/R expansion in toroidal coordinates, but in my toroidal 1/R doc I don't quite show that this is in fact the case, more work is needed there. [that work is now done!] The Selvaggi expansion is the toroidal 1/R expansion but some parts of it are expressed in cylindrical rather than toroidal coordinates so it is a bit of a hybrid thing.

Part A: Green's Function for Infinite cylinder using J_m functions

Pre 0. Introduction: Green's function for an infinite cylinder of radius b.

(I did this problem accidentally, but since I finished it, I will keep it).

Our point charge q is located at some point r' = (ρ',φ',z') inside a cylinder of radius b. We write the following atomic form for the potential

$$V(\mathbf{r} \mid \mathbf{r}') = \Sigma_{mk} A_{mk} \exp(-k|z-z'|) J_m(k\rho) e^{im\varphi}$$

We know this z stuff as follows. Draw a z-plane at z' which passes through our point charge. On either size of this plane, we must have expo decay as we go to large |z|. There is some discrete spectrum of k determined presumably by the zeros of J_m on our cylindrical boundary, but we leave it just as Σ_k. =

Notice that we are parameterizing our two points **r** and **r'** this way

$$\mathbf{r} = (\rho\cos\varphi, \rho\sin\varphi, z) \quad \Rightarrow \quad \hat{\mathbf{r}} = (\rho\cos\varphi, \rho\sin\varphi, z)/\sqrt{\rho^2+z^2}$$
$$\mathbf{r}' = (\rho'\cos\varphi',\rho\sin\varphi',z') \quad \Rightarrow \quad \hat{\mathbf{r}}' = (\rho'\cos\varphi',\rho'\sin\varphi',z')/\sqrt{\rho'^2+z'^2}$$

$$\Rightarrow \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = [\,\rho\rho'(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi') + zz']/[\,\sqrt{\rho^2+z^2}\,\sqrt{\rho'^2+z'^2}\,] = \cos(\gamma)$$

$$\Rightarrow \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = [\,\rho\rho'\cos(\varphi-\varphi') + zz']/[\,\sqrt{\rho^2+z^2}\,\sqrt{\rho'^2+z'^2}\,] = \cos(\gamma)$$

so that γ is the angle directly between our two vectors **r** and **r'**. But for our cylindrical case, we are really more interested (later) in these facts, simpler than the above,

$$\mathbf{r}\bullet\mathbf{r}' = \rho\rho'\cos(\varphi-\varphi') + zz' \qquad r^2 = \rho^2 + z^2 \qquad r'^2 = \rho'^2 + z'^2$$

$$R^2 = |\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2\,\mathbf{r}\bullet\mathbf{r}' = \rho^2 + z^2 + \rho'^2 + z'^2 - 2\rho\rho'\cos(\varphi-\varphi') - 2zz'$$

$$= \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

At this point there are several different things we can do.

Pre 1. Use the pillbox condition to find the A_{mn} coefficients for the infinite cylinder.

The general pillbox condition is this (Jackson units)

$$\partial_{q_1} V_i - \partial_{q_1} V_o = 4\pi q\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

where q_1 is the dimension perp to our partition of space into two regions at the point charge. I will assume for the moment that z' > 0 and that "outer" means z > z' which is farther away from the origin, etc. We then have (that is to say, the "inner" region contains the origin)

$$V_i = \Sigma_{mk} A_{mk} \exp(+k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z < z'$$
$$V_o = \Sigma_{mk} A_{mk} \exp(-k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z > z'$$

So $q_1 = z$. We can then take $q_2 = \rho$ and $q_3 = \varphi$ and use these facts

$h_1 = h_z = 1$
$h_2 = h_\rho = 1$
$h_3 = h_\varphi = \rho$

I take these facts from M&M page 178. I really should start with the three Cartesian equations and run my little program to compute the metric tensor, but OK for now. Our pillbox condition is then this:

$$\partial_z V_i - \partial_z V_o = 4\pi q\,(1/\rho)\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')$$

We then need to compute the derivatives

$$\partial_z V_i = \Sigma_{mk} A_{mk}\, k \exp(+k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z < z'$$
$$\partial_z V_o = \Sigma_{mk} A_{mk}\, (-k) \exp(-k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z > z'$$

then evaluate both on the bounding surface between the two regions (which means z = z')

$$\partial_z V_i = \Sigma_{mk} A_{mk}\, k\, J_m(k\rho)\, e^{im\varphi}$$
$$\partial_z V_o = -\Sigma_{mk} A_{mk} k J_m(k\rho)\, e^{im\varphi}$$

so

$$\partial_z V_i - \partial_z V_o = 2\,\Sigma_{mk} A_{mk}\, k\, J_m(k\rho)\, e^{im\varphi} = 4\pi q\,(1/\rho)\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')$$

We will use two orthogonalities to find the A_{mn}. First, apply $\int d\varphi\, e^{-im'\varphi}$ to both sides and use

$$(1/2\pi) \int_{-\pi}^{\pi} d\varphi\, e^{im\varphi} e^{-im'\varphi} = \delta_{mm'} \qquad \text{// orthogonality}$$

The LHS becomes

$$2\,\Sigma_{mk} A_{mk}\, k\, J_m(k\rho) \int d\varphi\, e^{-im'\varphi} e^{im\varphi} = 2\,\Sigma_{mk} A_{mk}\, k\, J_m(k\rho)\, 2\pi\delta_{mm'} = 4\pi\,\Sigma_k A_{m'k}\, k\, J_{m'}(k\rho)$$

and the RHS becomes $4\pi q\,(1/\rho)\,\delta(\rho-\rho')\,e^{-im'\varphi'}$ so we then have (replace m' with m)

$$4\pi\,\Sigma_k A_{mk}\, k\, J_m(k\rho) = 4\pi q\,(1/\rho')\,\delta(\rho-\rho')\,e^{-im\varphi'}$$

Now we reach into our prepared "transforms doc" and pull out this rabbit,

$$f(\rho) = \Sigma_{n=1}^{\infty} A_{n,\nu}\, J_\nu(k_n\rho) \qquad \text{// expansion}$$
$$A_{n,\nu} = 2/[b^2 J_{\nu+1}(bk_n)^2] \int_0^b d\rho\, \rho\, f(\rho)\, J_\nu(k_n\rho) \qquad \text{// projection}$$
$$[b\, J_{\nu+1}(bk_n)/\sqrt{2}\,]^{-2} \int_0^b d\rho\, \rho\, J_\nu(k_n\rho)\, J_\nu(k_{n'}\rho) = \delta_{n,n'} \qquad \text{// orthogonality}$$
$$\Sigma_{n=1}^{\infty} [b\, J_{\nu+1}(bk_n)/\sqrt{2}\,]^{-2}\, J_\nu(k_n\rho)\, J_\nu(k_n\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$

where we now assume that the k_n quantized values are set by $J_m(k_n b) = 0$ so the radius of whatever BC cylinder we have is $\rho = b$. So let's first rewrite our above equation showing the spectral k_n (we change then from A_{mk} to A_{mn})

$$4\pi \Sigma_n A_{mn} k_n J_m(k_n\rho) = 4\pi q (1/\rho') \delta(\rho-\rho') e^{-im\varphi'}$$

where n = 1,2,3... just enumerate the zeros of $J_m(k_n b)$, and we note that the k_n depend on m, but we don't want to over clutter our notation to show this fact right now.

So, the next step is to apply $\int_0^b d\rho\, \rho\, J_m(k_{n'}\rho)$ to both sides of the above equation.

$$\text{LHS} = 4\pi \Sigma_n A_{mn} k_n \int_0^b d\rho\, \rho\, J_m(k_{n'}\rho) J_m(k_n\rho)$$

$$= 4\pi \Sigma_n A_{mn} k_n \delta_{n,n'} [\, b\, J_{m+1}(bk_n)/\sqrt{2}\,]^2$$

$$= 4\pi\ A_{mn'}\, k_{n'}\ [b\, J_{m+1}(bk_{n'})/\sqrt{2}\,]^2$$

$$\text{RHS} = 4\pi q (1/\rho') e^{-im\varphi'} \int_0^b d\rho\, \rho\, J_m(k_{n'}\rho) \delta(\rho-\rho') = 4\pi q (1/\rho') e^{-im\varphi'} \rho' J_m(k_{n'}\rho')$$

$$= 4\pi q\, e^{im\varphi'} J_m(k_{n'}\rho')$$

The result is this (change n' to n)

$$4\pi\ A_{mn} k_n\ [b\, J_{m+1}(bk_n)/\sqrt{2}\,]^2 = 4\pi q\, e^{-im\varphi'} J_m(k_n\rho')$$

$$A_{mn} = q\, e^{=im\varphi'} J_m(k_n\rho') [b\, J_{m+1}(bk_n)/\sqrt{2}\,]^{-2} / k_n$$

So we now have our coefficients! We can install these into our expansion above

$$V(\mathbf{r} \mid \mathbf{r'}) = \Sigma_{mn} A_{mn} \exp(-k_n|z-z'|) J_m(k_n\rho)\, e^{im\varphi}$$

$$= \Sigma_{mn} \{\, q\, e^{-im\varphi'} J_m(k_n\rho') [b\, J_{m+1}(bk_n)/\sqrt{2}\,]^{-2} / k_n \,\} \exp(-k_n|z-z'|) J_m(k_n\rho)\ e^{im\varphi}$$

$$= q\, [b/\sqrt{2}\,]^{-2} \Sigma_{mn}\ [J_{m+1}(bk_n)]^{-2} (1/k_n) \exp(-k_n|z-z'|)\ J_m(k_n\rho) J_m(k_n\rho')\, e^{im\varphi} e^{-im\varphi'}$$

We notice that the result is symmetric under $\mathbf{r} \leftrightarrow \mathbf{r'}$. The *reason* this is true is not as obvious as it is in our 1/R free space problem where $R = |\mathbf{r}-\mathbf{r'}|$. If **r** is close to the cylinder and **r'** close to the center line, it is not obvious that switching these points should give the same result! Ie, that $V(\mathbf{r} \mid \mathbf{r'}) = V(\mathbf{r'} \mid \mathbf{r})$. There is no obvious transformation of space that swaps the two points and leaves the cylinder where it is. But we have just *proven* it is symmetric in this case, and we know from our Stakgold reading that all Green's

Functions have this symmetry. This is discussed in Chapter 6, and I will quote my entire little note section on this topic, it is worth rereading right now since we have an actual application

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Theorem 2: The function g(x|ξ) is symmetric. This is not as obvious, so let's do the work he outlines.

$-\nabla^2 g(x|\xi) = \delta(x-\xi)$ $\quad g = 0$ on σ // definitions
$-\nabla^2 g(x|\eta) = \delta(x-\eta)$ $\quad g = 0$ on σ

$- g(x|\eta)\nabla^2 g(x|\xi) = g(x|\eta)\delta(x-\xi)$ // process
$- g(x|\xi)\nabla^2 g(x|\eta) = g(x|\xi)\delta(x-\eta)$

$- g(x|\eta)\nabla^2 g(x|\xi) + g(x|\xi)\nabla^2 g(x|\eta) = g(x|\eta)\delta(x-\xi) - g(x|\xi)\delta(x-\eta)$ // subtract

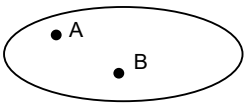
$\int_R dx \{ g(x|\xi)\nabla^2 g(x|\eta) - g(x|\eta)\nabla^2 g(x|\xi)\} = g(\xi|\eta) - g(\eta|\xi)$ // integrate over R

The purpose is to get the RHS as shown, which we would like to claim vanishes. Use Green #2 on the LHS and it becomes

$$LHS = \int_\sigma dS \{ g(x|\xi)\, \partial_n g(x|\eta) - g(x|\eta)\, \partial_n g(x|\xi) \} = 0$$

since both g's vanish on σ, and since we assume no singular behavior of $\partial_n g(x|\text{-})$ on σ. QED.

But if $g(x|\xi) = g(\xi|x)$, then this must be true as well for $v(x,\xi)$ in 6.75, since it is obviously true for E.



So if we put unit charge at B, we get a certain v(A,B). But if we put the unit charge at A, then at point B we see v(B,A). These numbers are the same, v is symmetric. This fact is not really obvious when surface σ is present (on which g = 0). The fact is called the **reciprocity principle of electrostatics**.

We now would like some verification of the above formula. I downloaded a paper that does this, but it uses sin in z and uses the K and I function approach. I think my result above is correct, but no one does it that way because it is easier to use the I and K method since orthog is simpler. So I guess I will just state my result with no verification. It was "the wrong problem" anyway. Now on to the right problem.

Part B: 1/R Green's Function using J_m functions

0. Introduction: Free space Green's function in cylindrical coordinates

The potential of a point charge at the origin is q/R with R = r. If we move the point charge to some other origin **r'**, the potential is q/R where R = |**r** - **r'**|. In either case, the potential is a solution of the Laplace equation (away from the point charge) and can therefore be written in terms of the Laplace "atoms" of whatever coordinate system we want to use. For example, if we think of R as f(**r**) we know we can expand it this way in cylindricals (the notation here means that **r'** is a parameter, **r** is the variable)

$$f(\mathbf{r} \mid \mathbf{r}') = \Sigma_m \int dk\, A_{mk} \exp(-k|z-z'|)\, J_m(k\rho)\, e^{im\varphi}$$

The k spectrum is continuous because there is no finite cylindrical surface on which we have V = 0. The sum Σ_m is over all integers, not just non-negative integers. We know the z stuff as follows. Draw a z-plane at z' which passes through our point charge. On either size of this plane, we must have expo decay as we go to large |z|. There is some discrete spectrum of k determined presumably by the zeros of J_m on some cylindrical boundary, but we leave it just as Σ_k. We are doing one of the "two ways" mentioned above.

Notice that we are parameterizing our two points **r** and **r'** this way

$$\mathbf{r} = (\rho\cos\varphi, \rho\sin\varphi, z) \quad \Rightarrow \quad \hat{\mathbf{r}} = (\rho\cos\varphi, \rho\sin\varphi, z)/\sqrt{\rho^2+z^2}$$

$$\mathbf{r'} = (\rho'\cos\varphi', \rho\sin\varphi', z') \quad \Rightarrow \quad \hat{\mathbf{r}}' = (\rho'\cos\varphi', \rho'\sin\varphi', z')/\sqrt{\rho'^2+z'^2}$$

$$\Rightarrow \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = [\,\rho\rho'(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi') + zz']/[\,\sqrt{\rho^2+z^2}\,\sqrt{\rho'^2+z'^2}\,] = \cos(\gamma)$$

$$\Rightarrow \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = [\,\rho\rho'(\cos(\varphi-\varphi') + zz']/[\,\sqrt{\rho^2+z^2}\,\sqrt{\rho'^2+z'^2}\,] = \cos(\gamma)$$

so that γ is the angle directly between our two vectors **r** and **r'**. But for our cylindrical case, we are really more interested (later) in these facts, simpler than the above,

$$\mathbf{r}\bullet\mathbf{r}' = \rho\rho'\cos(\varphi-\varphi') + zz' \qquad r^2 = \rho^2 + z^2 \qquad r'^2 = \rho'^2 + z'^2$$

$$R^2 = |\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2\,\mathbf{r}\bullet\mathbf{r}' = \rho^2 + z^2 + \rho'^2 + z'^2 - 2\rho\rho'\cos(\varphi-\varphi') - 2\,zz'$$

$$= \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

At this point there are several different things we can do.

1. Use the pillbox condition to find the A_{mn} coefficients.

The general pillbox condition is this (Jackson units)

$$\partial_{q_1} V_i - \partial_{q_1} V_o = 4\pi q\, (h_1/h_2h_3)\, \delta(q_2-q_2')\, \delta(q_3-q_3')$$

where q_1 is the dimension perp to our partition of space into two regions at the point charge. I will assume for the moment that z' > 0 and that "outer" means z > z' which is farther away from the origin, etc. We then have (that is to say, the "inner" region contains the origin)

$$V_i = \Sigma_m \int dk\, A_{mk} \exp(+k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z < z'$$

$$V_o = \Sigma_m \int dk\, A_{mk} \exp(-k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z > z'$$

So $q_1 = z$. We can then take $q_2 = \rho$ and $q_3 = \varphi$ and use these facts

$$h_1 = h_z = 1$$
$$h_2 = h_\rho = 1$$
$$h_3 = h_\varphi = \rho$$

I take these facts from M&M page 178. I really should start with the three Cartesian equations and run my little program to compute the metric tensor, but OK for now. Our pillbox condition is then this:

$$\partial_z V_i - \partial_z V_o = 4\pi q\, (1/\rho)\, \delta(\rho-\rho')\, \delta(\varphi-\varphi')$$

We then need to compute the derivatives

$$\partial_z V_i = \Sigma_m \int dk\, A_{mk}\, k \exp(+k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z < z'$$

$$\partial_z V_o = \Sigma_m \int dk\, A_{mk}\, (-k) \exp(-k(z-z'))\, J_m(k\rho)\, e^{im\varphi} \qquad z > z'$$

then evaluate both on the bounding surface between the two regions (which means z = z')

$$\partial_z V_i = \Sigma_m \int dk\, A_{mk}\, k\, J_m(k\rho)\, e^{im\varphi}$$

$$\partial_z V_o = -\Sigma_m \int dk\, A_{mk} k J_m(k\rho)\, e^{im\varphi}$$

so

$$\partial_z V_i - \partial_z V_o = 2\, \Sigma_m \int dk\, A_{mk}\, k\, J_m(k\rho)\, e^{im\varphi} = 4\pi q\, (1/\rho)\, \delta(\rho-\rho')\, \delta(\varphi-\varphi')$$

We will use two orthogonalities to find the A_{mn}. First, apply $\int d\varphi\, e^{-im'\varphi}$ to both sides and use

$$(1/2\pi) \int_{-\pi}^{\pi} d\varphi\, e^{im\varphi}\, e^{-im'\varphi} = \delta_{mm'} \qquad \text{// orthogonality}$$

The LHS becomes

$$2\,\Sigma_m \int dk\, A_{mk}\, k\, J_m(k\rho) \int d\varphi\, e^{-im'\varphi}\, e^{im\varphi} = 2\,\Sigma_m \int dk\, A_{mk}\, k\, J_m(k\rho)\, 2\pi\delta_{mm'}$$

$$= 4\pi\,\Sigma \int dk\, A_{m'k}\, k\, J_{m'}(k\rho)$$

and the RHS becomes $4\pi q\,(1/\rho)\,\delta(\rho-\rho')\,e^{-im'\varphi'}$ so we then have (replace m' with m)

$$4\pi \int dk\, A_{mk}\, k\, J_m(k\rho) = 4\pi q\,(1/\rho')\,\delta(\rho-\rho')\,e^{-im\varphi'}$$

Now we reach into our prepared "transforms doc" and pull out this rabbit, (Hankel Transform)

$f(\rho) = \int_0^\infty dk\, k\, J_\nu(k\rho)\, F_\nu(k)$ // expansion

$F_\nu(k) = \int_0^\infty dx\, x\, J_\nu(k\rho)\, f(\rho)$ // projection

$\int_0^\infty d\rho\, \rho\, J_\nu(k\rho)\, J_\nu(k'\rho) = \delta(k-k')/k$ // orthogonality

$\int_0^\infty dk\, k\, J_\nu(k\rho)\, J_\nu(k\rho') = \delta(\rho-\rho')/\rho$ // completeness

So, the next step is to apply $\int_0^\infty d\rho\, \rho\, J_m(k'\rho)$ to both sides of the above equation.

$$\text{LHS} = 4\pi \int dk\, A_{mk}\, k \int_0^\infty d\rho\, \rho\, J_m(k'\rho)\, J_m(k\rho) = 4\pi \int dk\, A_{mk}\, k\, \delta(k-k')/k = 4\pi\, A_{mk'}$$

$$\text{RHS} = 4\pi q\,(1/\rho')\,e^{-im\varphi'} \int_0^\infty d\rho\, \rho\, J_m(k'\rho)\delta(\rho-\rho') = 4\pi q\,(1/\rho')\,e^{-im\varphi'}\,\rho'\,J_m(k'\rho') = 4\pi q\,e^{-im\varphi'}\,J_m(k'\rho')$$

The result is this (change k' to k)

$$4\pi\, A_{mk} = 4\pi q\, e^{-im\varphi'}\, J_m(k\rho')$$

$$A_{mk} = q\, e^{-im\varphi'}\, J_m(k\rho')$$

So we now have our coefficients! We can install these into our expansion above

$$V(\mathbf{r} \mid \mathbf{r}') = q/R = \Sigma_m \int dk\, A_{mk} \exp(-k|z-z'|)\, J_m(k\rho)\, e^{im\varphi}$$

$$= \Sigma_m \int dk\, q\, e^{-im\varphi'}\, J_m(k\rho') \exp(-k|z-z'|)\, J_m(k\rho)\, e^{im\varphi}$$

$$= q \ \Sigma_m \int dk \exp(-k|z-z'|) \ J_m(k\rho) \ J_m(k\rho') \ e^{im\varphi} \ e^{-im\varphi'}$$

where Σ_m is over all integers. So here then is our interesting result:

$$1/R = \Sigma_m \int dk \exp(-k|z-z'|) \ J_m(k\rho) \ J_m(k\rho') \ e^{im\varphi} \ e^{-im\varphi'}$$

We know that the result is real, so the imaginary part vanishes, and this can also be written as

$$1/R = \Sigma_m \cos[m(\varphi-\varphi')] \int dk \exp(-k|z-z'|) \ J_m(k\rho) \ J_m(k\rho')$$

We can at this point do the usual "fold over" operation. We know $J_{-m}(z) = (-1)^m J_m(z)$ so everything is even in m. Thus, the negative sum will equal the positive sum, and we get at once

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m \ \cos[m(\varphi-\varphi')] \int dk \exp(-k|z-z'|) \ J_m(k\rho) \ J_m(k\rho')$$

This of course has the expected symmetric form under $\mathbf{r} \leftrightarrow \mathbf{r}'$.

We now want some verification of the above formula. Jackson on page 86 gives 1/R "the other way" which I am not doing right now, so we need to look elsewhere.

Smythe? On page 194 he does the I and K version, similar to Jackson. Does not give the above.

MF? On page 1263 they have it! I copy:

$$\frac{1}{R} = \sum_{m=0}^{\infty} \epsilon_m \cos[m(\phi - \phi_0)] \int_0^{\infty} J_m(kr) J_m(kr_0) e^{-k|z-z_0|} \, dk \quad (10.3.23)$$

Hurray! By the way, the toroidal Q function version is mentioned here

http://en.wikipedia.org/wiki/Green%27s_function_for_the_three-variable_Laplace_equation

2. Development of the Addition Theorem

Now, what happens if we put our point charge on the z axis? Then

$\mathbf{r}' = (0,0 ,z')$

We have then that $J_m(k\rho') = J_m(0) = \delta_{m,0}$ [Jackson p 72 or Schaum p 136] so we have

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m \ \cos[m(\varphi-\varphi')] \int dk \exp(-k|z-z'|) \ J_m(k\rho) \ \delta_{m,0}$$

$$= \int dk \exp(-k|z-z'|) \ J_0(k\rho)$$

which is a pretty simple result indeed. But in this case we also know that $1/R = 1/\sqrt{\rho^2 + (z-z')^2}$. So we have this interesting result

$$1/\sqrt{\rho^2 + (z-z')^2} = \int_0^\infty dk \exp(-k|z-z'|) J_0(k\rho) \qquad (*)$$

which is better stated as

$$1/\sqrt{\rho^2 + z^2} = \int_0^\infty dk \exp(-k|z|) J_0(k\rho)$$

Can I verify this result? GR p 694 says

6.61 Combinations of Bessel functions and exponentials

6.611

1. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, dx = \frac{\beta^{-\nu}\left[\sqrt{\alpha^2+\beta^2}-\alpha\right]^\nu}{\sqrt{\alpha^2+\beta^2}} \qquad [\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,(\alpha \pm i\beta) > 0]$$

EH II 49(18), WA 422(8)

So set $\nu = 0$ and $x = k$ and $\beta = \rho$ and $\alpha = |z|$ and only the denominator survives, QED. This result is similar to Jackson p 86 3.150.

Now I need something like my little Theorem 1 of the spherical 1/R document? Well, here it is simpler than that. Our equality (*) is valid for any quantity $\rho>0$ you want. To stress that point, write

$$1/\sqrt{a^2 + (z-z')^2} = \int_0^\infty dk \exp(-k|z-z'|) J_0(ka)$$

Now recall from above that we have

$$R^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

Therefore, if we set

$$a^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi') \qquad \text{// which is also a geometric law of cosines}$$

we then have

$$\sqrt{a^2 + (z-z')^2} = R$$

so we then have

$$1/R = \int_0^\infty dk \exp(-k|z-z'|) J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')})$$

This, then, is our result analogous to the sphericals result

$$1/R = \Sigma_n \ (r_<)^n (r_>)^{-n-1} \ P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))$$

But now consider that our more general 1/R result from above is this: (one form of it)

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m \ \cos[m(\varphi-\varphi')] \ \int dk \ \exp(-k|z-z'|) \ J_m(k\rho) \ J_m(k\rho')$$

Setting things equal we must have

$$\Sigma_{m=0}^{\infty} \varepsilon_m \ \cos[m(\varphi-\varphi')] \ \int dk \ \exp(-k|z-z'|) \ J_m(k\rho) \ J_m(k\rho')$$

$$= \int_0^{\infty} dk \ \exp(-k|z-z'|) \ J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')} \ )$$

Although exp(-k|z-z'|) functions are not really a complete set, we suspect these things can only really be equal for all values of z and z' if the following is true

$$J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')} \ ) \ = \ \Sigma_{m=0}^{\infty} \varepsilon_m \ \cos[m(\varphi-\varphi')] \ J_m(k\rho) \ J_m(k\rho')$$

Verification for this addition theorem comes from GR7 page 940,

1. $$J_0(mR) = J_0(m\varrho)\, J_0(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho)\, J_k(mr) \cos k\varphi$$ WA 391(1)

with $$R = \sqrt{r^2 + \varrho^2 - 2r\varrho\cos\varphi};$$

The variable names are very skewed, but we see it is exactly the same. The connection is this:

GR7	me
m	k
ρ	ρ
r	ρ'
k	m
φ	φ-φ'

This addition theorem is similar to 3.151, but his is for the I K world. I will be doing that soon. I am sure this addition theorem also has group theory support, but I have never read that stuff. On the list!

In the spherical case, we had $P_n(\cos\gamma)$ where γ was the angle between **r** and **r'**. Here we have in place of that our $J_0(ka)$ where a is the distance between the projections of **r** and **r'** onto the same z plane. No doubt this is an invariant of some symmetry operation the underlies the group interpretation.

Part C: 1/R Green's Function using K_m functions

0. Introduction: Free space Green's function in cylindrical coordinates

In this case, our two spaces will be an inside cylinder, and the complementary outside space. The boundary is a cylindrical surface which contains our Green's charge. The atomic form is this

$$f(\mathbf{r} \mid \mathbf{r}') = q/R = \Sigma_m \int_0^\infty dk\, A_{mk} \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)\, e^{im\varphi}$$

where m is over all integers. This is a "Smythian form" in that I have built in the known symmetry for both z,z' and ρ,ρ'. Since even under swap symmetry, we cannot have a sin[k(z-z')] term. And of course I is the function good near ρ = 0, etc.

Notice that we are parameterizing our two points **r** and **r'** this way

$$\mathbf{r} = (\rho\cos\varphi, \rho\sin\varphi, z) \quad \Rightarrow \quad \hat{\mathbf{r}} = (\rho\cos\varphi, \rho\sin\varphi, z)/\sqrt{\rho^2+z^2}$$

$$\mathbf{r}' = (\rho'\cos\varphi', \rho\sin\varphi', z') \quad \Rightarrow \quad \hat{\mathbf{r}}' = (\rho'\cos\varphi', \rho'\sin\varphi', z')/\sqrt{\rho'^2+z'^2}$$

$$\Rightarrow \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = [\,\rho\rho'(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi') + zz']/[\,\sqrt{\rho^2+z^2}\,\sqrt{\rho'^2+z'^2}\,] = \cos(\gamma)$$

$$\Rightarrow \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = [\,\rho\rho'(\cos(\varphi-\varphi') + zz']/[\,\sqrt{\rho^2+z^2}\,\sqrt{\rho'^2+z'^2}\,] = \cos(\gamma)$$

so that γ is the angle directly between our two vectors **r** and **r'**. But for our cylindrical case, we are really more interested (later) in these facts, simpler than the above,

$$\mathbf{r}\bullet\mathbf{r}' = \rho\rho'\cos(\varphi-\varphi') + zz' \qquad r^2 = \rho^2 + z^2 \qquad r'^2 = \rho'^2 + z'^2$$

$$R^2 = |\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2\,\mathbf{r}\bullet\mathbf{r}' = \rho^2 + z^2 + \rho'^2 + z'^2 - 2\rho\rho'\cos(\varphi-\varphi') - 2zz'$$

$$= \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

At this point there are several different things we can do.

1. Use the pillbox condition to find the A_{mn} coefficients.

The general pillbox condition is this (Jackson units)

$$\partial_{q_1} V_i - \partial_{q_1} V_o = 4\pi q\,(h_1/h_2h_3)\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

where q_1 is the dimension perp to our partition of space into two regions at the point charge. In our current application, $q_1 = \rho$ so we make these choices

$h_2 = h_z = 1$
$h_1 = h_\rho = 1$
$h_3 = h_\varphi = \rho$

I take these facts from M&M page 178. I really should start with the three Cartesian equations and run my little program to compute the metric tensor, but OK for now. Our pillbox condition is then this:

$$\partial_\rho V_i - \partial_\rho V_o = 4\pi q\, (1/\rho)\, \delta(z-z')\, \delta(\varphi-\varphi')$$

We then need to compute the derivatives

$$V_i = \Sigma_m \int_0^\infty dk\, A_{mk} \cos[k(z-z')]\, I_m(k\rho)\, K_m(k\rho')\, e^{im\varphi}$$

$$V_o = \Sigma_m \int_0^\infty dk\, A_{mk} \cos[k(z-z')]\, I_m(k\rho')\, K_m(k\rho)\, e^{im\varphi}$$

$$\partial_\rho V_i = \Sigma_m \int_0^\infty dk\, k\, A_{mk} \cos[k(z-z')]\, I_m(k\rho)'\, K_m(k\rho')\, e^{im\varphi}$$

$$\partial_\rho V_o = \Sigma_m \int_0^\infty dk\, k\, A_{mk} \cos[k(z-z')]\, I_m(k\rho')\, K_m(k\rho)'\, e^{im\varphi}$$

then evaluate both on the bounding surface between the two regions (which means $\rho' = \rho$)

$$\partial_\rho V_i = \Sigma_m \int_0^\infty dk\, k\, A_{mk} \cos[k(z-z')]\, I_m(k\rho)'\, K_m(k\rho)\, e^{im\varphi}$$

$$\partial_\rho V_o = \Sigma_m \int_0^\infty dk\, k\, A_{mk} \cos[k(z-z')]\, I_m(k\rho)\, K_m(k\rho)'\, e^{im\varphi}$$

so

$$\partial_\rho V_i - \partial_\rho V_o = \Sigma_m \int_0^\infty dk\, k\, A_{mk} \cos[k(z-z')]\, W[K_m(k\rho), I_m(k\rho)]\, e^{im\varphi}$$

$$= \Sigma_m \int_0^\infty dk\, k\, A_{mk} \cos[k(z-z')]\, (k\rho)^{-1}\, e^{im\varphi} \qquad \text{// Jackson p 86}$$

$$= (1/\rho)\, \Sigma_m \int_0^\infty dk\, A_{mk} \cos[k(z-z')]\, e^{im\varphi}$$

The pillbox condition is therefore,

$$(1/\rho)\, \Sigma_m \int_0^\infty dk\, A_{mk} \cos[k(z-z')]\, e^{im\varphi} = 4\pi q\, (1/\rho)\, \delta(z-z')\, \delta(\varphi-\varphi')$$

or

$$\Sigma_m \int_0^\infty dk\, A_{mk} \cos[k(z-z')]\, e^{im\varphi} = 4\pi q\, \delta(z-z')\, \delta(\varphi-\varphi') \qquad \text{LHS = RHS}$$

We will use two orthogonalities to find the A_{mn}. First, apply $\int d\varphi\, e^{-im'\varphi}$ to both sides and use

$$(1/2\pi) \int_{-\pi}^{\pi} d\varphi\, e^{im\varphi}\, e^{-im'\varphi} = \delta_{mm'} \qquad \text{// orthogonality}$$

The LHS becomes

$$\Sigma_m \int_0^{\infty} dk\, A_{mk} \cos[k(z-z')] \int d\varphi\, e^{-im'\varphi}\, e^{im\varphi} = \Sigma_m \int_0^{\infty} dk\, A_{mk} \cos[k(z-z')]\, 2\pi\, \delta_{mm'}$$

$$= 2\pi \int_0^{\infty} dk\, A_{m'k} \cos[k(z-z')]$$

and the RHS becomes $4\pi q\, \delta(z-z')\, e^{-im'\varphi'}$ so we then have (replace m' with m)

$$2\pi \int_0^{\infty} dk\, A_{mk} \cos[k(z-z')] = 4\pi q\, \delta(z-z')\, e^{-im\varphi'}$$

$$\int_0^{\infty} dk\, A_{mk} \cos[k(z-z')] = 2q\, \delta(z-z')\, e^{-im\varphi'}$$

or

$$\int_0^{\infty} dk\, A_{mk} \cos(kz) = 2q\, \delta(z)\, e^{-im\varphi'} \qquad \text{LHS = RHS}$$

Now we reach into our prepared "transforms doc" and pull out this rabbit, (Cosine Integral Transform)

Summary Case 2: For z in $(-\infty,\infty)$ and k in $(0,\infty)$ where f(z) is even in z:

$$f(z) = \sqrt{1/\pi} \int_0^{\infty} dk\, f^k \cos(kz) \qquad \text{// expansion pair 2}$$

$$f^k = \sqrt{1/\pi} \int_{-\infty}^{\infty} dz\, f(z) \cos(kzx) \qquad \text{// projection}$$

$$\int_{-\infty}^{\infty} dz \cos(kz) \cos(k'z) = \pi\, \delta(k-k') \qquad \text{// orthogonality}$$

$$\int_0^{\infty} dk \cos(kz) \cos(kz') = (\pi/2)[\, \delta(z-z') + \delta(z+z')] \qquad \text{// "completeness"}$$

So, the next step is to apply $\int_{-\infty}^{\infty} dz \cos(k'z)$ to both sides of the above equation LHS = RHS:

$$\text{LHS} = \int_0^{\infty} dk\, A_{mk} \int_{-\infty}^{\infty} dz \cos(k'z) \cos(kz) = \int_0^{\infty} dk\, A_{mk}\, \pi\, \delta(k-k') = \pi\, A_{mk'}$$

$$\text{RHS} = 2q \int_0^{\infty} dz \cos(k'z)\delta(z)\, e^{-im\varphi'} = 2q\, e^{-im\varphi'}$$

The result is this (change k' to k)

$$\pi A_{mk} = 2q\, e^{-im\varphi'}$$

$$A_{mk} = (2/\pi)q\, e^{-im\varphi'}$$

So we now have our coefficients! We can install these into our expansion above

$$f(\mathbf{r} \mid \mathbf{r}') = q/R = \Sigma_m \int_0^\infty dk\, A_{mk} \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)\, e^{im\varphi}$$

$$= (2/\pi)q\, \Sigma_m \int_0^\infty dk \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)\, e^{im\varphi}\, e^{-im\varphi'}$$

with the result then that

$$1/R = (2/\pi)\, \Sigma_m \int_0^\infty dk \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)\, e^{im\varphi}\, e^{-im\varphi'}$$

where as noted above m is summed over all integers. We get immediate verification of this result from Jackson page 86 (3.148). Bateman page 5 shows that $K_{-\nu} = K_\nu$ for general ν (page 5), and $I_{-m} = I_m$ for integer m only. This means that, if we first replace the expos with $\cos[m(\varphi-\varphi')]$, we can reflect the negative side of the series in the usual way, and we then get this alternate form:

$$1/R = (2/\pi)\, \Sigma_{m=0}^\infty \varepsilon_m \int_0^\infty dk \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>) \cos[m(\varphi-\varphi')]$$

and this appears as Jackson (3.149). As usual, ε_m is the Neumann factor which is 2 for $m > 0$ else 1.

2. Development of the Addition Theorem

Now, what happens if we put our point charge on the z axis? Then

$$\mathbf{r}' = (0,0,z')$$

Since $\rho' = 0$, surely $\rho' = \rho_<$ so we have in this case

$$1/R = (2/\pi)\, \Sigma_{m=0}^\infty \varepsilon_m \int_0^\infty dk \cos[k(z-z')]\, I_m(0)\, K_m(k\rho) \cos[m(\varphi-\varphi')]$$

and as before we get $I_m(0) = \delta_{m0}$ so this thing shrinks down to

$$1/R = (2/\pi) \int_0^\infty dk \cos[k(z-z')]\, K_0(k\rho)$$

which is a pretty simple result indeed. But in this case we also know that $1/R = 1/\sqrt{\rho^2 + (z-z')^2}$. So we have this interesting result

$$1/\sqrt{\rho^2 + (z-z')^2} = (2/\pi) \int_0^\infty dk \cos[k(z-z')] K_0(k\rho) \qquad (*)$$

which is better stated as

$$1/\sqrt{\rho^2 + z^2} = (2/\pi) \int_0^\infty dk \cos(kz) K_0(k\rho)$$

Can I verify this result? Jackson has it on page 86 as (3.150). Also GR7 page 719

$$14.^8 \quad \int_0^\infty K_0(\beta x) \cos \alpha x \, dx = \frac{\pi}{2\sqrt{\alpha^2 + \beta^2}} \qquad [\alpha > 0] \qquad \text{WA 425(10)a, MO 48}$$

and there it is.

Our equality (*) is valid for any quantity $\rho>0$ you want. To stress that point, write

$$1/\sqrt{a^2 + (z-z')^2} = (2/\pi) \int_0^\infty dk \cos[k(z-z')] K_0(ka)$$

Now recall from above that we have

$$R^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

Therefore, if we set

$$a^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi') \qquad \text{// which is also a geometric law of cosines}$$

we then have

$$\sqrt{a^2 + (z-z')^2} = R$$

so we then have

$$1/R = (2/\pi) \int_0^\infty dk \cos[k(z-z')] K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')})$$

This, then, is our result analogous to the sphericals result

$$1/R = \Sigma_n \ (r_<)^n (r_>)^{-n-1} \ P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))$$

But now consider that our more general 1/R result from above is this: (one form of it)

$$1/R = (2/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \int_0^{\infty} dk \cos[k(z-z')] I_m(k\rho_<) K_m(k\rho_>) \cos[m(\varphi-\varphi')]$$

Setting things equal we must have

$$(2/\pi) \int_0^{\infty} dk \cos[k(z-z')] K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')})$$

$$= (2/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \int_0^{\infty} dk \cos[k(z-z')] I_m(k\rho_<) K_m(k\rho_>) \cos[m(\varphi-\varphi')]$$

Although cos[k(z-z')] functions are maybe not really a complete set, we suspect these things can only really be equal for all values of z and z' if the following is true

$$K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}) = \Sigma_{m=0}^{\infty} \varepsilon_m I_m(k\rho_<) K_m(k\rho_>) \cos[m(\varphi-\varphi')]$$

and this is our addition theorem, verified as Jackson p 86 (3.151).

Finally, in Part D I summarize all results of this doc, for Parts A,B and C.

Part D: Summary of All Results

Part A : The Green's Function for a point charge q located at **r'** inside an **infinitely** long conducting **cylinder** of radius b can be written as,

$$g(\mathbf{r} \mid \mathbf{r'}) = q\,(2/b^2)\ \Sigma_m\ \Sigma_n\ [J_{m+1}(bk_n)]^{-2}\,(1/k_n)\exp(-k_n|z-z'|)\ J_m(k_n\rho)\,J_m(k_n\rho')\,e^{im\varphi}\,e^{-im\varphi'}$$

The literature usually provides an alternate formula using I and K functions which I could, but did not, derive in this doc. I did the harder way. $\beta_n^{(m)} = bk_n$ are the zeros of $J_m(k_n b) = 0$ which are enumerated by the index n = 1,2... so we have really $k_n = k_n^{(m)} = \beta_n^{(m)}/b$. The m sum is over all integers. Using methods shown above, this result can be written by replacing the two exponentials with cos[m(φ-φ')]. Once this is done, these same methods show that the negative half of the series can be reflected and we get the "all real" result:

$$g(\mathbf{r} \mid \mathbf{r'}) = q\,(2/b^2)\ \Sigma_{m=0}^{\infty}\ \varepsilon_m\ \cos[m(\varphi-\varphi')]\ \Sigma_n\ [J_{m+1}(bk_n)]^{-2}\,(1/k_n)\exp(-k_n|z-z'|)\ J_m(k_n\rho)\,J_m(k_n\rho')$$

Because I did it the hard way, I was unable to verify these results against any other sources. The results do in fact show the required symmetry

Part B, including the $Q_{m-1/2}$ expansion :

The free-space 1/R Green's function for a unit point charge can be written as

$$1/R = \Sigma_m\, e^{im\varphi}\, e^{-im\varphi'} \int_0^\infty dk\, \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')$$

where the m sum is over all integers, and the symmetry is manifest. As in part A, we can write this in its real form as follows:

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m\, \cos[m(\varphi-\varphi')] \int_0^\infty dk\, \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')$$

This is verified in MF as noted above. The related addition theorem formulas are these:

$$1/\sqrt{\rho^2 + z^2} = \int_0^\infty dk\, \exp(-k|z|)\, J_0(k\rho)$$

$$1/R = \int_0^\infty dk\, \exp(-k|z-z'|)\, J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,)$$

$$J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) = \Sigma_{m=0}^{\infty} \varepsilon_m\, \cos[m(\varphi-\varphi')]\, J_m(k\rho)\, J_m(k\rho')$$

The first and third are verified in GR7, while the second is a restatement of the first and must be true as well, so I regard it as verified.

In passing, I want to mention the Selvaggi result which I derived in my "integral doc" which is this:

$$1/R = 1/(\pi\sqrt{\rho\rho'}\,)\, \Sigma_{m=0}^{\infty} \varepsilon_m\, \cos[m(\varphi-\varphi')]\, Q_{m-1/2}(\beta) \qquad \beta = [\, \rho^2 + \rho'^2 + (z-z')^2\,]/(2\rho\rho')$$

This result was derived (by me) by doing a direct Cosine Fourier Series expansion of 1/R

$$1/R \equiv 1/|\mathbf{r}-\mathbf{r}'| = 1/\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi') + (z-z')^2}$$

So we can compare these two results for 1/R:

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m\, \cos[m(\varphi-\varphi')] \int_0^\infty dk\, \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')$$

$$1/R = 1/(\pi\rho\rho')\, \Sigma_{m=0}^{\infty} \varepsilon_m\, \cos[m(\varphi-\varphi')]\, Q_{m-1/2}(\beta) \qquad \beta = [\, \rho^2 + \rho'^2 + (z-z')^2\,]/(2\rho\rho')$$

The following would seem then to be true:

$$\int_0^\infty dk\, \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho') = Q_{m-1/2}(\beta)\, /\, (\pi\sqrt{\rho\rho'}\,)$$

Lo and behold, this result appears in GR page 696

3. $$\int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, J_\nu(\gamma x)\, dx = \frac{1}{\pi\sqrt{\gamma\beta}}\, Q_{\nu-\frac{1}{2}}\left(\frac{\alpha^2+\beta^2+\gamma^2}{2\beta\gamma}\right)$$
$$\left[\mathrm{Re}\,(\alpha \pm i\beta \pm i\gamma) > 0, \quad \gamma > 0, \quad \mathrm{Re}\,\nu > -\tfrac{1}{2}\right] \quad \text{WA 426(2), ET II 50(17)}$$

where x = k, β = ρ, γ = ρ', ν = m, α = |z-z'|. This $Q_{m-1/2}$ expansion is still in terms of cylindrical coordinates, we don't see any toroidal coordinates appearing here (except φ). Maybe at some point I will understand *why* a toroidal harmonic appears in this cylindrical problem. An opportunity will come soon when I do 1/R for toroidal coordinates, on the menu.

Part AB: Suppose we wanted to know the potential of just the induced charge in our Part A cylinder problem. We could find this by subtracting out the potential of the point charge. Thus,

$$V(r|r')_{\text{due to induced charge}}$$

$$= (2/b^2)\ \Sigma_{m=0}^{\infty}\varepsilon_m \cos[m(\varphi-\varphi')]\ \ \Sigma_{n=1}^{\infty}\ [J_{m+1}(bk_n)]^{-2}\,(1/k_n)\exp(-k_n|z-z'|)\ J_m(k_n\rho)\,J_m(k_n\rho')$$

$$-\ \Sigma_{m=0}^{\infty}\varepsilon_m \cos[m(\varphi-\varphi')]\ \ \int dk \exp(-k|z-z'|)\,J_m(k\rho)\,J_m(k\rho')$$

The two terms cannot be very will combined as they can in other problems, here because one is discrete and the other continuous.

Part C :

The free-space 1/R Green's function for a unit point charge can be written as

$$1/R = (2/\pi)\,\Sigma_m \int_0^\infty dk \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)\, e^{im\varphi}\, e^{-im\varphi'}$$

where the m sum is over all integers, and the symmetry is manifest. As in part A, we can write this in its real form as follows:

$$1/R = (2/\pi)\,\Sigma_{m=0}^{\infty}\,\varepsilon_m \int_0^\infty dk \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>) \cos[m(\varphi-\varphi')]$$

Both these forms are verified in Jackson page 86. The related addition theorem formulas are these:

$$1/\sqrt{\rho^2+z^2}\ = (2/\pi)\ \int_0^\infty dk \cos(kz)\, K_0(k\rho)$$

$$1/R\ = (2/\pi)\ \int_0^\infty dk \cos[k(z-z')]\, K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}\ \right)$$

$$K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}\ \right) = \Sigma_{m=0}^{\infty}\,\varepsilon_m\, I_m(k\rho_<)\, K_m(k\rho_>)\cos[m(\varphi-\varphi')]$$

These results are also verified on green Jackson page 86.

Part D (spherical results): Let's just add here the corresponding spherical coordinate results. The Green's function for a unit point charge is given by

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_m f(n,-m) (r_<)^n (r_>)^{-n-1} P_n^m(z) P_n^m(z') e^{im\varphi} e^{-im\varphi'}$$

where Σ_{nm} is the conventional spherical double sum. One can regard the m sum as being over all integers with the cutoff being provided by the P functions. This result is confirmed by Jackson as shown in my 1/R spherical doc when converted to his Y functions. In the usual way, we can reflect the negative m contributions and write this as

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} \varepsilon_m f(n,-m) (r_<)^n (r_>)^{-n-1} P_n^m(z) P_n^m(z') \cos[m(\varphi-\varphi')]$$

I don't know where this is stated in this manner but feel certain it is correct. The addition theorem related equations are these:

$$1/R = \Sigma_n (r_<)^n (r_>)^{-n-1} P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))$$

$$P_n[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] = \Sigma_{m=0}^{\infty} \varepsilon_m f(n,-m) P_n^m(\cos\theta) P_n^m(\cos\theta') \cos[m(\varphi-\varphi')]$$

The first line is verified Jackson page 62. The second from Jackson p 69, also Bateman p 169 and GR7.