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Jackson Section 3.6 Laplace in cylindricals: Bessel Functions

I start of course with Green Jackson. The subject first comes up on page 69 where he writes the Laplace in cylindricals, separates, and takes the z and φ solutions to be $e^{\pm kz}$ and $e^{\pm i\nu\varphi}$. The ρ equation has k and ν as separation constants then as in (3.75) p 69. Variable change to x = ρk gives 3.77 as the new radial equation in which only ν appears, and he then talks about $J_\nu(\rho k)$ as the solution, and we then get mention of the N and Hankel solutions as well. The Hankel in 3.86 are complex, the others are real for real parameters. We then get recursion relations and large and small argument forms for J and N on page 72, then a discussion of the zeros of J.

Then on page 73 he asks a good question: What is the orthogonality situation for these Bessel functions? Jackson pours out a long page of math, but I want to translate this into standard SL language.

The Bessel ODE eigenvalue problem.

I am encountering a riptide of confusion about what the Bessel ODE is, how it appears in Jackson, how it appears in Schaum, how it fits the Stakgold mold of a self-adjoint regular BC problem, of the variable change, of the EV problem and so on. Where to start to detangle this mess?

(a) Some forms of the Bessel ODE

Schaum is as good as any starting point. He has the **Frobenius standard form**

$$L_{bess1} = x^2\partial_x^2 + x\partial_x + (x^2-n^2) = x\partial_x(x\partial_x) + (x^2-n^2)$$

which more generally we write as

$L = (x-a)^2D^2 + (x-a)P_a\, D + Q_a$

in which a = 0 so we have

$L_{bess1} = x^2D^2 + xP_0\, D + Q_0$ with $P_0(x) = 1$ and $Q_0(x) = (x^2-n^2)$

We can divide by x^2 to arrive at what I like to call the **canonical form** (no leading function)

$L_{bess2} = \partial_x^2 + (1/x)\partial_x + (1-n^2/x^2) = D^2 + p(x)\, D + q(x)$ $p(x) = 1/x$ $q(x) = (1-n^2/x^2)$

Here p has a single pole and q a double pole, so x = 0 is a **regular singular point**. There is also an **irregular singular point** at x = ∞ as my "ODE notes1.doc" shows.

If we divide L_{bess1} by -x instead of x^2 we get

$L_{bess3} = (-1/x)\, L_{bess1} = -x\partial_x^2 - \partial_x + (n^2/x - x)) = -\partial_x(x\partial_x) + (n^2/x - x)$

$= -\partial_x(p(x)\partial_x) + q(x)$ with $p(x) = x$ $q(x) = (n^2/x - x)$

and this is **Stakgold standard form** for a regular BV problem, see page 268. The solutions of all these forms are of course the same Bessel functions like $J_n(x)$. Since we have Stak standard form, we know that L_{bess3} is self adjoint.

(b) Constructing a SL problem: the "scaled Bessel equation"

We could at this point try to impose some homo BC's such as u(0) = 0 and u(b) = 0 and then we have a self-adjoint ODE system whose EV equation $L_{bess3}\, u = \lambda u$ will have orthogonal eigenfunctions. In Stak we still (seem to) have the freedom to choose our weight function s(x) and then we can form $L_\lambda = L_{bess3} - \lambda s(x)$. But how then do we find the eigenfunctions? One idea is to *try* this, setting n = ν,

$u_i(x) = J_\nu(x\, [x_{\nu,i}/b])$ // parameter ν is a fixed bystander parameter

As long as Re(ν) > 0, Jackson p 72 tells us that $J_\nu(0) = 0$, so we meet the left BC. Since $J_\nu(x)$ is oscillatory, it has an infinite number of zeros which we call $x_{\nu,i}$ for i = 1,2,3.... and $J_\nu(x_{\nu,i}) = 0$. This then shows that our $u_i(b) = 0$ as well. So we have some candidate SL eigenfunctions, but something tricky has just happened before our very eyes:

NOTICE: if we decide to use the functions $u_i(x)$, we are no longer working with $J_\nu(x)$, we are working with J_ν(constant times x). So the $u_i(x)$ don't solve the Bessel equation as stated above, this is the point I was failing to see.

Suppose we generically write $u(x) = J_\nu(ax)$ where a is some constant. We have rescaled the variable. Suppose we start with

$L_{bess1(x)} = x^2\partial_x^2 + x\partial_x + (x^2-n^2) = x\partial_x(x\partial_x) + (x^2-n^2)$ $\qquad L_{bess1}\, J_n(x) = 0$

Then suppose we define x = at. Then we can write:

$L_{bess1(at)} = L_{bess1'(t)} = t^2\partial_t^2 + t\partial_t + (a^2t^2-n^2) = t\partial_t(t\partial_t) + (a^2t^2-n^2)$ $\qquad L_{bess1'}\, J_n(at) = 0$

All I did was replace x with at everywhere. The only appearance of a is then in the third term. We can then rename things x again and have

$L_{bess1'(x)} = x^2\partial_x^2 + x\partial_x + (a^2x^2-n^2) = x\partial_x(x\partial_x) + (a^2x^2-n^2)$ $\qquad L_{bess1'}\, J_n(ax) = 0$

This is NOT the Bessel operator $L_{bess1(x)}$, which is why I have been writing it with the little prime.

$[\, x\partial_x(x\partial_x) + (a^2x^2-n^2)]\, J_n(ax) = 0$ $\qquad$ ie $\qquad L_{bess1'}\, J_n(ax) = 0$

It is not the actual Bessel equation because it does not act on $J_n(x)$ to give 0. So what name do we give to this ODE? Let's call this **"the scaled Bessel equation"**. Our proposed eigenfunctions above have this form where $a = x_{\nu,i}/b$. We are forced to do this scaling in order to get $u_i(x) = J_\nu(x\,[x_{\nu,i}/b])$ with $u_i(0) = u_i(b) = 0$ so that we *have* a SL problem! [Some people refer to the above equation with $a^2 = \lambda$ as "the Bessel equation" or as the "ordinary Bessel equation".]

Now let's write our scaled Bessel equation in all three forms we know about:

$L_{bess1'} = x^2\partial_x^2 + x\partial_x + (a^2x^2-n^2) = x\partial_x(x\partial_x) + (a^2x^2-n^2)$ $\qquad$ Frobenius

$L_{bess2'} = \partial_x^2 + (1/x)\partial_x + (a^2-n^2/x^2)$ $\qquad$ Canonical

$L_{bess3'} = -x\partial_x^2 - \partial_x + (n^2/x - a^2x)) = -\partial_x(x\partial_x) + (n^2/x - a^2x)$ $\qquad$ Stakgold

(c) Statement of the Eigenvalue Equation : Orthogonality

We are associating $a_i = x_{\nu,i}/b$ with the eigenvalue "i" of our SL problem. Let's see this more explicitly. We have at this point that

$L_{bess3'}\, J_\nu(ax) = 0$

But now let's put the a^2x term on the RHS and define yet another L operator:

$L_{bess3''} \equiv -\partial_x(x\partial_x) + n^2/x$ $\qquad L_{bess3''}\, J_\nu(ax) = a^2\, x\, J_\nu(ax)$

We can make this identification with Stakgold page 268

$L = L_{bess3''}$

$L_\lambda = L_{bess3''} - \lambda s(x)$ $\qquad \lambda = a^2$ $\qquad s(x) = x$

$L_{\lambda}\, J_{\nu}(ax) = 0$

So when the dust has all settled, THIS is the L operator that interests us: $\nu \equiv n$

$L = -\partial_{x}(x\partial_{x}) + \nu^{2}/x$

and *this* is our eigenvalue equation

$L\, J_{\nu}(ax) = \lambda s(x)\, J_{\nu}(ax) = a^{2}x\, J_{\nu}(ax)$

and FINALLY we see where that weight function is coming from!

Recall how we did this same idea with the Legendre equation, but without the scaling. We wrote the Legendre ODE and it has a term n(n+1) in it. We regard this as λ and put it on the RHS, and the residual operator L is the Legendre operator without this term. It is the operator whose eigenvalues are n(n+1). Just so, in our Bessel case that operator is $L = -\partial_{x}(x\partial_{x}) + \nu^{2}/x$.

Now that we have made the contact with Stakgold, we know at once that this will be true:

$(u_{i}, u_{j}) = \delta_{i,j}\, K_{\nu,i}$ or $\int_{0}^{b} dx\, x\, J_{\nu}(x\,[x_{\nu,i}/b])\, J_{\nu}(x\,[x_{\nu,j}/b]) = \delta_{i,j}\, K_{\nu,i}$

When we write our Bessel eigenvalue equation as $L\, J_{\nu}(ax) = a^{2}x\, J_{\nu}(ax)$, we finally "see" a statement of our SL eigenvalue problem, and the weight s(x) = x is sitting right there staring at you, and $\lambda = a^{2}$. You don't get to "choose" the weight function, it is part of our little ODE system developed here. It comes from using those zeros of the Bessel functions to realize the homogeneous BC u(b) = 0 at the right end of our interval (0,b). Of course Jackson repeats Stakgold's more general processing with the ODE and shows that this weight function must be there.

Now this quote from a web PDF makes more sense, where $\lambda = a^{2}$:

> This sections summarizes the normalization of the Bessel functions (refer to Brown and Churchill for more details). The normalization depends critically on the boundary conditions for the problem. As discussed in previous sections, the singular Sturm-Liouville equation giving rise to the Bessel functions $L_{n}R_{\lambda}(\rho) = -\lambda\rho R_{\lambda}(\rho)$ where ρ is the radial coordinate, "n=0,1,...", and $0 \le \rho \le \rho_{o}$ can be written as
>
> $$\left[\rho R_{\lambda}'\right]' - \frac{n^{2}}{\rho}R_{\lambda} = -\lambda\rho R_{\lambda}$$

In my notation this last equation says $-L\, J_{\nu}(ax) = -a^{2}x\, J_{\nu}(ax)$. This author has used the non-Stakgold sign for the L operator which is fine, and is using the traditional ρ name for the variable.

So we get back to Jackson finally where he uses the following names for things:

me	Jackson

$x_{\nu,i}$	i=1,2,3	$x_{\nu n}$	n = 1,2,3...
x		ρ	
b		a	
$a = \sqrt{\lambda}$		$x_{\nu n}/a$	

The annoying thing is that Jackson never *states* the eigenvalue equation $L\, J_\nu(a\rho) = \lambda\, \rho\, J_\nu(a\rho)$, and he never even *mentions* that this is a Sturm-Liouville problem. He maintains this silence through the red edition. So I think I have finally detangled things.

(d) Completeness and associated Transforms

What did Stak have to say about this? I find that page 305 vol I has some good Stak stuff. Stak has λ where I put a^2 above, perfect. See Chapter 4 meta notes page 12 for my summary of this Stak example. Whether LP or LC depends on value of ν it turns out. The properly normalized eigenfunctions are shown p 307 J, and completeness is shown in p 307 I.

Then page 308 is the **Fourier Bessel Transform** which I will now write out:

projection: $$f_{k,\nu} = \int_0^1 dx\, \sqrt{x}\, f(x)\, J_\nu(x_{\nu,k} x)$$

recovery $$f(x) = 2 \sum_{k=1}^{\infty} \sqrt{x}\, f_{k,\nu}\; J_\nu(x_{\nu,k} x)/[\, J_\nu'(x_{\nu,k})]^2$$

I think to generalize to interval (0,b) you would just replace x with x/b in all the J's, my guess. (there is another factor, see below)

Now back to Jackson p 73-4. He has this: [note that $J_\nu(x_{\nu,k}) = 0$]

recovery $$f(x) = \sum_{k=1}^{\infty} A_{k,\nu}\; J_\nu(x_{\nu,k} x/a) \qquad (3.96)$$

projection: $$A_{k,\nu} = 2/[a^2 J_{\nu+1}(x_{\nu,k})^2] \int_0^a dx\, x\, f(x)\, J_\nu(x_{\nu,k} x/a) \qquad (3.97)$$

Now the mystery is why the normalizations seem different. Can we show that

$$J_\nu'(x_{\nu,k}) = \pm\, J_{\nu+1}(x_{\nu n})\ ?$$

Let's try playing with the Schaum recursions on page 137. The first would say:

$$J_{\nu+1}(x_{\nu n}) = -\, J_{\nu-1}(x_{\nu n}) \qquad \text{// since } J_\nu(x_{\nu n}) = 0$$

Then the second would say

$$J_\nu'(x_{\nu,k}) = (1/2)[J_{\nu-1}(x_{\nu n}) - J_{\nu+1}(x_{\nu n})] = (1/2)[\, -2\, J_{\nu+1}(x_{\nu n})] = -\, J_{\nu+1}(x_{\nu n})$$

So there it is, which I repeat

$J_\nu'(x_{\nu,k}) = -J_{\nu+1}(x_{\nu n})$

and then we see from Jackson how we generalize to (0,a). So this Jackson transform might prove useful to me eventually when I start doing problems.

Jackson now comments (p 74) on his transform. Clearly the expansion works fine if you know that your f(x) vanishes at x = a, since every term in the recovery formula has this property! (just look above)

Jackson points out that you can do another SL problem setting slope = 0 at your end points, and then you have $y_{\nu.n}$ being the zeros of $J_\nu'(y_{\nu,k}) = 0$. Details are shown in Problem 3.8.

He then notes that there are still other different Bessel expansions available, with names Neumann, Kapteyn and Schlomilch which I know are mentioned in Bateman.

This stuff appears in Bateman around page 70.

So this is for interval (0,a) and the 0 end is singular. If you take $a \to \infty$, then that end also becomes singular, the eigenvalues $\lambda_n = x_{\nu n}/a$ coalesce into a continuum and the transform becomes continuous, as we shall see below.

(e) Modified Bessel functions

If we go with oscillatory functions in z, then k^2 has the opposite sign, and the ODE has the reverse sign on its k^2 term (compare 3.98 with 3.75). So where you had $J_\nu(k\rho)$ before, you now have $J_\nu(ik\rho)$. Two independent functions are always chosen as the I and K he shows. This is just like my Legendre flip equation, but here we will get two independent functions K and I which are manifestly real.

Now jump back to page 71 top. You see that the power series for $J_\nu(x)$ only has even powers apart from the outside factor (which suggests to me the Frobenius exponents are $\pm\nu$). That is why the I function as defined in 3.100 is purely real! I guess the same thing happens with K.

For small argument, I is well behaved and is sort of the r^n radial function, while K blows up.

For large argument, I blows up expo, but K blows down. So in the distance you will be seeing K functions, and near origins you will see I functions, if the z direction is oscillating.

Jackson Section 3.7 BV problems in cylindricals p 75

Example 1: Jackson considers a cylinder with all metal except one end open with some Dirichlet potential applied on that end plate. In 3.108 he does a Smythian Form which has sinh(kz) so we are expo in z and V=0 on the lower end plate. Since the other end will have some arbitrary $V(\rho,\varphi)$, we include both $\sin(m\varphi)$ and $\cos(m\varphi)$. Then in the ρ direction we have our $J_\nu(x_{\nu,i}x/a)$. Of course we call $x_{\nu,i}/a = k_{\nu i}$ and then we have the usual atomic cross-linking so this same k appears in the sinh function. Recall this linkage scheme:

$[\, J_m(k\rho), N_m(k\rho)]\ e^{\pm kz}\, e^{\pm im\varphi}$ $\qquad$ $[\, I(k\rho), K_m(k\rho)]\ e^{\pm ikz}\, e^{\pm im\varphi}$

So, 3.108 is a Smythian form for this kind of problem, and we can use φ and ρ orthogonality to find the coefficients which he calls A_{mn} and B_{mn}. I understand all of this perfectly. Question: How did he know to use sinh(kz) instead of sin(kz) for the z direction? Both fix the bottom plate. The reason is that if you

do this, then either ρ or φ must be "expo". If ρ were expo, no zeros at the walls. If φ were expo, no matching at 2π. Therefore z must be the expo dimension.

Example 2: Zero on the end faces and some f(φ,z) on the cylinder. This is problem 3.6, but I would put sin(kz) in this case and that would quantize k and make z be oscillatory. How about this Smythian form

$$V(\rho,\varphi,z) = \Sigma_{mn}\, I_m(k_n\rho)\sin(k_n z)\,[\, A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi)\,]$$

where $k_n = \pi n/L$, n = 1,2,3..... Since z oscillates, we have to use I or K for our radial function, as per Jackson discussion page 77. Only I works since we have ρ = 0 in our domain. In this case, we could use φ and z orthogonality to get the coefficients for this Dirichlet problem.

$$V(\rho=a,\varphi,z) = \Sigma_{mn}\, I_m(k_n a)\sin(k_n z)\,[\, A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi)\,] = f(\varphi,z)$$

As usual, we have oscillatory in 2 dimensions, and expo in the third. A good problem.

Example 3:

(a) About the Hankel Transform.

Here we will take $a \to \infty$ as part of our domain, so we have to use the continuous version of the Fourier-Bessel transform. This continuous version is called the Hankel Transform and appears in Stakgold on page 316 in Exercise 4.27. We find that ∞ is a LP endpoint with $H^{(1)}(\sqrt{\lambda}\,\rho)$ as the single solution with finite-s norm for general complex λ (added note on this subject inside Stak Chap 4 notes p 26). Stak writes the 1D Green's function for the (0,∞) interval on page 316 top. Stak has a way of finding the constant out front from the usual jump condition. See page 271 C where this constant starts out as "A", but then quickly we learn that A = -1/C where C is p(x) times the Wronskian of the two end solutions, and this fact then lets you compute constant A for later cases. Once you know the Green's g(x|ξ; λ), you can get the δ(x-ξ) completeness statement as in 4.133 page 316, from which you can derive the transform. This is done for ν = 0 on Stak page 315 bottom two equations.

But what happens to orthogonality in this case? I realized this is a hole in my understanding of Stak and added some notes at the end of my Stak Chap 4 notes. There I consider this Bessel example and show that (1) the δ completeness statement is what you need to show that if you insert the projection into the expansion, you recover your function. (2) You can formally do things in the other order and obtain a continuum form of orthogonality, at least for an ODE which has a purely continuous spectrum like the Bessel one we consider here does. But Stak avoids talking about this.

So here is the Stakgold version of the Hankel Transform from pages 315-316

$$F_\nu(\mu) = \int_0^\infty dx\; xJ_\nu(\mu x)f(x) \qquad\qquad f(x) = \int_0^\infty d\mu\; \mu J_\nu(\mu x)\,F_\nu(\mu)$$

where ν is a fixed value, but Re(ν) > -1/2 is required (see ET II page 12). The above transform is equivalent to this Stakgold completeness relation which you see on Stak page 316.

$$\delta(x-x')/x = \int_0^\infty d\mu\; \mu J_\nu(\mu x')\,J_\nu(\mu x)$$

But as I show in my Chap 4 end notes, if you do things the other way, you get this continuous orthogonality relation:

$$\delta(\mu-\mu')/\mu = \int_0^\infty dx\, x J_\nu(\mu x)\, J_\nu(\mu' x) \qquad \text{// derived this myself}$$

This result is in fact obvious since the Hankel Transform is fully symmetric under $\mu \leftrightarrow x$ so each of the two delta representations above is equally valid! Stakgold never writes the second "orthogonality" one, but I am very happy to see it appear as 3.112 in Jackson who uses k for μ .

(b) Jackson's example using the Hankel Transform

He considers some generic situation where our world only consists of $z \geq 0$ and goes off to $z = \infty$. He does not quite say so, but he is also interested in ρ running in $(0,\infty)$. I would take this latter as the justification for the Smythian form 3.110 where we are now integrating over what he calls k (and I called μ above). However, he is incorporating the μ factor inside $F_\nu(\mu)$ which are his A and B coefficients. The form 3.110 does not need to know about the above formulas (yet). The e^{-kz} factor is in there for the reason he stated at the start. Now he applies to a Dirichlet problem: potential is specified the $z = 0$ plane! So this makes the x expo go away, and we get 3.111 and we can then solve for the coefficients using the usual for φ and the thing I derived above (Hankel Transform) which he shows as 3.112. So this was another Dirichlet problem.

Example PL 1. How about the potential outside a charged cylinder? We know $m = 0$, but we might have to make three "regions". All three regions go to large ρ which suggests using the K function as shown page 75, but then z will oscillate I think, so not sure what to do here. Jackson will do something like this soon with his "disk" problem, and Smythe might do the cylinder somehow. We would like z to oscillate in the cylinder central region so we can have the potential vanish on the two end plates. This is a good problem, but I defer it for now in order to continue reading Jackson.

Comment: Somehow I feel that my favorite word "diagonalization" should enter into this discussion, as in we are diagonalizing a certain integral equation 3.110 so what is the "group theory" involved here? What is the group involved, what are its representations, why do the J functions appear? Rainy day.

Jackson Section 3.10 Green's of point charge in cylindricals p 84

His first task here is to compute the Green's Function in cylindrical coordinates with no BC's other than 0 and ∞. This is what Stak calls a "fundy solution", but Stak only did sphericals in n dimensions. So we put a unit point charge at $\mathbf{x}'$ and observe if from $\mathbf{x}$. Jackson writes z and φ delta functions as expansions in the usual way, we note that the full range of z causes an integral for that one which he folds over to cosine only. He then expands the Green's as well onto these same expansion functions each of which forms a complete set in its dimension. This would be Stakgold's "partial eigenfunction expansion" method, though Stak only did 2D examples this way. So when we apply ∇^2 in cylindricals to our expanded form 3.140, the claim is that you get the (modified) Bessel equation as in 3.141 (this is the residual 1D Green's problem as in Stak). So we have reduced our problem from a 3D one to a standard 1D Green's problem on

the interval ρ in (0,∞). The Green's function is found in the usual Stakgold manner, the jump is figured out. We pick K on the outside for its large ρ expo decay, and I on the inside for good ρ = 0 behavior. One detail I would have done differently: in 3.140, the g_m function should really be g_{mk} with both k and m quantum number labels. The result he finds is that $g_{mk}(\rho,\rho') = 4\pi I_m(k\rho_<)K_m(k\rho_>)$ and this is then put into the expansion 3.140 and this yields the result 3.148, which he rewrites in "all reals" on the next line.

So again, put a point charge at some (ρ',z',φ') -- that is to say, in general NOT at the origin of your cylindrical coordinate system -- and observe it from (ρ,z,φ) and 3.148 or 149 tells you the potential you see at that point.

Now Jackson notes that certain "useful" math results come out of what we have just done.

(1) If you put your point charge at the origin, only m = 0 exists and $\rho_< = 0$ and $I_0(0) = 1$ so 3.149 gives 3.150.

(2) If you put your point charge somewhere in the z' = 0 plane at (ρ',φ') then 1/R on LHS 3.149 has the obvious form he quotes (law of cosines), and this produces the RHS of 3.151 but with a k cos integral. But then if you replace ρ^2 by this same value in 3.150, the RHS has this same k cos integral form and you identify the integrands and this is where 3.151 comes from. This certainly is a strange look expansion. It is reminiscent of the Legendre addition theorem shown Stak II page 398.

(3) He then takes k→∞ in this last result to get 3.152 which he says is Stakgold's 2D Green's function result, and it does look familiar.

Jackson Section 3.11 Eigenfunction method to find a Green's p 87

In Section 3.11 Jackson discusses Stakgold's "full eigenfunction expansion method" for a Green's function in general. The eigenfunctions quantize λ in order to vanish on your boundary, then the Green's comes out being the sum in 3.160. He then applies this to the Helmholtz ODE where f(x) = 0 in 3.153 and where $\lambda = k^2$. The result 3.164 is the usual expansion of 1/R in expo functions which is the 3D Fourier transform of 1/R. Stak did this stuff too in various places. Jackson's second example is Helmholtz in a Cartesian box. He first finds the eigenfunctions as in 3.166 and then the sum 3.160 becomes 3.167.

He then wants to make a connection between the full EF method solution and the partial EF method solution. I skip this section.

Jackson Section 3.12 The Charged Metal Disk Problem p 89

Has opening comments that "mixed" is usually a harder BV problem to solve. He then moves at once to the disk problem. He takes 3.110 p 77 as the Smythian Form, but with m = 0 and $e^{-k|z|}$ and this gives 3.170 where the "coefficient" is f(k). He now shows the "mixed" nature of this problem in terms of the ρ coordinate. For ρ < a we have $V = V_0$ Dirichlet. But for ρ > a we have $\partial_n V = 0$ Neumann since no charge out there. This is what 3.171 says. Jackson uses Φ for V, and V for V_0. When we apply the BC's to our expansion, we get the "dual integral equations" as shown in 3.173. We are trying to solve these two equations for a function f(k) that makes them both true. Jackson then lists off his famous list of references of people who have tried this problem in history, similar to my Canonical pdf. Mention of Abel method eg.

Jackson then does his great cheat on p 91. He writes a dual integral equation in 3.174 which is scaled to 1 instead of a, and to 1 instead of V, and he quotes the solution in 3.175. The reader could presumably

verify that somehow the solution solves both integral equations. Then he applies that thing for n = 0 and out pops the solution in 3.176 and we see that f(k) is a simple elementary sinc function. He then takes the usual large distance limit to get the familiar capacitance $C = 2a/\pi$.

The solution potential is then 3.177 as an integral over k. He then takes the on-axis and in-plane limits of this result and again everything is elementary functions. He writes the general solution in what he calls Weber's form, and this is done using one of my theorems, I have done all this stuff recently in other notes.

Aside 6.17.10: Where? See "Summary of Facts about Metal Objects.doc" for some stuff. See "2 Oblate Spheroidal Coordinates and the Metal Disk Problem.doc" for more. I never do the integral 3.177, but when I convert my oblate spheroidal disk result I get 3.178 as the result, and the ellipse theorem given in "ellipses.doc" is involved there in this form,

$$(1/2)\left(\sqrt{(\rho-a)^2+z^2}+\sqrt{(\rho+a)^2+z^2}\right)^2 = (\rho^2+z^2+a^2)+\sqrt{(\rho^2-z^2-a^2)^2+4\rho^2z^2} \quad .$$

His last action is to compute $\partial_z\Phi$ on the disk surface to get the charge density σ.

His last comment seems strange. He refers to oblate spheroidal coordinates as "elliptical coordinates", but OK, it is elliptical in 2D which you then rotate around z. And he quotes Smythe.

Now, back to the big cheat. This "cheat" appears in a more general way at the end of **Bateman's** Bessel function theory section on page 76 -- I just happened to notice it there yesterday. Jackson has the special case where the RHS is just a power x^n, but Bateman is more general in two ways (1) he has an arbitrary power y^α in place of just y; (2) he has a general function in place of x^n on the RHS. Bateman then gives the solution of his general dual integral equation system as an integral in (78). He then takes the special case $\alpha = 1$ and $g(x) = 1$ and $\nu = 1$. Jackson's case was this but $\nu = 0$ instead.

Bateman next considers a second dual integral equation pair (Trantor) which has $\alpha = 1$ but allows a general function on the RHS for both equations instead of one being 0 as in the previous pair. The solution this time is trickier: you have to first compute the integral H(y), then that becomes part of a double integral to get L(t).

Finally Bateman does a third pair which differs from the second pair by which of the pair is used for the (0,1) range of x.

Stak did something on page 192 like this but in 2D. I don't think Stak attempted the charged disk problem (in 3D) in his Chapter 6. Maybe MF or Smythe did.

Comment: There is something strange going on when you have a "discontinuous integral" and somehow this is what makes the dual integral equation stuff work.

References: We have MF, Byerly, MO, Watson, Bateman, Smythe. He then adds three references I have not looked at but have heard of: Durand, Jeans and Stratton.

PL: Comparison of the Bessel SL Problem to the Legendre SL Problem.

Much of the Bessel material is developed above. First, we can directly compare the eigenvalue equations:

$$L\,J_\nu(ax) = a^2 x\,J_\nu(ax) \qquad L = -\partial_x(x\partial_x) + \nu^2/x \qquad s(x) = x \quad \lambda = a^2$$

regular singular points: x = 0 irregular: x = ∞
interval =(0,b) 1st kind = $J_\nu(ax)$ 2nd kind = $N_\nu(ax)$

$L\,P_n^m(z) = n(n+1)P_n^m(z)$ $L = -(1-z^2)\,\partial_z^2 + 2z\,\partial_z + m^2/(1-z^2)]$ $s(z) = 1$ $\lambda = n(n+1)$
regular singular points: z = -1,1,∞
interval = (-1,1) 1st kind = $P_n^m(z)$ 2nd kind = $Q_n^m(z)$

There are two interesting differences here: (1) In the Legendre case, both endpoints of the interval are singular points, whereas in the Bessel case, only the left endpoint is singular. (2) The eigenvalue appears as a label "n" on the P and Q functions in the Legendre case, but it appears as an argument scale factor in Bessel. In terms of similarity, notice that m is a "bystander" parameter in the Legendre case, such that the functions $P_n^m(z)$ form a complete set on (-1,1) with a discrete spectrum for n. Similarly, in the Bessel case the parameter ν is also a "bystander" parameter, and the functions $J_\nu(a_n x)$ form a complete set on (0,b) where $a_n = x_{\nu,n}/b$.

In the usual treatment of the "regular" Bessel BV problem, we require u(0) = 0 and u(b) = 0, though Jackson mentions above various other possibilities for the homogeneous endpoint conditions. The u(0)=0 condition rules out the $N_\nu(ax)$ Bessel functions so we have $J_\nu(ax)$. Then u(b) = 0 causes quantization of the eigenvalue parameter a such that $\sqrt{\lambda_i} = a_i = x_{\nu,i}/b$ i = 1,2,3.... where $J_\nu(x_{\nu,i}) = 0$. We then have a clean BV problem with eigenfunctions that are orthogonal and complete as written in detail above.

Stak on page 307 obtains the Green's function g(x|ξ; λ) for the Bessel problem on the interval (0,1). He shows that the poles in λ of g are the zeros of $J_\nu(\sqrt{\lambda})$ and this lets him compute the δ completeness form using the usual dλ line integral of g, and he ends up with 4.101, and from this we can read off the normalization. His result is this:

$$\int_0^1 dx\, x\, J_\nu(x\, x_{\nu,i})\, J_\nu(x\, x_{\nu,j}) = \delta_{i,j}\, K_{\nu,i}$$

$$K_{\nu,i} = [J_\nu'(x_{\nu,i})]^2/2 = [-J_{\nu+1}(x_{\nu,i})]^2/2 = [J_{\nu-1}(x_{\nu,i})]^2/2$$

Well actually this integral does appear in GR7 p 664 as follows:

6.521

1. $$\int_0^1 x\, J_\nu(\alpha x)\, J_\nu(\beta x)\, dx = \frac{\beta\, J_{\nu-1}(\beta)\, J_\nu(\alpha) - \alpha\, J_{\nu-1}(\alpha)\, J_\nu(\beta)}{\alpha^2 - \beta^2} \qquad [\alpha \neq \beta, \quad \nu > -1]$$
$$= \frac{\alpha\, J_\nu(\beta)\, J_\nu'(\alpha) - \beta\, J_\nu(\alpha)\, J_\nu'(\beta)}{\beta^2 - \alpha^2} \qquad [\alpha \neq \beta, \quad \nu > -1]$$

WH

In this integral, if α and β are different zeros, the RHS expressions are both 0 so at least you get the orthogonality part. For α and β being the same 0, you have to take a limit. This is trivially done and you get the $K_{\nu,i}$ factor I show above (I did this on scratch). The WH reference is in fact my Whittaker and Watson, so it is derived somewhere in there!

The corresponding completeness will be this: (in my usual form, but see also Stak p 307 4.101)

$$\Sigma_{i=1}^{\infty}(\ 1/K_{\nu,i})\, J_\nu(x\, x_{\nu,i})\, J_\nu(x'\, x_{\nu,i}) = \delta(x-x')/x$$

So we can summarize orthogonality and completeness as follows:

$$\int_0^1 dx\, x\ J_\nu(x\, x_{\nu,i})\, J_\nu(x\, x_{\nu,j}) \ = \ \delta_{i,j}\ K_{\nu,i} \qquad\qquad K_{\nu,i} = [J_\nu'(x_{\nu,i})\,]^2/2$$

$$\Sigma_{i=1}^{\infty}(\ 1/K_{\nu,i})\, J_\nu(x\, x_{\nu,i})\, J_\nu(x'\, x_{\nu,i}) \ = \ \delta(x-x')/x$$

If you make the replacement x = y/b , it is easy to show that the above become

$$\int_0^b dy\ y\ J_\nu(y\, x_{\nu,i}/b)\, J_\nu(y\, x_{\nu,j}/b) \ = \ \delta_{i,j}\ b^2\ K_{\nu,i} \qquad\qquad K_{\nu,i} = [J_\nu'(x_{\nu,i})\,]^2/2$$

$$\Sigma_{i=1}^{\infty}(\ 1/K_{\nu,i})\, J_\nu(y\, x_{\nu,i}/b)\, J_\nu(y'\, x_{\nu,i}/b) \ = \ b^2\delta(y-y')/y$$

You can now think of y as having the dimensions of distance and this explains the b^2 factors alternatively. This is then orthogonality and completeness for the interval (0,b). I will now rewrite the above replacing y with x:

$$\int_0^b dx\, x\ J_\nu(x\, x_{\nu,i}/b)\, J_\nu(x\, x_{\nu,j}/b) \ = \ \delta_{i,j}\ b^2\ K_{\nu,i} \qquad \text{orthogonality}$$

$$\Sigma_{i=1}^{\infty}(\ 1/K_{\nu,i})\, J_\nu(x\, x_{\nu,i}/b)\, J_\nu(x'\, x_{\nu,i}/b) \ = \ b^2\delta(x-x')/x \qquad \text{completeness}$$

If we take $b\to\infty$, the right endpoint becomes singular as well as the left, and this takes us to the Hankel Transform. There is no analogous deal in the Legendre case which is (-1,1) all the time, except in the cone situation of Smythe where it might be (α,1). In problems where the radial ρ coordinate will go out to infinity, the Hankel transform will be needed to invert the Smythian form to find coefficients. We can compare the Hankel transform on $(0,\infty)$ to the "regular" deal on (0,b)

$$\int_0^\infty dx\ xJ_\nu(\mu x)\, J_\nu(\mu' x) \qquad = \delta(\mu-\mu')/\mu \qquad \text{orthogonality}$$

$$\int_0^\infty d\mu\ \mu J_\nu(\mu x')\, J_\nu(\mu x) \qquad = \delta(x-x')/x \qquad \text{completeness}$$

You see how the discrete eigenvalue ratio $x_{\nu,i}/b \to$ continuous eigenvalue μ. In both cases, the orthogonality condition is an integral over the interval of interest. But completeness is a sum in the case of the finite interval (since the spectrum is discrete), while it is an integral for the infinite interval. In general, the Hankel transform is simpler than the finite-interval version! As in the discrete case, the parameter ν is a bystander parameter. For any fixed ν, you have a Hankel transform!