The potential of a charged ellipsoid META PhL 11.26.09

The raw doc is 53 pages long and hugely convoluted, I will try to do a summary here of the main stuff. This doc ended up being 14 pages.

References: Byerly*, W&W*, Moon & Spencer, Hobson,

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0. Starting the search

I think I understand the ellipsoidal coordinates now and see in general how using the right orthogonal coordinates allow solution of electrostatics problems of corresponding geometry. I know the ellipsoidals are a separable system. I show the Wolfram coordinates which seem strange but later in section E I reconcile them.

A. The Surfaces of Ellipsoidal Coordinates and The Equations for Such Surfaces.

This section discusses the three "quadric surfaces" involved, and shows the focal distances a and b in some pretty good slice pictures. I am quite comfortable with this stuff now, and have gotten Maple to draw the surfaces "custom" for me. See "morphing" and "quick tour of coord systems" docs, the latter stored one level up. A major point to keep in mind: the *equations* for the three "surfaces" are really all the same when written in terms of a generic ξ. But for each *range* shown below, the surface is geometrically different.

0	ξ_3	b	ξ_2	a	ξ_1	∞
	hyper2sheet		hyper1sheet		ellipsoid	

Later we should not be surprised to find that the ODEs for the separated functions are basically the same.

B. How to Obtain Cartesian Coordinates from Ellipsoidal Coordinates

Here I just paste in the x,y,z equations from MF. This is the "direction" that is always easy. Going the other way is what I call the inversion problem, see section P below. [But where do these equations come from? In a separate doc, I show they come from the three quadric surface equations via Cramer's Rule.]

C. Statement of the Helmholtz (Laplace if $k_1=0$) Equation in Ellipsoidal Coordinates

If you write ∇^2 in ellipticals, you can write the Helmholtz equation $(\nabla^2 + k_1^2)\psi = 0$ in ellipticals. This is another pasted equation and of course ψ is not yet separated. The equation can be written this way

$$\Sigma_n \; [G_n/ (G_1G_2G_3)] \, f_n \, \partial_n \, (f_n \partial_n \psi) + k_1^2 \psi \; = 0 \qquad G_1 = \xi_2^2 - \xi_3^2 \text{ etc} \qquad f_n^2 = (\xi_n^2 - a^2)\,(\xi_n^2 - b^2)$$

D. Statement of the Separated Equations in Ellipsoidal Coordinates

Here we try $\psi = X_1(\xi_1)X_2(\xi_2)X_3(\xi_3)$ and we find this separated equation

$$(1/f_n)\, \partial_n \, [\, f_n \, (\partial_n \, X_n(\xi_n)) \,] \; + [\, k_1^2 + k_2^2/(\xi_n^2 - a^2) + k_3^2/\, [(\xi_n^2 - b^2)(a^2 - b^2)]\, X_n(\xi_n) = 0$$

where k_2 and k_3 are the separation constants. Note that f_1 is real since $\xi_1 > a,b$. Then f_2 is imaginary, and f_3 is real again. Of course the two imaginary i's of the f_2 case cancel out so all is real. Notice that each of the three separation equations involves *both* separation constants, a situation different from spherical coordinates, which we shall comment on below.

E. The connection between MF Ellipsoidal Coordinates and Wolfram Ellipsoidal

This section does what it says and seems a bit out of place here in the development, but OK. I refer to the Wolfram parameters as a', b', c' and his generic coordinate as ξ' (though not primed in this cut/paste)

$$\frac{x^2}{a^2+\xi}+\frac{y^2}{b^2+\xi}+\frac{z^2}{c^2+\xi}=1$$ with $c' \geq b' \geq a'$ (the reverse of MF) $\xi' = \xi', \eta', \zeta'$

We compare this to the MF equation where we have a different a,b,c (not primed)

$$\frac{x^2}{\xi^2-a^2}+\frac{y^2}{\xi^2-b^2}+\frac{z^2}{\xi^2-c^2}=1; \quad a \geq b \geq c \geq 0 \qquad (5.1.33)$$

$\xi = \xi_1, \xi_2, \xi_3$

The two systems are related obviously by (1) MF is squared; (2) shift by constant for each variable. It is possible, given a Wolfram system, to choose a,b,c such that the shifts are all the same. Pick any $C \geq c'$ and then you find

$a'^2 + a^2 = C^2$	$a^2 = C^2 - a'^2$	=>	$\xi_1^2 = \xi' + C^2$
$b'^2 + b^2 = C^2$	$b^2 = C^2 - b'^2$	=>	$\xi_2^2 = \eta' + C^2$
$c'^2 + c^2 = C^2$	$c^2 = C^2 - c'^2$	=>	$\xi_3^2 = \zeta' + C^2$

and on the right you see the constant shifts.

E1. The Byerly Coordinates compared to MF ones.

The Byerly λ,μ,ν are very much like ξ_1, ξ_2, ξ_3 but we have to do x↔z and a↔c on constants. Byerly defines the "tricky" coordinates he calls α,β,γ which allow for simpler inversion formulas, though in MF these are called λ,μ,ν -- just to add to the confusion.

F. Comments on the Remainder of MF Chap 5 on ODE's

Some obscure remarks about conjugate variables (think Fourier) being a rotation in Hilbert space. Then we have the idea of all systems being degenerate versions of the ellipsoidal system. This chapter then becomes a huge monograph on ODE's which I should someday read (or perhaps read it in W&W). [I later (circa 12/20/09) read and took notes on the 50-page Section 5.2 on ODE's, see elsewhere.]

G. Summary Page on Ellipsoidal Coordinates

Here I pasted in the MF "ellipsoidal summary page" followed by the MF generic curvilinear differential operators page. We now leave MF Chapter 5 and move to MF Chapter 10.

H. Statement of the ODE for Separated Functions in Ellipsoidals: Lamé Equation

Here we just rewrite the ODE's for the three separated functions with $k_1 = 0$ for Laplace, and we end up with

$$f_n \partial_n [f_n (\partial_n X_n(\xi_n))] = [m(m+1)\xi_n^2 - \kappa] X_n(\xi_n) \qquad f_n^2 = (\xi_n^2-a^2)(\xi_n^2-b^2)$$

All 3 equations are the same, BUT for ξ_2 we have f_n = imaginary, so we pick up a -1 on the LHS of the ξ_2 equation *when* we rewrite the square root integrands as positive quantities. He renames the three separated functions this way: $\psi = X_1(\xi_1)X_2(\xi_2)X_3(\xi_3) = F(\xi_1)G(\xi_2)H(\xi_3)$. Key idea: the three equations really are the same, all written the same way as above, so they have the same solutions. We shall find that these solutions are polynomials and square root radical combinations called crudely $E_m^{\kappa}(\xi_n)$ where κ is later replaced by an integer p. These functions, or perhaps the product of three, are the first kind ellipsoidal harmonics

I looked pretty hard for Lame information and there is very little on the web, see main doc. [But then I found and downloaded the Byerly book of 1898, and found the W&W Chapter 23 in new download, so I am now in good shape I think on Lame and ellipsoidal harmonic "data".] [I later 12/24/09 tried to read the WW chapter, but found it completely opaque and useless.] [Don't forget Moon and Spencer.]

I. Series Solutions to the Lamé Equation at z=0

Suppose you could find a solution $f(z) = \Sigma_n d_n z^n$ that converges for all z in $(0,\infty)$. [Note that any truncated series like $f(z) = z$ "converges", despite the fact that as $z\to\infty$ we have $f\to\infty$.] Then you would have found a Laplace solution of the form $u(\xi) = f_{m,\kappa}(\xi_1)\, f_{m,\kappa}(\xi_2)\, f_{m,\kappa}(\xi_3)$, because the range of each variable lies in the range $(0,\infty)$. Now MF go on to show that for general values of the separation constants, κ and m, you *cannot* find a viable f(x) with this property, so the hoped-for separated triple-product form fails to exist. The reason is that in general the Frobenius series diverges for $|z| > R$ (it must have $R_{conv} = \infty$), *or* has inverse powers that cause blow up at the origin (z=0 problem). These two conditions force m and κ to become quantized so we have actual solutions which are first-kind functions.

J. Simple First-Kind Solutions to the Lamé Equation for m = 0,1,2 : functions $E^p_m(z)$

But if we go with special values of m (low integers m = 0,1,2...), and corresponding special values of κ, then you can find some solutions called $E_m^p(\xi$, aka z). These are "first kind" since they are polynomials in z and thus blow up for large ξ. Here are some examples: (various "species")

$E_0^0(\xi_1)\, E_0^0(\xi_2)\, E_0^0(\xi_3) = 1 * 1 * 1 = 1$ m = 0 $\kappa = 0$

$E_1^0(\xi_1)\, E_1^0(\xi_2)\, E_1^0(\xi_3) = \xi_1\, \xi_2\, \xi_3 = abz$ m=1 $\kappa = a^2+b^2$

$E_1^1(\xi_1)\, E_1^1(\xi_2)\, E_1^1(\xi_3) = \sqrt{\xi_1^2 - a^2} * \sqrt{\xi_2^2 - a^2} * \sqrt{\xi_3^2 - a^2} = x\, a \sqrt{a^2-b^2} \sim x$

$E_2^2(\xi_1)\, E_2^2(\xi_2)\, E_2^2(\xi_3) = \xi_1 \sqrt{\xi_1^2 - a^2} * \xi_2 \sqrt{\xi_2^2 - a^2} * \xi_3 \sqrt{\xi_3^2 - a^2} = x\, a \sqrt{a^2-b^2}\, abz \sim xz$

From **Byerly** I learn that $\kappa = (a^2 + b^2)p$ where p is the upper label (so I think MF have maybe goofed up their labels above -- it is more complicated than that, see below). The general solution term is then this

$$[\, A_m^p\ E_m^p(\xi_1) + B_m^p\, F_m^p(\xi_1)]\, [\, A'^p_m\ E_m^p(\xi_2) + B'^p_m\, F_m^p(\xi_2)]\, [\, A''^p_m\ E_m^p(\xi_3) + B''^p_m\, F_m^p(\xi_3)]$$

and here is a form which is a special case of the above

$G_m^p(\xi_1)\, G_m^p(\xi_2)\, G_m^p(\xi_3)$ where G is *either* E or F.

Byerly explains on page 256 or so that you have to have m = integer to get truncation of your recursion relation, and then you need special values of "p" to avoid negative powers that would blow up at the origin, and this is how the m and p get "quantized". Byerly shows ("following Heine") that there really are four "species" of solutions for E which he calls K,L,M and N and here is his table for m = 1,2,3

$E_0(x)$

$$\begin{aligned} K_0(x) &= 1 \\ L_0(x) &= 0 \\ M_0(x) &= 0 \\ N_0(x) &= 0 \end{aligned}$$

$E_1(x)$

$$\begin{aligned} K_1(x) &= x \\ L_1(x) &= \sqrt{x^2 - b^2} \\ M_1(x) &= \sqrt{x^2 - c^2} \\ N_1(x) &= 0 \end{aligned}$$

$E_2(x)$

$$\begin{aligned} K_2^{p_1}(x) &= x^2 - \tfrac{1}{3}[b^2 + c^2 - \sqrt{(b^2 + c^2)^2 - 3b^2c^2}] \\ K_2^{p_2}(x) &= x^2 - \tfrac{1}{3}[b^2 + c^2 + \sqrt{(b^2 + c^2)^2 - 3b^2c^2}] \\ L_2(x) &= x\sqrt{x^2 - b^2} \\ M_2(x) &= x\sqrt{x^2 - c^2} \\ N_2(x) &= \sqrt{(x^2 - b^2)(x^2 - c^2)} \end{aligned}$$

$E_3(x)$

$$\begin{aligned} K_3^{p_1}(x) &= x^3 - \frac{x}{5}[2(b^2 + c^2) - \sqrt{4(b^2 + c^2)^2 - 15b^2c^2}] \\ K_3^{p_2}(x) &= x^3 - \frac{x}{5}[2(b^2 + c^2) + \sqrt{4(b^2 + c^2)^2 - 15b^2c^2}] \\ L_3^{q_1}(x) &= \sqrt{x^2 - b^2}[x^2 - \tfrac{1}{5}(b^2 + 2c^2 - \sqrt{(b^2 + 2c^2)^2 - 5b^2c^2})] \\ L_3^{q_2}(x) &= \sqrt{x^2 - b^2}[x^2 - \tfrac{1}{5}(b^2 + 2c^2 + \sqrt{(b^2 + 2c^2)^2 - 5b^2c^2})] \\ M_3^{r_1}(x) &= \sqrt{x^2 - c^2}[x^2 - \tfrac{1}{5}(2b^2 + c^2 - \sqrt{(2b^2 + c^2)^2 - 5b^2c^2})] \\ M_3^{r_2}(x) &= \sqrt{x^2 - c^2}[x^2 - \tfrac{1}{5}(2b^2 + c^2 + \sqrt{(2b^2 + c^2)^2 - 5b^2c^2})] \\ N_3(x) &= x\sqrt{(x^2 - b^2)(x^2 - c^2)} \end{aligned}$$

Notice that the L and M solutions have one radical while the N has two and the K have none, which I think is always the case. These are all first-kind solutions. He then goes on to show that every solution has a corresponding solution, and in general you can combine the two

$$z = AE_m^p(x) + BE_m^p(x)\int_x^\infty \frac{dx}{\sqrt{(x^2 - b^2)(x^2 - c^2)}[E_m^p(x)]^2}.$$

The above is what is quoted on the Wolfram ellipsoidal harmonics site.

I just downloaded Whittaker and Watson 4th edition which has the "new" (1920) chapter on Lame functions, and it has more of the above stuff.

K. Second-Kind Solutions to the Lamé Equation for m = 0,1,2 : functions $F^p_m(z)$

These are found by a nice ODE trick and we find that for each E there is a corresponding F which is goes I think to 0 at large z, so called second-kind functions.

$$F_m^p(z) = (2m+1)E_m^p(z)\int_z^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}\,[E_m^p(x)]^2} \quad (10.8.89)$$

If we stick in our super trivial $E_0{}^0(\xi_1) = 1$ here we get

$$F_0^0(z) = \int_z^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\frac{a}{z},\frac{b}{a}\right) \quad (10.3.90)$$

which is our baby!

L. The Potential of a Charged Metal Ellipsoid

Consider this obvious Lamé / Laplace solution of the form 1 * 1 * something:

$$\psi = AE_0^0(\xi_3)E_0^0(\xi_2)F_0^0(\xi_1) = V_0\left[\mathrm{sn}^{-1}\left(\frac{a}{\xi_1},\frac{b}{a}\right)\Big/\mathrm{sn}^{-1}\left(\frac{a}{c},\frac{b}{a}\right)\right] \quad (10.3.91)$$

This solves Laplace, and it is a constant V_0 on the ellipsoid $\xi_1 = c$, and vanishes at ∞. Had the first two functions not been constants, this would not be true. So we are LOOKING for a function of just this form. The first two functions MUST be $E_0{}^0$ otherwise there will be dependence on ξ_2 and ξ_3. But then the ξ_1 function must be a linear combination of $E_0{}^0$ and $F_0{}^0$. But the first, being 1, does not vanish for large ξ_1 so it must be the second. So, once you know the general *form* of the Lamé solution shown above, and once you know that you want potential ψ constant on a ξ_1 surface, this is the ONLY solution it can be!!! In sphericals outside a sphere of radius a we would say $\psi = Y_{00}(\theta,\varphi)\,(a/r) = 1 * 1 * (a/r)$ since $\ell = 0$ and $m = 0$. [In the limit ellipsoid → sphere, we have $a\to 0$ and $b\to 0$ so limit not totally obvious above, but I have done this elsewhere, see below.]

M. The alternate ellipsoidal coordinates (λ,μ,ν) and Jacobi function connection to (ξ_1,ξ_2,ξ_3)

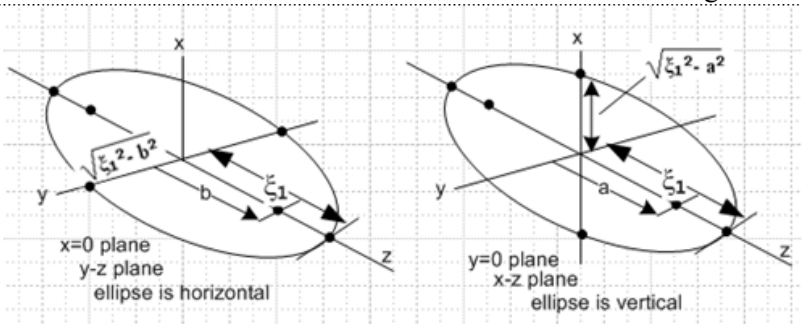
Here MF I think talk about that second Byerly choice of coordinates. So I think the heavy math theory has now ended and the rest is just technical details.

N. Comparison of the MF Solution of the Charged Ellipsoid to the solution of Kelvin

The MF solution is this (where $a \geq b$ are the two focal distances)

$\psi(\xi_1, \xi_2, \xi_3) = \psi(\xi_1) = V_0\, F[\sin^{-1}(a/\xi_1), b/a] \,/\, F[\sin^{-1}(a/c), b/a]$

$A = \sqrt{c^2-a^2}$	$B = \sqrt{c^2-b^2}$	$C = c$
smallest semi	middle semi	largest semi



I show (in raw notes) how you can trivially rewrite the MF solution above as the first line below:

$\psi(a) = V_0\, F[\sin^{-1}(\sqrt{a^2-c^2}/a), \sqrt{a^2-b^2}/\sqrt{a^2-c^2}\,] \,/\, F[\sin^{-1}(\sqrt{a^2-c^2}/a_1), \sqrt{a^2-b^2}/\sqrt{a^2-c^2}\,]$
$\psi(a) = V_0\, F[\sin^{-1}(\sqrt{a^2-b^2}/a), \sqrt{a^2-c^2}/\sqrt{a^2-b^2}\,] \,/\, F[\sin^{-1}(\sqrt{a^2-b^2}/a_1), \sqrt{a^2-c^2}/\sqrt{a^2-b^2}\,]$

where (!!) the meaning of a and b has now been *redefined*. Now, we have

c	b	a
smallest semi	middle semi	largest semi

and now the focal distances are $f^2 \equiv a^2 - b^2$ and $g^2 \equiv a^2 - c^2$ with $g > f$. We can regard the largest semi "a" as the variable = ξ_1. It of course labels ellipsoids. The second line of the pair above comes from $b \leftrightarrow c$ on the RHS, the proof of which's validity requires use of a certain F function transformation rule. We can rewrite the two lines above as

$\psi(a) = V_0\, F[\sin^{-1}(g/a), f/g] \,/\, F[\sin^{-1}(g/a_1), f/g]$ // restatement of MF above
$\psi(a) = V_0\, F[\sin^{-1}(f/a), g/f] \,/\, F[\sin^{-1}(f/a_1), g/f]$ // this line is "the Kelvin solution"

The first line is identical with the original MF solution, setting $\xi_1 = a$ and realizing that we just have new names $g > f$ for those focal distances that were a>b. The second line is the way the solution appears in the Kelvin paper. So we could rewrite these as: (reference one now has semi = a instead of a_1)

$\psi(\xi_1) = V_0\, F[\sin^{-1}(g/\xi_1), f/g] \,/\, F[\sin^{-1}(g/a), f/g]$ // restatement of MF above
$\psi(\xi_1) = V_0\, F[\sin^{-1}(f/\xi_1), g/f] \,/\, F[\sin^{-1}(f/a), g/f]$ // this line is "the Kelvin solution"

semis: $\xi_1 = a \geq b \geq c$ focals: $g^2 \equiv a^2 - c^2$ and $f^2 \equiv a^2 - b^2$ $g > f$

(a) If we take $b \to a$, we inflate our ellipsoid to become **oblate**, $f = 0$, and the first line above could be used to state the solution in this case, where we use

$F(\varphi,k) = F(\varphi|m) = F(\varphi\backslash\alpha)$ $k = \sin\alpha$ $m = k^2$ $F(\varphi,0) = \varphi$

AS p 589

AS p 594

to get [$a = \xi_1$]

$$\psi(a) = V_0 F[\sin^{-1}(g/a), 0] / F[\sin^{-1}(g/a_1), 0] = V_0 \sin^{-1}(g/a) / \sin^{-1}(g/a_1) \qquad g^2 \equiv b^2 - c^2$$

where g indicates the amount of oblateness (spherical when g =0). From here we could then take the limit $c \to a$ to inflate our pillow up to a sphere, so $g \to 0$ and we get

$$\psi(a) = V_0 \sin^{-1}(g/a) / \sin^{-1}(g/a_1) = V_0 (a_1/a)$$

giving the famous $\psi(r) = V_0 R/r$ outside an equipotential sphere of radius R.

On the other hand, we could instead take $c \to 0$ and crush the pillow down to a flat ellipse. Then we just have g = b and our solution is

$$\psi(a) = V_0 \sin^{-1}(b/a) / \sin^{-1}(b/a_1)$$
$$\psi(\xi_1) = V_0 \sin^{-1}(b/\xi_1) / \sin^{-1}(b/a)$$

This is the potential everywhere of a flat ellipse of semis a and b. And if a = b we get

$$\psi(\xi_1) = V_0 \sin^{-1}(a/\xi_1) / [\pi/2] \qquad \text{potential of circular disk of radius a}$$

(b) Alternatively, we could first take $c \to b$ taking us to a **prolate** football. Then g = f and we use the second form above (either form gives the same result) to get

$$F(\varphi,k) = F(\varphi|m) = F(\varphi\backslash\alpha) \qquad k = \sin\alpha \qquad m = k^2 \qquad F(\varphi,k)$$
$$F(\varphi,1) = F(\varphi|1) = F(\varphi\backslash 90) = \ln(\sec\varphi + \tan\varphi)$$

$$\psi(a) = V_0 F[\sin^{-1}(f/a),1] / F[\sin^{-1}(f/a_1), 1] \qquad f^2 = g^2 = a^2 - b^2$$

where now f is a measure of the prolateness and I won't bother to write it out as logs. If, however, we now go to the spherical limit where $f \to 0$, we can use this fact for small φ,

$$F(\varphi|1) = \ln(\sec\varphi + \tan\varphi) = \ln(1/\cos\varphi + \sin\varphi/\cos\varphi) \approx \ln(1+\varphi) \approx \varphi$$

and we then find that $\psi(a) = V_0(a_1/a)$, as before.

Summary:

Potential of an **ellipsoid** of semi's a > b > c with $g^2 \equiv a^2 - c^2$ and $f^2 \equiv a^2 - b^2$ so g > f :

$$\psi(\xi_1) = V_0 F[\sin^{-1}(g/\xi_1), f/g] / F[\sin^{-1}(g/a), f/g] \qquad \text{// restatement of MF above}$$
$$\psi(\xi_1) = V_0 F[\sin^{-1}(f/\xi_1), g/f] / F[\sin^{-1}(f/a), g/f] \qquad \text{// this line is "the Kelvin solution"}$$

Potential of **oblate spheroid** where a = b > c:

$\psi(\xi_1) = V_0 \sin^{-1}(g/\xi_1) / \sin^{-1}(g/a)$ where $g^2 \equiv a^2 - c^2$

Sublimit where a = b = c for **sphere**: $\psi(\xi_1) = V_0 (a/\xi_1)$

Sublimit where c = 0, flat **ellipse** a > b: $\psi(\xi_1) = V_0 \sin^{-1}(b/\xi_1) / \sin^{-1}(b/a)$

Sublimit c = 0 and a = b, flat **disk**: $\psi(\xi_1) = V_0 \sin^{-1}(a/\xi_1) / [\pi/2]$

Potential of **prolate spheroid** where a > b = c: F[

$\psi(\xi_1) = V_0 F[\sin^{-1}(f/\xi_1),1] / F[\sin^{-1}(f/a), 1]$ $f^2 = a^2 - b^2$

Sublimit a >>> b = c, cylinder: did not do this one, $a \approx f$

[Question: why in oblate spheroidals do we get $\tan^{-1}$ as the disk solution instead of $\sin^{-1}$?]
ANSWER added 1.10.10: In oblate you get $\cot^{-1}$ or $\tan^{-1}$ forms which you can convert to $\sin^{-1}$ using "the little triangle". See "the Smythe method for Green's Functions, cone exampls.doc" for details.

O. The charge density on a charged metal ellipsoid.

I show that the general result for any of the above cases is this (V = ψ)

$4\pi\sigma = E^{\perp} = -dV/dn = -[dV(\xi_1)/d\xi_1] (d\xi_1/dn) = -[dV(\xi_1)/d\xi_1] (1/h_{\xi 1}) =$

$-[dV(\xi_1)/d\xi_1] \{ \sqrt{(\xi_1^2 - a^2)(\xi_1^2 - b^2)} / \sqrt{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)} \}$

where $a \geq b$ are the two focal distances. So the "action" is all in the last factor, this tells how charge density varies over the surface of your ellipsoid, ellipse, disk or sphere.

But how could it vary over the sphere?? If we take the sphere limit a = b = 0 of our inversion formula results below we get this

$h = r^2$ $f = k = 0$ $R = h^3/27 = (h/3)^3$ $Q = -(h/3)^2$

$D = Q^3 + R^2 = -(h/3)^6 + (h/3)^6 = 0$

$S = T = R^{1/3} = h/3$ $S+T = 2h/3$ $S - T = 0$

$\xi_1^2 = h/3 + 2h/3 = h = r^2$ => $\xi_1 = r$
$\xi_2^2 = h/3 - h/3 = 0$
$\xi_3^2 = h/3 - h/3 = 0$ add up to get h which is correct

These are all "in range, we expect to see $\xi_2 = \xi_3 = 0$. Then

$1/\sqrt{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)} = 1/(\xi_1^2) = 1/r^2$

and of course the numerator radical is also ξ_1^2 so num/den = 1, and σ is then

$4\pi\sigma = -[\, d\psi(\xi_1)/d\xi_1]$ $\quad\psi(\xi_1) = V_0\,(a/\xi_1)$ $\quad\psi' = -\,V_0 a/\xi_1^2$

$4\pi\sigma = V_0 a/\xi_1^2 = V_0/a = Q/a^2$ $\quad$ and $\quad Q = 4\pi a^2\sigma$

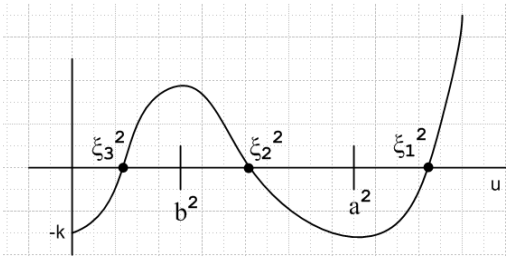
So the basic answer is that in the sphere limit, $\xi_2 = \xi_3 = 0$ and our "factor" shows no variation.

P. The inverse coordinate formulas for ellipsoidal coordinates

With the help of corrected Schaum, I write down the exact inversion formulas that give the three ξ_i in terms of x,y,z and a,b. We of course already know the forward formulas for x,y,z in terms of ξ_i and a,b. I state three relatively simple combination formulas like $\xi_1^2 + \xi_2^2 + \xi_3^2 = h = x^2+y^2+z^2+a^2+b^2$.

Q. Comments on Ellipsoidal Coordinates

(1) Here I argue that the mapping between the first octant of x-space and ξ-space is 1-to-1 and invertible and the inverse solutions for the ξ_i^2 are the roots of a certain cubic [$\alpha = a^2$, $\beta = b^2$] and the curve shows the appropriate ranges for the three variables.



$F(u) = u^3 - h\,u^2 + f\,u - k = 0$

$k = \alpha\beta z^2$ // all three are positive

$f = \beta x^2+\alpha y^2+(\alpha+\beta)z^2+\alpha\beta$

$h = x^2+y^2+z^2+(\alpha+\beta)$ // note that $g = h + k/f$

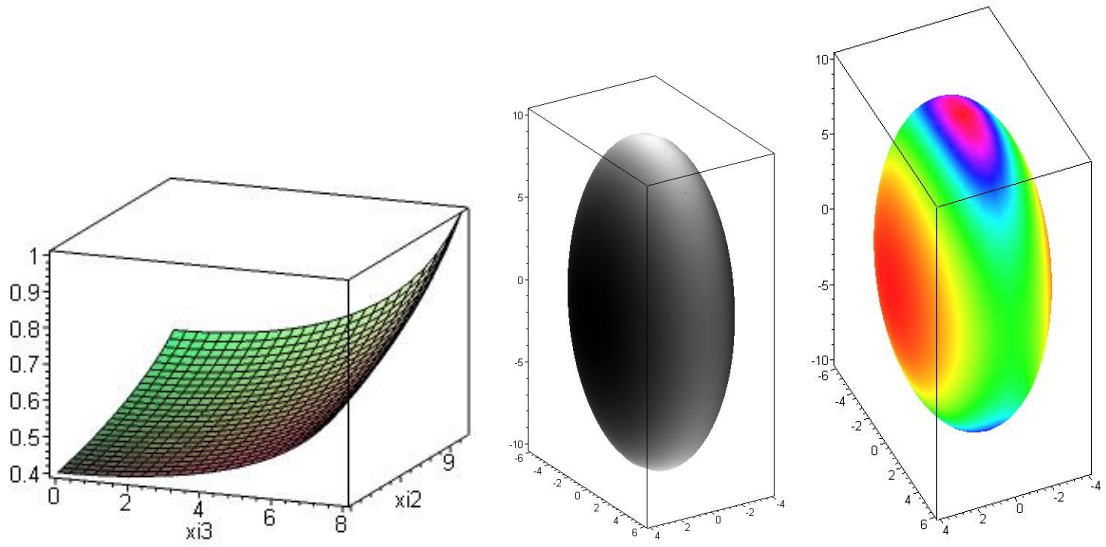
(2) How do we *know* that the ξ_i coordinates are orthogonal? Just one of my pet questions, I comment on this in a separate doc as well. ["about 2D ellipsoidal..."]

R. Express the charged ellipsoid potential in Cartesian coordinates.

I argue that it is a mess because the ellipsoid is labeled by ξ_1 and $\psi(\xi_1)$ is simple, but $\xi_1(x,y,z)$ is the big mess.

S. How would you PLOT the charge density on the ellipsoid using Maple?

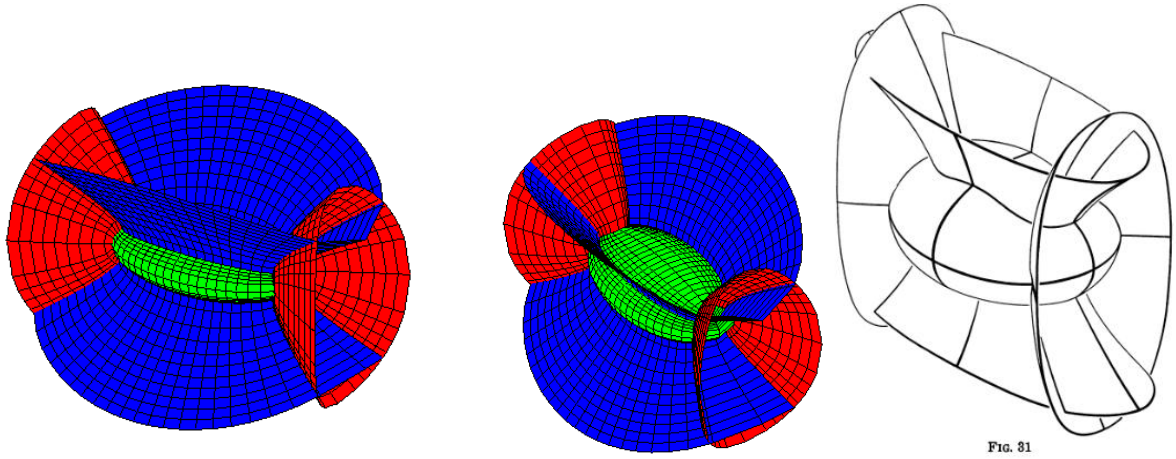
I first plot the "factor" ($\sim\sigma$) directly in ξ_2, ξ_3 and get the totally reasonable plot on the left:



I then figured out how to plot this on the ellipsoid using grayscale and false color, the latter being the best, as shown on the right above. [Another approach is to use the Kelvin p expression!]

T. Generate those nice fixed-ξ_i surfaces using Maple

Here for the first time I was able to get the generate for myself the famous pictures,



which I think Hilbert and Cohn-Vossen would have enjoyed if they had computers in 1898. I show their picture on the right above, which is pretty good.

U. Verifying The Cubic Coordinate Inversion Equations

Back in section P I summarize the corrected inversion formulas for ξ_i in terms of x,y,z. Here I verify using Maple that the results are now correct. At first they were wrong because Schaum was wrong in 1968. But Schaum got corrected and I pasted in the corrected page from the 2008 edition of the book.

V. Compare Kelvin and MF on dV/dn: interpretation of p

Here in clumsy roundabout fashion I prove the following very useful fact:

$$\sqrt{(\xi_1^2 - a^2)(\xi_1^2 - b^2)} \;/\; \sqrt{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)} \;=\; (1/\xi_1)\, p(x,y,z)$$

where

$p = 1/\sqrt{x^2/A^4 + y^2/B^4 + z^2/C^4}$ a,b are focal distances, and

semis: $A^2 = \xi_1^2 - a^2$ $B^2 = \xi_1^2 - b^2$ $C^2 = \xi_1^2$

$$4\pi\sigma = -[\, dV(\xi_1)/d\xi_1] \;\{\; \sqrt{(\xi_1^2 - a^2)(\xi_1^2 - b^2)} \;/\; \sqrt{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}\;\}$$

$$= -[\, dV(\xi_1)/d\xi_1]\,(1/\xi_1)\, p(x,y,z)$$

So although ξ_1 is a very messy function of x,y,z, it turns out that our σ-factor is a very simple function of x,y,z. So in retrospect, I could have used p to make my charge plots above. Moreover, p has a simple geometrical interpretation: p = the perp distance from ellipse center to the plane tangent to an ellipsoid at point x,y,z. Only Kelvin brought out this useful fact. I find the p expression here to be very "morph friendly". I keep feeling there is some "scaling method" of doing ellipsoidal coordinates that is related to a 2D conformal map. That would be an entertaining problem, but I have done enough!

W. More on σ for ellipsoid, ellipse and disc; capacitances of various objects (10.8.10)

I added this section much later than the rest of the doc, and subsection (e) contains a summary, so I just paste that right here, but first here are titles of the little subsections:

(a) Expression for p, and then limit of p when ellipsoid is crushed A→0. 60
(b) Differentiate the potential. 62
(c) Show cancellation of A factors when we go flat. 63
(d) Capacitances of various objects 65
(e) Summary of main facts of this last effort: 67

(a) We start with this basic information

$V(\xi_1) = V_0\, F[\sin^{-1}(a/\xi_1), b/a] \,/\, F[\sin^{-1}(a/c), b/a]$ // potential of charged ellipsoid

$4\pi\sigma = -[\, dV(\xi_1)/d\xi_1]\,(1/\xi_1)\, p(x,y,z)$ // charge density on same

$$(1/\xi_1)\, p(x,y,z) = \sqrt{(\xi_1^2 - a^2)(\xi_1^2 - b^2)} \;/\; \sqrt{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)} = AB/\sqrt{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}$$

semis: $C \geq B \geq A$. $A^2 = \xi_1^2 - a^2$ $B^2 = \xi_1^2 - b^2$ $C^2 = \xi_1^2 = c^2$

$p(x,y,z) = 1/\sqrt{x^2/A^4 + y^2/B^4 + z^2/C^4} = ABC \,/\, \sqrt{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}$ // Note: $V = (4\pi/3)ABC$

and if we flatten by taking A→0 we find that

$p = A / \sqrt{1 - y^2/B^2 - z^2/C^2}$ **// small A limit of p**

(b) We compute the derivative $\partial_{\xi 1} V(\xi_1)$ using the fact that

$\partial F(\varphi,k)/\partial\varphi = 1/\sqrt{1-k^2\sin^2\varphi}$ => $\partial_{\xi 1} F(\sin^{-1}(a/\xi_1),b/a) = - a\,(1/\sqrt{\xi_1^2-b^2})(1/\sqrt{\xi_1^2 - a^2})$

with the result

$\partial_{\xi 1} V(\xi_1) = - (aV_0/F[\sin^{-1}(a/c),b/a]) (1/AB)$ // "radial" derivative of potential

and if we insert this in to our general σ formula we get these equivalent results,

$4\pi\sigma(\xi_1 \text{ label}, \xi_2, \xi_3) = (aV_0/F[\sin^{-1}(a/c),b/a]) / \sqrt{(\xi_1^2- \xi_2^2)(\xi_1^2- \xi_3^2)}$ $\xi_1 = C = c$

$4\pi\sigma(\xi_1 \text{ label}, x, y,z) = (aV_0/F[\sin^{-1}(a/c),b/a]) (1/ABC)\, p(x,y,z)$

(c) Inserting the small A limit for p into this last gives our flat ellipse charge density in y-z plane:

$4\pi\sigma(\xi_1 \text{ label}, x, y,z) = (aV_0/F[\sin^{-1}(a/c),b/a]) (1/BC) (1 / \sqrt{1 - y^2/B^2 - z^2/C^2})$

But since a = c, we have $\sin^{-1}(a/c) = \sin^{-1}(1) = \pi/2$ so this simplifies to

$4\pi\sigma = (1/B) [V_0/ K(\sqrt{C^2-B^2}/C)] (1/\sqrt{1 - x^2/C^2 - y^2/B^2})$ **// σ on an ellipse in x-y plane**

If we then set B = C for a **disc** and define ρ as usual, this becomes, using K(0) = π/2, we get Jackson:

$\sigma = (1/C) (V_0/ K(0) (1/\sqrt{1 - \rho^2/C^2}) = (V_0/2\pi^2C) (1/\sqrt{1 - \rho^2/C^2}) = (q/4\pi C) /\sqrt{C^2 - \rho^2}$

(d) I compute the capacitance of various objects, results are

$A = \sqrt{c^2-a^2}$ $B = \sqrt{c^2-b^2}$ $C = c$ $C \geq B \geq A$

$cap(\text{ellipsoid}) = Q/V_0 = a / F[\sin^{-1}(a/c), b/a]$

$cap(\text{ellipse}) = a / K(b/c)$

$cap(\text{disc}) = a / K(0) = (2/\pi)a$

$cap(\text{prolate spheroid}) = a / F[\sin^{-1}(a/c), 1] = a / \ln [(C+a)/A]$ $A = B$ $a = b$

$cap(\text{prolate needle}) = C / \ln [(2C/A]$

cap(oblate spheroid) = Q/V_0 = $a / F[\sin^{-1}(a/c),0] = a / \sin^{-1}(a/c)$ B = C b = 0