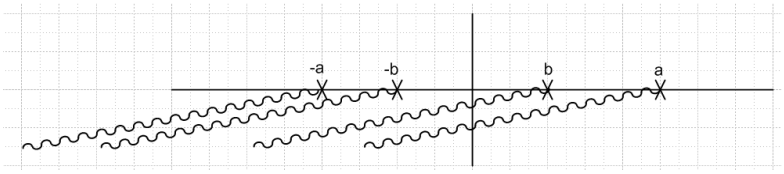
Finding Solutions to the Lamé Equation hL 12.25.09

Overview (10.9.10, 2 pages)

I don't yet have all the facts, but this is a good first cut at understanding the Lamé function world. I first draw the cut structure for the Lamé ODE,



and take note of the indices at each singular point. I then follow the MF discussion and think about the Frobenius series solutions about the origin $\xi=0$ which is an ordinary singular point with indices 1 and 0. It turns out this will give the "first species" solutions. If you let $f(\xi) = \Sigma_{n=0}^{\infty} d_n \xi^n$, the Lamé ODE gives you this recursion relation,

$$a^2b^2n(n-1)d_n = [\,(n-2)^2(a^2+b^2) - \kappa]\, d_{n-2} + [\,m(m+1) - (n-3)(n-4)]\, d_{n-4} \equiv A_n\, d_{n-2} + B_n\, d_{n-4}$$

where we note there are three terms and each is a double jump in n. Clearly we can think about even and odd power series as separate cases. MF somehow calculate the radius of convergence of these two series and conclude that one has radius a and the other radius b. Thus, *neither* series solution exists outside the disk $|\xi| = a$. But we *need* a solution "out there" for our ξ_1 ellipsoidal coordinate. So at least in terms of these two Frobenius solutions, the only way out is to have one of these series truncate. Recall that a truncated series such as $f(x) = A + Bx$ has an *infinite* radius of convergence, even though we might have $f(\infty) = \infty$. The point is that the series *exists*. This, then, is what leads to quantized values of the κ and m separation constants.

Somehow, if we want to get our series to truncate, we have to kill off two adjacent coefficients, so for example we might want to have $d_{n-4} = 0$ and $d_{n-2} = 0$. This would then assure that d_n and all subsequent coefficients vanish and I would have achieved truncation

In **Plan A** I attempt to achieve this goal in a particular way: I require that $A_{n-2} = B_{n-2} = 0$ to kill d_{n-2}, and at the same time I require $A_{n-4} = B_{n-4} = 0$ to kill d_{n-4}. But you can see that $B_{n-2} = B_{n-4} = 0$ requires m(m+1) to have to different values, so there is no solution discoverable by this Plan A.

In **Plan B** I engage more systematically in a search for m,κ pairs that will give $d_{n-4} = d_{n-2} = 0$. It is not very pleasant, and it seems you have to consider each value of n separately and thus build up your set of truncating m,κ pairs. I consider the even and odd cases separately.

The first pair I discover (Case I Even) is $\kappa = 0$ and $m(m+1) = 0$ which makes $d_2 = d_4 = 0$, and this is the "F=1" solution. It turns out that m(m+1) will always be an integer which kills off the coefficient $B_n = [\,m(m+1) - (n-3)(n-4)]$ for some value of n, and this in turn means that m is always an integer. Then κ ends up having some strange value.

The next solution I find (Case II Even) has $m(m+1) = 6$ and $\kappa = 2[\,(a^2+b^2) \pm \sqrt{(a^2+b^2)^2 - 3a^2b^2}\,]$ and this causes $d_4 = d_6 = 0$. We have then $d_0 = 1$ and $d_2 = -\kappa/(2a^2b^2)$ so in this way we have found the

Lamé solution $F_2^{\kappa}(\xi) = 1 + \{ -\kappa/(2a^2b^2)\}\xi^2$ where we use m = 2 to get m(m+1) = 6. MF rescale this solution to be $G_2^{\kappa}(\xi) = -(2a^2b^2)/\kappa + \xi^2$ with κ as shown above. Call these κ values κ_1, κ_2 if you like.

I next try to find m and κ such that $d_6 = d_8 = 0$, Case III Even. This leads to m(m+1) = 10, but we then have a nasty cubic equation we have to solve to find the right κ. I call the solutions $\kappa_{1,2,3}$. So here seems to be the trend for this class of even series solutions (κ's should be written $\kappa_i^{(m)}$). Notice that all these solutions are just polynomials of degree m/2 in variable ξ.

Even Solutions: // There seem to be m/2+1 solutions for each legal m value.

m = 0	κ
m = 2	κ_1, κ_2
m = 4	$\kappa_1, \kappa_2, \kappa_3$
m = 6	$\kappa_1, \kappa_2, \kappa_3, \kappa_4$
etc	

Turning to the odd solutions, I can get $d_3 = d_5 = 0$ by using m(m+1) = 2 and $\kappa = a^2 + b^2$ so F = ξ. Things go pretty much as above, and our Odd Solutions are like the above list but set m = 1,3,5... .

Since we started off expanding about ξ = 0 with its indices 0,1, we know we shall never get any strange factors like $\sqrt{\xi - a}$ in our Frobenius solution. Starting around ξ = 0 thus yields only the solutions without such factors which are called "first species" solutions, and those are the solutions we are finding by the above laborious method. It turns out that the solutions call into four species groups:

factor	1	$\sqrt{\xi^2-a^2}$	$\sqrt{\xi^2-b^2}$	$\sqrt{\xi^2-a^2}\sqrt{\xi^2-b^2}$
species	1st	2nd	3rd	4th
	K	L	M	N

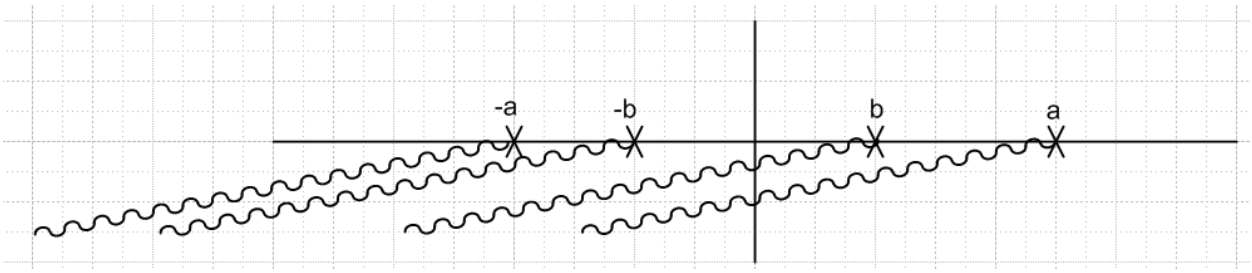
Although it is not exactly clear, these solutions must arise when you consider the Frobenius series solutions about the regular singular points ±a and ±b which recall have indices 0 and 1/2. These are all "first kind" solutions in the sense of $P_n^m(z)$. Those about the 5th regular singular point at ∞ somehow yield the second kind solutions, and we know how to generate a second kind solution from each first kind one using "the trick", so there are a whole set of solutions analogous to $Q_n^m(z)$.

Notice that we have solutions for each integer m in the list m = 0,1,2,3... . It is traditional to enumerate the first kind solutions as E_m^x . A second index upper index "x" is used to somehow indicate the appropriate κ_i value, but it would be too messy to write κ_i itself, so there is some system to just mark the solutions with a second index to distinguish them. If you scroll down and examine the Byerly yellow background solutions, you will see this notation. The corresponding second kind functions are called F_m^x.

It is hard to imagine a complete table of all the Lamé functions and general functional forms as we have for the P and Q functions, but maybe I will find something like this. Moreover, somehow we expect to find a S-L problem in ξ with oscillatory solutions (I guess the E_m) perhaps in the (-b,b) interval which will then have orthogonality, completeness, and an implied transform. To make this happen, all we need to know is that the Lamé differential operator is self-adjoint on (-b,b) which I suspect it is.

General Comments. Since studying ellipsoidal coordinates, I went off and did more general ODE reading, which culminated in "an ODE essay on MF Section 5.2.doc" where I give a pretty concise and I think readable summary of "how you solve an ODE". In particular, I finally drilled into myself how important the notion of "series solutions" is -- in a sense, that is really all there is! And that is the name of the game here with this Lamé equation.

If you do things according to my "essay", your first act is to draw up the cut structure of the ODE solutions, and you get this picture: [warning: the variable is really ξ here, but MF and I tend to call it z sometimes. But then that gets confused with Cartesian $z = \xi_1\xi_2\xi_3/ab$. So I will always use ξ.]



The indices associated with an ordinary point are 0,1 as usual, while for the regular singular points,

At $z = \pm b$; with indices 0 and $\frac{1}{2}$
At $z = \pm a$; with indices 0 and $\frac{1}{2}$
At $z = \infty$; with indices $m + 1$ and $-m$

I have never seen this cut picture drawn, by the way. If you expand around one of these four regular singular points, one solution is analytic, and the other has a square root multiplying function. For example at $\xi = a$ we will have a solution of the form $f_{1/2}(\xi) = \sqrt{\xi\text{-}a}\ v(\xi)$ where $v(\xi)$ is an analytic series about a. The other solution $f_0(\xi)$ will be an analytic series with no cut. MF try out a series expanding around $\xi = 0$, which is an ordinary point of the Lamé ODE. They find a triple recursor

$$a^2b^2n(n-1)d_n = [(n-2)^2(a^2+b^2) - \kappa]d_{n-2} + [m(m+1) - (n-3)(n-4)]d_{n-4} \quad (10.3.84)$$

which implies an even and odd terms only series -- that is, two different series can be formed, which no doubt are the two independent ODE solutions at $\xi = 0$. At first glance, you might expect that both these series have a radius of convergence of b, since that is where we hit the first singularity in the picture above. But MF (p 1306) compute the radius of convergence, and they find two *different* radii. One is b and the other is a. So somehow one of our series manages to not be singular at b. Perhaps that is reasonable since we know that one of the indices at b is 0.

Now back to a very major point. We need to find product form solutions of Lamé like so [note that F here is just a generic function symbol, it is not the second-kind Lamé function; note also κ as label.]

$E_m^{\kappa}(\xi_1)\ F_m^{\kappa}(\xi_2)\ F_m^{\kappa}(\xi_3)$ $\qquad\qquad$ $0 \le \xi_3 \le b \le \xi_2 \le a \le \xi_1$

One of our series solutions at $\xi = 0$ converges out to $\xi = a$, so that solution is viable for the ξ_2 and ξ_3 problems, but not for the ξ_1 problem which has $\xi > a$. The problem is then that the function $F_m^{\kappa}(\xi_1)$

diverges in the entire ξ_1 variable range. You might think there was some analytic continuation of it that is finite there. One vague question is whether you have to worry about the entire (a,∞) range for ξ_1 or not. You might have some "interior problem" where you only care on (a,a_{max}), for example. Regardless, I think our series solution $F_m^{\kappa}(\xi)$ is "all there is" for a first kind solution. So the answer is that for general values of m and κ, THERE ARE NO Lamé equation solutions, and we are going to have quantization caused in the usual way by **truncation** of the recursion series. In that case, our series will be finite and then we get convergence out beyond $|\xi| = a$ and we are viable.

Plan A Fails: Cannot find general solutions by killing separate coefficients.

First, let's look at the triple recursor away from low values of n. How could we get truncation? If we want $d_n = 0$, we have to kill off BOTH earlier terms, that is, we need both these things to vanish: (the other possibility is that we arrange to have the two terms *cancel*, and as we shall see later, that is how you get the solutions!)

$$[\ (n-2)^2(a^2+ b^2) - \kappa]/[n(n-1)] = 0 \qquad d_n$$
$$[\ m(m+1) - (n-3)(n-4)]\ /[n(n-1)] = 0$$

But this is not enough! Suppose $d_n = 0$. If we then look at d_{n+2}, it picks up d_{n-2} and the series could then propagate through our zeroed term and still be an infinite series. So I think we have to kill two terms in a row! So let's kill off the d_n term as above, and the d_{n+2} term as below:

$$[\ (n)^2(a^2+ b^2) - \kappa]\ /[(n+2)(n+1)] = 0 \qquad d_{n+2}$$
$$[\ m(m+1) - (n-1)(n-2)]\ /[(n+2)(n+1)] = 0$$

But this is not right because, when considering d_{n+2}, we have already killed of d_n, so we don't need the first condition above in the second group. So I think we truncate if we can satisfy these three conditions:

$$[\ (n-2)^2(a^2+ b^2) - \kappa]/[n(n-1)] = 0$$
$$[\ m(m+1) - (n-3)(n-4)]\ /[n(n-1)] = 0$$
$$[\ m(m+1) - (n-1)(n-2)]\ /[(n+2)(n+1)] = 0$$

These conditions then "kill off" d_n and d_{n-2} so that last non-vanishing series tern is d_{n-2}. Let's now try to find any solution of these conditions. We need to have $n-2 \geq 0$ since our first series coefficient is d_0 so I think that means $n \geq n$. So try $n = 2$. First equation says $\kappa = 0$. But I don't see how m(m+1) can then have two different values for some m, so $n = 2$ has no solutions. But this is then true for any larger n. So we conclude that if we try to achieve truncation out in the series somewhere away from the start, we cannot do it by requiring the two terms to be separately zero! There is no value of m that will work. So this particular "method" gives no solutions.

Plan B. But now let's think about the above recursor as things start off,

Now there is more opportunity at the start to kill something off. Our recursor condition is this:

$$a^2b^2n(n-1)d_n = [\ (n-2)^2(a^2+ b^2) - \kappa]\ d_{n-2} + [\ m(m+1) - (n-3)(n-4)]\ d_{n-4}$$

Let's consider separately the even and odd solutions.

Even solutions: Assume $d_0 = 1$ and $d_1 = 0$. Then our lowest non-trivial 0 = 0 equations are these:

$n = 2 \quad a^2b^2 2*1\, d_2 = [\,(0)^2(a^2+b^2) - \kappa]\, d_0$
$n = 4 \quad a^2b^2 4*3 d_4 = [\,(2)^2(a^2+b^2) - \kappa]\, d_2 + [\,m(m+1) - (1)(0)]\, d_0$
$n = 6 \quad a^2b^2 6*5 d_6 = [\,(4)^2(a^2+b^2) - \kappa]\, d_4 + [\,m(m+1) - (3)(2)]\, d_2$
$n = 8 \quad a^2b^2 8*7 d_8 = [\,(6)^2(a^2+b^2) - \kappa]\, d_6 + [\,m(m+1) - (5)(4)]\, d_4$

Now can we get truncations?

Case I Even: Use $\kappa = 0$ and this kills off d_2 then need $m(m+1) = 0$ so m = 0 or -1 and d_4 is dead as well. Our corresponding series solution is then simply $F_m^{\kappa}(\xi) = 1 = F_0^0(\xi)$.

Case II Even: Suppose $\kappa \neq 0$. Then we have $a^2b^2 2*1\, d_2 = [-\kappa]\, d_0 = -\kappa$ or $d_2 = -\kappa/(2a^2b^2)$. The n=4 equation then reads

$$a^2b^2 4*3 d_4 = [\,(2)^2(a^2+b^2) - \kappa]\, \{-\kappa/(2a^2b^2)\} + [\,m(m+1) - (1)(0)]$$

Suppose we pick κ and m to kill off d_4 via this equation. Then we would also have to kill of d_6 at the same time to get two toasted muffins in a row. That would require

$$a^2b^2 6*5 d_6 = + [\,m(m+1) - (3)(2)]\, \{-\kappa/(2a^2b^2)\}$$

So we have to pick m = 2 (or -3) to kill d_6. Then the d_4 equation reads

$$a^2b^2 4*3 d_4 = [\,(2)^2(a^2+b^2) - \kappa]\, \{-\kappa/(2a^2b^2)\} + [\,6\,]$$

So to kill d_4 we need to have

$$[\,(2)^2(a^2+b^2) - \kappa]\, \{-\kappa/(2a^2b^2)\} + 6 = 0$$
$$[\,(2)^2(a^2+b^2) - \kappa]\, \{-\kappa\} + 12a^2b^2 = 0$$
$$\kappa^2 - 4(a^2+b^2)\kappa + 12a^2b^2 = 0$$
$$2\kappa = 4(a^2+b^2) \pm \{16(a^2+b^2)^2 - 48\, a^2b^2\}^{1/2}$$
$$2\kappa = 4(a^2+b^2) \pm 4\{(a^2+b^2)^2 - 3\, a^2b^2\}^{1/2}$$
$$\kappa = 2[\,(a^2+b^2) \pm \sqrt{(a^2+b^2)^2 - 3\, a^2b^2}\,]$$

So for our even Case II we choose m = 2 or -3 (totally equivalent) and then there are two possible values for κ that achieve truncation. Our corresponding solutions are now

$$F_2^{\kappa}(\xi) = 1 + \{-\kappa/(2a^2b^2)\}\xi^2 \qquad \kappa = 2[\,(a^2+b^2) \pm \sqrt{(a^2+b^2)^2 - 3\, a^2b^2}\,]$$

We can rescale this to get

$$G_2^{\kappa}(\xi) = -(2a^2b^2)/\kappa + \xi^2 \qquad \kappa = 2[\,(a^2+b^2) \pm \sqrt{(a^2+b^2)^2 - 3\,a^2b^2}\,]$$

Now I think we can rewrite this. Consider our $d_4 = 0$ equation above which we rewrite as

$$[\,(2)^2(a^2+b^2) - \kappa] = [\,6\,](2a^2b^2)/\kappa$$
$$[\,(2/3)(a^2+b^2) - \kappa/6] = (2a^2b^2)/\kappa$$
$$(1/3)[\,2(a^2+b^2) - \kappa/2] = (2a^2b^2)/\kappa$$
$$(1/3)[\,\kappa/2 - 2(a^2+b^2)\,] = -(2a^2b^2)/\kappa$$
$$-(2a^2b^2)/\kappa = (1/3)[\,\kappa/2 - 2(a^2+b^2)\,]$$
$$= (1/3)[\,(a^2+b^2) \pm \sqrt{(a^2+b^2)^2 - 3\,a^2b^2} - 2(a^2+b^2)\,]$$
$$= (1/3)[\,-(a^2+b^2) \pm \sqrt{(a^2+b^2)^2 - 3\,a^2b^2}\,]$$
$$= -(1/3)[\,(a^2+b^2) \mp \sqrt{(a^2+b^2)^2 - 3\,a^2b^2}\,]$$

Thus our Case II solutions are

$$G_2^{\kappa}(\xi) = \xi^2 - (1/3)[\,(a^2+b^2) \mp \sqrt{(a^2+b^2)^2 - 3\,a^2b^2}\,]$$

which agrees with (10.3.86).

Case III Even.

In Case II, we managed to make d_4 and d_6 vanish. Here let's try instead to make d_6 and d_8 vanish. Here are our equations, where I have filled in for d_2 in several places

$$n = 2 \quad a^2b^2 2*1\, d_2 = [-\kappa]$$
$$n = 4 \quad a^2b^2 4*3 d_4 = [4(a^2+b^2) - \kappa][-\kappa] + [\,m(m+1)]$$
$$n = 6 \quad a^2b^2 6*5 d_6 = [\,16(a^2+b^2) - \kappa]\, d_4 + [\,m(m+1) - 6][-\kappa]$$
$$n = 8 \quad a^2b^2 8*7 d_8 = [36(a^2+b^2) - \kappa]\, d_6 + [\,m(m+1) - 20]\, d_4$$

Assume we have $d_6 = 0$ somehow (and $d_4 \neq 0$). Then we make $d_8 = 0$ by taking m = 4. So we are done killing off d_8 . Now how do we kill off d_6? We need

$$0 = [\,16(a^2+b^2) - \kappa]\, d_4 + [\,m(m+1) - 6][-\kappa]$$

$$0 = [\,16(a^2+b^2) - \kappa]\,\{[4(a^2+b^2) - \kappa][-\kappa] + [\,m(m+1)]\}/(12a^2b^2) + [\,m(m+1) - 6][-\kappa]$$

$$0 = [\,16(a^2+b^2) - \kappa]\,\{[4(a^2+b^2) - \kappa][-\kappa] + [20]\}/(12a^2b^2) + [20 - 6][-\kappa]$$

$$0 = [\,16(a^2+b^2) - \kappa]\,\{[4(a^2+b^2) - \kappa][-\kappa] + [20]\}/(12a^2b^2) + 14\kappa$$

$$0 = [\,16(a^2+b^2) - \kappa]\,\{[2(a^2+b^2) - \kappa][-\kappa] + [10]\}/(12a^2b^2) + 7\kappa$$

This is a **cubic for κ** so I presume there will be 3 solutions and we could then write them out in some manner having this form:

$$F_2^{\kappa}(\xi) = 1 + A\xi^2 + B\xi^4$$

which we might rescale like this

$$G_4^{\kappa}(\xi) = \xi^4 + \alpha\xi^2 + \beta$$

where we get three sets of (α,β) coefficients [or (A,B)] corresponding to our three κ values.

So let's try to find the general trend here:

Even Solutions:

m = 0	κ
m = 2	κ_1, κ_2
m = 4	$\kappa_1, \kappa_2, \kappa_3$
m = 6	$\kappa_1, \kappa_2, \kappa_3, \kappa_4$
etc	

There seem to be m/2+1 solutions for each legal m value.

For odd series, we will find solutions similar to these, but we now have $d_1 = 1$ so

$$a^2b^2n(n-1)d_n = [\,(n-2)^2(a^2+b^2) - \kappa]\, d_{n-2} + [\,m(m+1) - (n-3)(n-4)]\, d_{n-4}$$

n = 3 $\quad a^2b^2 3*2 d_3 = [\,(1)^2(a^2+b^2) - \kappa]\, d_1$

n = 5 $\quad a^2b^2 5*4 d_5 = [\,(3)^2(a^2+b^2) - \kappa]\, d_3 + [\,m(m+1) - (2)(1)]\, d_1$

n = 7 $\quad a^2b^2 7*6 d_7 = [\,(5)^2(a^2+b^2) - \kappa]\, d_5 + [\,m(m+1) - (4)(3)]\, d_3$

Case I odd: Here we can kill off d_3 and d_5 using $\kappa = a^2+b^2$ and then m = 1. Solution is

solution = ξ $\qquad$ one solution $\qquad$ m = 1

Case II odd: Now want to kill off d_5 and d_7. This means m = 3. Then we have $d_5 = 0$ saying

$$0 = [\,(3)^2(a^2+b^2) - \kappa]\, d_3 + [\,m(m+1) - (2)(1)]\, 1$$
$$0 = [\,(3)^2(a^2+b^2) - \kappa]\, \{\,[\,(1)^2(a^2+b^2) - \kappa]\, d_1\} + [\,m(m+1) - (2)(1)]\, 1$$
$$0 = [\,(3)^2(a^2+b^2) - \kappa]\, \{\,[\,(1)^2(a^2+b^2) - \kappa]\, 1\} + [\,12 - (2)(1)]\, 1$$
$$0 = [\,(3)^2(a^2+b^2) - \kappa]\, \{\,[\,(a^2+b^2) - \kappa]\, 1\} + 10$$

This is a quadratic for κ so there will be two solutions of the form

solution = $\xi + A\xi^3$ $\qquad$ two solutions $\qquad$ m = 3

Here is my guess for how this will work

Odd Solutions:

m = 1	κ
m = 3	κ_1, κ_2
m = 5	$\kappa_1, \kappa_2, \kappa_3$
m = 7	$\kappa_1, \kappa_2, \kappa_3, \kappa_4$
etc	

So if we combine the even and odd solutions, we find solutions for each positive integer m, and the number of solutions for each m is m/2+1

ALL these solutions are **"first species"** because none of them have a square root leading factor. The other three species are found by assuming leading factors of :

factor	1	$\sqrt{\xi\text{-a}}$	$\sqrt{\xi\text{-b}}$	$\sqrt{\xi\text{-a}}\sqrt{\xi\text{-b}}$
species	1st	2nd	3rd	4th

It certainly seems reasonable that you might have such leading factors, given the indices of the regular singular points noted above.

It happens that, when you multiply the three functions together, you obtain simple Cartesian coordinate forms like Ax and Axz and so on. These Cartesian functions are "harmonic" since $\nabla^2(xz) = 0$, for example. Any function which is linear in one of the Cartesian variables will be harmonic, and recall that -- after all -- we are looking for Laplace solutions.

Conclusion and Main Point. When we seek solutions to the Lamé equation in our ellipsoidal coordinate framework, we need to find series solutions that converge for all ξ, since our three product functions have three separate ranges relative to a and b and these ranges then fill all of $(0,\xi)$, This rules out any non-truncating series solutions. Our two separation constants are named m and κ. What we find is a two-dimensional quantization of these separation constants. Because the recursor is triple, we find that BOTH m and κ must have particular discrete values. When I first learned this, it seemed strange, but having done the work above, I see exactly how this quantization arises. The value of m is always an integer, and κ I think is always a set m%2 + 1 discrete values, at least for the first species solutions. I did not look much into the other species, but I get the point.

It should be kept in mind that for each first kind solution of the type noted here, we can get "the other solution" using the usual "trick", and all Lamé authors mention this fact. That is why I write

$$[\, A_m^p\ E_m^p(\xi_1) + B_m^p\ F_m^p(\xi_1)]\ [\, A'^p_m\ E_m^p(\xi_2) + B'^p_m\ F_m^p(\xi_2)]\ [\, A''^p_m\ E_m^p(\xi_3) + B''^p_m\ F_m^p(\xi_3)]$$

as my general solution term, where I am now using E for all species. The index "p" enumerates the various κ values you have for each m. The F's are the second kind solutions from "the trick".

Here now is a page of Byerly :

$E_0(x)$

$$K_0(x) = 1$$
$$L_0(x) = 0$$
$$M_0(x) = 0$$
$$N_0(x) = 0$$

$E_1(x)$

$$K_1(x) = x$$
$$L_1(x) = \sqrt{x^2 - b^2}$$
$$M_1(x) = \sqrt{x^2 - c^2}$$
$$N_1(x) = 0$$

$E_2(x)$

$$K_2^{p_1}(x) = x^2 - \tfrac{1}{3}[b^2 + c^2 - \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$$
$$K_2^{p_2}(x) = x^2 - \tfrac{1}{3}[b^2 + c^2 + \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$$
$$L_2(x) = x\sqrt{x^2 - b^2}$$
$$M_2(x) = x\sqrt{x^2 - c^2}$$
$$N_2(x) = \sqrt{(x^2 - b^2)(x^2 - c^2)}$$

$E_3(x)$

$$K_3^{p_1}(x) = x^3 - \frac{x}{5}[2(b^2 + c^2) - \sqrt{4(b^2 + c^2)^2 - 15b^2c^2}]$$

$$K_3^{p_2}(x) = x^3 - \frac{x}{5}[2(b^2 + c^2) + \sqrt{4(b^2 + c^2)^2 - 15b^2c^2}]$$

$$L_3^{q_1}(x) = \sqrt{x^2 - b^2}[x^2 - \tfrac{1}{5}(b^2 + 2c^2 - \sqrt{(b^2 + 2c^2)^2 - 5b^2c^2})]$$
$$L_3^{q_2}(x) = \sqrt{x^2 - b^2}[x^2 - \tfrac{1}{5}(b^2 + 2c^2 + \sqrt{(b^2 + 2c^2)^2 - 5b^2c^2})]$$
$$M_3^{r_1}(x) = \sqrt{x^2 - c^2}[x^2 - \tfrac{1}{5}(2b^2 + c^2 - \sqrt{(2b^2 + c^2)^2 - 5b^2c^2})]$$
$$M_3^{r_2}(x) = \sqrt{x^2 - c^2}[x^2 - \tfrac{1}{5}(2b^2 + c^2 + \sqrt{(2b^2 + c^2)^2 - 5b^2c^2})]$$
$$N_3(x) = x\sqrt{(x^2 - b^2)(x^2 - c^2)}$$

You can see he uses letters K,L,M,N for species 1,2,3,4. But at a higher level he just organizes the solutions according to the m value. The even Case II solutions I found above (and which MF found) appear inside the E_2 box. So in Byerly's scheme, the letter indicates the species, the lower index is m, and the upper index is an enumerator index. The set of solutions goes on forever, with each box getting larger and larger. For each species, when you insert the form, you get a slightly different recursor equation, so

things are slightly different and the κ values are different. Byerly does it all. He goes on to mention the second kind stuff, and writes

$$\text{It is} \qquad z = AE_m^p(x) + BE_m^p(x)\int_x^\infty \frac{dx}{\sqrt{(x^2-b^2)(x^2-c^2)}[E_m^p(x)]^2}. \qquad (1)$$

$$F_m^p(x) = (2m+1)E_m^p(x)\int_x^\infty \frac{dx}{\sqrt{(x^2-b^2)(x^2-c^2)}[E_m^p(x)]^2}. \qquad (2)$$

Byerly does not come up with a general formula for the general solution, I imagine that is what the methods of W&W (that I did not like) talk about.

Closing Comments: I looked first at spherical, then at spheroidal coordinates, pondering the question of the cross-linkage of the separation constants and how they got quantized. In these two cases, quantization started with the azimuthal world, usually an index called m taking equally spaced values. For each such m, we found that the second separation constant ν on P_ν^m had *also* to be quantized. This happens a little differently in different cases. For a full sphere or spheroid, ν is the simple string ν = m, m+1, m+2... as this causes P to be finite at z = -1 and +1. For an orange slice of a sphere, ν = a similar string but is not integers. For a spherical section with one or two conical surfaces having V = 0, ν = a string that causes the P function to have zeroes on the conical surface(s).

When we turn to ellipsoidals, there is no azimuth to start quantization with, but as we have seen, it turns out that there is still an integral quantum number m = 0,1,2,3... just as in the azimuthal systems. In ellipsoidals, however, the second separation constant κ takes rather strange values, not just an equally spaced string of values as in the other two cases and the ν coordinate. The mechanism which causes the double quantization in ellipsoidals is still that the solution functions be finite on their intervals, or shall we say, the solution functions exist on their intervals. In the ellipsoidal case, the three separated ODE's are all the same, so we need a solution valid on the entire interval (0,∞) and this requires truncation, and that requires double quantization.

Finally, I have to say there is really no mystery about these Lamé solutions and the four species and the second kind partners. It is all completely straightforward, and there is no need to bring in the Weierstrass P function or Neville's method and all that stuff that WW get into.

Just a reminder about the exterior potential of a metal ellipsoid held at constant V. We find the solution with the lowest level functions $E_0^0 = 1$ for ξ_2 and ξ_3, and the F_0^0 second partner for ξ_1. This partner happens to be a Jacobi type elliptic integral function sn^{-1} of F(z,k) thing. Both MF and Kelvin and I am sure Byerly do this solution, and compress to get the elliptical plate as well. It is not rocket science, I have to keep saying that over and over.