Derive Hobson p 479 B,C and D and p 480 A PhL 11.10.10

I have flailed at this for perhaps 6 hours, so it is time to move it this separate document where I can find out what I am doing wrong. // It is all done right now, there was a lot to do. I summarize the work done here in the main Hobson notes doc.

Overview (2 pages) **1**

Problem 1: **2**

Problem 2: **10**

Overview (2 pages)

The conical harmonics are given in Hobson p 478 B as:

$$r^n E_{n,p}(\mu) E_{n,p}(\nu)$$

$$= \{ 1 \quad x \quad y \quad z \quad xy \quad xz \quad yz \quad xyz \} \; \kappa'' \Pi_{s=1}^{J} [x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2)]$$

K	K	L	M	M	L	N	N	// class
e	o	o	o	e	e	e	o	// n = even or odd
1	2	2	2	3	3	3	4	// species s

// Hobson p 478 A

where J = (n + s - 1)/2 which is always an integer for the allowed n even or odd values shown. For any given n, there will be 2n+1 harmonics and those will be associated with 4 of the 8 forms shown, depending on the parity of n. The number of harmonics associated with each class is of course the number of E functions associated with each class, and those counts are given by (Hobson notation here)

K	r+1	where r = n/2 if n even, else (n-1)/2
L	n-r	
M	n-r	
N	r	
	2n+1 solutions!!	

and these are labeled in some manner by the p index on $E_{n,p}(\mu)$. We know that we can write the E functions in this manner

n even:

$K_{n,p} = \Pi_{s=1}^{J} (\mu^2 - \rho_s^2)$ J = (n-0)/2 = n/2

$L_{n,p} = \sqrt{\mu^2 - h^2}\ \mu \Pi_{s=1}^{J} (\mu^2 - \rho_s^2)$ J = (n-1-1)/2 = (n-1)/2

$M_{n,p} = \sqrt{\mu^2 - k^2}\ \mu \Pi_{s=1}^{J} (\mu^2 - \rho_s^2)$ J = (n-1-1)/2 = (n-1)/2

$N_{n,p} = \sqrt{\mu^2 - h^2}\ \sqrt{\mu^2 - k^2}\ \Pi_{s=1}^{J} (\mu^2 - \rho_s^2)$ J = (n-2)/2 = (n-2)/2

n odd:

$K_{n,p} = \mu \prod_{s=1}^{J} (\mu^2 - \rho_s^2)$ $\quad J = (n-0-1)/2 = (n-1)/2$

$L_{n,p} = \sqrt{\mu^2 - h^2} \prod_{s=1}^{J} (\mu^2 - \rho_s^2)$ $\quad J = (n-1)/2 = (n-1)/2$

$M_{n,p} = \sqrt{\mu^2 - k^2} \prod_{s=1}^{J} (\mu^2 - \rho_s^2)$ $\quad J = (n-1)/2 = (n-1)/2$

$N_{n,p} = \sqrt{\mu^2 - h^2}\sqrt{\mu^2 - k^2}\ \mu \prod_{s=1}^{J} (\mu^2 - \rho_s^2)$ $\quad J = (n-2-1)/2 = (n-3)/2$

so we "know all there is to know" about the Lamé functions once we know the roots ρ_s. We know the exact form for all the classes, and we know the number of harmonics (number of E functions) in each class. The purpose of the calculation here is to determine the θ_s values which make ∇^2(harmonic) = 0, and then we know the values $\rho_s^2 = a^2 + \theta_s$ of the (squared) roots. There is perhaps some issue with how we allocate the roots to the various functions in a given class. [See META notes about $\boldsymbol{\theta}^{(k)}$ for clarification of this issue, there is no ambiguity at all about how to allocate the roots.]

In **Problem 1** we compute this quantity

$$\nabla^2 [\{1\} \{ \prod_{s=1}^{J} [x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)] \}]$$

which is simply a product of quadratic factors There are J quadratic factors. J is a function of harmonic degree n as shown above, but that connection is not needed here. The above [ ] shows the first family of "conical harmonics" showing in Hobson p 478B (K class, n even). We shall then set the result to 0 to get a system of equations which we can then solve to find the root values θ_s and then $\rho_s^2 = a^2 + \theta_s$ which are the actual squared roots of the Lamé functions in the K class for even n. In the main doc, we do cases n = 2 and n = 4 explicitly showing results for all classes. Problem 1 is fairly complicated, so it is presented in 13 numbered subsections.

In **Problem 2** we show how the result gets modified to apply to any of the other 7 harmonic families. There we need to compute

$$\nabla^2 [\{ x\ y\ z\ xy\ xz\ yz\ xyz \} \{ \prod_{s=1}^{J} [x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)] \}]$$

and set to 0 and deduce the θ_s in these cases. We treat n=3 explicitly in the main notes, for all four classes K,L,M,N. Problem 2 is much easier than Problem 1 since it uses the hard-earned results of Problem 1. Also, we are able really to do all the cases at once assuming a certain $x_\alpha x_\beta x_\gamma$ form for the primitives.

Problem 1: What happens if we apply ∇^2 to $\{ \prod_{s=1}^{J} [x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)] \}$?

(1) Let's define a_i = a,b,c and write this as

$$\nabla^2 \prod_{s=1}^{J} [\Sigma_i x_i^2/(a_i^2 + \theta_s)] = \Sigma_j \partial_j^2 \{ \prod_{s=1}^{J} [\Sigma_i x_i^2/(a_i^2 + \theta_s)] \}$$

(2) now define

$f_s = \Sigma_i \, x_i^2/(a_i^2 + \theta_s)$

Then our problem is to compute

$\nabla^2 \, \Pi_{s=1}^J \, [f_s] = \Sigma_j \, \partial_j^2 \, \{ \, \Pi_{s=1}^J \, [f_s] \}$

(3) Compute the first derivative. We want to compute this object,

$\partial_j \{ \, \Pi_{s=1}^J \, [f_s] \}$

It seems to me this is just a multifactor product rule and we get this

$\partial_j \, (\Pi_{s=1}^J \, f_s) \; = \partial j \, (\, f_1 f_2 ....f_J) = f_1' \, f_2 ..... \; + f_1 f_2' f_3 .... \text{ etc } \; = \Sigma_{k=1}^J \, (\partial j \, f_k) \, (\Pi_{s \neq k} \, f_s)$

Just to make sure, let's write out the last sum

$= \; (\partial j \, f_1) \, f_2 f_3 .... \; + \; (\partial j \, f_2) \, f_1 f_3 f_4 ... \; + \text{etc}$

I am pretty sure I have step 3 correct.

(4) Apply the second derivative:

$\nabla^2 \, \Pi_{s=1}^J \, [f_s] = \Sigma_j \, \partial_j^2 \, \{ \, \Pi_{s=1}^J \, [f_s] \}$

$= \Sigma_j \, \partial_j \, \{ \, \partial_j \, (\Pi_{s=1}^J \, [f_s]) \, \}$

Now insert into this our result from step 3

$= \Sigma_j \, \partial_j \, \{ \; \Sigma_{k=1}^J \, (\partial_j \, f_k) \, (\Pi_{s \neq k} \, f_s) \; \} \; = \; \Sigma_{k=1}^J \, \Sigma_j \, \partial_j \, \{ \; (\partial_j \, f_k) \, (\Pi_{s \neq k} \, f_s) \; \}$

For the moment, we just think of {..} containing just two factors, and we use the product rule to get

$= \Sigma_{k=1}^J \, \Sigma_j \; [(\partial_j^2 \, f_k) \; (\Pi_{s \neq k} \, f_s) \; + \; (\partial_j \, f_k) \, \partial_j \, (\Pi_{s \neq k} \, f_s)$

$= \Sigma_{k=1}^J \; [\nabla^2 f_k \; (\Pi_{s \neq k} \, f_s) \; + \; \Sigma_j \, (\partial_j \, f_k) \, \partial_j \, (\Pi_{s \neq k} \, f_s) \,]$

(5) We now need to very carefully compute this quantity

$\partial_j \, (\Pi_{s \neq k} \, f_s) \; = \partial_j (\, f_1 f_2 f_3 \;\; f_5 f_6 ....)$

where we assume for the moment that k = 4. We continue:

$= (\partial_j f_1) f_2 f_3 \;\; f_5 f_6 ... \; + \; f_1 (\partial_j f_2) f_3 \;\; f_5 f_6 ... \; + ...$ but the $(\partial_j f_4)$ term is missing.

$= \Sigma_{n\neq 4} (\partial_j f_n) (\Pi_{s\neq n, s\neq 4} f_s)$

To verify this result, let's go the other way and expand it

$= (\partial_j f_1) (\Pi_{s\neq 1, s\neq 4} f_s) + (\partial_j f_2) (\Pi_{s\neq 2, s\neq 4} f_s) + ...$

This certainly *seems* right. It seems then that instead of using k = 4, we could use general k = k and say

$\partial_j (\Pi_{s\neq k} f_s) = \Sigma_{n\neq k} (\partial_j f_n) (\Pi_{s\neq n, s\neq k} f_s)$

(6) If we install this result back into our computation we get

$\nabla^2 \Pi_{s=1}^{J} [f_s] = \Sigma_{k=1}^{J} [\nabla^2 f_k (\Pi_{s\neq k} f_s) + \Sigma_j (\partial_j f_k) \partial_j (\Pi_{s\neq k} f_s)]$

$= \Sigma_{k=1}^{J} [\nabla^2 f_k (\Pi_{s\neq k} f_s) + \Sigma_j (\partial_j f_k) \Sigma_{n\neq k} (\partial_j f_n) (\Pi_{s\neq n, s\neq k} f_s)]$

$= \Sigma_k \nabla^2 f_k (\Pi_{s\neq k} f_s) + \Sigma_k \Sigma_j (\partial_j f_k) \Sigma_{n\neq k} (\partial_j f_n) (\Pi_{s\neq n, s\neq k} f_s)$

That second term is certainly looking pretty opaque. There are three sums and a product. Only the j sum is 1,2,3. It certainly seems that we should rearrange the sums in this way

$= \Sigma_k (\nabla^2 f_k) (\Pi_{s\neq k} f_s) + \Sigma_k \Sigma_{n\neq k} \Sigma_j (\partial_j f_k)(\partial_j f_n) (\Pi_{s\neq n, s\neq k} f_s)$

and we could write $(\partial_j f_k)(\partial_j f_n) = \nabla f_k \bullet \nabla f_n$.

(7) Let's work only on the **first term** for the moment. We have

$\nabla^2 f_k = \nabla^2[\Sigma_i x_i^2/(a_i^2 + \theta_s)] = \Sigma_i (\partial_j^2 x_i^2)/(a_i^2 + \theta_s)$

$\partial_j x_i^2 = 2x_i\delta_{ij}$ => $\partial_j^2 x_i^2 = \Sigma_j \partial_j 2x_i\delta_{ij} = \Sigma_j 2\delta_{ij}\delta_{ij} = 2\delta_{ii} = 2$

In passing, we have just shown that

$\partial_j f_s = \Sigma_i (\partial_j x_i^2)/(a_i^2 + \theta_s) = \Sigma_i 2x_i\delta_{ij}/(a_i^2 + \theta_s) = 2x_j / (a_j^2 + \theta_s)$

Then we have

$\nabla^2 f_k = \Sigma_i (\partial_j^2 x_i^2)/(a_i^2 + \theta_k) = \Sigma_i 2/(a_i^2 + \theta_k) = 2\{\Sigma_i 1/(a_i^2 + \theta_k)\}$

Therefore our "first term" is this:

$\nabla^2 \Pi_{s=1}^{J} [f_s]_{\text{first term}} = \Sigma_k \nabla^2 f_k (\Pi_{s\neq k} f_s) =$

$= \Sigma_k 2\{\Sigma_i 1/(a_i^2 + \theta_k)\} (\Pi_{s\neq k} f_s)$

$$= 2\,\Sigma_k\ \Sigma_i\ [1/(a_i^2 + \theta_k)]\quad (\Pi_{s\neq k}\ f_s)$$

Let's write out some terms:

$$= 2\ 1/(a_i^2 + \theta_1) *\ \ f_2 f_3 ....\ +\ 2\ 1/(a_i^2 + \theta_2)\ f_1\quad f_3 f_4 ....\ + ...$$

There are a total of J terms (J = Hobson's m). So my first term is exactly twice the first line of Hobson p 477 B.

(8) If his result and my results so far are both correct, it must be possible to show that

$$\nabla^2\ \Pi_{s=1}^{J}\ [f_s]_{\mathbf{second\ term}}\ = \Sigma_k\ \Sigma_{n\neq k}\ \Sigma_j\ (\partial_j\ f_k)(\partial_j f_n)\ (\Pi_{s\neq n, s\neq k}\ f_s)$$

is equal to twice Hobson's second line in B. Now we know that from step 7 that

$$\partial_j f_k = \ 2x_j\ /\ (a_j^2 + \theta_k)$$
$$\partial_j f_n = \ 2x_j\ /\ (a_j^2 + \theta_n)$$

so we then have

$$\nabla^2\ \Pi_{s=1}^{J}\ [f_s]_{\mathbf{second\ term}}\ = \Sigma_k\ \Sigma_{n\neq k}\ \Sigma_j\ (\partial_j\ f_k)(\partial_j f_n)\ (\Pi_{s\neq n, s\neq k}\ f_s)$$

$$= \Sigma_k\ \Sigma_{n\neq k}\ \Sigma_j\ [2x_j\ /\ (a_j^2 + \theta_k)][\ 2x_j\ /\ (a_j^2 + \theta_n]\ (\Pi_{s\neq n, s\neq k}\ f_s)$$

$$= 4\ \ \Sigma_k\ \Sigma_{n\neq k}\ \Sigma_j\quad [x_j^2\ /\ [(a_j^2 + \theta_k)\ (a_j^2 + \theta_n)\ ]\ (\Pi_{s\neq n, s\neq k}\ f_s)$$

At least we are getting denominators that look like Hobson's second line B denominators. Let's now define

$$F_{kn} \equiv \Sigma_j\quad [x_j^2\ /\ [(a_j^2 + \theta_k)\ (a_j^2 + \theta_n)\ ]$$

and we note that this thing is symmetric under n↔k. So we have

$$\nabla^2\ \Pi_{s=1}^{J}\ [f_s]_{\mathbf{second\ term}}\ = 4\ \ \Sigma_k\ \Sigma_{n\neq k}\ F_{kn}\ (\Pi_{s\neq n, s\neq k}\ f_s)$$

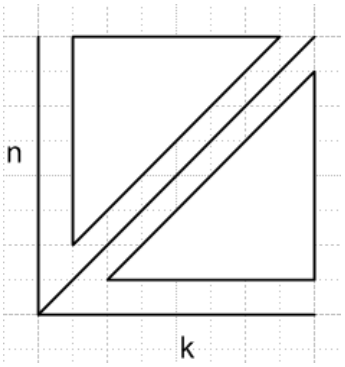
The product $(\Pi_{s\neq n, s\neq k}\ f_s)$ is also symmetric $(\Pi_{s\neq n, s\neq k}\ f_s)$.

(9) So what can we say about

$$\text{sum} = \Sigma_k\ \Sigma_{n\neq k}\ S_{nk}$$

where S is symmetric? If we draw a 2D sum space picture with axes k and n, we get a sum over the entire square less the diagonal. We can think of this as two triangles. [Note: in usual matrix pictures, the n axis

points down but here it points up. In a matrix picture, symmetric is about a \ directed diagonal, but in our picture symmetric is about the / directed axis. So the numbers in the lower triangle are the same as those in the upper, reflected in the / diagonal.]



Breaking the sum into the two triangular contributions. We then get two ways to write each triangle

lower triangle sum = $\Sigma_{k=1}^{J} \Sigma_{n=1}^{k-1} S_{nk} = \Sigma_{n=1}^{J} \Sigma_{k=n+1}^{J} S_{nk}$

upper triangle sum = $\Sigma_{k=1}^{J} \Sigma_{n=k+1}^{J} S_{nk} = \Sigma_{n=1}^{J} \Sigma_{k=1}^{n-1} S_{nk}$

We can then pick two and say

sum = lower triangle + upper triangle $= \Sigma_{k=1}^{J} \Sigma_{n=1}^{k-1} S_{nk} + \Sigma_{n=1}^{J} \Sigma_{k=1}^{n-1} S_{nk}$

We can swap the dummy summation indices in the second sum and say

sum = $\Sigma_{k=1}^{J} \Sigma_{n=1}^{k-1} S_{nk} + \Sigma_{k=1}^{J} \Sigma_{n=1}^{k-1} S_{kn}$

$= \Sigma_{k=1}^{J} \Sigma_{n=1}^{k-1} [S_{nk} + S_{kn}]$

But at this point we can use the assumed symmetry of S to get

sum = $2 \Sigma_{k=1}^{J} \Sigma_{n=1}^{k-1} S_{kn} = 2 \Sigma_{k} \Sigma_{n<k} S_{kn}$

Suppose we swap the summation indices here to get

sum = $2 \Sigma_{n} \Sigma_{k<n} S_{nk}$

If we then use the symmetry of S, we find that

sum = $2 \Sigma_{k} \Sigma_{n<k} S_{kn} = 2 \Sigma_{n} \Sigma_{k<n} S_{kn}$ (*)

which just tells us we can swap the two indices with abandon.

(10) Now the big question: does this help us in some way? We were doing this sum, which we then rewrite:

$$4\ \Sigma_k \Sigma_{n\neq k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s) = 8 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s)$$

Let's write out some of the terms:

$$\Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s) =$$

We see that there is no term with k= 1 because you cannot then have n < 1. So start with k = 2

$$\begin{array}{lll} k=2\ \ n=1 & k=3\ \ n=1 & k=3\ \ n=2 \\ = F_{21} (\ * * f_3 f_4 ..... f_J) & + F_{31} (\ * f_2 * f_4 ..... f_J) & + F_{32} (\ f_1 * * f_4 ..... f_J) + ... \end{array}$$

This does look promising. Recall that

$$F_{kn} \equiv \Sigma_j \ [x_j^2 / [(a_j^2 + \theta_k) (a_j^2 + \theta_n)]$$

so on Hobson's second like, that first {...} thing is exactly F_{12} . His notation really is sufficient to show the full result I think.

How many terms are there? How many ways can you pick 2 out of J? So (J,2) is the answer. That would be (1/2) J!/(J-2)! which is J(J-1)/2 terms.

So we how have

$$\nabla^2 \Pi_{s=1}^J [f_s]_{\text{second term}} = 4 \Sigma_k \Sigma_{n\neq k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s)$$

$$= 8 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s)$$

And in total we have

$$\nabla^2 \Pi_{s=1}^J [f_s] =$$

$$2 \Sigma_k \Sigma_i [1/(a_i^2 + \theta_k)] (\Pi_{s\neq k} f_s) + 8 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s)$$

If we require this be 0, our condition is then

$$\Sigma_k \Sigma_i [1/(a_i^2 + \theta_k)] (\Pi_{s\neq k} f_s) + 4 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s) = 0 \qquad (*)$$

which we write as

$$\Sigma_k \{ \underbrace{\Sigma_i [1/(a_i^2 + \theta_k)] (\Pi_{s\neq k} f_s)}_{J \text{ terms}} + \underbrace{4 \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s)}_{J(J-1)/2 \text{ terms}} \} = 0$$

where $F_{kn} \equiv \Sigma_j \ [x_j^2 / [(a_j^2 + \theta_k) (a_j^2 + \theta_n)]$

This is Hobson p 479 B, finally !

(11) While we are here, let's also derive Hobson p 479 C.

$$f_n - f_k = \Sigma_i\, x_i^2/(a_i^2 + \theta_n) - \Sigma_i\, x_i^2/(a_i^2 + \theta_k)$$

$$= \Sigma_i\, x_i^2\, [(a_i^2 + \theta_k) - (a_i^2 + \theta_n)\ ]\ /\ [(a_i^2 + \theta_k)\,(a_i^2 + \theta_n)]$$

$$= \Sigma_i\, x_i^2\, [\theta_k - \theta_n]\ /\ [(a_i^2 + \theta_k)\,(a_i^2 + \theta_n)]$$

$$= [\theta_k - \theta_n]\ \{\ \Sigma_i\, x_i^2\ /\ [(a_i^2 + \theta_k)\,(a_i^2 + \theta_n)]\ \}$$

$$= [\theta_k - \theta_n]\, F_{kn}$$

So we have that

$$F_{nk} = \{\ \Sigma_i\, x_i^2\ /\ [(a_i^2 + \theta_n)\,(a_i^2 + \theta_k)]\ \} = (f_n - f_k)/(\theta_k - \theta_n) \qquad // \ n \leftrightarrow k \ \text{symmetric}$$

and if we set n = 1 and k = 2, we get Hobson p 479 C.

(12) Now we want to get to equation D. Suppose we stuff in this result (11) into (10) (*):

$$\Sigma_k\ \Sigma_i\, [1/(a_i^2 + \theta_k)]\ \ (\Pi_{s\neq k}\, f_s) + 4\,\Sigma_k\ \Sigma_{n<k}\, F_{kn}\, (\Pi_{s\neq n,\, s\neq k}\, f_s) = 0$$

so

$$\Sigma_k\ \Sigma_i\, [1/(a_i^2 + \theta_k)]\ \ (\Pi_{s\neq k}\, f_s) + 4\,\Sigma_k\ \Sigma_{n<k}\, [(f_n - f_k)/(\theta_k - \theta_n)]\, (\Pi_{s\neq n,\, s\neq k}\, f_s) = 0$$

Consider this second term some more:

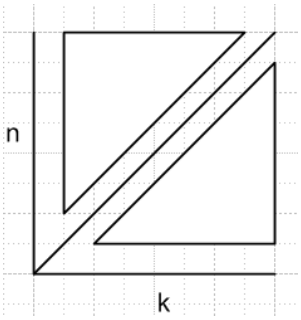
$$4\Sigma_k\ \Sigma_{n<k}\, [(f_n - f_k)/(\theta_k - \theta_n)]\, (\Pi_{s\neq n,\, s\neq k}\, f_s)$$

$$= 4\Sigma_k\ \Sigma_{n<k}\, [f_n\ /(\theta_k - \theta_n)]\, (\Pi_{s\neq n,\, s\neq k}\, f_s) - 4\Sigma_k\ \Sigma_{n<k}\, [f_k/(\theta_k - \theta_n)]\, (\Pi_{s\neq n,\, s\neq k}\, f_s)$$

$$= 4\Sigma_k\ \Sigma_{n<k}\, [1/(\theta_k - \theta_n)]\, (f_n\, \Pi_{s\neq n,\, s\neq k}\, f_s) - 4\Sigma_k\ \Sigma_{n<k}\, [1/(\theta_k - \theta_n)]\, (f_k\, \Pi_{s\neq n,\, s\neq k}\, f_s)$$

$$= 4\Sigma_k\ \Sigma_{n<k}\, [1/(\theta_k - \theta_n)]\, (\Pi_{s\neq k}\, f_s) - 4\Sigma_k\ \Sigma_{n<k}\, [1/(\theta_k - \theta_n)]\, (\Pi_{s\neq n}\, f_s)$$

If we look back at the k-n triangle situation again,



we see that $\Sigma_k\ \Sigma_{n<k}$ is a sum over the lower right triangle. We can rewrite this same sum in this way

$$\Sigma_k\ \Sigma_{n<k} \quad = \Sigma_n\ \Sigma_{k>n}$$

Lets do this in our second term above to get

$$= 4\Sigma_k\ \Sigma_{n<k}\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq k}\ f_s)\ -\ 4\ \Sigma_n\ \Sigma_{k>n}\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq n}\ f_s)$$

Now swap the dummy index names in the second term and rewrite

$$= 4\Sigma_k\ \Sigma_{n<k}\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq k}\ f_s)\ -\ 4\ \Sigma_k\ \Sigma_{n>k}\ [1/(\theta_n-\theta_k)]\ (\Pi_{s\neq k}\ f_s)$$

$$= 4\Sigma_k\ \Sigma_{n<k}\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq k}\ f_s)\ +\ 4\ \Sigma_k\ \Sigma_{n>k}\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq k}\ f_s)$$

$$= 4\Sigma_k\ (\Sigma_{n<k}\ +\ \Sigma_{n>k}\ )\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq k}\ f_s)$$

$$= 4\Sigma_k\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq k}\ f_s)$$

So now lets rejoin our two terms again to say

$$\Sigma_k\ \Sigma_i\ [1/(a_i^2 + \theta_k)]\ \ (\Pi_{s\neq k}\ f_s) + \ 4\ \Sigma_k\ \Sigma_{n<k}\ [(f_n - f_k)/(\theta_k-\theta_n)]\ (\Pi_{s\neq n,\, s\neq k}\ f_s)\ = 0 \qquad \text{// before}$$

$$\Sigma_k\ \Sigma_i\ [1/(a_i^2 + \theta_k)]\ \ (\Pi_{s\neq k}\ f_s) + \ 4\Sigma_k\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)]\ (\Pi_{s\neq k}\ f_s)\ \ = 0 \qquad \text{// now}$$

We can factor out some things and write this now as

$$\Sigma_k\ \ (\Pi_{s\neq k}\ f_s)\ \ \{\ \Sigma_i\ [1/(a_i^2 + \theta_k)]\ +\ 4\Sigma_k\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)]\ \}\ \ = 0$$

And now, once again -- finally --, we can solve this equation if we require that {...} = 0, that is,.

$$\Sigma_i\ [1/(a_i^2 + \theta_k)]\ +\ 4\Sigma_k\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)]\ = 0$$

Notice how we have avoided the zero denominator situation in the second term. Rewrite as

$$\Sigma_i\ [1/(a_i^2 + \theta_k)]\ +\ 4\Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)]\ = 0 \qquad\qquad k = 1...J$$

where we have slightly changed our meaning of $\Sigma_{n\neq k}$ so it now means a double sum excluding the diagonal elements.

The first of these equations is

$$\Sigma_i\,[1/(a_i^2+\theta_1)] \;+\; 4/(\theta_1-\theta_2)+4/(\theta_1-\theta_3)\;+\ldots 4/(\theta_1-\theta_J)\;=0$$

and this then is the first equation you see in Hobson p 479 D. There are J of these equations, and there are J variables which are $\theta_1....\theta_J$, so it would seem that there is a unique solution for the set $\{\theta_i\}$, and that of course is what we have been trying to do. This is not a simple Cramer's rule situation since our variables are all in denominators, so each equation is highly non-linear.

Now let's backtrack just a bit. We had

$$\nabla^2\,\Pi_{s=1}^{J}\;[f_s] \;=\; 2\,\Sigma_k\;\Sigma_i\,[1/(a_i^2+\theta_k)]\;\;(\Pi_{s\neq k}\,f_s)+\;8\,\Sigma_k\;\Sigma_{n<k}\,F_{kn}\,(\Pi_{s\neq n,\,s\neq k}\,f_s)$$

We then showed that

$$4\;\Sigma_k\,\Sigma_{n<k}\,F_{kn}\,(\Pi_{s\neq n,\,s\neq k}\,f_s) \;=\; 4\Sigma_k\;\Sigma_{n\neq k}\,[1/(\theta_k-\theta_n)]\;(\Pi_{s\neq k}\,f_s)\;=0$$

Therefore we know that

$$\nabla^2\,\Pi_{s=1}^{J}\;[f_s] \;=\; 2\,\Sigma_k\;\Sigma_i\,[1/(a_i^2+\theta_k)]\;\;(\Pi_{s\neq k}\,f_s)+\;\;8\Sigma_k\;\Sigma_{n\neq k}\,[1/(\theta_k-\theta_n)]\;(\Pi_{s\neq k}\,f_s)$$

$$=2\,\Sigma_k\,(\Pi_{s\neq k}\,f_s)\;\{\,\Sigma_i\,[1/(a_i^2+\theta_k)]+\;4\Sigma_k\;\Sigma_{n\neq k}\,[1/(\theta_k-\theta_n)]\;\}$$

so we can record this as the action of ∇^2 on our $\Pi_{s=1}^{J}\;[f_s]$, which we will use below.

Problem 2: We want here to look at all the other cases $\{\,x\;\;y\;\;z\;\;xy\;\;xz\;\;yz\;\;xyz\,\}\;\Pi_{s=1}^{J}\;[f_s]$.

In each case, we can thing or our harmonic as the product of a first term {...} times the second term. We already know certain very useful facts from Problem 1:

$$\partial_j\,\Pi_{s=1}^{J}[f_s] \;=\; \Sigma_k(\partial j\,f_k)\,(\Pi_{s\neq k}\,f_s) \qquad\qquad \text{// from (3)}$$

$$\partial_j^2\,\Pi_{s=1}^{J}[f_s] \;=\; 2\,\Sigma_k\,(\Pi_{s\neq k}\,f_s)\;\{\,\Sigma_i\,[1/(a_i^2+\theta_k)]+\;4\Sigma_k\;\Sigma_{n\neq k}\,[1/(\theta_k-\theta_n)]\;\}$$

In the first result we can make use of

$$\partial_j f_k = 2x_j\,/\,(a_j^2+\theta_k)$$

$$\Rightarrow\;\partial_j\,\Pi_{s=1}^{J}[f_s] \;=\; 2\Sigma_k x_j\,/\,(a_j^2+\theta_k)\;*\;(\Pi_{s\neq k}\,f_s)$$

Now let's roll out our vector identity which says (you can here it being trundled out of TK)

$$\nabla^2(uv) = u\nabla^2(v) + v\nabla^2(u) + 2\nabla u \bullet \nabla v$$

and apply it here to say

$$\nabla^2(\{..\}\Pi_{s=1}^{J}\ [f_s]) = \{..\}\nabla^2(\Pi_{s=1}^{J}\ [f_s]) + \Pi_{s=1}^{J}\ [f_s]\nabla^2(\{..\}) + 2\nabla\{..\} \bullet \nabla\ \Pi_{s=1}^{J}\ [f_s]$$

Now the good news is that $\nabla^2(\{..\}) = 0$ for all these primitive function, so that term goes away,

$$\nabla^2(\{..\}\Pi_{s=1}^{J}[f_s]) = \{..\}\nabla^2(\Pi_{s=1}^{J}[f_s]) + 2\nabla\{..\} \bullet \nabla\ \Pi_{s=1}^{J}[f_s]$$

$$= \{..\}\ [\quad 2\ \Sigma_k\ (\Pi_{s\neq k}\ f_s)\ \{\ \Sigma_i\ [1/(a_i^2+\theta_k)] + 4\Sigma_k\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)]\ \}\quad]$$

$$+ 2\ (\partial_j\ \{..\})\ 2\Sigma_k x_j\ /\ (a_j^2+\theta_k)\ *\ (\Pi_{s\neq k}\ f_s)$$

where we have now installed our "very useful facts" from above. We can move the k sum and Π factor off to the left and write this as

$$\nabla^2(\{..\}\Pi_{s=1}^{J}[f_s]) = \Sigma_k\ (\Pi_{s\neq k}\ f_s)\ \{..\}\ *$$

$$\{\ 2\ \Sigma_i\ [1/(a_i^2+\theta_k)] + 8\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)] + 4\ [(\partial_j\ \{..\})/\{...\}]\ x_j\ /\ (a_j^2+\theta_k)\ \}$$

Rewrite as

$$\nabla^2(\{..\}\Pi_{s=1}^{J}[f_s]) = 2\ \Sigma_k\ (\Pi_{s\neq k}\ f_s)\ \{..\}\ *$$

$$\{\ \Sigma_i\ [1/(a_i^2+\theta_k)] + 2\ [(\partial_j\ \{..\})/\{...\}]\ x_j\ /\ (a_j^2+\theta_k) + 4\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)]\ \}$$

And if we require this to be 0, we simply have

$$\Sigma_i\ [1/(a_i^2+\theta_k)] + 2\ [(\partial_j\ \{..\})/\{...\}]\ x_j\ /\ (a_j^2+\theta_k) + 4\ \Sigma_{n\neq k}\ [1/(\theta_k-\theta_n)] \qquad k=1.2..J$$

We can see right away that the last term is the same as we had before, so it is not affected by our changing from function {1} to function {...}. So now finally we are forced to consider the individual cases. But here is possible idea to treat them all uniformly. Let

$\{...\} = x_\alpha x_\beta x_\gamma$ where $x_0 = 1$ and the others are the usuals.

Two of the indices can be equal only if they are 0. Two non-zero indices must be different.

Then we have

$$\partial_j \{..\} = \partial_j \{ x_\alpha x_\beta x_\gamma \} = \delta_{j\alpha} x_\beta x_\gamma + x_\alpha \delta_{j\beta} x_\gamma + x_\alpha x_\beta \delta_{j\gamma}$$

$$\partial_j \{..\}/\{...\} = [\delta_{j\alpha} x_\beta x_\gamma + x_\alpha \delta_{j\beta} x_\gamma + x_\alpha x_\beta \delta_{j\gamma}]/ (x_\alpha x_\beta x_\gamma)$$

$$= \delta_{j\alpha}/x_\alpha + \delta_{j\beta}/x_\beta + \delta_{j\gamma}/x_\gamma$$

Our middle term is then

$$\Sigma_{j=1}^{3} 2 [(\partial_j \{..\})/\{...\}] x_j / (a_j^2+\theta_k) = 2 \Sigma_{j=1}^{3} [\delta_{j\alpha}/x_\alpha + \delta_{j\beta}/x_\beta + \delta_{j\gamma}/x_\gamma] x_j / (a_j^2+\theta_k)$$

But now for example $(\delta_{j\alpha}/x_\alpha)x_j = (\delta_{j\alpha}/x_j)x_j = \delta_{j\alpha}$ so we can write this as

$$= \Sigma_{j=1}^{3} [2\delta_{j\alpha} + 2\delta_{j\beta} + 2\delta_{j\gamma}] / (a_j^2+\theta_k)$$

Our first term is just

$$\Sigma_{j=1}^{3} 1 / (a_j^2+\theta_k)$$

so we can combine these two sums as

$$\Sigma_{j=1}^{3} [1 + 2\delta_{j\alpha} + 2\delta_{j\beta} + 2\delta_{j\gamma}] / (a_j^2+\theta_k)$$

$$= [1 + 2\delta_{1\alpha} + 2\delta_{1\beta} + 2\delta_{1\gamma}]/ (a_1^2+\theta_k)$$
$$+ [1 + 2\delta_{2\alpha} + 2\delta_{2\beta} + 2\delta_{2\gamma}]/ (a_2^2+\theta_k)$$
$$+ [1 + 2\delta_{3\alpha} + 2\delta_{3\beta} + 2\delta_{3\gamma}]/ (a_3^2+\theta_k)$$

Now we know from our starting $\{...\} = x_\alpha x_\beta x_\gamma$ the no two indices in the 1..3 range can be the same. Therefore, in each [....] above we have at most one 2δ term that will not vanish. Therefore, each [...] can be either a 1 or a 3.

We have three terms above written on separate lines.

In the first term, if one of the three indices αβγ is a 1, the bracket is 3, otherwise it is 1. Of course only one of the three indices can be a 1 as noted above. This says "if {...} contains x, then first [] = 3, otherwise first [] = 1.

The second term has [] = 3 only if one of the indices is a 2, meaning our {...} has a y in it.

So here is our conclusion. The sum of the first two terms in our Laplace = 0 result are these

$$k_1/(a_1^2+\theta_k) + k_2/(a_2^2+\theta_k) + k_3/(a_3^2+\theta_k)$$

where $k_i = 3$ if x_i appears in {...}, otherwise $k_i = 1$ // Hobson p 480 A

For example, if {} = xyz, then all three k's are 3's.