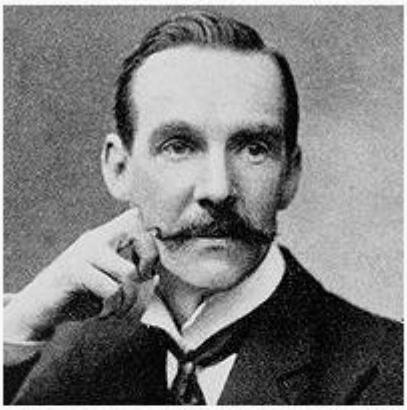
Hobson Ellipsoidal META META Notes PhL 11.16.10

This is a 6-page high-level summary of the ellipsoidal Chapter of Hobson's 1931 book which I checked out from Marriott library. There are not a lot of sources on ellipsoidal harmonics. This source in many ways is very excellent. The chapter consists of Sections 269-292, about 40 pages. There are lots of equations and none are numbered, so I used my usual A,B,C lettering scheme on the hard copy. I do not have a PDF of this chapter or the book. As with most of my docs, this one has a short overview.

I spent pretty much a whole month (started Oct 14) reading this chapter and doing quite a few support tasks which included writing other docs on related topics (such as conicals) and writing many Maple programs which resulted in updates to the Maple User Guide. I knew this ellipsoidal stuff was the hardest of all the classical curvilinear systems, and I wanted to get it nailed down once and for all.

E. W. Hobson



Ernest William Hobson (1856–1933)

Born	27 October 1856 Derby
Died	19 April 1933
Residence	UK
Nationality	British
Fields	Mathematician
Institutions	Christ's College, Cambridge
Alma mater	Royal School of Mines Christ's College, Cambridge
Notable students	Philippa Fawcett John Maynard Keynes
Known for	Real analysis

E. W. Hobson

From Wikipedia, the free encyclopedia

Ernest William Hobson FRS (27 October 1856 – 19 April 1933) was an English mathematician, now remembered mostly for his books, some of which broke new ground in their coverage in English of topics from mathematical analysis. He was Sadleirian Professor at the University of Cambridge from 1910 to 1931.

He was born in Derby, and was educated at Derby School, the Royal School of Mines, and Christ's College, Cambridge, graduating Senior wrangler in 1878.[1] He became a Fellow of Christ's almost immediately after graduation. He made his way into research mathematics only gradually, becoming an expert in the theory of spherical harmonics.

His 1907 work on real analysis was something of a watershed in the British mathematical tradition; and was lauded by G. H. Hardy.[2] It included material on general topology and Fourier series that was topical at the time; and included mistakes that were picked up later (for example by R. L. Moore).

Contents [hide]

1 Works
2 See also
3 References
4 External links

Works [edit]

- *A Treatise on Trigonometry* (1891)
- *Theory of Functions of a Real Variable* (1907)
- *Squaring the Circle* (1913)
- *The Domain of Natural Science* (1923) Gifford Lectures
- *The Theory of Spherical and Ellipsoidal Harmonics* (1931)

Overview (1/2 page)

Hobson first uses conical coordinates to show that the conical atoms are $r^n E_{n,p}(\mu)E_{n,p}(\nu)$ where the E functions are first kind solutions of an ODE known as the Lamé equation. He uses η and ζ to simplify the metric tensor and ∇^2 and finds some solutions to certain conical potential problems. He then advances to ellipsoidal coordinates ρ,μ,ν and shows that the atoms are $E_{n,p}(\rho)E_{n,p}(\mu)E_{n,p}(\nu)$ where the exact same functions are involved! These E functions are the "Lamé functions" (first kind). I learned here for the first time the connection to the Jacobi functions like sn and dn, it is very clear now.

He then shows that the EE are linear combinations of the Y_{nm} and this leads to an elucidation of the precise forms for the E functions which we learn fall into four simple classes K,L,M,N. The only thing missing are the coefficients of the simple series, but then he goes on to show how these are calculated, and I have clarified his method I think a lot. He computes the lowest E functions that Byerly shows. Here are those E function forms where each function has a series part which descends to a 1 or μ term.

$$K_{n,p}(\mu) = \mu^n + b\mu^{n-2} + c\mu^{n-4} + ..$$
$$L_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\,[\,\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$
$$M_{n,p}(\mu) = \sqrt{\mu^2 - k^2}\,[\,\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$
$$N_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\,\sqrt{\mu^2 - k^2}\,[\,\mu^{n-2} + b\mu^{n-4} + c\mu^{n-6} + ..]$$

We are made aware that the $E_{n,p}(\mu)E_{n,p}(\nu)$ situation is a 2D Sturm Liouville problem which quantizes the n and p parameters all at once, and cannot be decomposed into a product of 1D systems.

He then rewrites the E function series in terms of factored polynomials which have zeros, since this form get used later in the Niven stuff. He then mentions the second-kind F functions, and shows how some Dirichlet problems can be solved by inspection for prescription on the ellipsoidal surfaces. He is not very clear how you really expand f(μ,ν) on the EE functions, but does one example.

He then gets into "the Niven world" where both the conical and ellipsoidal harmonics r^nEE and EEE are converted to a fascinating all-Cartesian form that was found by Ferrars and Niven in 1891. Using the conical versions, he shows a whole new method of computing the Lamé function coefficients that involves solving a well-defined system of simple equations.

The rest of the chapter consists of certain advanced topics involving these Cartesian EEE functions which he denotes G_n . For example, he shows how they can be obtained by operating on the spherical harmonics with a certain morphed Laplacian. He mentions the Weierstrass version of the Lamé in his very last section. Two chapter sections concern reducing the ellipsoidal stuff to spheroidal stuff.

In **Section 269** Hobson opens the show, stating the ellipsoidal surfaces and the x,y,z equations. He comments he will follow Heine's 1878 book which I don't think exists in English. He mentions a certain non-orthogonal system ρ,θ,φ that I wasted a bunch of time on and which is irrelevant.

In **Section 270** he suddenly starts talking conical coordinates r,μ,ν which I then had to go off and learn. He then introduces speed-adjusted variables η and ζ and shows how the metric tensor (already diagonal) gets very simple, and so also the Laplacian. This simple form allows solutions of constant potential on conical surface problems. He then does separation and finds the atoms to be $r^n E_{n,p}(\mu)E_{n,p}(\nu)$ where the E are solutions of the Lamé equation. So we learn his conical motivation!

In **Section 271** he returns to the ellipsoidals with ρ,μ,ν (different μ,ν). He then introduces speed-adjusted variables ξ, η, ζ and shows how the metric tensor (already diagonal) gets very simple, and so also the Laplacian. This simple form allows solutions of constant potential on ellipsoidal surface problems. He then does separation and finds the atoms to be $E_{n,p}(\rho)E_{n,p}(\mu)E_{n,p}(\nu)$. The section ends with detailed data on how the ξ, η, ζ are related to the ρ,μ,ν, we have elliptic integrals and Jacobi functions.

In **Section 272** I give my own explanation of why the conical harmonics $r^n E_{n,p}(\mu)E_{n,p}(\nu)$ are linear combinations of the spherical harmonics of the same n. This leads to

$$P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)] = \Sigma_p f_{n,p}(m) E_{n,p}(\mu) E_{n,p}(\nu)$$

By writing the LHS "tessorals" in x,y,z form and then using the conical x,y,z equations to replace things with r,μ,ν (a lot of work I might say) we are able to *deduce* the basic forms of the E functions. We find that there are basically four kinds (four classes), and they have these quite simple forms:

$$K_{n,p}(\mu) = \mu^n + b\mu^{n-2} + c\mu^{n-4} + ..$$
$$L_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\,[\,\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$
$$M_{n,p}(\mu) = \sqrt{\mu^2 - k^2}\,[\,\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$
$$N_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\,\sqrt{\mu^2 - k^2}\,[\,\mu^{n-2} + b\mu^{n-4} + c\mu^{n-6} + ..]$$

The only thing we don't know are the coefficients. We now that the total number of independent functions must be 2n+1, but we don't yet know how many there are of each class. The deduction takes advantage of the nice way the tesserals break down in EE sums, which is this

$$P_n^m(\cos\theta)\cos(m\varphi) = \Sigma_p\, f_{n,p}(m)\, K_{n,p}(\mu)\, K_{n,p}(\nu) \qquad \text{m even}$$
$$P_n^m(\cos\theta)\cos(m\varphi) = \Sigma_p\, f_{n,p}(m)\, M_{n,p}(\mu)\, M_{n,p}(\nu) \qquad \text{m odd}$$
$$P_n^m(\cos\theta)\sin(m\varphi) = \Sigma_p\, f_{n,p}(m)\, N_{n,p}(\mu)\, N_{n,p}(\nu) \qquad \text{m even}$$
$$P_n^m(\cos\theta)\sin(m\varphi) = \Sigma_p\, f_{n,p}(m)\, L_{n,p}(\mu)\, L_{n,p}(\nu) \qquad \text{m odd}$$

I then express the E functions in a general way using my own notation

$$E_{n,p}(\mu) = \{\, 1\,,\ \sqrt{k^2-\mu^2}\,,\ \sqrt{h^2-\mu^2}\,,\ \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\,\}\ [\,1,\mu]\ \kappa\ \Pi_{s=1}^{J}(\mu^2-\rho_s^2)$$

K	M	L	N	// class
I = 0	I=1	I=1	I=2	// class dependent

J = (n-I)/2	n-I = even	=> select 1 in [1, μ]
J = (n-I-1)/2	n-I = odd	=> select μ in [1, μ]

where I introduce for the first time my special J integer (Hobson's m in places), which is just a function of n and class. The ellipsoidal harmonics then appear as

$$E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu) = \kappa^3$$
$$\{\, 1\,,\ \sqrt{k^2-\rho^2}\,,\ \sqrt{h^2-\rho^2}\,,\ \sqrt{h^2-\rho^2}\ \sqrt{k^2-\rho^2}\,\}\ [\,1,\rho]$$
$$\{\, 1\,,\ \sqrt{k^2-\mu^2}\,,\ \sqrt{h^2-\mu^2}\,,\ \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\,\}\ [\,1,\mu]$$
$$\{\, 1\,,\ \sqrt{k^2-\nu^2}\,,\ \sqrt{h^2-\nu^2}\,,\ \sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2}\,\}\ [\,1,\nu]$$
$$\Pi_{s=1}^{J}(\rho^2-\rho_s^2)\,(\mu^2-\rho_s^2)\,(\nu^2-\rho_s^2)$$

Although Hobson does it in later sections, I shall point out here that we learn how the above EEE can be written in two other forms which I call partial-Niven and Niven,

$$E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu)$$
$$= \{\,1\ \ x\ y\ z\ xy\ xz\ yz\ xyz\,\}\ \ \kappa'\,\Pi_{s=1}^{J}(\rho^2-\rho_s^2)\,(\mu^2-\rho_s^2)\,(\nu^2-\rho_s^2)$$

K	K	L	M	M	L	N	N	
e	o	o	o	e	e	e	o	// values of n for which this form applies (n even or odd)

$$E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu)$$
$$= \{\,1\ \ x\ y\ z\ xy\ xz\ yz\ xyz\,\}\ \ \kappa''\ \ \{\,\Pi_{s=1}^{J}\ [\,x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1]\,\}$$

K	K	L	M	M	L	N	N	
e	o	o	o	e	e	e	o	n = even or odd
1	2	2	2	3	3	3	4	species s

where in the full Niven form, the ellipsoidal harmonics appear entirely in Cartesian coordinates. Later Hobson refers to these Niven functions as the G_n harmonics. These are all "first kind" ellipsoidal functions which would be used for "internal" problems which include the origin.

In **Section 273** Hobson shows **how to find the coefficients** of the Lamé functions. He does this in a peculiar way I think which makes you think the method is different for the K class than for the other classes, since his K method involves a determinant. I eventually wrote "Understanding the Hobson Lamé Recursion Relation Solutions.doc" which clears everything up.

The method is simply this. The E function of any class has this form

$$E_{n,p}(\mu) = a_0\mu^{n-I} + a_1\mu^{n-I-2} + a_2\mu^{n-I-4} + ... + \; a_J\,[1, \mu]$$

The equation which determines the legal p values is this:

$$a_{J+1}(p) = 0 \qquad \text{where LHS} = \text{a poly}_{J+1}(p)$$

and you determine the LHS $a_{J+1}(p)$ from the roll-up of the appropriate recursors for each case. Once you have the p values, for each p the rollups give you the coefficients $a_i(p)$ which appear in E functions. In any class there are always J p values, and J+1 coefficients including a_0 which is usually set to 1. Since J is a function of n and I, J and J+1 a different number for the different classes for the same n. Here is a table showing J+1 which is the number of functions in each class for several n values,

n =	0	1	2	3	4	5	6	7	8
K	1	1	2	2	3	3	4	4	5
L	0	1	1	2	2	3	3	4	4
M	0	1	1	2	2	3	3	4	4
N	0	0	1	1	2	2	3	3	4
	1	3	5	7	9	11	13	15	17

and one sees that there are always a total of 2n+1 E functions for any n. The functions of the first four columns are tabulated by Byerly and also by Hobson in a less clear display.

You could alternatively (and for each class) form the special recursor determinant which will be (J+1)x(J+1) and there will be J+1 solutions. Here are the three recursors:

K: $2s(2n+1-2s)a_s = \alpha\,\{\,p - (n-2s+2)^2\}\,a_{s-1} + \beta(n-2s+4)(n-2s+3)a_{s-2}$

L,M: $2s(2n+1-2s)a_s = [\alpha\{\,p - (n-2s+1)^2\} - (2n-4s+3)k^2]\,a_{s-1} + \beta(n-2s+2)(n-2s+3)a_{s-2}$

N: $2s(2n+1-2s)a_s = \alpha\,\{\,p - (n-2s+1)^2\}\,a_{s-1} + \beta(n-2s+2)(n-2s+1)a_{s-2}$

I explain where the determinant method comes from, and why it is probably not the preferred way to do things in the modern world where we have computer algebra programs like Maple which are happy to "roll up" recursion relations for any order n you want. The determinant method asks you to compute det M for a nearly diagonal matrix M of recursor values to determine the J+1 p values. Then you still have to use the roll-ups to find the coefficients, they don't just fall out from the matrix equation M**a** = **0**. I almost think Hobson did not have a clear view that all classes are done the same way.

In **Section 274** he shows how the E functions become trig and Legendre P functions in the spheroidal limits of the ellipsoidal harmonics. My atoms are the harmonics! Harmonics are things that solve the Laplace equation in some coordinate system.

In **Section 275** he derives those E functions which appear in my Byerly yellow display.

In **Section 276** we get wind of the strange fact that in the conical world, the μ and ν coordinates which are oscillatory in the atoms of the form $r^n E_{n,p}(\mu)E_{n,p}(\nu)$ don't separate into two independent Sturm-Liouville problems, but instead form one glommed 2D Sturm-Liouville problem which has two quantum numbers n and p and causes them both to be set. Even today, this is a bit of a specialty subject, but there are papers out there. This means that completeness and orthogonality are going to be in this 2D sense and you don't have these features separately in the μ and ν variables as you do in most other systems. So this section is Hobson's exploration of this 2D orthogonality condition which looks like this

$$\int\int [E_{np}(\mu) E_{np}(\nu)] [E_{n'p'}(\mu) E_{n'p'}(\nu)] d\eta\, d\zeta\, (\mu^2-\nu^2) = \delta_{nn'}\delta_{ss'} * \text{stuff}$$

where the measure here when written entirely in μ and ν gives ν dμ dν where ν is the Jacobian as I compute in my Maple metric tensor file.

In **Section 277** we then consider the whole notion of doing **expansions and projections** in this 2D SL world, the basic idea being

$$F(\mu,\nu) = \Sigma_{n=0}^{\infty} \Sigma_p\ F_{np} E_{np}(\mu)E_{np}(\nu) \qquad \text{// expansion}$$

$$F_{np} = \int\int [E_{np}(\mu) E_{np}(\nu)] F(\mu,\nu)\, d\eta\, d\zeta\, (\mu^2-\nu^2) \qquad \text{// projection}$$

Hobson is not too clear about the conditions that the function F(μ,ν) might have to satisfy. He seems in general not too sharp on this subject, and the year of his writing might be the 1915-25 era and maybe things really were less clear then on these subjects which are so organized and documented nowadays.

In **Section 278** we just note that you can write any of the E function as a radical times a series, and you can factor the zeros into $\Pi_i\ (\mu-\alpha_i)$ form and when you do, the α_i will be "**zeros of the E functions**". So the α_i are alternate data for the coefficients a_i. The roots occur in pairs, so it is really $\Pi_i\ (\mu^2-\alpha_i^2)$, and I show why this is so.

Section 279 brings up the **second-kind** E functions called F functions which would be used in any dimension of a problem that was "external" and going off to infinity. These functions decay for large argument. Hobson uses a method that seems different from the usual "trick" M&F use, but it is doubtless the same method.

In **Section 280** he points out that if you have a Dirichlet potential prescribed on an ellipsoid which is some simple linear combination of the EE harmonics, you know pretty much "by inspection" the full solution to the interior and exterior **Dirichlet problems**. In the case of the charged metal ellipsoid of course the interior problem is V = 0, this being the only Dirichlet problem in this geometry I have ever done. You could solve similar Dirichlet problems for prescription on either hyperboloid type surface.

In **Section 281** he expands $P_n(\cos\gamma)$ on EE functions, but I am not sure what angle this is. He just wants to show that at least in theory, you should be able to project a reasonable function onto the EE harmonics.

In **Section 282** we enter what I might call "the Niven world" (Ferrars 1877, then Niven 1891) where we write the ellipsoidal harmonics in Cartesian coordinates. Remember that Hobson and Niven were work contemporaries, Hobson being 1856-1933. Hobson shows how we can write the partial-Niven and then full-Niven forms I quoted already above. When I first tried to read this stuff in W&W, I had no idea what they were doing, but now it is completely clear, I owe Hobson his due. Here are some of the equations that go with this world:

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \;\sim\; \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\}\; \Theta_1(\theta_1)\Theta_2(\theta_1)....\Theta_J(\theta_J)$$

$$\text{where } \Theta_s \equiv \left[\, x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s) - 1 \,\right] \qquad \theta_s = \rho_s^2 - a^2$$

For later use, Hobson then defines

$$G_n^s(\rho,\mu,\nu) = E_{n,s}(\rho)E_{n,s}(\mu)E_{n,s}(\nu) \qquad s = 1,2...2n+1$$
$$\mathcal{G}_n^s(\rho,\mu,\nu) = F_{n,s}(\rho)F_{n,s}(\mu)F_{n,s}(\nu) \qquad s = 1,2...2n+1$$

and uses H_n to refer to a generic spherical harmonic.

In **Section 283** the Niven path is followed again this time for conicals, and we get

$$r^n\, E_{n,p}(\mu)\, E_{n,p}(\nu)$$

$$= \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\}\; \kappa''\, \Pi_{s=1}^{J} \left[\, x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s) \,\right]$$

K	K	L	M	M	L	N	N	// class
e	o	o	o	e	e	e	o	// n

// Hobson p 478 B

In **Section 284** we learn a new method for finding the Lamé function coefficients using the Niven Cartesian form for the harmonics. The method is just this: set $\nabla^2(\text{form}) = 0$ and show this results in a system of equations which you can solve for the parameters θ_s which are related to the roots ρ_s^2 as shown above, and which are thus related to the desire coefficients a_i. The system of equations has J equations in J unknowns θ_s. I explain (Hobson did not) how if you refer to a solution of these equations as a vector $\boldsymbol{\theta}$, there will in fact be J+1 distinct solution vectors $\boldsymbol{\theta}^{(k)}$. The equation set is very slightly different for the different classes K,L,M,N and each class then has different solutions $\boldsymbol{\theta}^{(k)}$ and this then determines the Lamé function coefficients for each class. The equations have the θ_s in denominator factors and are thus highly non-linear in these variables, so Cramer's Rule has nothing to do with this situation. I show how this method also determines the J+1 dimensionless p values. I then do some Examples of using this method for n = 0,1,2,3,4.

The *derivation* of this root finding method is written up now in "Derive Hobson p 479 B,C and D.doc" and it did take a bit of work. Many things in Hobson take a bit of work.

Section 285 comments on some Klein theorem about how the roots θ_s are distributed in $(-a^2,-c^2)$.

We now come to Hobson's final sections which are sort of "extra credit" sections on fairly obscure but related development paths in the Niven vein.

In **Section 286** we have a brief comment on orthogonality of the G functions on an ellipsoid. Hobson then does a few things to prepare for the next section.

In **Section 287**, which is "the next section", Hobson reprieves his own published derivation of a way compute the G_n functions from the spherical harmonics he calls H_n. Wall to wall equations it is, and there is probably no other source for this material derived in this way. The end result is this

$G_n(x,y,z)$ = (series of even powers of operator D acting on) $H_n(x,y,z)$

where $D = a^2\partial_x^2 + b^2\partial_y^2 + c^2\partial_z^2$

where D is some kind of morphed Laplacian operator! This result appears in W&W as well. It was first shown by Niven, and the proof was refined as shown here by Hobson.

In **Section 288** Hobson comes with a similar, but more horrible, way to write the external $\mathcal{G}_n$ functions also in terms of the H_n. We have delightful things like $H_n(\partial_x, \partial_y, \partial_z)$ appearing.

In **Section 289** Hobson ponders how one might do Dirichlet problems for ellipsoidal systems directly using these Cartesian ellipsoidal harmonics Gn . This stuff is very ugly. It seems to me that this sort of defeats the whole benefit of even having ellipsoidal coordinates in the first place. I think this is the kind of academic problem people get involved in. So much is invested in the Niven form, you want to try to do something with it. I has a certain pleasantness about it, but I just doubt it is practical in any way. Maybe I am wrong.

Sections 290 and 291 show now the Cartesian G_n harmonics reduce in the prolate spheroidal cases. This is not of great interest to me, but again the kind of academic problem it is hard to ignore. You know it has to work out somehow, and you want to show how.

Section 292 talks about a different "speed adjusted" variable one can use for the E functions. This would be an alternative to the η(μ) elliptic integral he used earlier to replace μ with η and thus simplify the Lamé equation when written in terms of both μ and η. The alternative discussed *here* is called ρ(u) where ρ is the **Weierstrass elliptic function**. This time the simple form Lamé equation has the derivative in u, but the function ρ(u) appears in the constant term. In effect we have this for the Lamé equation,

$$\partial_u^2 E_{np}(u) - \{ n(n+1)\rho(u) + B\} E_{np}(u) = 0$$

I may be oversimplifying the situation, but the point is that there are other alternate variables and other ways to write the Lamé equation. Hobson is just saying he is aware of these other ideas.