Hobson Elliptical Coordinates Meta Notes PhL 10.28.10

The raw notes were supposed to be simply by-section notes on Hobson's 42 page chapter, but I had to fill in sub detail so much that the doc ballooned out into a huge mess, so we now have some meta notes here. These notes include references to other docs that I threaded through in the process. This meta doc is 24 pages long, whereas the raw note doc is 83 pages, and there are perhaps 40 pages of other docs involved.

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269: The ρ,μ,ν ellipsoidal coordinates, and then the weird ρ,θ,φ ones.

Here Hobson declares the ellipsoidal coordinates to be ρ,μ,ν instead of the ones ξ_1, ξ_2, ξ_3 used by MF (suggesting the three solutions of a cubic equation). Also k>h are the MF a>b.

Then suddenly without warning Hobson defines a totally new set of ellipsoidal coordinates ρ,θ,φ which have the following x,y,z definitions

$$x = C\ \cos\theta = \rho\cos\theta \qquad // \rho, \mu, \nu = \xi_1, \xi_2, \xi_3$$
$$y = B\ \sin\theta\cos\varphi = \sqrt{\rho^2-h^2}\ \sin\theta\cos\varphi \qquad s^2 = k^2-h^2$$
$$z = A\ \sin\theta\sin\varphi = \sqrt{\rho^2-k^2}\ \sin\theta\sin\varphi$$

He does this and then says NOTHING about why he has done it. I thought maybe this was going to feed into his later discussion involving spherical harmonics, so I spent a lot of time pondering ρ,θ,φ.

In retrospect, Hobson only spent 1/4 of one page on this and never used it again, but I took off with it, dog with a bone. I recommend the reader just skip to Section 270 below.

Details on the weird "distorted spherical angle" coordinates θ and φ.

(1) Inversion and level surfaces.

The ρ coordinate is exactly the same in ρ,μ,ν and in ρ,θ,φ so you can regard this as μ,ν ↔ θ,φ. I then looked into the matter of "inverting" the above equations (trivial) and drawing the "level surfaces" all at the same time. Here are what the surfaces look like:

First, I consider ρ. I know the level surfaces are ellipsoids. And I know the inversion is this

$$\rho^2(x,y,z) = \text{poly}_2 + \sqrt{\text{poly}_4} * \cos\{\,[1/3]\cos^{-1}(\text{poly}_6/\sqrt{\text{poly}_{12}}\,)\,\}$$

Second, I consider φ. The inversion is this

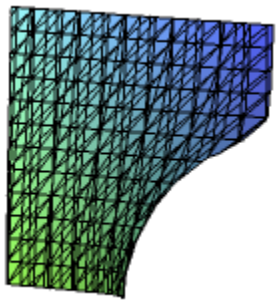
$$\varphi(x,y,z) = \tan^{-1}\{\ (z/y)\,[\sqrt{\rho^2-h^2}\,/\sqrt{\rho^2-k^2}\,]\,\} \qquad \text{where } \rho = \rho(x,y,z) \text{ as above.}$$

The level surfaces for φ = const are hugely complicated functions, so I had Maple just plot one and the surface looks like an azimuthal plane with the part near the origin bent out in some curved manner. (If h = k then the bend goes away and you have a tangent plane as in prolates.)

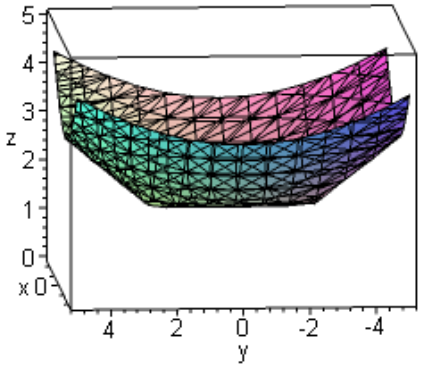
Third, I consider θ. The inversion here is this

$\theta(x,y,z) = \cos^{-1}(x/\rho)$ where $\rho = \rho(x,y,z)$ as above.

Again, the level surfaces for θ = const are hugely complicated functions, so I had Maple just plot one and the surface looks like a vertical cone which has been warped into an upside down circus tent (note that the up direction should be called x to match the Hobson notation, not z as shown in the picture)



surface of constant φ



surface of constant θ

I then went on to compute the "tangent base vectors" for the ρ,θ,φ system as rows of the matrix Tup. I discovered that $\mathbf{e}_\varphi$ lies in the horizontal plane (the y-z plane for Hobson), and that $e_\rho \bullet e_\theta = 0$ and $e_\rho \bullet e_\varphi = 0$. Since ρ,θ,φ is a non-orthogonal system (next item below) these base vectors are not surface normal vectors, so the pictures don't contradict the horizontal $\mathbf{e}_\varphi$ claim. I never found a way to explain the three base vector facts just quoted other than they come out of the equations defining the ρ,θ,φ system.

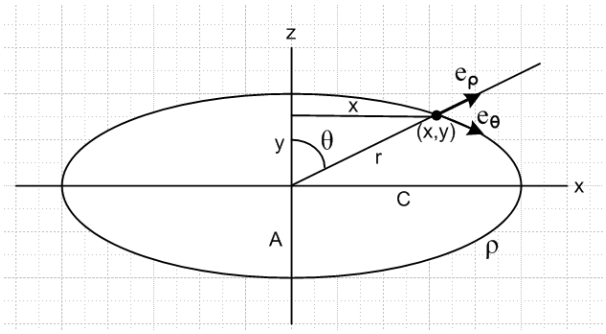
(2) Mystery about which metric tensor elements vanish

So yes, I next computed the metric tensor and found that

$$g_{dn} = \begin{pmatrix} e_\rho \bullet e_\rho & e_\rho \bullet e_\theta & e_\rho \bullet e_\varphi \\ e_\theta \bullet e_\rho & e_\theta \bullet e_\theta & e_\theta \bullet e_\varphi \\ e_\varphi \bullet e_\rho & e_\varphi \bullet e_\theta & e_\varphi \bullet e_\varphi \end{pmatrix} = \begin{pmatrix} e_\rho \bullet e_\rho & 0 & 0 \\ 0 & e_\theta \bullet e_\theta & e_\theta \bullet e_\varphi \\ 0 & e_\varphi \bullet e_\theta & e_\varphi \bullet e_\varphi \end{pmatrix}$$

so g is almost diagonal but not quite. Expressions were pretty messy for the non-zero elements.

Next, I tried comparing the non-orthogonal ρ,θ,φ system to a 2D non-orthogonal ρ,θ system that I invented and that involves the following picture (see "rho theta ellipse system.doc")



I was very confused as to why $e_\rho \bullet e_\theta \neq 0$ for my ρ,θ system, and $e_\rho \bullet e_\theta = 0$ in the ρ,θ,φ system. But then I realized that the two θ angles are just completely unrelated except perhaps at large distances from the origin. For example, if my ρ,θ picture above were the cross section of an oblate spheroid, the constant θ surface implied by my picture would be a cone, but I already know that the constant θ surface for ρ,θ,φ is an upside down circus tent. So this resolved the paradox of $e_\rho \bullet e_\theta$ being 0 in one system and not in the other.

I then briefly (very briefly) considered what ∇^2 might look like in the non-orthogonal ρ,θ,φ system but quickly gave up because $\nabla^2(f) = (1/\sqrt{g'})\ \partial_i\ [\sqrt{g'}\ g'^{ij}\ (\partial_j f)\ ]$ would be incredibly messy. I concluded that you would never find in here the simple $\nabla_{\theta\varphi}^2$ that define the spherical harmonics. I reached no conclusion about whether or not such a non-orthogonal system might allow for a separated Laplace solution. MF only deal with separable solutions for orthogonal systems, I think. I just saved a paper on this subject, but won't look at it now.

It is true that the ρ,θ,φ system is "nice" in the following sense. We can write

$$x = C\ \cos\theta$$
$$y = B\ \sin\theta\ \cos\varphi$$
$$z = A\ \sin\theta\ \sin\varphi$$

where C>B>A are the semi-major axes of the ellipsoid. This means the θ,φ are some kind of **"distorted spherical angles"** which "become" the usual spherical angles when A=B=C. This somewhat aligns with an idea I had which was to start with a sphere and **"morph"** it into an ellipsoid. As you did this, the surfaces of constant θ and constant φ would morph from cone and plane to the surfaces plotted above. Perhaps the $\nabla_{\theta\varphi}^2$ operator morphs into some kind of more complicated animal and perhaps separation is possible perhaps at least for ρ, and there then might be some separated "harmonics" of the form $R(\rho)Y'(\theta,\varphi)$ where Y' are morphed versions of the usual Y's. I imagine that historically various people must have pondered the three equations shown above with regard to ellipsoidal systems, perhaps there are papers out there on the subject.

270: Conical Coordinates

Again rather suddenly Hobson launches off (with no motivation given) into a discussion of *conical* coordinates. I had never really seen these before, so I started a "conicals" folder and wrote up some details

there in "conical coordinates.doc" which are summarized in the main raw Hobson notes. I will summarize that summary here.

The coordinates are called r,μ,ν where r labels spheres and μ,ν label two elliptical cones. The μ,ν are NOT the same as the those appearing in the ρ,μ,ν ellipsoidal coordinates. The surfaces and xyz are:

$x^2+ y^2+ z^2 = r^2$ red spheres

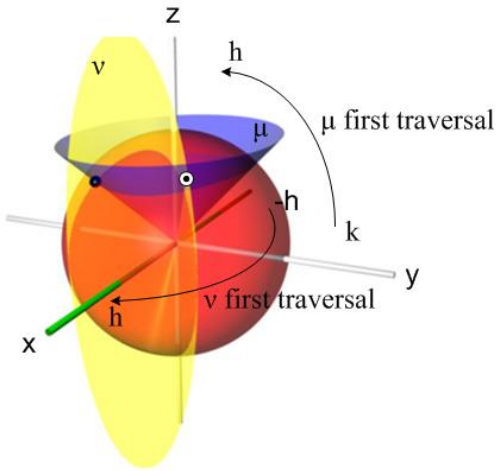
$x^2/\mu^2 + y^2/(\mu^2-h^2) = z^2/(k^2-\mu^2)$ blue μ elliptical cones

$x^2/\nu^2 = y^2/(h^2-\nu^2) + z^2/(k^2-\nu^2)$ yellow ν elliptical cones

$x = r\,\mu\nu/(hk)$

$y = r\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}/(hs)$ $\quad s = \sqrt{k^2-h^2}$ $\quad 0 < \nu^2 < h^2 < \mu^2 < k^2$

$z = r\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}/(ks)$



I checked M&F on conicals and they had only 3 confusing column inches of stuff which I now ignore. I then went off and wrote a separate doc " Inversion of Conical Coordinates.doc" and found this inversion of the above x,y,z equations: (which I never used for anything)

$r = \sqrt{x^2+y^2+ z^2}$ $\quad s^2 = k^2- h^2$

$\nu = \sqrt{W - \sqrt{W^2 - 4h^2k^2(x/r)^2}}\ /\sqrt{2}$ $\quad W = k^2(x/r)^2 + s^2(y/r)^2 + h^2$

$\mu = h\,k(x/r)\,(1/\nu)$

Hobson then writes out the Laplacian in conical r,μ,ν coordinates and makes it very clear that if you replace μ,ν with "alternative" coordinates η,ζ that the metric tensor and the Laplacian become simple. These new coordinates are not mixtures of μ and ν, they are just speed-rescalings of μ and ν.

Aside: Specifically, η,ζ are given by (one understands the integration variables should be primed):

$\eta(\mu) = \int_{\mathbf{h}}^{\boldsymbol{\mu}} d\mu/\sqrt{\mu^2-h^2}\sqrt{k^2-\mu^2}$ // see Hobson p 458B

$\zeta(\nu) = \int_{\mathbf{0}}^{\boldsymbol{\nu}} d\nu/\sqrt{h^2-\nu^2}\sqrt{k^2-\nu^2}$

and we later learn that the inversions of the above are

$\mu(\eta) = k\ dn(K - k\eta, k_1)$ $\quad k_1' \equiv h/k \quad k_1 = \sqrt{1-(h/k)^2} \quad K = K(k_1^2)$
$\nu(\zeta) = h\ sn(k\zeta, k_1')$ $\quad$ dn and sn are Jacobi elliptic functions

Thus we could also write

$\eta(\mu) = (1/k)[K + dn^{-1}(\mu/k, k_1)]$
$\zeta(\nu) = (1/k)\ sn^{-1}(\nu/h, k_1')$

The η,ζ functions have the form of "elliptic integrals" of μ,η and the other direction are inverse Jacobi.

Now, the metric tensor is diagonal in both r,μ,ν and in r,η,ζ (1,2,3), but in the latter it is very simple

$(ds)^2 = (dr)^2 + (\mu(\eta)^2 - \nu(\zeta)^2)\ r^2\ \{\ (d\eta)^2 + (d\zeta)^2\ \} \quad Q_1 = 1 \quad Q_2 = Q_3 = r\sqrt{\mu^2 - \nu^2}$

Using these Q_i, I use my general formula to compute ∇^2 with this result

$\nabla^2 f = (1/h_{\eta\eta}^2)\ \{\ \partial_r [h_{\eta\eta}^2\ (\partial_r f)] + (\partial_\eta^2 f) + (\partial_\zeta^2 f)\ \} \quad h_{\eta\eta} = Q_2 = r\sqrt{\mu^2 - \nu^2}$

So the Laplace equation $\nabla^2 f = 0$ may be written in this very simple (hybrid) form

$(\mu^2 - \nu^2)\partial_r [r\ (\partial_r f)] + (\partial_\eta^2 f) + (\partial_\zeta^2 f) = 0$

This ODE allows for some simple solutions of this form

$V = A\eta + B$
$V = A\zeta + B$

and these would be the solutions for the potential between two concentric elliptic cones held at constant potentials (of blue and yellow type), and of course would tell you the capacitance of one cone. You could convert this solution from η to μ using $\eta(\mu) = (1/k)[K + dn^{-1}(\mu/k, k_1)]$, and then to Cartesian coordinates using the "conical inversion" formula I showed earlier. Yes, it would be messy in Cartesians.

Now if we make the *ansatz* that the solution of this equation has the form $f = r^n F(\mu)G(\nu)$, we find F and G must solve the following separated equations (which have one sign different)

$(\partial_\eta^2 F(\mu)) + [\ n(n+1)\ \mu^2 - p(h^2+k^2)]\ F(\mu) = 0$
$(\partial_\zeta^2 G(\nu)) - [\ n(n+1)\ \nu^2 - p(h^2+k^2)]\ G(\nu) = 0$ $\quad$ // which is p **457 C**

When we convert the derivatives from η,ζ to μ,ν, we find that both F and G solve the *exact same* ODE in their respective coordinates μ and ν ! That ODE is this: (p 457 D)

$$(\mu^2-h^2)(\mu^2-k^2)\,\partial_\mu{}^2 F(\mu) + \mu\,(2\mu^2-h^2-k^2)\,\partial_\mu F(\mu) + [p(h^2+k^2) - n(n+1)\,\mu^2]\;F(\mu) = 0$$

Obviously F(μ) will depend on the separation constants n and p (neither quantized at this point). Since F = G, we write both as F(μ) = G(μ) = $E_{n,p}(\mu)$ and we have then confirmed our ansatz with

$$f = r^n\, E_{n,p}(\mu)\, E_{n,p}(\nu) \qquad \text{// an atomic form for conicals = "conical harmonics"}$$

This is the Lamé equation (ODE) and the E are the Lamé functions. We thus see *why* Hobson went off on this conicals digression: the functions E are the same functions that will appear as EEE when we soon separate the ellipsoidal equations. We shall see that the conicals is an easier world in which to study the E functions. It will turn out that n gets quantized to positive integers, and p to certain specific real values. An atomic form like that shown above satisfies the Laplace equation and is thus "harmonic" in 3D, so people refer to atoms like the above as "conical harmonics". Every coordinate system has its harmonics.

271: Ellipsoidal Coordinates Resumed

Finally we get back to our ellipsoidal coordinates ρ,μ,ν. This is a very meat and potatoes Section.

(1) How everything simplifies if we use the alternate ξ,η,ζ coordinates.

We pursue the same program we did with the conicals in order to simplify the metric tensor and the Laplace equation. As before, we introduce certain helper variables for ρ,μ,ν which are called ξ,η,ζ .The last two are *exactly the same* as they were in the conical presentation, and the third ξ is some other elliptic function looking integral. We find first that (after much algebra) our metric tensor has this orthogonal and simple form

$$(ds)^2 = Q_1{}^2(d\xi)^2 + Q_2{}^2(d\eta)^2 + Q_3{}^2(d\zeta)^2$$

$$Q_1 = \sqrt{\rho^2-\mu^2}\,\sqrt{\rho^2-\nu^2}$$
$$Q_2 = \sqrt{\rho^2-\mu^2}\,\sqrt{\mu^2-\nu^2}$$
$$Q_3 = \sqrt{\rho^2-\nu^2}\,\sqrt{\mu^2-\nu^2}$$

The Laplacian becomes

$$\nabla^2 f = \qquad = [(\rho^2-\nu^2)\,(\rho^2-\mu^2)\,(\mu^2-\nu^2)]^{-1} \;*$$
$$\{\;\; (\mu^2-\nu^2)\,\partial_\xi{}^2 f \;+ (\rho^2-\nu^2)\,\partial_\eta{}^2 f \;+\; (\rho^2-\mu^2)\partial_\zeta{}^2 f \;\}$$

so the Laplace equation becomes (yes, it is hybrid)

$$\nabla^2 V = 0 \quad \Leftrightarrow \quad (\mu^2-\nu^2)\,\partial_\xi{}^2 V \;+ (\rho^2-\nu^2)\,\partial_\eta{}^2 V \;+\; (\rho^2-\mu^2)\partial_\zeta{}^2 V$$

which is simpler even than the conical case!

(2) some simple solutions of the Laplace equation

At once we have these three candidate single-variable solutions

$V = A\xi+B$
$V = C\eta+D$
$V = E\zeta + F$

In line with our example done above, we could immediately solve for the potential between two metal ellipsoids at V_1 and V_2, or between two one-sheeted metal hyperboloids, or between two two-sheeted metal ones! These very difficult sounding problems are now all solved in a trivial fashion.

The pair connections $\xi\leftrightarrow\rho$, $\eta\leftrightarrow\mu$, $\zeta\leftrightarrow\nu$ are (see below)

$\rho = k\ sn(ik\xi, k_1)$ $\quad k_1^2 = 1 - (h/k)^2$ $\quad k_1'^2 = (h/k)^2$ $\quad K = K(k_1^2)$
$\mu = k\ dn(K - k\eta, k_1)$ $\quad h,k$ = our usual friends, $\quad k \geq h$
$\nu = k\ sn(k\zeta, k_1')$

$\xi(\rho) = (1/ik)sn^{-1}(\rho/k, k_1)$ // these all come out real if ρ,μ,ν are in their ranges.
$\eta(\mu) = (1/k)[\ K + dn^{-1}(\mu/k, k_1)]$
$\zeta(\nu) = (1/k)\ sn^{-1}(\nu/h, k_1')$

For example, the potential between two constant-potential ellipsoids would be $V = A\xi+B$, and we can replace $\xi(\rho) = (1/ik)sn^{-1}(\rho/k, k_1)$ in favor or ρ, the more familiar ellipsoidal coordinate.

Writing things in Cartesian coordinates is a bit messy because the inversion formulas are messy, to wit:

$\rho^2 = e/3 + 2\sqrt{-Q}\ \cos(\theta/3)$ // $poly_2 + \sqrt{poly_4} * \cos\{\ [1/3]\ \cos^{-1}(poly_6/\sqrt{poly_{12}}\ )\ \}$
$\mu^2 = e/3 + 2\sqrt{-Q}\ \cos(\theta/3 + 2\pi/3)$
$\nu^2 = e/3 + 2\sqrt{-Q}\ \cos(\theta/3 + 4\pi/3)$

$e = x^2+y^2+z^2 +(k^2+h^2)$ $\quad L^2$ // all these are $poly_2$
$f = h^2x^2+ k^2y^2+ (k^2+h^2)z^2 +k^2 h^2$ $\quad L^4$
$g = k^2 h^2z^2$ $\quad L^6$
$R = (-9ef +27g + 2e^3)/54$ // $poly_6$ (ie, poly of 6th degree in x,y,z)
$Q = (3f - e^2)/9$ // $poly_4$
$\theta = \cos^{-1}(R/\sqrt{-Q^3}\ )$ // $\cos^{-1}(poly_6/\sqrt{poly_{12}}\ )$

Next, Hobson claims, and I have verified in full detail, that the Laplace equation shown above

$\nabla^2V = 0 \quad \Leftrightarrow \quad (\mu^2-\nu^2)\ \partial_\xi{}^2V + (\rho^2- \nu^2)\ \partial_\eta{}^2V + (\rho^2- \mu^2)\partial_\zeta{}^2V$

(3) the separated solutions of the Laplace equation

has a separated solution of the form

$V = E_{n,p}(\rho/\xi)\ E_{n,p}(\mu/\eta)\ E_{n,p}(\nu/\zeta)$

where these E functions solve the *same* Lamé ODE we obtained in our conical discussion. Again *he* holds off on calling these Lamé functions and calling that ODE the Lamé equation. Remember that the conical μ.ν for a point in space won't be the same as the ellipsoidal μ,ν, but the atoms for both show the same functions.

(4) Comparing conicals to ellipsoidals

After this *tour de force*, I pondered momentarily how the conicals and ellipsoidals might be directly related to each other. We know the connection at least through the intermediary x,y,z coordinates. This discussion starts off like this:

The conical equations were: (where I now use r,U,V)

$$x = r\,UV/(hk)$$
$$y = r\sqrt{U^2 - h^2}\sqrt{h^2 - V^2}/(hs) \qquad s = \sqrt{k^2-h^2} \qquad 0 < V < h < U < k$$
$$z = r\sqrt{k^2 - U^2}\sqrt{k^2 - V^2}/(ks)$$

The ellipsoidal equations were

$$x = \rho\mu\nu/(hk) \qquad // \ \rho, \mu, \nu = \xi_1, \xi_2, \xi_3$$
$$y = \sqrt{\rho^2-h^2}\sqrt{\mu^2- h^2}\sqrt{h^2- \nu^2}\,/\,(hs) \qquad s^2 = k^2-h^2$$
$$z = \sqrt{\rho^2-k^2}\sqrt{k^2- \mu^2}\sqrt{k^2- \nu^2}/(ks)$$

I was able to invert the conicals above to get

$$r = \sqrt{x^2+y^2+ z^2} \qquad s^2 = k^2- h^2$$
$$V = \sqrt{\ W - \sqrt{W^2 - 4h^2k^2(x/r)^2}\ ]}\ /\sqrt{2} \qquad W = k^2(x/r)^2 + s^2(y/r)^2 + h^2$$
$$U = h\,k(x/r)\,(1/V)$$

The coordinates μ,ν are obviously NOT the same as the coordinates U,V of conicals, but as we have seen, there is much similarly in the manipulations you do, and this is why Hobson used the same μ,ν symbols for both the r,μ,ν conical coordinates, and for the ρ,μ,ν ellipsoidal coordinates.

I was able to obtain two simple connections directly between conicals and ellipsoidals,

$$r = \sqrt{\rho^2+ \mu^2+ \nu^2- s^2}$$
$$W = (\mu^2\rho^2 + p^2\nu^2+ u^2\nu^2- h^2k^2)/(\rho^2+ \mu^2+ \nu^2- h^2 - k^2)$$

but beyond that things are too messy to deal with.

(5) Computation of the elliptic integrals which define the alternate coordinates

At this point in the raw notes, I studied the three pairings like $\xi \leftrightarrow \rho$ and "did the integrals". This was a LOT of work for me. In all cases, I used a set of elliptic integral formulas which appear on page 280 of GR7, and then used Jacobi function relations also in GR7, and here are my results:

$$k_1' \equiv h/k \qquad k_1 = \sqrt{1-(h/k)^2} \qquad K = K(k_1^2)$$

The ρ situation:

$$\xi(\rho) = \int_k^\rho d\rho \,/\sqrt{\rho^2-h^2}\,\sqrt{\rho^2-k^2} = (1/k)\, sn^{-1}\left[\sqrt{(\rho^2-k^2)/(\rho^2-h^2)}\,,\, h/k\right]$$

$$\Rightarrow \rho^2 = (k^2 - h^2 sn^2)\,/\,(1-sn^2) \Rightarrow \rho = (k/cn)\sqrt{1 - k_1'^2\, sn^2} \qquad \text{where} \quad sn = sn(k\xi, k_1')$$

Then using properties of the Jacobi functions I was able to message this into Hobson's form

$$\rho(\xi) = k\, dn(ik\xi, k_1^2)$$

The μ situation:

$$\eta(\mu) = \int_h^\mu d\mu \,/\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2} = (1/k) sn^{-1}\left[\sqrt{(\mu^2-h^2)}\,/(k_1\mu),\, k_1\right]$$

$$\Rightarrow \mu^2 = h^2 \,/\, (1 - k_1^2 sn^2) \qquad \text{where } sn = sn(k\eta, k_1)$$

Using identifies I was able to get this into some other forms

$$\mu = h\,/dn\,(k\eta, k_1) = k\, dn(K - k\eta, k_1) \qquad \text{where } K = K(k_1^2)$$

The ν situation:

$$\zeta(\nu) = \int_0^\nu d\nu \,/\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2} = (1/k)\, sn^{-1}(\nu/h, k_1')$$

$$\Rightarrow \nu = h\, sn(k\zeta, k_1')$$

Thus, I fully verified Hobson's claims, which I now summarize

$\rho = k\, sn(ik\xi, k_1)$ $\qquad k_1^2 = 1 - (h/k)^2 \qquad k_1'^2 = (h/k)^2$

$\mu = k\, dn(K[k_1] - k\eta, k_1)$ $\qquad h,k$ = our usual friends

$\nu = k\, sn(k\zeta, k_1')$

$\xi = (1/k)\, sn^{-1}\left[\sqrt{(\rho^2-k^2)/(\rho^2-h^2)}\,,\, k_1'\right] = (1/ik) sn^{-1}(\rho/k, k_1)$

$\eta = (1/k)\, sn^{-1}\left[\sqrt{(\mu^2-h^2)}\,/(k_1\mu),\, k_1\right] = k\, dn(K - k\eta, k_1)$

$\zeta = (1/k)\, sn^{-1}(\nu/h, k_1')$

272. How the E functions are related to the spherical harmonics?

Claim: the functions $Y_{nm}(\theta,\varphi)$ and $E_{np}(\mu)$ $E_{np}(\nu)$ are linear combinations of each other, in some sense "within the n subspace".

(a) Sphericals

This is a tricky subject, I will try to do better here than in the raw notes. First, a review of spherical coordinates. We can write

$$\nabla^2_{r\theta\varphi} = (1/r^2)\,\partial_r(r^2\partial_r) + (1/r^2)\,\nabla^2_{\theta\varphi} = (1/r^2)\,\partial_r(r^2\partial_r) - (1/r^2)\,L^2$$

$$L^2 = -\,[\,(1/S\theta)\,\partial_\theta\,[\,S\theta\,\partial_\theta\,] + (1/\sin\theta)^2\,\partial_\phi^2\,] = -\,r^2\nabla^2{}_{\theta\varphi}$$

We know that we can represent L_x, L_y, L_z in terms of θ,φ such that the usual SO(3) Lie Algebra is "represented". Once we do this, we know that we can find eigenfunctions that diagonalize L^2 and L_z with eigenvalues n(n+1) and m, and these eigenfunctions are the $Y_{nm}(\theta,\varphi)$. We know we can expand functions of θ,φ like this

$$f(\theta,\varphi) = \Sigma_{n=0}^{\infty}\,\Sigma_{m=-n}^{n} f_{nm}\,Y_{nm}(\theta,\varphi)$$

We can regard the set of L_2 functions defined on the surface of a sphere as a Hilbert Space $\mathcal{H}$. What we see from the above expansion formula is that we can *decompose* the Hilbert Space $\mathcal{H}$ into a direct sum of subspaces

$$\mathcal{H} = \oplus\,\Sigma_{n=0}^{\infty}\,\mathcal{H}^n$$

where the subspace $\mathcal{H}^n$ has the finite dimension 2n+1 and the basis functions $Y_{nm}(\theta,\varphi)$. This is the thing referred to vaguely in the opening claim as "the n subspace". The functions f in space $\mathcal{H}^n$ are those that have these properties: $L^2 f = n(n+1)f$ and $L_z f = mf$.

(b) Conicals

Now we switch to the conical coordinates. We have

$$\nabla^2_{r\mu\nu} = (1/h_{\eta\eta}^2)\;\{\;\partial_r\,[h_{\eta\eta}^2\,(\partial_r)] + (\partial_\eta{}^2 f) + (\partial_\zeta{}^2)\;\} \qquad h_{\eta\eta}^2 = (\mu^2-\nu^2)\,r^2$$

$$(1/r^2)\partial_r\,(r^2\partial_r) + (1/r^2)[\;(\partial_\eta{}^2) + (\partial_\zeta{}^2)\;]/\,(\mu^2-\nu^2)$$

On comparison with the spherical Laplacian, we can identify the angular momentum operator expressed in terms of the conical "angles" μ and ν,

$$L^2 = -\,[\ (\partial_\eta{}^2) + (\partial_\zeta{}^2)\ ]/(\mu^2-\nu^2)$$

which is of course in our hybrid form; we could write it all out in terms of μ and ν if we had to. Somehow it is possible to represent the Lie algebra in these coordinates. We could get to the expressions by starting with either the Cartesian or spherical forms for the L_i. I have never done this, but could do it. We would then find that we can again diagonalize L^2 and L_z with some eigenfunctions we might call $C_{np}(\mu,\nu)$ where index p can take on 2n+1 values. It turns out that $C_{np}(\mu,\nu) = E_{np}(\mu)\ E_{np}(\nu)$ as I show in item (c) below. Now consider again

$$\mathcal{H} = \oplus \Sigma_{n=0}{}^{\infty}\ \mathcal{H}^n$$

The (2n+1) functions $E_{np}(\mu)\ E_{np}(\nu)$ span the space $\mathcal{H}^n$. This is exactly the same subspace discussed in the spherical discussion above. It is the subspace with angular momentum n, so to speak. We have merely taken the L_i operators in θ,φ and rewritten them in the different variables μ,ν. This does not change the Hilbert Space $\mathcal{H}^n$. The functions f in space $\mathcal{H}^n$ are those that have these properties: $L^2 f = n(n+1)f$ and $L_z f = mf$. We can express L^2 and L_z in any pair of coordinates we want to replace θ,φ.

Since the functions $Y_{nm}(\theta,\varphi)$ and $E_{np}(\mu)E_{np}(\nu)$ both span $\mathcal{H}^n$, which is to say, both sets of 2n+1 functions form a basis for space $\mathcal{H}^n$, they must be linear combinations of each other. This is the claim I am here trying to justify. One can state this claim as follows, where I make up names for the real coefficients:

$$E_{n,p}(\mu)\ E_{n,p}(\nu) = \Sigma_m\ g_{n,m}(p)\ P_n{}^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$$
$$P_n{}^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)] = \Sigma_p\ f_{n,p}(m)\ E_{n,p}(\mu)\ E_{n,p}(\nu)$$

The μ,ν expansion theorem for functions defined in a sphere (space $\mathcal{H}$) would be

$$F(\mu,\nu) = \Sigma_{n=0}{}^{\infty}\ \Sigma_p\ F_{np}\ E_{np}(\mu)E_{np}(\nu)$$

where Σ_p is over 2n+1 distinct values of p. These values are not simply -n..n, but are the special values of p caused by Frobenius truncation of series. We know that the 2n+1 functions $E_{np}(\mu)$ are distributed over the K,L,M,N classes in a well defined manner.

I comment that when we write the conical atoms as $r^n\ E_{np}(\mu)E_{np}(\nu)$, the two E functions don't form a pair of independent Sturm-Liouville problems, but in fact form one *2D* SL problem, and this is different from most coordinate systems I have dealt with. I presume the ellipsoidal atoms $E_{np}(\rho)E_{np}(\mu)E_{np}(\nu)$ form a 3D SL problem.

(c) Separation in μ,ν

In this section I consider L^2 as shown above in terms of μ,ν and apply it to $f(\mu,\nu) = a(\mu)b(\nu)$ and show that the separated equations are these

$$\partial_\eta{}^2 a(\mu) + [\ n(n+1)\ \mu^2 - ps^2]\ a(\mu) = 0$$
$$\partial_\zeta{}^2 b(\nu) - [n(n+1)\ \nu^2 - ps^2]\ b(\nu) = 0$$

We know from earlier Hobson work that when these equations are written entirely in terms of μ or ν, they become the same Lamé equation and have the same solution so $a(\mu)= E_{\mathbf{np}}(\mu)$ and $b(\mu)= E_{\mathbf{np}}(\nu\mu)$

(d) bra-ket notation

Here I state items of section (a) above but in QM bra-ket notation . Here are some examples:

$$E_{\ell\mathbf{p}}(\mu)\, E_{\ell\mathbf{p}}(\nu) = \langle\mu\nu|\ell\mathbf{p}\rangle = \langle\mu|\ell\mathbf{p}\rangle\langle\nu|\ell\mathbf{p}\rangle$$

$$= \langle\theta\varphi|\ell\mathbf{p}\rangle = \Sigma_{\mathbf{m}} \langle\theta\varphi|\ell\mathbf{m}\rangle\langle\ell\mathbf{m}|\ell\mathbf{p}\rangle = \Sigma_{\mathbf{m}} \langle\ell\mathbf{m}|\ell\mathbf{p}\rangle\, Y_{\ell\mathbf{m}}(\theta,\varphi)$$

which shows the EE as linear combinations of the Y, with $n = \ell$. The key idea is $\langle\mu\nu| = \langle\theta\varphi| = \langle x|$ where a point on the sphere denoted by θ,φ has coordinates μ,ν in the conical system. Note that the coefficients $\langle\ell\mathbf{m}|\ell\mathbf{p}\rangle$ are complex numbers, since the Y are complex and the EE are real.

(e) "Using Conical and Spherical Coordinates to explore Lamé functions.doc"

At this point, the thread of my Hobson notes temporarily moves to "Using Conical and Spherical Coordinates to explore Lamé functions.doc". The reader is invited to read the Overview at the start of that doc. The main idea is this. Above we showed that

$$P_{\mathbf{n}}^{\mathbf{m}}(\cos\theta)[\sin(m\varphi),\cos(m\varphi)] = \Sigma_{\mathbf{p}}\, f_{\mathbf{n,p}}(m)\, E_{\mathbf{n,p}}(\mu)\, E_{\mathbf{n,p}}(\nu)$$

If we go off and write out the tesseral harmonics in terms of μ and ν coordinates instead of θ,φ, we are able to *deduce* the form that the functions $E_{\mathbf{n,p}}(\mu)$ must have! This is fairly clever little path to follow. We first learn that the E functions come in four "classes" called K,L,M,N. Each of these classes contains a certain subset of the 2n+1 E functions with certain p values. If we partition the tesserals into four groups, we find that the above equation becomes these four equations

$$P_{\mathbf{n}}^{\mathbf{m}}(\cos\theta)\cos(m\varphi) = \Sigma_{\mathbf{p}}\, f_{\mathbf{n,p}}(m)\, K_{\mathbf{n,p}}(\mu)\, K_{\mathbf{n,p}}(\nu) \qquad \text{m even}$$
$$P_{\mathbf{n}}^{\mathbf{m}}(\cos\theta)\cos(m\varphi) = \Sigma_{\mathbf{p}}\, f_{\mathbf{n,p}}(m)\, M_{\mathbf{n,p}}(\mu)\, M_{\mathbf{n,p}}(\nu) \qquad \text{m odd}$$
$$P_{\mathbf{n}}^{\mathbf{m}}(\cos\theta)\sin(m\varphi) = \Sigma_{\mathbf{p}}\, f_{\mathbf{n,p}}(m)\, N_{\mathbf{n,p}}(\mu)\, N_{\mathbf{n,p}}(\nu) \qquad \text{m even}$$
$$P_{\mathbf{n}}^{\mathbf{m}}(\cos\theta)\sin(m\varphi) = \Sigma_{\mathbf{p}}\, f_{\mathbf{n,p}}(m)\, L_{\mathbf{n,p}}(\mu)\, L_{\mathbf{n,p}}(\nu) \qquad \text{m odd}$$

In other words, for each case only certain p values appear in the sum -- those corresponding to the E functions of one of the four classes. Our deduction leads to the following forms for the E functions of the four classes:

$$K_{\mathbf{n,p}}(\mu) = \mu^{\mathbf{n}} + b\mu^{\mathbf{n-2}} + c\mu^{\mathbf{n-4}} + ..$$
$$L_{\mathbf{n,p}}(\mu) = \sqrt{\mu^2 - h^2}\,[\,\mu^{\mathbf{n-1}} + b\mu^{\mathbf{n-3}} + c\mu^{\mathbf{n-5}} + ..]$$
$$M_{\mathbf{n,p}}(\mu) = \sqrt{\mu^2 - k^2}\,[\,\mu^{\mathbf{n-1}} + b\mu^{\mathbf{n-3}} + c\mu^{\mathbf{n-5}} + ..]$$
$$N_{\mathbf{n,p}}(\mu) = \sqrt{\mu^2 - h^2}\,\sqrt{\mu^2 - k^2}\,[\,\mu^{\mathbf{n-2}} + b\mu^{\mathbf{n-4}} + c\mu^{\mathbf{n-6}} + ..]$$

where we know everything except the coefficients, and we have set the leading coefficient to 1 in each series (the b,c... are generic, not meant to be the same in the different functions). The leading power in the series part of the above descending series is μ^{n-I} where I=0,1,1,2 for K,L,M,N. I then combine the four forms shown above into this single way of writing an E function

$$E_{n,p}(\mu) = \{ 1 , \sqrt{k^2 - \mu^2} , \sqrt{h^2 - \mu^2} , \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2} \} [1, \mu] \kappa \Pi_{s=1}^{J} (\mu^2 - \rho_s^2)$$

K	M	L	N	// class
I = 0	I=1	I=1	I=2	// class dependent

J = (n-I)/2	n-I = even	=> select 1 in [1, μ]
J = (n-I-1)/2	n-I = odd	=> select μ in [1, μ]

and this leads to a form for the triple product which is the form of the ellipsoidal harmonics

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu) = \kappa^3$$
$$\{ 1 , \sqrt{k^2 - \rho^2} , \sqrt{h^2 - \rho^2} , \sqrt{h^2 - \rho^2}\ \sqrt{k^2 - \rho^2} \} [1, \rho]$$
$$\{ 1 , \sqrt{k^2 - \mu^2} , \sqrt{h^2 - \mu^2} , \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2} \} [1, \mu]$$
$$\{ 1 , \sqrt{k^2 - \nu^2} , \sqrt{h^2 - \nu^2} , \sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \nu^2} \} [1, \nu]$$
$$\Pi_{s=1}^{J} (\rho^2 - \rho_s^2)\, (\mu^2 - \rho_s^2)\, (\nu^2 - \rho_s^2)$$

This is then transformed into the "partial Niven form"

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$$

$$= \{ 1 \quad x \; y \; z \; xy \; xz \; yz \; xyz \} \; \kappa' \Pi_{s=1}^{J} (\rho^2 - \rho_s^2)\, (\mu^2 - \rho_s^2)\, (\nu^2 - \rho_s^2)$$

K	K	L	M	M	L	N	N	
e	o	o	o	e	e	e	o	// values of n for which this form applies (n even or odd)

and finally we get to the "full Niven form" where EEE is expressed entirely in Cartesian coordinates:

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$$

$$= \{ 1 \quad x \; y \; z \; xy \; xz \; yz \; xyz \} \; \kappa'' \; \{ \Pi_{s=1}^{J} [x^2/\rho_s^2 + y^2/(\rho_s^2 - h^2) + z^2/(\rho_s^2 - k^2) - 1] \}$$

K	K	L	M	M	L	N	N	
e	o	o	o	e	e	e	o	n = even or odd
1	2	2	2	3	3	3	4	species s

And this concludes subsection (e) where I review work done in ""Using Conical and Spherical Coordinates to explore Lamé functions.doc". Now it happens that this "work" really has jumped ahead of the Hobson presentation, and we now have to *back up* to where we were in his Notes.

273. Elaboration of the four Frobenius solutions K, L, N, M.

Note: I finally wrote "Understanding the Hobson Lamé Recursion Relation Solutions.doc" which makes the meta notes below pretty much old hat. See summary at the end of that doc.

Hobson has at this point presented the tesseral work quoted above and has presented the "deduced" forms of the K,L,M,N functions as shown above, but he has not started yet into the remaining stuff above involving the roots and the Niven form and all that business. Instead, at this point he looks at the power series forms quoted above and shows how you can determine the coefficients by inserting the power series into the Lamé ODE which of course creates recursion relations. Later we shall learn a completely different way of doing this that has to do with those Niven forms above.

So, Hobson has in hand the K,L,M,N general forms shown above, he knows that n is an integer (due to the connection to the $\mathcal{H}^n$ subspace we noted above). He also knows that all the series must truncate at either the μ^1 term or the μ^0 term to avoid negative powers.

He starts with the assumed K function form and concludes that you will get proper series truncation only if the determinant of a certain matrix vanishes. He defines r ≡ (n even)? n/2 else (n-1)/2 and the matrix has dimension r+1. The matrix has only a main diagonal and first off diagonals on each side, all other elements are 0. A matrix row thus has 3 adjacent non-zero elements (2 if at the ends) and these elements are the coefficients which appear in the recursion relations. Since the Lamé equation contains n and p, these matrix elements are certain simple functions of n and p (linear in p). Setting detM=0 produces r+1 values of p, since detM is a polynomial of degree r+1 in p. The matrix equation here was M**a** = 0 where **a** is the vector of series coefficients a_i. If detM≠0, we get **a** = $M^{-1}0$ = 0 and we have no series. Once you know the values of p, call them p_i for i = 1 to r+1, you can construct the coefficients one at a time starting with a_0 and so on. So we then have the r+1 K solutions for a given n.

For the L functions, it is easier. You build up an expression for the coefficients using the new recursion relations that arise in this case (the square root is accounted for, etc). You then just insist that the series truncate, meaning a_{n-r}= 0, and by that time you have built up a polynomial in p of degree n - r so you will get n-r values p_i. For each such p_i, you have a coefficient set a_i and then you have constructed the n-r solutions of class L. For some reason I don't quite understand, this method fails to work for the K class case due to the nature of the recursion relations in that case. I could figure it out, but don't want to now.

The M case is just like the L case. The N class is treated similarly, and of course has yest another form of the recursion relations. We get r functions of type N. Here then is a summary

K	r+1	r ≡ (n even)? n/2 else (n-1)/2
L	n-r	
M	n-r	
N	r	
	2n+1 solutions!!	

I think I could make Maple do all this computation without too much trouble, so I could make my own table of the Lamé functions for any n. At this point in my notes, I insert the Hobson table of E functions for n = 0,1,2,3 but Hobson has not computed them yet in his flow.

274. How things reduce in the spheroidal limits of ellipsoidal.

This is a pretty technical topic I don't care much about now, but am glad he included it. Roughly here is what happens: [Lame ODE becomes Legendre or trig ODE, etc]

Oblate h=0:

$E(\mu) \rightarrow P_n^m(\mu')$ $E(\rho) \rightarrow P_n^m(i\rho')$ $E(\nu) \rightarrow \sin/\cos\varphi'$

where μ',ρ',φ' are the oblate spheroidal coordinates

Prolate h=k:

$E(\nu) \rightarrow P_n^m(\cos\theta')$ $E(\rho) \rightarrow P_n^m(\rho')$ $E(\nu) \rightarrow \sin/\cos\varphi'$

where ρ',θ',φ' are the prolate spheroidal coordinates

275. The Lamé functions of degree n= 0,1,2,3.

Here Hobson derives exactly those E function which appear in Byerly and which I quoted above. He then reviews Lamé's proof that the 2n+1 values of p are all real and distinct. In this proof, Hobson presents certain equations which look like orthogonality of certain E functions. The whole subject of orthogonality and completeness and the 2D Sturm Liouville problem is not presented in any systematic way by Hobson, we just get peeks here and there. Maybe the theory didn't exist in the early 1900's.

276. The Evaluation of a Certain Integral (the 4-E orthogonality)

The integral shown here must be the orthogonality of the EE functions in the 2D SL theory. It has the form

$$\int\int [\, E_{np}(\mu)\, E_{np}(\nu)\,]\,[\, E_{n'p'}(\mu)\, E_{n'p'}(\nu)\,]\, d\eta\, d\zeta\, (\mu^2-\nu^2) \;=\; \delta_{nn'}\delta_{ss'} * \text{stuff}$$

where you see two of the 2D EE functions integrated together, and you see the unusual integration measure which couples the two coordinates together (this is a hybrid thing as we are used to now). We know from our dη and dζ expressions that

$$d\eta\, d\zeta\, (\mu^2-\nu^2) = (d\mu/\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,)(d\nu/\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,)(\mu^2-\nu^2)$$

so we expect that our Jacobian deal is

$$v = (\mu^2-\nu^2)/\,[\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,]$$

which is confirmed in my Maple metric tensor file for conicals

$$q123 := \frac{(\mu^2-\nu^2)\,r^2}{\sqrt{(k^2-\mu^2)\,(\mu^2-h^2)\,(k^2-\nu^2)\,(h^2-\nu^2)}}$$

So the orthogonality integral has just the form I would expect.

277. Double Fourier in the μν world.

Above I claimed that we can expand

$$F(\mu,\nu) = \Sigma_{n=0}^{\infty} \Sigma_p \; F_{np} \, E_{np}(\mu) E_{np}(\nu)$$

This section used the orthogonality integral of the previous section so claim that the coefficient will be

$$F_{np} = \int\int \; [\, E_{np}(\mu) \, E_{np}(\nu) \,] \, F(\mu,\nu) \, d\eta \, d\zeta \, (\mu^2-\nu^2)$$

However, Hobson puts a lot of conditions on F(μ,ν). He basically wants to assume it has the square root factors we know the E have, and then the above happens in each class, then you can write an F that is a sum of such terms. Fine.

278. The Zeros of the E(μ) functions.

Claims the zeros are real and never larger than k, and that you never get degenerate roots.

Although I did not realize it on first reading, the zeros of E are significant when you write the polynomial part of E(u) in factored form $\Pi_i \, (\mu-\alpha_i)$. It turns out roots are in ± pairs so $\Pi_i \, (\mu^2-\alpha_i^2)$. I showed above how these roots come into play in various "forms" of the E functions, where the roots are called ρ_s^2.

279. The Second Kind Lamé Functions

He writes these as integrals not quite in the form I am used to, but fine. The functions are called F.

280. Potential Problems for an Ellipsoid (473).

This section is quite clear. If on an ellipsoid you happen to have a prescribed Dirichlet potential that is the product E(μ)E(ν) (indices suppressed), then you *at once* know the exact solution of both the interior and exterior Dirichlet problems! If your Dirichlet potential is some weighted sum of this EE, then of course your two Dirichlet problems are as shown in p 473 B. If the Dirichlet BV potential is written f(μ,ν), this weighted sum has the form p 473 C. Hobson then rewrites the two Dirichlet problem solutions as p 474A where now he shows my $E_{n,p}(\rho)$ as $E_n^{(m)}(\rho)$. So know how to solve internal and external Dirichlet problems for ellipsoids providing the prescribed Dirichlet potential f(μ,ν) can be written as a weighted sum of $E_{np}(\mu) \, E_{np}(\nu)$. Hobson is pretty quiet about how general this class of functions is. It no doubt corresponds to the simpler g(θ,φ) functions on a sphere. You have square roots introduced in the transformation between θ,φ and μ,ν and you have to worry about them.

At this point I noticed on the web that there is a world of multi-dimensional SL problems, and this case with $E_{np}(\mu) \, E_{np}(\nu)$ is one such situation where we are 2D. I have a paper on this general topic.

281. Expanding a function on Lamé products: the example of $P_n(\cos\gamma)$

Hobson shows how this particular function can be expanded on the $E_{np}(\mu) \, E_{np}(\nu)$.

282. Ellipsoidal harmonics expressed in Cartesian coordinates

This is the Niven theory which I have already summarized above, since I did it in the separate doc noted above. I seem to have some overlap between the mainline Hobson notes and the doc notes. Here is a quick outline of the mainline notes:

(a) a new way to write $E_{n,p}(\mu)$ and $E_{n,p}(\mu)\,E_{n,p}(\nu)$

I give the same E form shown earlier, and I prove that the roots occur in ±pairs. I then write the EE expression not quoted above but which is completely obvious by now.

$$E_{n,p}(\mu)\,E_{n,p}(\nu) = \kappa^2$$
$$\{\,1\,,\sqrt{k^2-\mu^2}\,,\sqrt{h^2-\mu^2}\,,\sqrt{h^2-\mu^2}\;\sqrt{k^2-\mu^2}\,\}\;[\,1,\mu]$$
$$\{\,1\,,\sqrt{k^2-\nu^2}\,,\sqrt{h^2-\nu^2}\,,\sqrt{h^2-\nu^2}\;\sqrt{k^2-\nu^2}\,\}\;[\,1,\nu]$$
$$\Pi_{i=1}^{J}\;(\mu^2-\alpha_i^2)\,(\nu^2-\alpha_i^2)$$

I talk about the dimensions of things, noting that h,k,ρ,μ,ν,x,y,z all have dimensions L^1, they are all true distances.

(b) a new way to write $E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu)$: { 1 x y z xy xz yz xyz } Π (roots)

This is the partial Niven form noted above.

(c) a second way to write $E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu)$: full Cartesian coordinates

And this is the full Niven form noted above. I have a comment on the meaning of "degree n". I show that the following is true

$$E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu) = \text{poly}_n(x,y,z;\text{ by 2})$$

and I argue that

$$E_{n1,p}(\rho)\,E_{n1,p}(\mu)\,E_{n1,p}(\nu) = \Sigma^{n=n1}{}_{0or1,by2}\;\;\Sigma_{m=-n}^{n}\,a_{n,m}\,r^n\,Y_{n,m}(\theta,\varphi)$$

so that the EEE product is a mix spherical atoms at various descending n levels as shown.

At this point, Hobson introduces the notation Θ_s for the full-Niven-form factors, and the new roots variable θ_s which makes things easier to write,

$$\Theta_s \equiv [\,x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s) - 1\,] \qquad \theta_s = \rho_s^2 - a^2$$

and we end up with the famous general form

$$E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu) \;\sim\; \{\,1\;\;x\;y\;z\;xy\;xz\;yz\;xyz\,\}\;\Theta_1(\theta_1)\Theta_2(\theta_1)....\Theta_J(\theta_J)$$

He goes on to define

$$G_n^s(\rho,\mu,\nu) = E_{n,s}(\rho)E_{n,s}(\mu)E_{n,s}(\nu) \qquad s = 1,2...2n+1$$
$$\mathcal{G}_n^s(\rho,\mu,\nu) = F_{n,s}(\rho)F_{n,s}(\mu)F_{n,s}(\nu) \qquad s = 1,2...2n+1$$

283. Conical harmonics expressed in Cartesian coordinates

We now go through the whole song and dance again, this time for EE products which form part of the conical harmonics r^nEE. Things are a little different here because we have various r factors running around, but the main conclusion we get to is this:

$$r^n E_{n,p}(\mu) E_{n,p}(\nu)$$

$$= \{ 1 \quad x \quad y \quad z \quad xy \quad xz \quad yz \quad xyz \} \ \kappa'' \ \Pi_{s=1}^{J} [x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)]$$

K K L M M L N N // class

e o o o e e e o // n // Hobson p 478 B

with our same old J formula I won't rewrite. This is how you can write any conical harmonic in Cartesian coordinates. The -1 is no longer present in the Θ_s factor, and this can be traced to the fact that the surface equation for an elliptic cone has no 1 on the RHS, whereas an ellipsoid does have the 1.

284. Finding the roots.

(a) Review of the Old Method

Way back in Section 273 we learned how to find the "coefficients" for the "series part" of the Lamé functions in each class. In a separate doc, I was able to unify the results of all classes into a single method which was this, where J is the same integer function of n and I I have been using all along,

$$E(\mu) = a_0\mu^{n-I} + a_1\mu^{n-I-2} + a_2\mu^{n-I-4} + ... + a_J [1, \mu]$$

$$a_{J+1}(p) = 0$$

The second equation is the $(J+1)^{st}$ coefficient obtained by rolling up the recursion relation from a_0, and this roll-up is different for the four classes since the reduced Lamé equations are different for each class. The equation $a_{J+1}(p) = 0$ has for its LHS $poly_{J+1}(p)$ and we can obtain the J+1 dimensionless roots called the p_i. For classes K,LM,N there are r+1,n-r,n-r,r roots where "r" is a simple function of n. Each integer in the list just given is J+1. So this unified method let's use write a program which can compute the coefficients of all classes for any n, problem solved.

I find it useful to replicate this information from the "understanding " doc. We know that

n-I = even => J = (n-I)/2 and select 1 in [1, μ]

n-I = odd => J = (n-I-1)/2 and select μ in [1, μ]

and from this we construct a little table

	class	I	n-I	J	J	J+1
n even	K	0	even	(n-I)/2	n/2	n/2+1
n even	L	1	odd	(n-I-1)/2	n/2-1	n/2
n even	M	1	odd	(n-I-1)/2	n/2-1	n/2
n even	N	2	even	(n-I)/2	n/2-1	n/2
						2n+1

	class	I	n-I	J	J	J+1
n odd	K	0	odd	(n-I-1)/2	(n-1)/2	(n+1)/2
n odd	L	1	even	(n-I)/2	(n-1)/2	(n+1)/2
n odd	M	1	even	(n-I)/2	(n-1)/2	(n+1)/2
n odd	N	2	odd	(n-I-1)/2	(n-3)/2	(n-1)/2
						2n+1

and this more specific table of function counts in each class (this is a table of J+1 values)

n =	0	1	2	3	4	5	6	7	8
K	1	1	2	2	3	3	4	4	5
L	0	1	1	2	2	3	3	4	4
M	0	1	1	2	2	3	3	4	4
N	0	0	1	1	2	2	3	3	4
	1	3	5	7	9	11	13	15	17

(b) The Cartesian Method

Now in this new Cartesian method we instead end up, for each class, with a system of J equations (Hobson's m) shown on Hobson page 479 (which is for K class). This system of equations has J unknowns called θ_i which we can think of as a vector $\boldsymbol{\theta}$ having J components. If this system of equations were a *linear* system in the sense of Cramer's rule, meaning $M\boldsymbol{\theta} = \mathbf{q}$, we would expect one solution vector $\boldsymbol{\theta}$. But this is *not* the form these equations have, and the equation set in fact has several solution vectors we call $\boldsymbol{\theta}^{(k)}$, where we use k to distinguish the multiple solution vectors. For each class, it turns out that there are J+1 solution vectors (J varies with class for given n), so we have k = 1,2....(J+1). Each solution vector $\boldsymbol{\theta}^{(k)}$ implies a vector of "squared roots" $(\rho_1^2, \rho_2^2, ... \rho_J^2)$ since $\theta_i = \rho_i^2 - a^2$ (where a is an unrelated constant), which vector we might just call $[\boldsymbol{\rho}^2]$ so $\boldsymbol{\theta} = [\boldsymbol{\rho}^2] - [\mathbf{a}^2]$ where $[\mathbf{a}^2]$ is a vector each of whose components is just a^2. Thus, we write $\boldsymbol{\theta}^{(k)} = [\boldsymbol{\rho}^2]^{(k)} - [\mathbf{a}^2]$ and we then have J+1 solution vectors of the form $[\boldsymbol{\rho}^2]^{(k)}$. But we know that a set of roots ρ_i^2 gives us a set of coefficients a_i so we then associate a vector of coefficients $\mathbf{a}^{(k)}$ with $[\boldsymbol{\rho}^2]^{(k)}$ and thus with $\boldsymbol{\theta}^{(k)}$. Therefore, if for a given class we solve the Hobson system of equations for the J+1 solution vectors $\boldsymbol{\theta}^{(k)}$, we will have found the J+1 Lamé functions of that class, since each such function is fully characterized by the coefficient vector $\mathbf{a}^{(k)}$.

The most complicated example I have done is n = 3 which has J = 1 for K class and J=0 for the other classes. For K class, we have J=1 Hobson equation $3/(a^2+\theta_s) + 1/(b^2+\theta_s) + 1/(c^2+\theta_s) = 0$ where a,b,c

are certain constants, and we can see this will have J+1 = 2 solutions θ_s. Since J = 1, the solution vector has only one component. So this example is a bit lightweight and does not bring out the complexity of the situation very well.

So where do these Hobson equation systems come from which we compute that solution vectors $\boldsymbol{\theta}^{(k)}$? The way this is done is that we take our above Cartesian representation of $r^n E_{n,p}(\mu) E_{n,p}(\nu)$, the conical harmonic, and require that it satisfy Laplace. So we apply Cartesian ∇^2 to this harmonic and it must be 0. Doing this for K class produces the set of J non-linear equations shown as p 479 D. The set of equations for any of the classes can be written as p 480 A where we let index p = 1,2....J and set the k_i appropriately for a given class. In my raw notes, I verify that this is correct.

I have to say, it seems that the Old Method is a much simpler way to find the Lamé functions.

One wonders how, from this Cartesian method, one might deduces the set of dimensionless p_i values? Here is a simple way. We have our set of solution vectors $\mathbf{a}^{(k)}$ as just noted above. From each vector we select the coefficient $a_1^{(k)}$. We know the lowest recursor in any class has the form $a_1 = (Ap+B)a_0$ and since we probably have $a_0 = 1$ and we know $a_1^{(k)}$, we can solve for the corresponding $p^{(k)}$.

(c) Cartesian root calculation examples.

In the raw notes, I go through the cases n = 0,1,2,3,4 applying this Cartesian root theory. The cases n = 0 and 1 are pretty trivial and we easily obtain the Byerly functions E_0 and E_1 for all classes.

For n = 2 the L,M,N classes all have J=0 so we get no "θ_s action" and the solutions are just the three M,L,N solutions are just the functions xy,xz,yz. For K class J = 1 so our solution vectors $\boldsymbol{\theta}^{(k)}$ have only one component, there are J+1 = 2 solution vectors, and we find the two values of θ_s and we then correctly obtain the two Byerly K_2 functions.

For n=3, the table above in fact shows for K,L,M,N we have J = 1,1,1,0. The single N class function is just xyz and has no θ_s action. For each of the other classes, the $\boldsymbol{\theta}^{(k)}$ vector J=1 component θ_s and we get a quadratic equation from which we find the $\theta_s^{(k)}$ and thus the two solutions.

The n=4 case is more interesting. The table shows for K,L,M,N we have J = 2,1,1,1. The L,M,N classes will all have pairs of functions (J+1=2) each of which will have a quadratic equation to solve for θ_s. We know how to do all that, so we focus on the K class situation where we expect J+1=3 solution vectors $\boldsymbol{\theta}^{(k)}$, each having J=2 components. The Hobson equations are these

$$1/(a^2+\theta_1)+ 1/(b^2+\theta_1) + 1/(c^2+\theta_1) + 4/(\theta_1-\theta_2) = 0$$
$$1/(a^2+\theta_2)+ 1/(b^2+\theta_2) + 1/(c^2+\theta_2) + 4/(\theta_2-\theta_1) = 0$$

I then expend some time trying to solve this system. You rationalize each equation. The first is then linear in θ_2 so you solve that for θ_2 and stick that into the second equation. We end up with $poly_9(\theta_1) = 0$. We have Maple compute the 9 roots. If you think about the rationalization process, we multiply everything in the first equation by the product $(a^2+\theta_1)(b^2+\theta_1)(c^2+\theta_1)(\theta_1-\theta_2)$, so this introduces "bogus" solutions which have $\theta_1 = -a^2,b^2,c^2$, each of which will have some partner θ_2 value. Similarly, rationalizing the second equation gives bogus solutions with $\theta_2 = -a^2,b^2,c^2$ each with a partner θ_1 value. When we ask Maple to solve $poly_9(\theta_1) = 0$, it finds the three bogus solutions $\theta_1 = -a^2,b^2,c^2$, but the remaining 6 solutions are

left embedded in a RootOf ($poly_6(\theta_1)$). We can imagine numerically solving this for the six θ_1 values. We would find that three of these values have partner values $\theta_2 = -a^2,b^2,c^2$ and we would know to throw those out. The remaining three solution vectors are then the ones we want.

(d) Derivation of the Cartesian root equations for general n

This section has been replaced by "Derive Hobson p 479 B,C and D.doc". The goal is to derive the set of equations Hobson 480 A for p = 1,2..J. These are the equation systems described above. The doc has an 2 page Overview you can go read now if you want. I break this goal down into Problem 1 and Problem 2.

285. More on finding the roots.

Here is where Hobson actually states the general equation system already explained above. He talks a little about how the θ_i roots are distributed in their range $(-a^2,-c^2)$, a theorem of Klein. I take note that although Maple knows about conical and ellipsoidal coordinates, it does not know about Lamé functions.

286. How are the ellipsoidal harmonics related to the spherical harmonics?

This is not a very enlightening section. He talks about G_n being Cartesian ellipsoidal harmonics and H_n being Cartesian spherical harmonics (which I now know a lot about). He then pulls out of nowhere a claim that the G_n are orthogonal when integrated over the ellipsoidal surface. This seems reasonable based on his "octants argument" looking at p 481A, but I am unhappy about the meaning of $\bar{p}\, dS$ as the area element in such an integral. I imagine that $dA = Q_\mu Q_\nu\, d\mu d\nu$ for a patch on an ellipsoid. I do have the feeling that all this has a bearing on my ellipsoid as a morphed sphere concept.

We now have a change in subject. In my notes, I show the meaning of equations B,C,D,E on page 482.

" If you select any point x,y,z located on the fundamental a,b,c ellipsoid and you insert that point into the expression that is the RHS of equation B (which is a K factor), you get expression D. "

I then show this equals expression E if you define $x_1 = x/a$ etc. I then tried to derive equation F but could not do it, but then I realized that he is just previewing something derived in the next section.

287. Expanding EEE on the spherical harmonics.

I think I convinced myself of this fact

$G_{n1}(x,y,z)$ = sum[n = n_1 down by 2 to 0 or 1, $H_n(x,y,z)$]

He is going to explore relationships between the G_n and H_n functions. I am able to prove then his first claim F which says

$G_n(x,y,z) = (-1)^s\ \Pi_{i=1}^{s}\theta_i\ \{ 1\ \ a\ \ b\ \ z\ \ ab\ \ ac\ \ bc\ \ abc \}\ H_n(x_1,y_1,z_1)$ Hobson p 483 F

Then on the top of page 484 we encounter, as equation A, a fascinating theorem about the Cartesian sphericals H_n where you compute an expression for $H_n(\partial_x, \partial_y, \partial_z)$ in terms of $H_n(x,y,z)$. This theorem is proved in Section 80 of Hobson which I don't have a copy of, and appears as a problem in W&W. So at this point, I stopped doing algebra and I cut to the chase to see his final goal. The result is this

$G_n(x,y,z)$ = (series of even powers of operator D acting on) $H_n(x,y,z)$

where $D = a^2\partial_x^2 + b^2\partial_y^2 + c^2\partial_z^2$

where D is some kind of morphed Laplacian operator! This result appears in W&W as well. It was first shown by Niven, and the proof was refined as shown here by Hobson.

Since $H_n(x,y,z)$ is $Poly_n(x,y,z)$, there will be a finite number of terms in the operator series, and you will grind out a specific G_n function if you start with a specific H_n one. There are 2n+1 $H_n \sim Y_{nm}$ of course, and there are 2n+1 ellipsoidal harmonics $E_{np}E_{np}E_{np}$ so I guess basically this is a way to compute the Cartesian ellipsoidal harmonics from the spherical ones in Cartesian coordinates.

288. Expanding EEE on the spherical harmonics, continued.

This is way beyond my interest. Doing 4 pages of Hobsonian math, the end result is page 489 J which is a way to write the "external elliptical harmonics" $\mathcal{G}_n$ (which I presume is F_{np} F_{np} F_{np}) as another fancy operator involving the H_n, but this time $H_n(\partial_x, \partial_y, \partial_z)$ appears. This is another Niven result with derivation improved by Hobson. The section ends with some comments about convergence of things.

289. The ellipsoidal Dirichlet problem all done in Cartesian coordinates.

The subject here is how you would do a Dirichlet problem using all these Cartesian ellipsoidal tools he has been developing. The end result is page 492 A which shows a Cartesian function f(x,y,z) expanded in some horrible way on the G_n functions. It is a thing of true ugliness. I suppose it is interesting to at least have a method of treating such a problem. The alternative is to try to convert the prescribed potential to ellipsoidal coordinates, and try to do it there using our standard methods. The problem there of course is that it is not each to express this Cartesian specified Dirichlet potential as a sum of $E(\mu)E(\nu)$ functions.

290 and 291. Reduction of Ellipsoidal harmonics to Spheroidal ones.

He starts with the "roots" θ_s from the ellipsoidal functions and tracks what happens to them as you take the prolate limit. He comes up with some ODE's and of course ends up with the right results. I will keep this section in mind if I ever write a treatise on spheroidal reductions of ellipsoidal things. See raw notes for a longer summary.

These two sections really are strange. He claims to be "reducing ellipsoidal to the prolate" but we never see the thing we start with. It is all done in terms of the roots θ_i. We thought maybe he would start with G somehow and reduce this [well yes, the θ_i determine the G_n] . He does show G bottom of page 494 in terms of H. I guess he puts $H = P_n$ somehow. I think this section was a "last gasp" of Hobson, a footnote detail that was not worth explaining in more detail.

292. Lamé relation to other functions.

We earlier showed that the helper *coordinates* ξ,η,ζ were given as elliptic integrals of the ρ,μ,ν coordinates. Here he suggests that the Lamé *functions* might be written as elliptic integrals of the Jacobi and Weierstrass type. [recall that the Weierstrass elliptic function is that funny **ρ**(u) thing that you see in various places like W&W and Bateman] He says W&W talk about this a little (4th Ed was 1927 which predates Hobson's book by 4 years). This is a version of the Lamé equation using this funny function

$$\frac{d^2\Lambda}{du^2} = \{n(n+1)\wp(u) + B\}\,\Lambda,$$

Hobson is just saying he knows this stuff exists, but he won't be dealing with it. Obviously the Lamé ODE written in terms of this function (think coordinate) ρ(u) is very simple, certainly its motivation. This would be an alternative coordinate different from the ξ,η,ζ type coordinates. In W&W we see lots of such coordinates in use.

His final comments are that someone has fiddled with Lamé for n being half integers, and he thinks that using $n = i\tau - 1/2$ might be useful in elliptic cone boundary stuff, which I could believe. Recall the closely related conical, toroidal and Mehler functions that I comment on somewhere.