Hobson on Ellipsoidal Harmonics PhL 10.14.10

The amount of information in this chapter (all of it new to me) is simply massive!

Section 269. Ellipsoidal coordinates, and distorted spherical coordinates. 3
Distorted Spherical Coordinates ρ,θ,φ 3
Inversion of the distorted spherical coordinate equations, and level curves. 5
(1) Surfaces of constant ρ 5
(2) Surfaces of constant φ 5
(3) Surfaces of constant θ 6
(4) The tangent vectors. 7
(5) The metric tensor. 8
(6) Comparison of the ρ,θ,φ ellipsoidal system to the ρ,θ ellipse system 8
(7) Resume Hobson notes. 9
Section 270: Conical Coordinates 11
(a) conical coordinates (see separate doc) 11
(b) MF on conical coordinates, not very useful 12
(c) inverting the conical coordinates ? (separate doc) 13
(d) the Laplacian in conical coordinates 13
(e) introduction of the "alternate" variables η,ζ for μ,ν; the new simple ∇^2 form! 13
(f) Why is this p 456 B ∇^2 form so simple? Because the metric tensor is simple! 14
(g) Computation of ∇^2 in terms of η and ζ 14
(g) Attempting separation of $\nabla^2 V = 0$ by assuming $V = r^n u$ with $u = E(\mu/\eta)E(\nu/\zeta)$ 15
Section 271. Back to ellipsoidal coordinates. 18
(b) introduction of the "alternate" variables ξ,η,ζ for ρ,μ,ν; the new simple metric tensor. 19
(c) computation of ∇^2 in the alternate coordinates 20
(d) Attempting separation of $\nabla^2 V = 0$ by assuming $V = E(\rho/\xi)E(\mu/\eta)E(\nu/\zeta)$ 21
Plan A: does not get very far. 22
Plan B: follow Hobson more closely: 23
Write the three separated equations in the alternate coordinates 23
Convert the three separated equations from alternate into original coordinates 24
Review of what has happened here! 25
(1) Conical connection to Ellipsoidal 25
(2) Step by Step Review: 25
Try to directly relate conicals and ellipsoidals 26
Find expressions for ρ,μ,ν in terms of the alternates ξ, η, ζ 27
Solve for ρ, given $\xi(\rho)$ = elliptic integral 27
Solve for μ, given $\eta(\mu)$ = elliptic integral 30
Solve for ν, given $\zeta(\nu)$ = elliptic integral 32
272. How E are related to the spherical harmonics? 33
Items related to Hilbert Space and its $\boldsymbol{\ell}$ subspace 33
(a) Angular momentum EV equation in sphericals; doing expansions on θ,φ; subspace $\mathcal{H}^{\ell}$ 33
(b) Same stuff in conicals; the L^2 operator expressed in μ,ν conicals 34
(c) Conical separation in these μ,ν coordinates 35

(d) Resume the eigenket discussion: how Y and E are related 36
(e) What can we learn from the connection between Y and E? 36
"Using Conical and Spherical Coordinates to explore Lamé functions.doc" goes here! 37
273. Elaboration of the four Frobenius solutions K, L, N, M. 38
(1) The K case : have to set detM=0 to find the r+1 distinct p values. 38
(2) The L case: set $a_{n-r} = 0$ to truncate poly, solve this for the n-r p values that work. 40
(3) The M case: same idea as L case, there are n-r values of p. 41
(4) The N case. this time there are r solutions. 41
(5) Summary, add them up to find 2n+1 solutions $E_{n,p}(x)$ 41
274. How things reduce in the spheroidal limits of ellipsoidal. 43
(a) oblate. 43
(b) prolate. 43
275. The Lamé functions of degree n= 0,1,2,3. 43
(a) Derivation of all the first-kind Lamé functions with n = 0,1,2 and 3. 43
(b) Proof that the (2n+1) p values are real and distinct. 44
276. The Evaluation of a Certain Integral (the 4-E orthogonality) 45
277. Double Fourier in the μν world 45
278. The Zeros of the E(μ) functions. 45
279. The Second Kind Lamé Functions 46
280. Potential Problems for an Ellipsoid (473). 46
Comment on SL separation. 46
281. Expanding a function on Lamé products: the example of $P_n(\cos\gamma)$ 47
282. Ellipsoidal harmonics expressed in Cartesian coordinates 48
(a) a new way to write $E_{n,p}(\mu)$ and $E_{n,p}(\mu)\, E_{n,p}(\nu)$ 48
General form for E. 48
Roots occur in ± pairs. 49
General form for EE 50
Comment on Dimensions: (long overdue) 50
(b) a new way to write $E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$: { 1 x y z xy xz yz xyz } Π (roots) 50
(c) a second way to write $E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$: full Cartesian coordinates 51
Aside on degree of EE: 52
Lengthy Aside on degree of EEE: 52
Introduction of the θ and Θ and a,b,c symbols 53
Quote from W&W concerning Niven and "species" 55
The G and $\mathcal{G}$ functions 56
283. Conicals in Cartesian coordinates 56
(a) a new way to write $E_{n,p}(\mu)\, E_{n,p}(\nu)$: { 1 x/r y/r z/r xy/$r^2$ xz/r^2 yz/r^2 xyz/r^3 } Π (roots) 56
(b) a second way to write $E_{n,p}(\mu)\, E_{n,p}(\nu)$ 57
(c) how to write $r^n\, E_{n,p}(\mu)\, E_{n,p}(\nu)$: full Cartesian form 59
284. Finding the roots. 60
The cases n = 0 and n = 1 61
The case n = 2 61

The case n = 3 and also some general results. 63
The n = 2 case revisited 67
The n = 4 case. 68
The case of general n. 71
285. More on finding the roots. 74
286. How are the ellipsoidal harmonics related to the spherical harmonics? 75
287. Expanding Cartesian EEE = G_n on the Cartesian spherical harmonics = H_n. 78
287. Expanding Cartesian EEE = G_n on the Cartesian spherical harmonics = H_n, continued. 79
289. The ellipsoidal Dirichlet problem all done in Cartesian coordinates. 80
290. Reduction of Ellipsoidal harmonics to Spheroidal ones. 81
291. Reduction of Ellipsoidal harmonics to Spheroidal ones continued. 81
292. Lamé relation to other functions. 82
Appendix A. Finding the θ_s system of equations for the case n = 3 82

Section 269. Ellipsoidal coordinates, and distorted spherical coordinates.

He starts out pretty clearly talking about ellipsoidal coordinates which he calls ρ, μ, ν , and we have these connections to MF:

Hobson	MF	
k > h	a > b	focal distances of ellipsoid
ρ, μ, ν	ξ_1, ξ_2, ξ_3	ellipsoidal coordinates

Hobson writes out the x,y,z equations mid page 455 commenting on the interesting complex-plane parameter ranges for μ and ν which I learn more about below. In his world, the ellipsoid goes flat when ρ → k from above.

Distorted Spherical Coordinates ρ,θ,φ

I am spending a lot of effort on this strange system. The main reason is that I allowed myself to get confused about it, and that required lots of back and fix to reconcile things. I still don't understand why he even mentioned this non-orthogonal system!

Then on the bottom of page 455 we get something completely new. He transforms from μ,ν to θ,φ in a manner that dimly resembles spherical coordinates, but not quite.

How does he do this?? Well, here is one way to understand it. We start with the known **ellipsoidal** x,y,z equations which are these [Hobson's x is what M&F call z, so x and z are swapped from M&F]

$x = \rho\mu\nu/(hk)$ // $\rho, \mu, \nu = \xi_1, \xi_2, \xi_3$

$y = \sqrt{\rho^2-h^2}\,\sqrt{\mu^2-h^2}\,\sqrt{h^2-\nu^2}\,/\,(hs)$ $s^2 = k^2-h^2$

$$z = \sqrt{\rho^2-k^2}\,\sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2}\,/(ks)$$

I derived these expressions in a separate doc directly from the ellipsoidal conicoid equations using a simple Cramer's Rule method. In that doc, I also point out that that the three semi-major axes of the ellipsoid labeled by ρ (or ξ_1) are A,B,C where

$A^2 = \xi_1^2-a^2 = \rho^2-k^2$ z
$B^2 = \xi_1^2-b^2 = \rho^2-h^2$ y
$C^2 = \xi_1^2-c^2 = \rho^2-0^2 = \rho^2$ x => $C \geq B \geq A$

so we can rewrite our x,y,z equations this way

$x = C\ \mu\nu/(hk)$ // $\rho, \mu, \nu = \xi_1, \xi_2, \xi_3$
$y = B\ \sqrt{\mu^2-h^2}\,\sqrt{h^2-\nu^2}\,/(hs)$ $s^2 = k^2-h^2$
$z = A\ \sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2}\,/(ks)$

Since C>B>A are semi major axes in the x,y,z directions, we know right away that |...| < 1 for the quantities multiplying C,B,A in the x,y,z equations above! Each of these factors is a function of μ,ν. Suppose we just *try* this association:

$\cos\theta = \mu\nu/(kh) = \xi_2\xi_3/(ab)$
$\sin\theta\cos\varphi = \sqrt{\mu^2-h^2}\,\sqrt{h^2-\nu^2}\,/(hs)$
$\sin\theta\sin\varphi = \sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2}\,/(ks)$

The first line determines cosθ and thus θ which will be in the range (-π,π) if we let ν have negative values, see blue and yellow cone pictures elsewhere, so this is a "reasonable" θ. Then the next two lines will be OK *if* we can show that they are consistent with the first line. Let's check that now by squaring and then adding the last two lines:

$\sin^2\theta = [\,(\mu^2-h^2)(h^2-\nu^2)/(hs)^2 + (k^2-\mu^2)(k^2-\nu^2)/(ks)^2\,]$

$= [\,(\mu^2-h^2)(h^2-\nu^2)/h^2 + (k^2-\mu^2)(k^2-\nu^2)/k^2\,]/s^2$

$= [\,k^2(\mu^2-h^2)(h^2-\nu^2) + h^2(k^2-\mu^2)(k^2-\nu^2)\,]/(shk)^2$

$= (k^2-h^2)(k^2h^2-\mu^2\nu^2)/(shk)^2$ // Maple did the numerator for me

$= (k^2h^2-\mu^2\nu^2)/(hk)^2$

$= 1 - (\mu\nu/hk)^2$

and this then is indeed consistent with the first line.

So we end up then with this connection between ρ,θ,φ and x,y,z:

$x = C\ \cos\theta = \rho\cos\theta$ // $\rho, \mu, \nu = \xi_1, \xi_2, \xi_3$

$y = B\ \sin\theta\cos\varphi = \sqrt{\rho^2-h^2}\ \sin\theta\cos\varphi$ $s^2 = k^2-h^2$

$z = A\ \sin\theta\sin\varphi = \sqrt{\rho^2-k^2}\ \sin\theta\sin\varphi$

This is totally new to me.

Inversion of the distorted spherical coordinate equations, and level curves.

What are the level curves? For example, if θ = constant, what surface do you get in x,y,z space?
To find out, we really have to invert the above system which is easy to do. Summary:

$x^2/\rho^2 + y^2/(\rho^2-h^2) + z^2/(\rho^2-k^2) = 1$ Solve for ρ = ugly cubic formula. Then:

$\tan\varphi = (z/y)\ [\sqrt{\rho^2-h^2}\ /\sqrt{\rho^2-k^2}\ ]$ $\cos\theta = x/\rho$

(1) Surfaces of constant ρ

We know the inversion for ρ already from work done elsewhere. Using the above x,y,z equations it is very obvious that this is true

$x^2/\rho^2 + y^2/(\rho^2-h^2) + z^2/(\rho^2-k^2) = 1$ // since $\cos^2\theta + \sin^2\theta = 1$

so we know that ρ = constant gives us a family of **confocal ellipsoids**. You can see that this thing gives a cubic equation for quantity ρ^2. The solution to that cubic is a real mess, which I found to have this form

$$\rho^2(x,y,z) = \text{poly}_2 + \sqrt{\text{poly}_4} * \cos\{\ [1/3]\ \cos^{-1}(\text{poly}_6/\sqrt{\text{poly}_{12}}\ )\ \}$$

where poly_n means a polynomial of degree n in variables x,y,z which includes constants and h and k such that dim[poly_n] = L^n.

(2) **Surfaces of constant φ**

Well, we can see that

$\tan\varphi = (z/y)\ [\sqrt{\rho^2-h^2}\ /\sqrt{\rho^2-k^2}\ ]$ where $\rho^2 = \rho^2(x,y,z)$ as shown above

If we set φ = constant, we get some **incredibly complicated surface** in x,y,z space! Only in the special case that h = k (prolate) do we get the simple result tanφ = z/y which gives the usual azimuthal planar surfaces we are used to.

I might be able to plot this surface using implicitplot3d. In my notes I have the ellipse equation written in this alternate manner

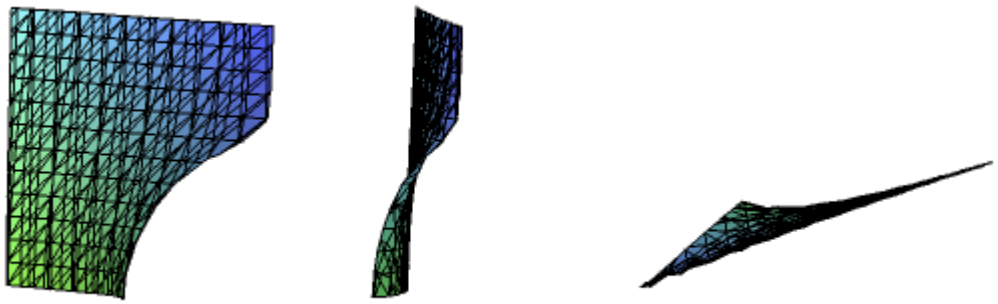
$x^2/(\xi_1^2 - a^2) + y^2/(\xi_1^2 - b^2) + z^2/\xi_1^2 = 1$ $\xi_1 > a$ ellipsoids

where we identify $\xi_1^2 = \rho^2$ and a = k and b = h and z ↔ x compared to my equation above. So in the language of this other doc, the surfaces of constant φ would be written

$$\tan\varphi = (x/y)\left[\sqrt{\rho^2-b^2}\,/\sqrt{\rho^2-a^2}\,\right] = \text{constant}$$

Here we look at this thing being constant in the first x,y,z quadrant, for these parameters:

a = 3 b=1 constant = 0.5 scan range (0,5) for x,y,z



first view rotate around vert top view

This is done in **hobson1.mws** and here is the last line of the code:

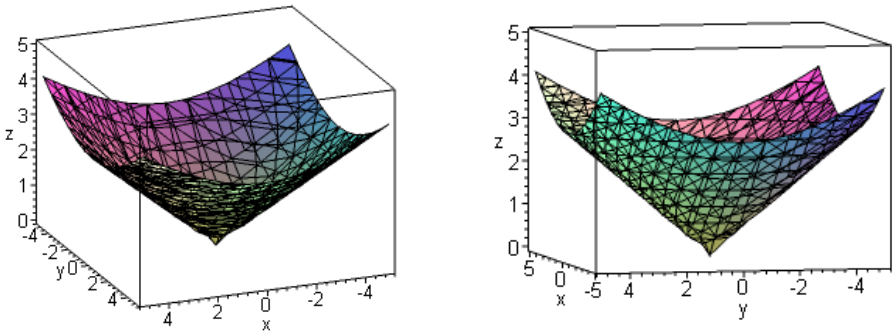
implicitplot3d((x/y)*sqrt(rho2-beta)/sqrt(rho2-alpha) = .5, x=0.1..5,y=0.1..5, z=0.1..5,grid=[13,13,13]);

If you set b = a = 3, you get the flat azimuthal plane of the prolate system. But you can see that otherwise you get a plane that has a strange bend in it. So this is what a surface of constant φ looks like!

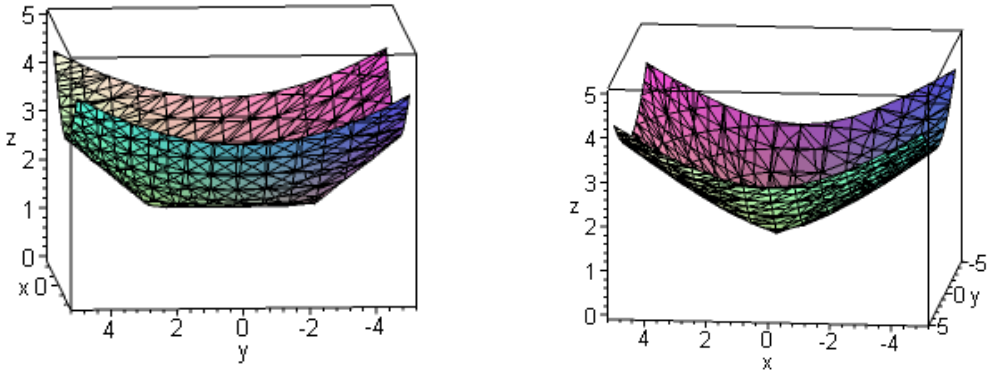
(3) **Surfaces of constant θ**

We have $\cos\theta = x/\rho$ which we translate into z/ρ, and we could then do another implicit plot to examine it. The surfaces are then x/ρ = constant.

If we first set a = b = 0, we have a spherical system and this surface is a cone with a pointed tip:



But now if we set a = 3 and b = 1, we find a cone far away, but close in the shape of the tip has been changed to what you see here:



So these are what the surfaces of constant θ look like.

For a = b= 3 (the prolate case), the θ surface is a rounded off cone that is a surface of revolution.

NOTE: Both the constant φ and constant θ surfaces have the same general distorted shape if we set a = 3 and b = 0, which is the situation of an oblate spheroid. Maple shows this to be true.

(4) **The tangent vectors.**

These are the rows of the Tup matrix, and in this case we find (ordering x,y,z)

```
Tup := matrix(3,3,T);
```

$$Tup := \begin{bmatrix} \cos(\theta) & \dfrac{\sin(\theta)\cos(\phi)\,\rho}{\sqrt{\rho^2-h^2}} & \dfrac{\sin(\theta)\sin(\phi)\,\rho}{\sqrt{\rho^2-k^2}} \\ -\rho\sin(\theta) & \sqrt{\rho^2-h^2}\cos(\theta)\cos(\phi) & \sqrt{\rho^2-k^2}\cos(\theta)\sin(\phi) \\ 0 & -\sqrt{\rho^2-h^2}\sin(\theta)\sin(\phi) & \sqrt{\rho^2-k^2}\sin(\theta)\cos(\phi) \end{bmatrix}$$

The three rows of this matrix are the three tangent base vectors $\mathbf{e}_\rho$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$. The last line tells us that

$$\mathbf{e}_\varphi = \sin\theta\,(\,0,\, -\sqrt{\rho^2-h^2}\,\sin\varphi,\, \sqrt{\rho^2-k^2}\,\cos\varphi)$$

Recall that our "up" direction is x. For the limit h=k=0 (spherical coords), we are used to having $\mathbf{e}_\varphi$ be parallel to the equatorial plane! But here we see that this is also true for our ρ,θ,φ coordinates. If you look at our φ = constant surface above, you can see that the *normal* vector points in a general 3D direction. However, since ρ,θ,φ is a non-orthogonal system, $\mathbf{e}_\varphi$ is not the normal vector, so we don't have a contradiction here.

The other two tangent vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\theta$ appear to have all three components non zero. If you stare at the matrix above, you can see that $e_\rho \bullet e_\theta = 0$ and $e_\rho \bullet e_\varphi = 0$. This is a *somewhat startling* fact (which reappears below), considering the strange shape of the constant parameter surfaces shown above. If only we also had $e_\varphi \bullet e_\theta = 0$, we would have an orthogonal system, but that one is not zero!

(5) **The metric tensor.**

If we continue in "metric tensor rho theta phi.mws", we find this for the metric tensor g_{dn} :

$$\begin{bmatrix} \frac{-\cos(\theta)^2\rho^2k^2+\cos(\theta)^2h^2k^2-\cos(\phi)^2\rho^2k^2+\cos(\phi)^2\rho^2k^2\cos(\theta)^2+\rho^4-h^2\rho^2+h^2\rho^2\cos(\phi)^2-h^2\rho^2\cos(\theta)^2\cos(\phi)^2}{\rho^4-\rho^2k^2-h^2\rho^2+h^2k^2}, & 0, & 0 \\ 0, & \rho^2-\cos(\theta)^2\cos(\phi)^2h^2-\cos(\theta)^2k^2+\cos(\theta)^2k^2\cos(\phi)^2, & \cos(\theta)\cos(\phi)\sin(\theta)\sin(\phi)h^2-\cos(\theta)\cos(\phi)\sin(\theta)\sin(\phi)k^2 \\ 0, & \cos(\theta)\cos(\phi)\sin(\theta)\sin(\phi)h^2-\cos(\theta)\cos(\phi)\sin(\theta)\sin(\phi)k^2, & \rho^2-\rho^2\cos(\theta)^2-h^2+h^2\cos(\phi)^2+h^2\cos(\theta)^2-\cos(\theta)^2\cos(\phi)^2h^2-\cos(\phi)^2k^2+\cos(\theta)^2k^2\cos(\phi)^2 \end{bmatrix}$$

We know we can think of this matrix in the following manner

$$g'_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j = e_i e_j \, \hat{\mathbf{e}}_i \bullet \hat{\mathbf{e}}_j$$

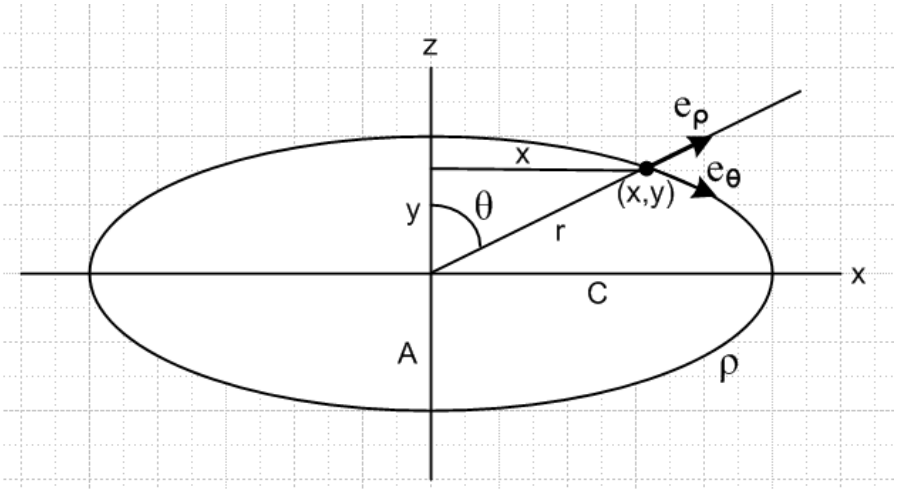
which in the above case we can write as

$$g_{dn} = \begin{pmatrix} e_\rho \bullet e_\rho & e_\rho \bullet e_\theta & e_\rho \bullet e_\varphi \\ e_\theta \bullet e_\rho & e_\theta \bullet e_\theta & e_\theta \bullet e_\varphi \\ e_\varphi \bullet e_\rho & e_\varphi \bullet e_\theta & e_\varphi \bullet e_\varphi \end{pmatrix} = \begin{pmatrix} e_\rho \bullet e_\rho & 0 & 0 \\ 0 & e_\theta \bullet e_\theta & e_\theta \bullet e_\varphi \\ 0 & e_\varphi \bullet e_\theta & e_\varphi \bullet e_\varphi \end{pmatrix}$$

so we see our startling fact that $e_\rho \bullet e_\theta = 0$ and $e_\rho \bullet e_\varphi = 0$ (for general h and k!). I don't have a physical explanation for why this is so, but surely one could be concocted based on the above facts.

(6) Comparison of the ρ,θ,φ ellipsoidal system to the ρ,θ ellipse system

This ρ,θ system is a non-orthogonal 2D coordinate system discussed in "rho theta ellipse system.doc". In that 2D system (in 2D a level curve is the same as a coordinate line) we obtain two tangent base vectors as shown here: [I have changed a few labels in the picture to make it better fit into the current discussion.]



My interest in this subject arose due to the fact that I can see that $e_\rho \bullet e_\theta \neq 0$ in this picture, but I know that in fact $e_\rho \bullet e_\theta = 0$ in the ρ,θ,φ system discussed above.

When I first pondered the ρ,θ,φ Hobson system, I was thinking of the above picture as a slice of the ρ,θ,φ ellipsoidal situation. A simple case would be to take the y=0 slice. C is the semi-major axis along x, and A is that along z. We could specialize more by taking B = C so the above ellipse would then be a central slice through an oblate spheroid. Then we could tentatively regard the ellipse picture above as an accurate slice of this oblate spheroid, and the surface of constant θ would definitely be a perfect cone. BUT, we know that the surface of constant θ is NOT a cone, we just showed that above (see NOTE). Therefore, the above ρ,θ picture is quite misleading if you think of it as such a slice of the ρ,θ,φ Hobson system!!

So if we compare the ρ,θ,φ system to my private ρ,θ system, we might argue that the ρ coordinate is "the same", at least for the slice discussed here (and for other slices it would just be scaled). But the two θ coordinates are NOT the same. The slice of a θ=constant surface in the ρ,θ,φ system is not a "V" whose right side is shown in the ellipse picture. It is some kind of curve that curves up above the origin and does not touch it. Similarly, the constant φ surface of the ρ,θ,φ system is not a plane even for the oblate spheroid case.

So the point is simply this: the θ which appears in the ρ,θ,φ 3D coordinate system is *completely different from* the θ that appears in my ρ,θ 2D system. Although ρ is "the same" for our z = 0 slice, it is the interaction between coordinates that results in the shape of the level surfaces and the tangent base vectors implied by the intersections of these surfaces. Therefore, we should not be surprised, in retrospect, that we have $e_\rho \bullet e_\theta \neq 0$ in our ρ,θ system (which is clear from the ellipse picture), but $e_\rho \bullet e_\theta = 0$ in our ρ,θ,φ system (which is clear from the metric tensor). If nothing else, it is just not the same θ.

I think then it is good terminology to refer to θ,φ as "distorted" spherical angle coordinates.

(7) Resume Hobson notes.

(these notes were written before the previous section was written) When h = k, we have A=B < C and this metric tensor does become diagonal as in prolate spheroidal. It just seems unlikely to me that the Laplacian ∇^2 will have any kind of simple form in ρ,θ,φ coordinates.

So what are these θ and φ angles physically? I am pretty sure that if you exhaust the range of these angles, you will at the same time exhaust the range of μ and ν, so in some sense these are spherical angles

going to a point on the ellipsoid ρ, but they are somehow distorted from the usual spherical angles because I presume the ellipsoid is a distorted sphere. As our ellipsoid morphs into a sphere, these coordinates morph into spherical coordinates. So think of θ and φ as some sort of **"distorted spherical coordinate angles"** which become the true angles in the sphere limit A=B=C.

Right now, I see how these θ,φ guys appear, but I don't see what you can do with them.

Note Added: Let's try anyway to separate Laplace in these distorted coordinates ρ,θ,φ. The first step is to write out the Laplace equation in these coordinates. For a general system, we know that is given by

$$\nabla^2(f) = \text{div grad}(f) = (1/\sqrt{g'}\,)\,\partial_i\,[\sqrt{g'}\,g'^{ij}\,(\partial_j f)\,] \qquad \text{where } g' = \det(g')$$

But using Maple, this is the best I can do for g' = det(g'):

$$-\frac{(\cos(\theta)-1)(\cos(\theta)+1)(-\cos(\theta)^2\rho^2k^2+\cos(\theta)^2h^2k^2-\cos(\phi)^2\rho^2k^2+\cos(\phi)^2\rho^2k^2\cos(\theta)^2+\rho^4-h^2\rho^2+h^2\rho^2\cos(\phi)^2-h^2\rho^2\cos(\theta)^2\cos(\phi)^2)^2}{(\rho-k)(\rho+k)(\rho-h)(\rho+h)}$$

which I can write out as

$$g' = -\,\{\,-\sin^2\theta\,[\,\text{mess}(\theta,\varphi,\rho,h,k)\,]^2\,\}\,/\,[(\rho^2-k^2)(\rho^2-h^2)]$$

So once again, this seems to make a horrible mess for $\nabla^2(f)$. Am I missing something here? If ∇^2 is a complete mess in these coordinates ρ,θ,φ, it seems unlikely that you even CAN find a separated solution. So making the claim that solutions are somehow $f_{nm}(\rho)\,Y_{nm}(\theta,\varphi)$ does not seem justified.

Suppose we treat ρ as a constant, so we are talking only about the surface of a *specific* ellipsoid. Then A,B,C are constants and we have there equations:

$$x = C\,\cos\theta$$
$$y = B\,\sin\theta\,\cos\varphi$$
$$z = A\,\sin\theta\,\sin\varphi$$

Only two of the three x,y,z are independent since they satisfy the ellipsoid equation, so this is really a mapping from say x,y to θ,φ.

Can we talk about the Laplace equation "on a surface" ? That is pretty unclear to me really, but we *can* talk about a complete set of functions spanning a surface.

Each point on our ellipsoid is characterized by a pair θ,φ. We could regard θ,φ as the normal spherical coordinates say on an inscribed sphere, then just run those sphere points out to the ellipsoid on rays. [but those θ,φ coordinates would not be Hobson's θ,φ coordinates.] So I guess if you have some general function f(θ,φ) defined on the ellipsoidal surface, you could expand f(θ,φ) on $Y_{nm}(\theta,\varphi)$. So I conclude that the spherical harmonics do form a complete set for a specific ellipsoidal surface, and this has nothing to do with the Laplace equation. Of course the Y functions do satisfy the usual angular spherical coordinates Laplace equation.

OK, go back to the above equations where A,B,C are functions of ρ. Question: are these angles θ,φ the same angles that appear in spherical coordinates?? I have been calling them "distorted spherical angles", but now I think they are in fact exactly the spherical angles of spherical coordinates. [wrong!]

My conclusion is that the θ,φ coordinates in the ρ,θ,φ Hobson system are completely different (except perhaps at large distances) from the spherical system θ,φ coordinates. The Hobson ρ,θ,φ Laplacian would be a huge mess and you would in no way shape or form see anything looking like the angular Laplacian you see in the spherical system, so even if you could do a separation (which I think you cannot), you would not find that the ρ,θ,φ atoms included the spherical harmonics. I really am puzzled now as to why Hobson even introduced this ρ,θ,φ system on page 455 right on his chapter's second page! Yes, his θ and φ *are* transformations of μ,ν of the ellipsoidal system ρ,μ,ν. I just see nothing useful or even instructive with this ρ,θ,φ system!!

Section 270: Conical Coordinates

(a) conical coordinates (see separate doc)

At once, Hobson launches off on what appears to be *another* huge digression. He wants to discuss what he calls **sphero-conical coordinates** r,μ,ν which we now just call conical coordinates. Since this system was new to me, I went off and spent some time learning about conicals and wrote up at least one **separate doc** on the subject and started a **new folder** "conicals" to hold such docs. He uses the same symbol names μ,ν for the last two coordinates here in the conical r,μ,ν but these are NOT the same objects as the μ,ν in the ρ,μ,ν ellipsoidal coordinates, so the reader should not think that.

In the conical system, which has two parameters h and k, the level surfaces are spheres and two sets of elliptical cones, one set pointing up (blue with μ label) and another set pointing to the side (yellow with ν label). Here are the equations of these level surfaces:

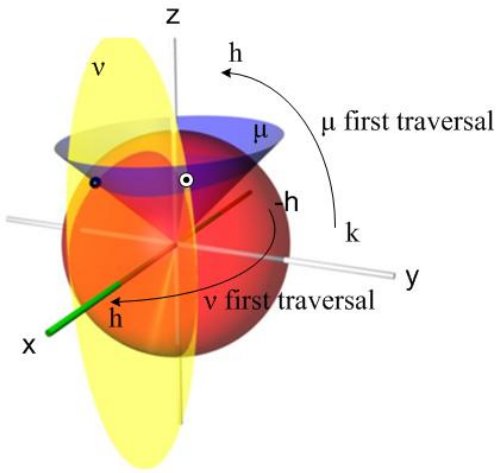
$x^2+ y^2+ z^2 = r^2$	red spheres
$x^2/\mu^2 + y^2/(\mu^2-h^2) = z^2/(k^2-\mu^2)$	blue μ cones
$x^2/\nu^2 = y^2/(h^2-\nu^2) +z^2/(k^2-\nu^2)$	yellow ν cones

Each particular cone has cross sections which form a family of similar ellipses, all having the same B/A ratio. For the blue μ cones we find that B/A = 1-h/μ and for the yellow ν cones we find B/A = 1-k/ν and we need k>h. We may thus interpret h and k as determining the eccentricity of the cross section ellipses which make the cones.

The x,y,z equations for conicals are these

$$x = r\,\mu\nu/(hk)$$
$$y = r\sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\,/(hs) \qquad s = \sqrt{k^2-h^2} \qquad 0 < \nu^2 < h^2 < \mu^2 < k^2$$
$$z = r\sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\,/(ks)$$

If you let ν lie in (-h,h) and μ in (h,k), you obtain x,y,z values for two octants only of x,y,z space corresponding to y>0, z>0 and either sign for x. Here is the picture



In a separate doc I show that if you examine the complex μ and ν planes, you find that these two octants correspond to certain "first traversals" of μ and ν, and the other octants are reached by combinations of first and second traversals. A simpler way to say all this is to allow for both square root values. But the point is that you can reach any point in the x,y,z space by taking a smooth path in μ and ν planes.

Conical coordinates r,μ,ν, which I point out elsewhere are a limit so to speak of the ellipsoidal coordinates where the RHS 1 → 0.

(b) MF on conical coordinates, not very useful

M&F at least mention these conical coordinates on p 659 from which I now quote:

$$x = (\xi_1/\alpha)\sqrt{(\alpha^2 - \xi_2^2)(\alpha^2 + \xi_3^2)}; \quad y = (\xi_1/\beta)\sqrt{(\beta^2 + \xi_2^2)(\beta^2 - \xi_3^2)}$$
$$z = \xi_1\xi_2\xi_3/\alpha\beta; \quad \alpha^2 + \beta^2 = 1$$
$$h_1 = 1; \quad h_2 = \xi_1\sqrt{\frac{\xi_2^2 + \xi_3^2}{(\alpha^2 - \xi_2^2)(\beta^2 + \xi_2^2)}}; \quad h_3 = \xi_1\sqrt{\frac{\xi_2^2 + \xi_3^2}{(\alpha^2 + \xi_3^2)(\beta^2 - \xi_3^2)}}$$
$$f_1 = \xi_1^2; \quad f_2 = \sqrt{(\alpha^2 - \xi_2^2)(\beta^2 + \xi_2^2)}; \quad f_3 = \sqrt{(\alpha^2 + \xi_3^2)(\beta^2 - \xi_3^2)}$$

$\xi_1 = r$; $\xi_2 = \alpha\ \mathrm{cn}(\lambda.\alpha)$; $\xi_3 = \beta\ \mathrm{cn}(\mu,\beta)$; cn is one of the elliptic functions so that

$$x = r\ \mathrm{dn}(\lambda,\alpha)\ \mathrm{sn}(\mu,\beta); \quad y = r\ \mathrm{sn}(\lambda,\alpha)\ \mathrm{dn}(\mu,\beta); \quad z = r\ \mathrm{cn}(\lambda,\alpha)\ \mathrm{cn}(\mu,\beta)$$

First we need to align these with Hobson, so:

MF conical	Hobson conical
ξ_1, ξ_2, ξ_3	r, μ, iν
α, β	k, ih

I think this makes the metric tensors match. MF then define λ and μ as shown using the cn function. I am not happy with this MF presentation however, the $\alpha^2 + \beta^2 = 1$ makes no sense, I don't like having some items being imaginary, and MF never really use them, and don't show the equations of the elliptic cones, etc etc. [We never have any Hobson world condition saying $k^2 - h^2 = 1$, for example.]

(c) inverting the conical coordinates ? (separate doc)

I would like to know the inverse of these conical coordinates. Can MF help?

At this point, I spent several days computing the inverse of Hobson's conical coordinate x,y,z equations, see " Inversion of Conical Coordinates.doc" stored in the curvilinear/conicals folder. Here are the forward and reverse directions:

Forward conical:

$x = r\,\mu\nu/(hk)$

$y = r\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}/(hs)$ $\qquad s = \sqrt{k^2-h^2}$ $\qquad 0 < \nu < h < \mu < k$

$z = r\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}/(ks)$

Inverse conical:

$r = \sqrt{x^2+y^2+z^2}$ $\qquad s^2 = k^2 - h^2$

$\nu = \sqrt{W - \sqrt{W^2 - 4h^2k^2(x/r)^2}}\ /\sqrt{2}$ $\qquad W = k^2(x/r)^2 + s^2(y/r)^2 + h^2$

$\mu = h\,k(x/r)\,(1/\nu)$

(d) the Laplacian in conical coordinates

Next, in p 456A Hobson writes $\nabla^2 V = 0$ in terms of these r,μ,ν conical coordinates. It looks different from the way Wolfram presents it using $r = \lambda$, but OK.

(e) introduction of the "alternate" variables η,ζ for μ,ν; the new simple ∇^2 form!

Now in the next step, Hobson defines $\eta(\mu) = \eta(\xi_2)$ as a certain integral of two positive square roots in the denominator. This looks to me *like* a first kind elliptic function in its second form here (pendulum stuff)

$$F(\varphi,k) = \int_0^{\varphi} d\theta\ \frac{1}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin(\varphi)} dt\ \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad \text{first kind} \qquad K(k) = F(\pi/2,k)$$

He also defines ζ(ν) by a similar integral, and it looks even more like the above elliptic integral. Something very important to notice is this

$d\eta = 1/[\sqrt{\mu^2-h^2}\sqrt{k^2-\mu^2}]\,d\mu$ $\qquad$ and a similar result for dζ

So now we have our conical coordinates not as r,μ,ν but as r,ζ,η . He then rewrites $\nabla^2 V = 0$ in p 456B and it is quite simple, but it is a hybrid equation because all four μ,ν,η,ζ appear along with r.

(f) Why is this p 456 B ∇^2 form so simple? Because the metric tensor is simple!

Here is one answer. Looking at what I just wrote above, we see that

$$(d\eta)^2 = 1/[\ (\mu^2-h^2)(k^2-\mu^2)]\ (d\mu)^2$$

$$(d\zeta)^2 = 1/[\ (h^2-\nu^2)(k^2-\nu^2)]\ (d\nu)^2$$

If we insert these into the metric tensor shown p 456 E, we get on the RHS this very simple result:

$$(dr)^2 + (\mu^2-\nu^2)\ r^2\ \{\quad (d\eta)^2 + (d\zeta)^2\ \ \}$$

Now the *tradeoff* to get this simple metric tensor result is that when you integrate dη and dζ to get η(μ) and ζ(ν), with the implied inverses μ(η) and ζ(ν), all these functions are fairly complicated, though at least they are closed-form known special functions. For example, we later learn that $\mu(\eta) = k\ dn[K(k_1^2)-k\eta, k_1]$ where k and k_1 are just functions of h and k, where dn is a Jacobi function.

So, we can rewrite the above metric tensor $(ds)^2$ as

$$(dr)^2 + (\mu(\eta)^2-\nu(\zeta)^2)\ r^2\ \{\quad (d\eta)^2 + (d\zeta)^2\ \ \}$$

This basically says that if we take these very strange coordinates η and ζ, then we obtain a diagonal metric tensor when we go from x,y,z to r,η,ζ and the diagonal elements are quite simple:

$$h_{rr}^2 = 1 \qquad Q_1 = 1 \qquad r, \eta, \zeta = 1,2,3$$

$$h_{\eta\eta}^2 = h_{\zeta\zeta}^2 = [\mu(\eta)^2-\nu(\zeta)^2]\ r^2 \qquad Q_2 = Q_3 = r\sqrt{\mu^2-\nu^2}$$

So (r,η,ζ) is in fact **some strange orthogonal curvilinear coordinate system** which as far as I know has **no name**.

(g) Computation of ∇^2 in terms of η and ζ

Because the metric tensor elements are so simple, the Laplacian comes out simple. Recall our general formula

$$\nabla^2 f = (1/[Q_1Q_2Q_3])\ \{\ \partial_1\ [\ (Q_2Q_3/Q_1\ )\ (\partial_1 f)]\ + \text{cyclic}\ \}$$

$$\begin{aligned} &= \quad (1/[Q_1Q_2Q_3])\ \{\ \partial_1\ [\ (Q_2Q_3/Q_1\ )\ (\partial_1 f)] \\ &+ (1/[Q_2Q_3Q_1])\ \{\ \partial_2\ [\ (Q_3Q_1/Q_2\ )\ (\partial_2 f)] \\ &+ (1/[Q_3Q_1Q_2])\ \{\ \partial_3\ [\ (Q_1Q_2/Q_3\ )\ (\partial_3 f)] \end{aligned}$$

$$= (1/[Q_2Q_3])\{\partial_1[(Q_2Q_3)(\partial_1 f)] \quad \text{// set } Q_1 = 1$$
$$+ (1/[Q_2Q_3])\{\partial_2[(Q_3/Q_2)(\partial_2 f)]$$
$$+ (1/[Q_3Q_2])\{\partial_3[(\partial_3 f)]$$

$$= (1/[h_{\eta\eta}^2])\{\partial_1[h_{\eta\eta}^2(\partial_1 f)] \quad \text{// set } (Q_3/Q_2) = 1 \text{ etc}$$
$$+ (1/[h_{\eta\eta}^2])\{\partial_2[(\partial_2 f)]$$
$$+ (1/[h_{\eta\eta}^2])\{\partial_3[(\partial_3 f)]$$

$$= (1/h_{\eta\eta}^2)\{\partial_1[h_{\eta\eta}^2(\partial_1 f)] + (\partial_2^2 f) + (\partial_3^2 f)\}$$

$$= (1/h_{\eta\eta}^2)\{\partial_r[h_{\eta\eta}^2(\partial_r f)] + (\partial_\eta^2 f) + (\partial_\zeta^2 f)\}$$

For the Laplace equation $\nabla^2 f = 0$ we can ignore the common factor, and we have

$$\partial_r[h_{\eta\eta}^2(\partial_r f)] + (\partial_\eta^2 f) + (\partial_\zeta^2 f) = 0$$

$$[\mu(\eta)^2 - \nu(\zeta)^2]\partial_r[r^2(\partial_r f)] + (\partial_\eta^2 f) + (\partial_\zeta^2 f) = 0 \quad \text{// verifying } \textbf{p 456 B}$$

The point then is that we first compute the metric tensor for conical coordinates and it comes out having those ugly denominators as shown in p 456 E. We then *invent* our η and ζ *differential* coordinates for the express purpose of cancelling out those messy factors. We then integrate those differential definitions and get the strange Jacobi functions if we want to express η and ζ in terms of μ and ν. This is a very interesting technique I think.

He then sets $V = r^n u$ and rewrites Laplace in terms of u in **p 456 B**, still a hybrid thing.

Hobson's next comment is very mysterious to me. He claims that our new conical-coordinates simple Laplace equation p 456C is "a transformation of" the θ,φ equation defining spherical harmonics $Y_{nm}(\theta,\varphi)$. At first I thought he meant θ,φ to be those same distorted angles we had in the ellipsoidal variant coordinates ρ,θ,φ, but now I don't *think* that is true. In general, if you start with spherical coordinates r,θ,φ and you do a transformation to some other coordinates like conicals r,μ,ν (r stays the same), then of course all the differential operators like ∇^2 will have some new form. If you start with $\nabla^2_{r,\theta,\varphi} f(r,\theta,\varphi) = 0$, you will end up with some $\nabla^2_{r,\mu,\nu} F(r,\mu,\nu) = 0$. So in this sense, these two Laplace equations are "transformations" of each other. But right now we still have no connection between the μ,ν of ellipsoidals and the μ,ν of conicals, other than "same range".

Question: are the μ,ν coordinates Hobson uses for his opening ellipsoidal coordinates discussion the same as those of the same name which appear in his conical coordinates discussion? I think there is no connection whatsoever. His logic thread is confusing me a lot right now!

(g) Attempting separation of $\nabla^2 V = 0$ by assuming $V = r^n u$ with $u = E(\mu/\eta)E(\nu/\zeta)$.

Comment: In this section we show that, if you assume this form perhaps with u = **E(μ/η)E'(ν/ζ),** you find that when you do the separation, the functions E and E' are solutions of the *exact same* Lamé ODE with the same values of separation constants n and p. So V = r^n $E_{n,p}(\mu)$ $E_{n,p}(\nu)$. Yes, we could replace either E with its second kind function, but ignore that for now because we are thinking "interior harmonics". A lot of detail follows.

He is now back to the equation p 456 C with its hybrid form and after assuming r^n for the r dependence. A accept that ansatz for now. So we get for this C equation

$$(\partial_\eta{}^2 u) + (\partial_\zeta{}^2 u) + n(n+1)(\mu^2 - \nu^2)u = 0$$

and now install u = E(μ/η)E(ν/ζ) to get

$$(\partial_\eta{}^2 E(\mu/\eta)E(\nu/\zeta)) + (\partial_\zeta{}^2 E(\mu/\eta)E(\nu/\zeta)) + n(n+1)(\mu^2 - \nu^2) E(\mu/\eta)E(\nu/\zeta) = 0$$

$$E(\nu/\zeta)(\partial_\eta{}^2 E(\mu/\eta)) + E(\mu/\eta)(\partial_\zeta{}^2 E(\nu/\zeta)) + n(n+1)(\mu^2 - \nu^2) E(\mu/\eta)E(\nu/\zeta) = 0$$

$$E(\nu/\zeta)(\partial_\eta{}^2 E(\mu/\eta)) + E(\mu/\eta)(\partial_\zeta{}^2 E(\nu/\zeta)) + n(n+1)\mu^2 E(\mu/\eta)E(\nu/\zeta) - n(n+1)\nu^2 E(\mu/\eta)E(\nu/\zeta) = 0$$

$$E(\nu/\zeta)\{(\partial_\eta{}^2 E(\mu/\eta)) + n(n+1)\mu^2 E(\mu/\eta)\} + E(\mu/\eta)\{(\partial_\zeta{}^2 E(\nu/\zeta)) - n(n+1)\nu^2 E(\nu/\zeta)\} = 0$$

which is **p 457 B** and we see where the minus sign came from. The next step would be

$$\{(\partial_\eta{}^2 E(\mu/\eta)) + n(n+1)\mu^2 E(\mu/\eta)\}/E(\mu/\eta) = -\{(\partial_\zeta{}^2 E(\nu/\zeta)) - n(n+1)\nu^2 E(\nu/\zeta)\}/E(\nu/\zeta)$$

and each side must be a constant which we arbitrarily call $p(h^2+k^2)$, so we get

$$\{(\partial_\eta{}^2 E(\mu/\eta)) + n(n+1)\mu^2 E(\mu/\eta)\}/E(\mu/\eta) = p(h^2+k^2)$$
$$-\{(\partial_\zeta{}^2 E(\nu/\zeta)) - n(n+1)\nu^2 E(\nu/\zeta)\}/E(\nu/\zeta) = p(h^2+k^2)$$

$$\{(\partial_\eta{}^2 E(\mu/\eta)) + n(n+1)\mu^2 E(\mu/\eta)\} = p(h^2+k^2) E(\mu/\eta)$$
$$\{(\partial_\zeta{}^2 E(\nu/\zeta)) - n(n+1)\nu^2 E(\nu/\zeta)\} = -p(h^2+k^2) E(\nu/\zeta)$$

$$(\partial_\eta{}^2 E(\mu/\eta)) + n(n+1)\mu^2 E(\mu/\eta) - p(h^2+k^2) E(\mu/\eta) = 0$$
$$(\partial_\zeta{}^2 E(\nu/\zeta)) - n(n+1)\nu^2 E(\nu/\zeta) + p(h^2+k^2) E(\nu/\zeta) = 0$$

$$(\partial_\eta{}^2 E(\mu/\eta)) + [n(n+1)\mu^2 - p(h^2+k^2)] E(\mu/\eta) = 0$$
$$(\partial_\zeta{}^2 E(\nu/\zeta)) - [n(n+1)\nu^2 - p(h^2+k^2)] E(\nu/\zeta) = 0 \qquad \text{// which is p } \mathbf{457\ C}$$

Next, we use

$$d\eta = 1/[\sqrt{\mu^2-h^2}\sqrt{k^2-\mu^2}]\, d\mu$$
$$d\zeta = 1/[\sqrt{h^2-\nu^2}\sqrt{k^2-\nu^2}]\, d\nu$$

$$\partial_\eta = \sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,\partial_\mu$$
$$\partial_\zeta = \sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\;\partial_\nu$$

$$\partial_\eta\, E(\mu/\eta) = \sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,\partial_\mu\, E(\mu/\eta)$$
$$\partial_\zeta\, E(\nu/\zeta) = \sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\;\partial_\nu\, E(\nu/\zeta)$$

Now things are complicated so we do one equation at a time. The first equation becomes:

$$\begin{aligned}\partial_\eta^2\, E(\mu/\eta) &= \partial_\eta[\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,\partial_\mu\, E(\mu/\eta)]\\ &= [\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,\partial_\mu]\,[\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,\partial_\mu\, E(\mu/\eta)]\\ &= [\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,]\,[\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,\partial_\mu^2 E(\mu/\eta)]\\ &\quad + [\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,]\partial_\mu[\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,]\,\partial_\mu\, E(\mu/\eta)]\end{aligned}$$

$$= (\mu^2-h^2)(k^2-\mu^2)\,\partial_\mu^2 E(\mu/\eta) + [\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,]\partial_\mu[\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,]\,\partial_\mu\, E(\mu/\eta)]$$

But

$$\begin{aligned}&\partial_\mu[\sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}\,] = \sqrt{\mu^2-h^2}\,(1/2)\,[1/\sqrt{k^2-\mu^2}\,]\,(-2\mu) + \sqrt{k^2-\mu^2}\,(1/2)[1/\sqrt{\mu^2-h^2}\,]\,(2\mu)\\ &= \mu\,[\,-\,\sqrt{\mu^2-h^2}\,/\sqrt{k^2-\mu^2} + \sqrt{k^2-\mu^2}\,/\sqrt{\mu^2-h^2}\,]\\ &= \mu\,[\,-\,(\mu^2-h^2)+(k^2-\mu^2)]\,/\,[\sqrt{k^2-\mu^2}\,\sqrt{\mu^2-h^2}\,] = \mu\,(k^2+h^2-2\mu^2)/\,[\sqrt{k^2-\mu^2}\,\sqrt{\mu^2-h^2}\,]\end{aligned}$$

so we end up then with

$$\partial_\eta^2\, E(\mu/\eta) = (\mu^2-h^2)(k^2-\mu^2)\,\partial_\mu^2 E(\mu/\eta) + \mu\,(k^2+h^2-2\mu^2)\,\partial_\mu\, E(\mu/\eta)$$

and then our equation has become

$$(\partial_\eta^2\, E(\mu/\eta)) + [\,n(n+1)\,\mu^2 - p(h^2+k^2)]\;E(\mu/\eta) = 0$$

$$(\mu^2-h^2)(k^2-\mu^2)\,\partial_\mu^2 E(\mu/\eta) + \mu\,(k^2+h^2-2\mu^2)\,\partial_\mu\, E(\mu/\eta) + [\,n(n+1)\,\mu^2 - p(h^2+k^2)]\;E(\mu/\eta) = 0$$

$$(\mu^2-h^2)(\mu^2-k^2)\,\partial_\mu^2 E(\mu/\eta) + \mu\,(-k^2-h^2+2\mu^2)\,\partial_\mu\, E(\mu/\eta) + [\,-n(n+1)\,\mu^2 + p(h^2+k^2)]\;E(\mu/\eta) = 0$$

$$(\mu^2-h^2)(\mu^2-k^2)\,\partial_\mu^2 E(\mu/\eta) + \mu\,(2\mu^2-h^2-k^2)\,\partial_\mu\, E(\mu/\eta) + [p(h^2+k^2) - n(n+1)\,\mu^2]\;E(\mu/\eta) = 0$$

which is **p 457 D**.

Now I will copy, paste, and edit to get the second equation -- I see no way to avoid this

---

$$\partial_\zeta = \sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\;\partial_\nu$$

$$\partial_\zeta E(\nu/\zeta) = \sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\;\partial_\nu E(\nu/\zeta)$$

$$\begin{aligned}\partial_{\zeta^2} E(\nu/\zeta) &= \partial_\zeta [\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,\partial_\nu E(\nu/\zeta)] \\ &= [\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,\partial_\nu][\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,\partial_\nu E(\nu/\zeta)] \\ &= [\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,][\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,\partial_{\nu^2} E(\nu/\zeta)] \\ &\quad + [\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,]\partial_\nu[\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,]\,\partial_\nu E(\nu/\zeta)]\end{aligned}$$

$$= (h^2-\nu^2)(k^2-\nu^2)\,\partial_{\nu^2} E(\nu/\zeta) + [\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,]\partial_\nu[\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,]\,\partial_\nu E(\nu/\zeta)]$$

But

$$\begin{aligned}&\partial_\nu[\sqrt{h^2-\nu^2}\,\sqrt{k^2-\nu^2}\,] = \sqrt{h^2-\nu^2}\;(1/2)\,[1/\sqrt{k^2-\nu^2}\,](-2\nu) + \sqrt{k^2-\nu^2}\;(1/2)[1/\sqrt{h^2-\nu^2}\,](-2\nu) \\ &= \nu\,[\,-\,\sqrt{h^2-\nu^2}/\sqrt{k^2-\nu^2} - \sqrt{k^2-\nu^2}/\sqrt{h^2-\nu^2}\,] \\ &= \nu\,[\,-\,(h^2-\nu^2) - (k^2-\nu^2)]\,/\,[\sqrt{k^2-\nu^2}\,\sqrt{h^2-\nu^2}\,] \;=\; \nu(-k^2-h^2+2\nu^2)/\,[\sqrt{k^2-\nu^2}\,\sqrt{h^2-\nu^2}\,] \\ &\qquad\qquad = -\nu(k^2+h^2-2\nu^2)/\,[\sqrt{k^2-\nu^2}\,\sqrt{h^2-\nu^2}\,]\end{aligned}$$

so we end up then with

$$\partial_{\zeta^2} E(\nu/\zeta) = (h^2-\nu^2)(k^2-\nu^2)\,\partial_{\nu^2} E(\nu/\zeta) - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E(\nu/\zeta)$$

and then our equation has become

$$(\partial_{\zeta^2} E(\nu/\zeta)) - [\,n(n+1)\,\nu^2 - p(h^2+k^2)]\;E(\nu/\zeta) = 0$$

$$(h^2-\nu^2)(k^2-\nu^2)\,\partial_{\nu^2} E(\nu/\zeta) - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E(\nu/\zeta) - [\,n(n+1)\,\nu^2 - p(h^2+k^2)]\;E(\nu/\zeta) = 0$$

$$(\nu^2-h^2)(\nu^2-k^2)\,\partial_{\nu^2} E(\nu/\zeta) + \nu(2\nu^2-k^2-h^2)\,\partial_\nu E(\nu/\zeta) + [p(h^2+k^2) - n(n+1)\,\nu^2\,]\;E(\nu/\zeta) = 0$$

Thus, as claimed, the ν equation is EXACTLY the same including all signs! Therefore, if we could solve this ODE for our functions, we would find that

$$V = r^n\,E_{n,p}(\mu/\eta)\,E_{n,p}(\nu/\zeta)$$

was a possible separation of variables solution for our r,μ,ν conical world.

Section 271. Back to ellipsoidal coordinates.

(a) Maple computes the ellipsoidal metric tensor

Recall that our original *ellipsoidal* coordinates were ρ,μ,ν. I have never actually made Maple compute the metric tensor for ellipsoidals, so maybe now would be a good time to do that. And I should regularize this

program so I can do it easily for all systems, I was lazy and never did that, so again, now is the time. // It is done in file "metric tensor ellipsoidal.mws" and here are the results: 1,2,3 = ρ,μ,ν

```
factor(g[1,1]);
```

$$\frac{(\nu-\rho)(\nu+\rho)(\mu-\rho)(\mu+\rho)}{(k-\rho)(k+\rho)(-\rho+h)(h+\rho)}$$

```
factor(g[2,2]);
```

$$\frac{(\mu-\rho)(\mu+\rho)(\mu-\nu)(\mu+\nu)}{(\mu-k)(\mu+k)(\mu-h)(\mu+h)}$$

```
factor(g[3,3]);
```

$$-\frac{(\nu-\rho)(\nu+\rho)(\mu-\nu)(\mu+\nu)}{(\nu-k)(\nu+k)(\nu-h)(\nu+h)}$$

So this says

$g'_{\rho\rho} = (\mu^2-\rho^2)(\nu^2-\rho^2)/[(k^2-\rho^2)(h^2-\rho^2)] = (\rho^2-\mu^2)(\rho^2-\nu^2)/[(\rho^2-k^2)(\rho^2-h^2)] = Q_1^2$ // agrees E
$g'_{\mu\mu} = (\mu^2-\rho^2)(\mu^2-\nu^2)/[(\mu^2-k^2)(\mu^2-h^2)] = (\rho^2-\mu^2)(\mu^2-\nu^2)/[(k^2-\mu^2)(\mu^2-h^2)] = Q_2^2$ // agrees E
$g'_{\nu\nu} = -(\nu^2-\rho^2)(\mu^2-\nu^2)/[(\nu^2-k^2)(\nu^2-h^2)] = (\rho^2-\nu^2)(\mu^2-\nu^2)/[(k^2-\nu^2)(h^2-\nu^2)] = Q_3^2$ // agrees E

So I have now verified Hobson **p 457 E.** So at this point we know that (note orthogonality)

$$(ds)^2 = Q_1^2(d\rho)^2 + Q_2^2(d\mu)^2 + Q_3^2(d\nu)^2$$

(b) introduction of the "alternate" variables ξ,η,ζ for ρ,μ,ν; the new simple metric tensor.

The next step is to repeat the trick we did above with the conical coordinates. We use the exact same η and ζ equations, and we add a new ξ equation as the first line (p 458 B)

$d\xi = 1/[\sqrt{\rho^2-h^2}\sqrt{\rho^2-k^2}]\, d\rho$ $\quad (\rho^2-k^2)(\rho^2-h^2) = (d\rho)^2/(d\xi)^2$
$d\eta = 1/[\sqrt{\mu^2-h^2}\sqrt{k^2-\mu^2}]\, d\mu$ $\quad (\mu^2-h^2)(k^2-\mu^2) = (d\mu)^2/(d\eta)^2$
$d\zeta = 1/[\sqrt{h^2-\nu^2}\sqrt{k^2-\nu^2}]\, d\nu$ $\quad (h^2-\nu^2)(k^2-\nu^2) = (d\nu)^2/(d\zeta)^2$

Then we can write

$$(ds)^2 = Q_1^2(d\rho)^2 + Q_2^2(d\mu)^2 + Q_3^2(d\nu)^2$$

$= Q_1^2\,(\rho^2-k^2)(\rho^2-h^2)\,(d\xi)^2$
$+ Q_2^2\,(\mu^2-h^2)(k^2-\mu^2)\,(d\eta)^2$
$+ Q_3^2\,(h^2-\nu^2)(k^2-\nu^2)\,(d\zeta)^2$

$= (\rho^2-\mu^2)(\rho^2-\nu^2)/[(\rho^2-k^2)(\rho^2-h^2)] *(\rho^2-k^2)(\rho^2-h^2)\,(d\xi)^2$
$+ (\rho^2-\mu^2)\,(\mu^2-\nu^2)/[(k^2-\mu^2)(\mu^2-h^2)] *(\mu^2-h^2)(k^2-\mu^2)\,(d\eta)^2$
$+ \rho^2-\nu^2)(\mu^2-\nu^2)/[(k^2-\nu^2)(h^2-\nu^2)] *(h^2-\nu^2)(k^2-\nu^2)\,(d\zeta)^2$

$= (\rho^2-\mu^2)(\rho^2-\nu^2)\,(d\xi)^2$

$+ \ (\rho^2-\mu^2)\,(\mu^2-\nu^2)\,(d\eta)^2$
$+ \ (\rho^2-\nu^2)(\mu^2-\nu^2)\ (d\zeta)^2$

NOTE: the expressions above for $d\xi,d\eta,d\zeta$ were *expressly chosen* to create the simple $(ds)^2$ form you see here. The goal was to cancel as many factors as possible to achieve a simple $(ds)^2$.

So now think of new coordinates ξ,η,ζ to replace ρ,μ,ν and we then have for our NEW 1,2,3 meaning:

$Q_1^2 = (\rho^2-\mu^2)(\rho^2-\nu^2)$
$Q_2^2 = (\rho^2-\mu^2)\,(\mu^2-\nu^2)$
$Q_3^2 = (\rho^2-\nu^2)(\mu^2-\nu^2)$

$Q_1 = \sqrt{\rho^2-\mu^2}\,\sqrt{\rho^2-\nu^2}$
$Q_2 = \sqrt{\rho^2-\mu^2}\,\sqrt{\mu^2-\nu^2}$
$Q_3 = \sqrt{\rho^2-\nu^2}\,\sqrt{\mu^2-\nu^2}$

where these are NOT the same Q's as before. These are fairly simple forms for the Q_i and only three distinct radicals are appearing here!

A quick review: we had coordinates ρ,μ,ν which we knew were orthogonal. We then basically "reparameterize" all three coordinates to get $\xi(\rho),\eta(\mu),\zeta(\nu)$. What does this do to the metric tensor? It starts out being this: (it is orthogonal because no terms like $d\rho d\mu$, for example. Ie, g_{ij} is diagonal)

$(ds)^2 = Q_1^2(d\rho)^2 + Q_2^2(d\mu)^2 + Q_3^2(d\nu)^2$

Then for example we have $\xi = \xi(\rho)$ which says $d\xi = \xi'(\rho)d\rho = \xi'(\rho[\xi])d\rho$ and $(d\xi)^2 = [\xi'(\rho[\xi])]^2(d\rho)^2$ and finally $(d\rho)^2 = [\xi'(\rho[\xi])]^{-2}(d\xi)^2$. We then end up with this

$(ds)^2 = Q_1^2[\xi'(\rho[\xi])]^{-2}(d\xi)^2 + Q_2^2[\eta'(\mu[\eta])]^{-2}(d\eta)^2 + Q_3^2[\xi'(\nu[\zeta])]^{-2}(d\zeta)^2$

The point is that changing the "velocity" of each coordinate does not affect orthogonality! If we instead defined $\xi = \xi(\rho,\mu)$ say, this would not be true.

(c) computation of ∇^2 in the alternate coordinates

Now anticipating our ∇^2 formula, let's compute

$(Q_3Q_2/Q_1) = \sqrt{\rho^2-\nu^2}\,\sqrt{\mu^2-\nu^2}\,\sqrt{\rho^2-\mu^2}\,\sqrt{\mu^2-\nu^2} \,/\, [\sqrt{\rho^2-\mu^2}\,\sqrt{\rho^2-\nu^2}\,] = (\mu^2-\nu^2)$

$(Q_1Q_3/Q_2) = \sqrt{\rho^2-\mu^2}\,\sqrt{\rho^2-\nu^2}\,\sqrt{\rho^2-\nu^2}\,\sqrt{\mu^2-\nu^2} \,/\, [\sqrt{\rho^2-\mu^2}\,\sqrt{\mu^2-\nu^2}\,] = (\rho^2-\nu^2)$

$(Q_2Q_1/Q_3) = \sqrt{\rho^2-\mu^2}\,\sqrt{\mu^2-\nu^2}\,\sqrt{\rho^2-\mu^2}\,\sqrt{\rho^2-\nu^2} \,/\, [\sqrt{\rho^2-\nu^2}\,\sqrt{\mu^2-\nu^2}\,] = (\rho^2-\mu^2)$

$Q_3Q_2Q_1 = \sqrt{\rho^2-\nu^2}\,\sqrt{\mu^2-\nu^2}\,\sqrt{\rho^2-\mu^2}\,\sqrt{\mu^2-\nu^2}\,\sqrt{\rho^2-\mu^2}\,\sqrt{\rho^2-\nu^2} = (\rho^2-\nu^2)\,(\rho^2-\mu^2)\,(\mu^2-\nu^2)$

Then we roll out our formula

$$\nabla^2 f = (1/[Q_1Q_2Q_3])\{\partial_1[(Q_2Q_3/Q_1)(\partial_1 f)] + \text{cyclic}\}$$

$$= (1/[Q_1Q_2Q_3])\,\partial_1[(Q_2Q_3/Q_1)(\partial_1 f)] + (1/[Q_2Q_3Q_1])\,\partial_2[(Q_3Q_1/Q_2)(\partial_2 f)] + (1/[Q_3Q_1Q_2])\,\partial_3[(Q_1Q_2/Q_3)(\partial_3 f)]$$

$$= [(\rho^2-\nu^2)(\rho^2-\mu^2)(\mu^2-\nu^2)]^{-1} * \{\partial_\xi[(Q_2Q_3/Q_1)(\partial_\xi f) + \partial_\eta[(Q_3Q_1/Q_2)(\partial_\eta f) + \partial_\zeta[(Q_1Q_2/Q_3)(\partial_\zeta f)\}$$

$$= [(\rho^2-\nu^2)(\rho^2-\mu^2)(\mu^2-\nu^2)]^{-1} * \{\partial_\xi[(\mu^2-\nu^2)(\partial_\xi f) + \partial_\eta[(\rho^2-\nu^2)(\partial_\eta f) + \partial_\zeta[(\rho^2-\mu^2)(\partial_\zeta f)\}$$

$$= [(\rho^2-\nu^2)(\rho^2-\mu^2)(\mu^2-\nu^2)]^{-1} * \{(\mu^2-\nu^2)\partial_\xi^2 f + (\rho^2-\nu^2)\partial_\eta^2 f + (\rho^2-\mu^2)\partial_\zeta^2 f\}$$

which is miraculously simple. So our Laplace equation is just this for $\xi(\rho),\eta(\mu),\zeta(\nu)$

$$(\mu^2-\nu^2)\partial_\xi^2 f + (\rho^2-\nu^2)\partial_\eta^2 f + (\rho^2-\mu^2)\partial_\zeta^2 f = 0 \qquad \text{// which is } \textbf{p 458 A}$$

Or more completely, we have this form for the Laplacian in our ξ,η,ζ coordinates

$$\nabla^2 = [(\rho^2-\nu^2)(\rho^2-\mu^2)(\mu^2-\nu^2)]^{-1}\{(\mu^2-\nu^2)\partial_\xi^2 + (\rho^2-\nu^2)\partial_\eta^2 + (\rho^2-\mu^2)\partial_\zeta^2\}$$

where it is understood that ρ,μ,ν are certain functions $\rho(\xi),\mu(\eta),\nu(\zeta)$ that we know. As Hobson comments later on page 458, we then know that we *can have* Laplace solutions of this form

$$f(\xi,\eta,\zeta) = f_1(\xi) = A\xi + B$$

and similarly in the other two coordinates. For example, if we have two ellipsoids ξ_1 and ξ_2 which are at constant potentials V_1 and V_2, the solution for the potential between will be $A\xi + B$ where $V_1 = A\xi_1 + B$ and where $V_2 = A\xi_2 + B$ so we subtract to get $V_1-V_2 = A(\xi_1-\xi_2)$ => $A = \Delta V/\Delta\xi$ and then $B = V_1 - A\xi_1$ and then you have a complete solution to the problem !!!!! Of course you need to know $\xi(\rho)$ if you want the solution in terms of the usual ellipsoidal coordinate labels. So this is a **huge payoff** for going to these fancy rescaled orthogonal coordinates that have no name! You could find the potential between two hyperboloids of the same kind in this same manner.

(d) Attempting separation of $\nabla^2 V = 0$ by assuming $V = E(\rho/\xi)E(\mu/\eta)E(\nu/\zeta)$.

Now we come to the issue of separation of coordinates to get a solution, and I think this is going to take some serious work, but maybe not. We are going to try this form

$$f = [\, E(\rho/\xi)\, E(\mu/\eta) E(\nu/\zeta)]$$

$$\nabla^2 f = (\mu^2-\nu^2)\, \partial_\xi^2 f + (\rho^2-\nu^2)\, \partial_\eta^2 f + (\rho^2-\mu^2)\, \partial_\zeta^2 f = 0$$

$$(\mu^2-\nu^2)\, \partial_\xi^2[\, E(\rho/\xi)\, E(\mu/\eta) E(\nu/\zeta)] + (\rho^2-\nu^2)\, \partial_\eta^2[\, E(\rho/\xi)\, E(\mu/\eta) E(\nu/\zeta)]$$
$$+(\rho^2-\mu^2)\, \partial_\zeta^2[\, E(\rho/\xi)\, E(\mu/\eta) E(\nu/\zeta)] = 0$$

$$(\mu^2-\nu^2)\, E(\mu/\eta) E(\nu/\zeta) \partial_\xi^2[\, E(\rho/\xi)] + (\rho^2-\nu^2)\, E(\rho/\xi)\, E(\nu/\zeta) \partial_\eta^2[E(\mu/\eta)]$$
$$+(\rho^2-\mu^2)\, E(\rho/\xi)\, E(\mu/\eta) \partial_\zeta^2[E(\nu/\zeta)] = 0$$

$$(\mu^2-\nu^2)\, (\partial_\xi^2 E(\rho/\xi))/\, E(\rho/\xi) + (\rho^2-\nu^2)\, (\partial_\eta^2 E(\mu/\eta))\, /\, E(\mu/\eta) + (\rho^2-\mu^2)\, (\partial_\zeta^2 E(\nu/\zeta))/\, E(\nu/\zeta) = 0$$

$$(\mu(\eta)^2-\nu(\zeta)^2)\, (\partial_\xi^2 E(\rho/\xi))/\, E(\rho/\xi) + (\rho(\xi)^2-\nu(\zeta)^2)\, (\partial_\eta^2 E(\mu/\eta))\, /\, E(\mu/\eta)$$
$$+(\rho(\xi)^2-\mu(\eta)^2)\, (\partial_\zeta^2 E(\nu/\zeta))/\, E(\nu/\zeta) = 0$$

Plan A: does not get very far.

It is not immediately obvious how you do this separation! Let's just plug our assumed form into the simple ODE and see what happens.

$$(\mu^2-\nu^2)\, E(\mu/\eta) E(\nu/\zeta) \partial_\xi^2[\, E(\rho/\xi)] + (\rho^2-\nu^2)\, E(\rho/\xi)\, E(\nu/\zeta) \partial_\eta^2[E(\mu/\eta)]$$
$$+(\rho^2-\mu^2)\, E(\rho/\xi)\, E(\mu/\eta) \partial_\zeta^2[E(\nu/\zeta)] = 0$$

Let's go for a more compact notation. First, we rewrite the ODE:

$$\partial_\xi^2 E_\rho = (h^2-\rho^2)(k^2-\rho^2)\, \partial_\rho^2 E_\rho - \rho(k^2+h^2-2\rho^2)\, \partial_\rho E_\rho$$

$$\partial_\eta^2 E_\mu = (\mu^2-h^2)(k^2-\mu^2)\, \partial_\mu^2 E_\mu + \mu\, (k^2+h^2-2\mu^2)\, \partial_\mu E_\mu$$

$$\partial_\zeta^2 E_\nu = (h^2-\nu^2)(k^2-\nu^2)\, \partial_\nu^2 E_\nu - \nu(k^2+h^2-2\nu^2)\, \partial_\nu E_\nu$$

$$(\mu^2-\nu^2)\, E_\mu E_\nu\, \partial_\xi^2 E_\rho + (\rho^2-\nu^2)\, E_\xi E_\nu\, \partial_\eta^2 E_\mu + (\rho^2-\mu^2)\, E_\xi E_\mu\, \partial_\zeta^2 E_\nu = 0$$

So now do the plug ins:

$$(\mu^2-\nu^2)\, E_\mu E_\nu\, [\partial_\xi^2 E_\xi]$$
$$+ (\rho^2-\nu^2)\, E_\rho E_\nu\, [\partial_\eta^2 E_\mu]$$
$$+(\rho^2-\mu^2)\, E_\rho E_\mu\, [\partial_\zeta^2 E_\zeta] = 0$$

$$(\mu^2-\nu^2)\, E_\mu E_\nu\, [(h^2-\rho^2)(k^2-\rho^2)\, \partial_\rho^2 E_\rho - \rho(k^2+h^2-2\rho^2)\, \partial_\rho E_\rho]$$
$$+ (\rho^2-\nu^2)\, E_\rho E_\nu\, [(\mu^2-h^2)(k^2-\mu^2)\, \partial_\mu^2 E_\mu + \mu\, (k^2+h^2-2\mu^2)\, \partial_\mu E_\mu]$$
$$+(\rho^2-\mu^2)\, E_\rho E_\mu\, [(h^2-\nu^2)(k^2-\nu^2)\, \partial_\nu^2 E_\nu - \nu(k^2+h^2-2\nu^2)\, \partial_\nu E_\nu] = 0$$

$$(\mu^2-\nu^2)\, [(h^2-\rho^2)(k^2-\rho^2)\, \partial_\rho^2 E_\rho - \rho(k^2+h^2-2\rho^2)\, \partial_\rho E_\rho]\, /E_\rho$$

$$+ (\rho^2 - \nu^2)\,[(\mu^2-h^2)(k^2-\mu^2)\,\partial_\mu{}^2 E_\mu + \mu\,(k^2+h^2-2\mu^2)\,\partial_\mu E_\mu]\,/E_\mu$$
$$+ (\rho^2 - \mu^2)\,[(h^2-\nu^2)(k^2-\nu^2)\,\partial_\nu{}^2 E_\nu - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E_\nu]/\,E_\nu = 0$$

I just **don't see** how this separates!

Plan B: follow Hobson more closely:

Write the three separated equations in the alternate coordinates

But let's try doing what Hobson says. Let's first go back to our earlier result which was this

$$(\mu^2-\nu^2)\,E(\mu/\eta)E(\nu/\zeta)\partial_\xi{}^2[\,E(\rho/\xi)] + (\rho^2 - \nu^2)\,E(\rho/\xi)\,E(\nu/\zeta)\partial_\eta{}^2[E(\mu/\eta)]$$
$$+(\rho^2 - \mu^2)\,E(\rho/\xi)\,E(\mu/\eta)\partial_\zeta{}^2[E(\nu/\zeta)] = 0$$

which I recoded in this way

$$(\mu^2-\nu^2)\,E_\mu\,E_\nu\,\partial_\xi{}^2 E_\rho + (\rho^2 - \nu^2)\,E_\rho\,E_\nu\,\partial_\eta{}^2 E_\mu + (\rho^2 - \mu^2)\,E_\rho\,E_\mu\,\partial_\zeta{}^2 E_\nu = 0$$

We are now suppose to *assume* that our three functions satisfy these three (non-identical) equations

$$\partial_\xi{}^2 E_\rho - [n(n+1)\rho^2 - (h^2+k^2)p\,]\,E_\rho = 0 \qquad \text{p 458C}$$
$$\partial_\eta{}^2 E_\mu + [n(n+1)\mu^2 - (h^2+k^2)p\,]\,E_\mu = 0 \qquad \text{p 457C}$$
$$\partial_\zeta{}^2 E_\nu - [n(n+1)\nu^2 - (h^2+k^2)p\,]\,E_\nu = 0 \qquad \text{p 457C}$$

where we note the sign difference in the middle equation. So *if* we can find three distinct E functions that satisfy these three equations, our overall Laplace equation will be

$$(\mu^2-\nu^2)\,E_\mu\,E_\nu\,\partial_\xi{}^2 E_\rho$$
$$+ (\rho^2 - \nu^2)\,E_\rho\,E_\nu\,\partial_\eta{}^2 E_\mu$$
$$+(\rho^2 - \mu^2)\,E_\rho\,E_\mu\,\partial_\zeta{}^2 E_\nu = 0$$

and we replace the derivatives using the three previous equations,

$$(\mu^2-\nu^2)\,E_\mu\,E_\nu\,[n(n+1)\rho^2 - (h^2+k^2)p\,]\,E_\rho$$
$$- (\rho^2 - \nu^2)\,E_\rho\,E_\nu\,[n(n+1)\mu^2 - (h^2+k^2)p\,]\,E_\mu$$
$$+(\rho^2 - \mu^2)\,E_\rho\,E_\mu\,[n(n+1)\nu^2 - (h^2+k^2)p\,]\,E_\nu = 0 \quad ?$$

$$(\mu^2-\nu^2)\,[n(n+1)\rho^2 - (h^2+k^2)p\,]$$
$$- (\rho^2 - \nu^2)\,[n(n+1)\mu^2 - (h^2+k^2)p\,]$$
$$+(\rho^2 - \mu^2)\,[n(n+1)\nu^2 - (h^2+k^2)p\,] = 0 \quad ?$$

$$n(n+1)\,\{\,(\mu^2-\nu^2)\,\rho^2 - (\rho^2 - \nu^2)\,\mu^2 + (\rho^2 - \mu^2)\,\nu^2\,\}$$
$$-(h^2+k^2)p\,\{\,(\mu^2-\nu^2) - (\rho^2 - \nu^2) + (\rho^2 - \mu^2)\,\} = 0\ ?$$

$$n(n+1)\ \{0\}$$
$$-(h^2+k^2)p\{\ 0\ \}\ = 0\ ?$$

So yes, if we can find three E's solving the three equations above, their product will satisfy Laplace and we will have our separated solution. Note that the three equations as shown above are not the same, so it is not clear that the same function "E" will solve all three equations.

Convert the three separated equations from alternate into original coordinates

Now let's examine our three equations and see if, when we get change from the alternate to the original coordinates, maybe all three equations are the same! But let's try to re-use work we did earlier in the conical business where we found that

$$\partial_\eta^2 E_\mu = (\mu^2-h^2)(k^2-\mu^2)\,\partial_\mu^2 E_\mu + \mu\,(k^2+h^2-2\mu^2)\,\partial_\mu E_\mu$$

$$\partial_\zeta^2 E_\nu = (h^2-\nu^2)(k^2-\nu^2)\,\partial_\nu^2 E_\nu - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E_\nu$$

If we compare our three new coordinate definitions,

$$d\xi = 1/[\ \sqrt{\rho^2-h^2}\ \sqrt{\rho^2-k^2}\ ]\ d\rho$$
$$d\eta = 1/[\ \sqrt{\mu^2-h^2}\ \sqrt{k^2-\mu^2}\ ]\ d\mu$$
$$d\zeta = 1/[\ \sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2}\ ]\ d\nu$$

it seems to me that the first and last are pretty much the same, so we should be able to take our second result above and translate it by doing these replacements: $\zeta \rightarrow \xi$, $\nu \rightarrow \rho$. So I claim this

$$\partial_\zeta^2 E_\nu = (h^2-\nu^2)(k^2-\nu^2)\,\partial_\nu^2 E_\nu - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E_\nu$$

$$\Rightarrow \quad \partial_\rho^2 E_\rho = (h^2-\rho^2)(k^2-\rho^2)\,\partial_\rho^2 E_\rho - \rho(k^2+h^2-2\rho^2)\,\partial_\rho E_\rho$$

Then I am claiming these three results:

$$\partial_\xi^2 E_\rho = (h^2-\rho^2)(k^2-\rho^2)\,\partial_\rho^2 E_\rho - \rho(k^2+h^2-2\rho^2)\,\partial_\rho E_\rho$$
$$\partial_\eta^2 E_\mu = (\mu^2-h^2)(k^2-\mu^2)\,\partial_\mu^2 E_\mu + \mu\,(k^2+h^2-2\mu^2)\,\partial_\mu E_\mu$$
$$\partial_\zeta^2 E_\nu = (h^2-\nu^2)(k^2-\nu^2)\,\partial_\nu^2 E_\nu - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E_\nu$$

So let's plug these three expressions into our three equations

$$\partial_\xi^2 E_\rho - [n(n+1)\rho^2 - (h^2+k^2)p\ ]\ E_\rho = 0 \qquad \text{p 458C}$$
$$\partial_\eta^2 E_\mu + [n(n+1)\mu^2 - (h^2+k^2)p\ ]\ E_\mu = 0 \qquad \text{p 457C}$$
$$\partial_\zeta^2 E_\nu - [n(n+1)\nu^2 - (h^2+k^2)p\ ]\ E_\nu = 0 \qquad \text{p 457C}$$

and see what we get:

$$[n(n+1)\rho^2 - (h^2+k^2)p\,]\,E_\rho = (h^2-\rho^2)(k^2-\rho^2)\,\partial_\rho^2 E_\rho - \rho(k^2+h^2-2\rho^2)\,\partial_\rho E_\rho$$
$$-[n(n+1)\mu^2 - (h^2+k^2)p\,]\,E_\mu = (\mu^2-h^2)(k^2-\mu^2)\,\partial_\mu^2 E_\mu + \mu\,(k^2+h^2-2\mu^2)\,\partial_\mu E_\mu$$
$$[n(n+1)\nu^2 - (h^2+k^2)p\,]\,E_\nu = (h^2-\nu^2)(k^2-\nu^2)\,\partial_\nu^2 E_\nu - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E_\nu$$

$$[n(n+1)\rho^2 - (h^2+k^2)p\,]\,E_\rho = (h^2-\rho^2)(k^2-\rho^2)\,\partial_\rho^2 E_\rho - \rho(k^2+h^2-2\rho^2)\,\partial_\rho E_\rho$$
$$[n(n+1)\mu^2 - (h^2+k^2)p\,]\,E_\mu = (h^2-\mu^2)(k^2-\mu^2)\,\partial_\mu^2 E_\mu - \mu\,(k^2+h^2-2\mu^2)\,\partial_\mu E_\mu$$
$$[n(n+1)\nu^2 - (h^2+k^2)p\,]\,E_\nu = (h^2-\nu^2)(k^2-\nu^2)\,\partial_\nu^2 E_\nu - \nu(k^2+h^2-2\nu^2)\,\partial_\nu E_\nu$$

Now in each of these equations, replace the variable with x

$$[n(n+1)x^2 - (h^2+k^2)p\,]\,E_x = (h^2-x^2)(k^2-x^2)\,\partial_x^2 E_x - x(k^2+h^2-2x^2)\,\partial_x E_x$$
$$[n(n+1)x^2 - (h^2+k^2)p\,]\,E_x = (h^2-x^2)(k^2-x^2)\,\partial_x^2 E_x - x\,(k^2+h^2-2x^2)\,\partial_x E_x$$
$$[n(n+1)x^2 - (h^2+k^2)p\,]\,E_x = (h^2-x^2)(k^2-x^2)\,\partial_x^2 E_x - x(k^2+h^2-2x^2)\,\partial_x E_x$$

<u>So indeed, all three equations are the same!</u> Therefore, our separated solution is the product of three E functions where each of them satisfies this equation

$$(h^2-x^2)(k^2-x^2)\,\partial_x^2 E_x - x(k^2+h^2-2x^2)\,\partial_x E_x - [n(n+1)x^2 - (h^2+k^2)p\,]\,E_x = 0$$

which we rewrite as

$$(x^2-h^2)(x^2-k^2)\,\partial_x^2 E_x + x(2x^2-k^2-h^2)\,\partial_x E_x + [(h^2+k^2)p - n(n+1)x^2\,]\,E_x = 0 \qquad (*)$$

But this equation matches both p 457D and p 458D!

Review of what has happened here!

(1) Conical connection to Ellipsoidal

We first showed that, if you use conical coordinates r,μ,ν, the separated Laplace solution has the form $r^n E_\mu E_\nu$ where both E's solves the above ODE (*). Then we showed that if you use ellipsoidal coordinates ρ,μ,ν, the separated Laplace solution has the form $E_\rho E_\mu E_\nu$ where all three E's solve this same ODE (*).

(2) Step by Step Review:

• We start off with the equation of a confocal family of ellipsoids which of course contains the Cartesian coordinates in squared form such as x^2 and also contains things like (ρ^2-h^2).

• We use Cramer's Rule to solve for x^2, y^2, z^2. We then take the square roots to get x,y,z and in the resulting expressions we of course find things like $\sqrt{\rho^2-h^2}$. So this is the origin of "all those square roots" that one sees in the x,y,z equations in ellipsoidal coordinates.

• Thus, we have "all those square roots" in the transformation matrix $T_a{}^b$ and these then propagate into the metric tensor, so factors like $(\rho^2-\mu^2)(\rho^2-\nu^2)/[(\rho^2-k^2)(\rho^2-h^2)]$ appear.

• In order to simplify the metric tensor, we rescale the coordinates into ξ,η,ζ and we then regard these in some sense as the "natural coordinates" for the ellipsoidal system. This rescaling in order to take out all those square roots, such as $d\xi = 1/[\sqrt{\rho^2-h^2}\sqrt{\rho^2-k^2}]\, d\rho$, is what generates the "elliptic integrals". This is why the natural coordinates for ellipsoidal coordinates are elliptic integrals of the simple geometric coordinates ρ,μ,ν.

Try to directly relate conicals and ellipsoidals

The conical equations were: (where I now use r,U,V)

$x = r\,UV/(hk)$
$y = r\sqrt{U^2 - h^2}\sqrt{h^2 - V^2}/(hs)$ $\qquad s = \sqrt{k^2-h^2}$ $\qquad 0 < V < h < U < k$
$z = r\sqrt{k^2 - U^2}\sqrt{k^2 - V^2}/(ks)$

The ellipsoidal equations were

$x = \rho\mu\nu/(hk)$ $\qquad //\ \rho, \mu, \nu = \xi_1, \xi_2, \xi_3$
$y = \sqrt{\rho^2-h^2}\sqrt{\mu^2- h^2}\sqrt{h^2- \nu^2}\,/\,(hs)$ $\qquad s^2 = k^2-h^2$
$z = \sqrt{\rho^2-k^2}\sqrt{k^2- \mu^2}\sqrt{k^2- \nu^2}/(ks)$

I was able to invert the conicals above to get

$r = \sqrt{x^2+y^2+ z^2}$ $\qquad s^2 = k^2- h^2$
$V = \sqrt{W - \sqrt{W^2 - 4h^2k^2(x/r)^2}}\ ]\ /\sqrt{2}$ $\qquad W = k^2(x/r)^2 + s^2(y/r)^2 + h^2$
$U = h\,k(x/r)\,(1/V)$

So if we were to insert our ellipsoidal x,y,z here, we would find out how ρ,μ,ν and r,V,U are related, when they both describe the same x,y,z point in space. Here is a start with Maple:

```
restart;
x := p*u*v/(h*k);
```

$$x := \frac{p\,u\,v}{h\,k}$$

```
y := sqrt(p^2-h^2)*sqrt(u^2-h^2)*sqrt(h^2-v^2)/(h*s);
```

$$y := \frac{\sqrt{p^2-h^2}\,\sqrt{u^2-h^2}\,\sqrt{h^2-v^2}}{h\,s}$$

```
z := sqrt(p^2-k^2)*sqrt(k^2-u^2)*sqrt(k^2-v^2)/(k*s);
```

$$z := \frac{\sqrt{p^2-k^2}\,\sqrt{k^2-u^2}\,\sqrt{k^2-v^2}}{k\,s}$$

```
s := sqrt(k^2-h^2);
```

$$s := \sqrt{k^2-h^2}$$

```
r := sqrt(x^2+y^2+z^2):simplify(%);
```

$$\sqrt{-h^2+p^2+u^2+v^2-k^2}$$

```
W := (k*x/r)^2 + (s*y/r)^2 + h^2: simplify(%);
```

$$\frac{-h^2k^2+u^2v^2+p^2u^2+p^2v^2}{-h^2+p^2+u^2+v^2-k^2}$$

We see that r and W are at least reasonable. We have thus

$$r = \sqrt{\rho^2+\mu^2+\nu^2-s^2}$$

$$W = (\mu^2\rho^2 + p^2\nu^2 + u^2v^2 - h^2k^2)/(\rho^2+\mu^2+\nu^2-h^2-k^2)$$

But the quantity $W^2 - 4h^2k^2(x/r)^2$ is not a perfect square, says Maple (otherwise it could factor it). For this quantity we find something with a horrible 8th degree polynomial in the numerator,

$$\frac{h^4k^4-2k^2h^2u^2v^2-2k^2p^2h^2v^2+4h^2p^2u^2v^2-2k^2p^2u^2h^2+p^4u^4-2p^2u^4v^2-2p^2v^4u^2-2p^4u^2v^2+p^4v^4+u^4v^4+4p^2u^2v^2k^2}{(-h^2+p^2+u^2+v^2-k^2)^2}$$

So we know that $V = \sqrt{W - \sqrt{W^2 - 4h^2k^2(x/r)^2}}\ /\sqrt{2}$ is going to be a huge mess and there is no point in doing more. The main point should be obvious: the μ,ν of conicals are not the same as μ,ν of ellipsoidals, in case you were wondering.

Find expressions for ρ,μ,ν in terms of the alternates ξ, η, ζ

Solve for ρ, given ξ(ρ) = elliptic integral

Let's start with our differential fact which is all we have ever used to this point:

$$d\xi = 1/[\sqrt{\rho^2-h^2}\,\sqrt{\rho^2-k^2}\,]\,d\rho$$

Since this is ellipsoidals, we do know that $\rho \geq k$. If we integrate the above, we could write

$$\xi(\rho) = \int_{\mathbf{k}}^{\boldsymbol{\rho}} d\rho / \sqrt{\rho^2 - h^2}\sqrt{\rho^2 - k^2}$$

This just sets the zero point such that $\xi(k) = 0$. The zero point should not matter in anything we do. We might also just say

$$\xi(\rho) = \int^{\boldsymbol{\rho}} d\rho / \sqrt{\rho^2 - h^2}\sqrt{\rho^2 - k^2} + \text{constant}$$

It turns out that these integrals appear only in the GR7 definite integral section. Here is a whole wad of integrals of which several might be useful from page 280,

7. $$\int_0^u \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}F(\eta,t) \qquad [a > b \geq u > 0]$$

8. $$\int_u^b \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a}F(\zeta,t) \qquad [a > b > u \geq 0]$$

9. $$\int_b^u \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a}F(\kappa,q) \qquad [a \geq u > b > 0]$$

10. $$\int_u^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a}F(\lambda,q) \qquad [a > u \geq b > 0]$$

11. $$\int_a^u \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}F(\mu,t) \qquad [u > a > b > 0]$$

12. $$\int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}F(\nu,t) \qquad [u \geq a > b > 0]$$

3.152 **Notation**: In **3.152–3.163** we set: $\alpha = \arctan\frac{u}{b}$, $\beta = \operatorname{arccot}\frac{u}{a}$

$$\gamma = \arcsin\frac{u}{b}\sqrt{\frac{a^2+b^2}{a^2+u^2}}, \quad \delta = \arccos\frac{u}{b}, \quad \varepsilon = \arccos\frac{b}{u}, \quad \xi = \arcsin\sqrt{\frac{a^2+b^2}{a^2+u^2}},$$

$$\eta = \arcsin\frac{u}{b}, \quad \zeta = \arcsin\frac{a}{b}\sqrt{\frac{b^2-u^2}{a^2-u^2}}, \quad \kappa = \arcsin\frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-b^2}},$$

$$\lambda = \arcsin\sqrt{\frac{a^2-u^2}{a^2-b^2}}, \quad \mu = \arcsin\sqrt{\frac{u^2-a^2}{u^2-b^2}}, \quad \nu = \arcsin\frac{a}{u}, \quad q = \frac{\sqrt{a^2-b^2}}{a},$$

$$r = \frac{b}{\sqrt{a^2+b^2}}, \quad s = \frac{a}{\sqrt{a^2+b^2}}, \quad t = \frac{b}{a}.$$

Now I am presently interested in this integral

$$\xi(\rho) = \int_{\mathbf{k}}^{\boldsymbol{\rho}} d\rho \,/\sqrt{\rho^2-h^2}\,\sqrt{\rho^2-k^2}$$

which seems to fit the mold of #11 above where I will set u = ρ, a = k, b = h so ρ > k > h > 0 which matches the conditions. So our answer is

$$\xi(\rho) = (1/k)\, F(\mu,t) = (1/k)\, F(\sin^{-1}[\sqrt{(\rho^2-k^2)/(\rho^2-h^2)}\,], h/k)$$

$$= (1/k)\, sn^{-1}[\sqrt{(\rho^2-k^2)/(\rho^2-h^2)}\,, h/k] \qquad \text{// according to my ellipse doc}$$

This tells us then that

$$\sqrt{(\rho^2-k^2)/(\rho^2-h^2)} = sn(k\xi, h/k)$$

$$(\rho^2-k^2)/(\rho^2-h^2) = sn^2(k\xi, h/k)$$

If I manually solve this I get

$$\rho^2 = (k^2 - h^2 sn^2) / (1-sn^2) \qquad \text{where} \qquad sn = sn(k\xi, h/k)$$

I will now adopt Hobson's suggested symbol names

$$k_1' \equiv h/k \qquad k_1 = \sqrt{1-(h/k)^2}$$

Then my solution is this

$$\rho^2 = (k^2 - h^2 sn^2) / (1-sn^2) \qquad sn = sn(k\xi, k_1')$$
$$= k^2(1 - k_1'^2\, sn^2)/ (1-sn^2)$$
$$= k^2(1 - k_1'^2\, sn^2) /cn^2 \qquad \rho = (k/cn)\sqrt{1 - k_1'^2\, sn^2}$$

But *he* is claiming that

$$\rho^2 = k^2\, dn^2(ik\xi, k_1^2)$$

I so see this GR7 claim

$$\mathrm{dn}(iu, k) = \frac{\mathrm{dn}\,(u, k')}{\mathrm{cn}\,(u, k')}$$

which I can translate into

$$dn(ik\xi,k_1) = dn(k\xi, k_1') / cn(k\xi, k_1')$$

so *his result* would then become

$\rho^2 = k^2\ dn^2(k\xi, k_1') / cn^2(k\xi, k_1') = k^2\ dn^2/cn^2$ all $(k\xi, k_1')$.

So to make this match my result, I have to show that

$k^2\ dn^2/cn^2 = k^2(1 - k_1'^2\ sn^2)/cn^2$?

$dn^2 = (1 - k_1'^2\ sn^2)$?

But I next see this result

$$\mathrm{dn}^2 u + k^2\,\mathrm{sn}^2 u = 1$$

which in my case says

$dn^2 + k_1'^2 sn^2 = 1$

$dn^2 = 1 - k_1'^2 sn^2$

and the clinches the deal! I have now derived his result for ρ !!! But now let's go back to my result which was this:

$\rho = (k/cn)\sqrt{1 - k_1'^2\ sn^2}$ $k_1' \equiv h/k$

Using the above identify, I can write this like so, which I think is pretty good

$\rho = k\ (dn/cn)$ all $(k\xi, k_1')$ $k_1' \equiv h/k$

His result once again was this

$\rho = k\ dn(ik\xi, k_1^2)$

so his solution has the disadvantage of an imaginary argument, the disadvantage of k_1^2, but the advantage of being a single Jacobi function!

Solve for μ, given η(μ) = elliptic integral

Our integral of interest is

$\eta(\mu) = \int_h^\mu d\mu / \sqrt{\mu^2 - h^2}\ \sqrt{k^2 - \mu^2}$ // see Hobson p 458B

$= \int_h^\mu dx / \sqrt{x^2 - h^2}\ \sqrt{k^2 - x^2}$

Here is integral # 9 from our list above

$$9. \qquad \int_b^u \frac{dx}{\sqrt{(a^2-x^2)\,(x^2-b^2)}} = \frac{1}{a}\,F\,(\kappa, q) \qquad\qquad [a \geq u > b > 0]$$

and we now use u = μ, b = h, a = k and we have k > μ > h > 0 which is correct for variable μ if we recal our rule from earlier that 0 < ν < h < μ < k. So we find that

$\eta(\mu) = (1/k)\, F(\kappa,q) = (1/k)\, F(\sin^{-1}[\,(k/\mu)\sqrt{(\mu^2-h^2)/(k^2-h^2)}\,, \sqrt{1-(h/k)^2}\,)$

But $k/\sqrt{k^2-h^2} = 1/\sqrt{1-(h/k)^2} = 1/k_1$, so we have

$\eta(\mu) = (1/k)\, F(\kappa,q) = (1/k)\, F(\sin^{-1}[\,(1/(k_1\mu)\sqrt{(\mu^2-h^2)}\,, k_1)$

$= (1/k)\mathrm{sn}^{-1}[\sqrt{(\mu^2-h^2)}\,/(k_1\mu),\, k_1]$

Therefore we have found that

$\sqrt{(\mu^2-h^2)}\,/(k_1\mu) = \mathrm{sn}(k\eta, k_1)$

Now square this and solve for μ to get (scratch)

$\mu^2 = h^2 / (1 - k_1^2\mathrm{sn}^2)$ args = $(k\eta, k_1)$

But we use our identity from above to replace $(1 - k_1^2\mathrm{sn}^2) = \mathrm{dn}^2$ so we have

$\mu^2 = h^2/\mathrm{dn}^2$

$\mu = h/\mathrm{dn}\,(k\eta, k_1)$

which I think is a pretty simple result. But his result is this:

$\mu = k\,\mathrm{dn}(K - k\eta, k_1)$ $K = K(k_1^2)$

OK, let's look at this piece of a GR7 table (left)

$u^* = u + \boldsymbol{K}$
$\operatorname{sn} u^* = \dfrac{\operatorname{cn} u}{\operatorname{dn} u}$
$\operatorname{cn} u^* = -k' \dfrac{\operatorname{sn} u}{\operatorname{dn} u}$
$\operatorname{dn} u^* = k' \dfrac{1}{\operatorname{dn} u}$

$u^* = 0$	$-u$
$\operatorname{sn} u^* = 0$	$-\operatorname{sn} u$
$\operatorname{cn} u^* = 1$	$\operatorname{cn} u$
$\operatorname{dn} u^* = 1$	$\operatorname{dn} u$

SI 19, SI 18(13), WH,

which tells us that

dn(u + K, k_1) = k_1' /dn(u)

dn(K - u, k_1) = k_1' /dn(-u)

dn(K - u, k_1) = k_1' /dn(u) // from right box above

So his answer is then

μ = k dn(K - kη,k_1) = k k_1'/ dn(kη,k_1) = k (h/k) / dn(kη,k_1) = h / dn(kη,k_1)

and this agrees with my solution! So two down, one to go!

Solve for ν, given ζ(ν) = elliptic integral

$\zeta(\nu) = \int_0^{\nu} d\nu / \sqrt{h^2-\nu^2}\sqrt{k^2-\nu^2}$

$= \int_0^{\nu} dx / \sqrt{h^2-x^2}\sqrt{k^2-x^2}$

The mold here matches integral # 7

$$7. \qquad \int_0^u \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a} F(\eta, t) \qquad [a > b \geq u > 0]$$

where u = ν, a = k, b = h and we have k > h > ν > 0 which is correct for this variable. Thus we have

$\zeta(\nu)$ = (1/k) F(η,t) = (1/k) F($\sin^{-1}$(ν/h),h/k) = (1/k) F($\sin^{-1}$(ν/h), k_1')

= (1/k) sn^{-1}(ν/h, k_1')

Then we have

$\nu/h = sn(k\zeta, k_1')$

$\nu = h\ sn(k\zeta, k_1')$ agrees, and all done!!!!

So this brings us to the end of the section top page 459. The last three items tell us $\rho(\xi)$, $\mu(\eta)$ and $\nu(\zeta)$ and all are closed form results. These results are just the inverses of $\xi(\rho)$, $\eta(\mu)$ and $\zeta(\nu)$ which are given as integrals on the top of page 458. So far then I have read exactly 5.2 pages of Hobson and have generated 23 pages of notes! Here again are *his* solutions:

$\rho = k\ sn(ik\xi, k_1)$ $k_1^2 = 1 - (h/k)^2$ $k_1'^2 = (h/k)^2$

$\mu = k\ dn(K[k_1] - k\eta, k_1)$ h,k = our usual friends

$\nu = k\ sn(k\zeta, k_1')$

So this relates the actual ellipsoidal coordinates ρ,μ,ν to our helper coordinates ξ,η,ζ . I remember reading about these "alternate coordinates" in M&F and perhaps elsewhere, so here they are in full gore.

272. How E are related to the spherical harmonics?

Old paragraph: I will come back to this section later. The claim is that we know from conical coordinates that the set of solutions coded as $r^n E_{n,p}(\mu)E_{n,p}(\nu)$ are somewhat like the spherical solutions $r^n P_n^m(\cos\theta)e^{im\varphi}$. If in this notation you go to the spherical surface r = 1, you know that the functions $Y_{nm}(\theta,\varphi)$ form a complete expansion set for the two dimensions θ,φ. In fact, we can think of two 1D S-L problems and then in each dimension we have a complete set with m and n quantized for their respective reasons. Just so, in the conical coordinate case on the surface r = 1 we are going to have $V_{np}(\mu,\nu) = E_{n,p}(\nu)E_{n,p}(\mu)$ act as a complete set on the sphere, and again it is two (this time identical) 1D SL problems. [See note below!] The difference is that we have μ,ν instead of θ,φ, and the quantum numbers are "linked" in a different manner. Since both spherical coordinates and conical coordinates have spheres as constant-parameter surfaces, we are able to get this connection between the two sets of functions. For sphericals, we know there are (2n+1) independent functions $Y_{nm}(\theta,\varphi)$ which span the n-subspace, so we suspect this is also true for the $V_{np}(\mu,\nu)$ functions.

[NOTE: it turns out that $E_{n,p}(\nu)E_{n,p}(\mu)$ does NOT in fact reduce to two 1D S-L problems, as we shall see below. A hint that this might be the case is given by the linkage of the quantum numbers n and p. You cannot isolate one of the quantum numbers to one of the functions and then have it just be a bystander on the other, as you can with the Y_{nm}.]

Items related to Hilbert Space and its ℓ subspace

(a) Angular momentum EV equation in sphericals; doing expansions on θ,φ; subspace $\mathcal{H}^\ell$.

Consider the eigenvalue equation $\nabla^2_{\theta\varphi}\ f(\theta,\varphi) = \ell(\ell+1)\ f(\theta,\varphi)$ which has the form $L^2 f = \lambda f$. We know the eigenvalues of ℓ are integers. Each eigenvalue is degenerate and has $(2\ell+1)$ different eigenfunctions. We can decompose the space of functions $F(r,\theta,\varphi)$ on L^2 into a direct sum of subspaces in a famous matrix sense.

$$\langle\theta\varphi|f\rangle = \Sigma_{\ell m}\langle\theta\varphi|\ell m\rangle\langle\ell m|f\rangle \qquad |f\rangle = \Sigma_{\ell m}\ f_{\ell m}\ |\ell m\rangle$$
$$f(\theta,\varphi) = \Sigma_{\ell m}\ f_{\ell m}\ Y_{\ell m}(\theta,\varphi)$$

The equation $|f\rangle = \Sigma_{\ell m}\ f_{\ell m}\ |\ell m\rangle$ says we can take a vector in our Hilbert Space and decompose it into first its projections on the subspaces $\mathcal{H}^\ell$ and within each such subspace it will have $(2\ell+1)$ coefficients $f_{\ell m}$. Solutions to the 3D Laplace problem have the form $F(r,\theta,\varphi) = \Sigma_{\ell m}\ F_{\ell m}\ (r/a)^{\ell}\ Y_{\ell m}(\theta,\varphi)$ and if we evaluate this on a sphere of radius r = a, we find $F(a,\theta,\varphi) = \Sigma_{\ell m}\ F_{\ell m}\ Y_{\ell m}(\theta,\varphi)$. We know that

$$\nabla^2_{r\theta\varphi} = (1/r^2)\ \partial_r(r^2\partial_r) + (1/r^2)\ \nabla^2_{\theta\varphi}$$

so the Laplace equation is

$$\partial_r(r^2\partial_r f) + \nabla^2_{\theta\varphi} f = 0$$

and this leads to a the usual separated atomic forms here claimed.

(b) Same stuff in conicals; the L^2 operator expressed in μ,ν conicals

Now how does all this play out in conicals? We found above that

$$\nabla^2_{r\mu\nu}\ f = (1/\ h_{\eta\eta}{}^2)\ \{\ \partial_r\ [h_{\eta\eta}{}^2\ (\partial_r f)] + (\partial_\eta{}^2 f) + (\partial_\zeta{}^2 f)\ \} \qquad h_{\eta\eta}{}^2 = (\mu^2-\nu^2)\ r^2$$

so the Laplace equation reads

$$\partial_r(r^2\partial_r f) + (\mu^2-\nu^2)^{-1}[(\partial_\eta{}^2 f) + (\partial_\zeta{}^2 f)] = 0 \qquad \text{conicals}$$
$$\partial_r(r^2\partial_r f) + \nabla^2_{\theta\varphi} f = 0 \qquad \text{sphericals}$$

The r part is the same and we want $f = r^{\ell}\ G$ for our atomic form there. We then get

$$\partial_r(r^2\partial_r f) = \partial_r(r^2\partial_r\ r^{\ell}\ G) = \ell(\ell+1)\ f$$

So our two reduced Laplace equations read

$$-(\mu^2-\nu^2)^{-1}(\partial_\eta{}^2 + \partial_\zeta{}^2) f = \ell(\ell+1)\ f \qquad \text{conicals}$$
$$-\nabla^2_{\theta\varphi} f = \ell(\ell+1)\ f \qquad \text{sphericals}$$

So we define the following operator, which must be (-) the angular momentum operator in conicals!

$$\nabla^2_{\mu\nu} = (\mu^2-\nu^2)^{-1}(\partial_\eta^2 + \partial_\zeta^2)$$

and we then have the following eigenvalue problem:

$$-\nabla^2_{\mu\nu} f = \ell(\ell+1)\, f$$

(c) Conical separation in these μ,ν coordinates

We can separate this by writing f(μ,ν) = a(μ)b(ν) to get

$$(\mu^2-\nu^2)^{-1}(\partial_\eta^2 + \partial_\zeta^2)\, a(\mu)b(\nu) = -\ell(\ell+1)\, a(\mu)b(\nu)$$

$$(\mu^2-\nu^2)^{-1}(b(\nu)\partial_\eta^2\, a(\mu) + a(\mu)\partial_\zeta^2 b(\nu)) = -\ell(\ell+1)\, a(\mu)b(\nu)$$

$$b(\nu)\partial_\eta^2\, a(\mu) + a(\mu)\partial_\zeta^2 b(\nu) = -\ell(\ell+1)\, a(\mu)b(\nu)\,(\mu^2-\nu^2)$$

$$\partial_\eta^2 a(\mu)/\, a(\mu) + \partial_\zeta^2 b(\nu) / b(\nu) = -\ell(\ell+1)\,(\mu^2-\nu^2) = -\ell(\ell+1)\,\mu^2 + \ell(\ell+1)\,\nu^2$$

$$\partial_\eta^2 a(\mu)/\, a(\mu) + \ell(\ell+1)\,\mu^2 = -\partial_\zeta^2 b(\nu) / b(\nu) + \ell(\ell+1)\,\nu^2$$

We are now separated and we set the separation constant to ps^2 as Hobson does to get

$$\partial_\eta^2 a(\mu)/\, a(\mu) + \ell(\ell+1)\,\mu^2 = ps^2$$
$$\partial_\zeta^2 b(\nu) / b(\nu) - \ell(\ell+1)\,\nu^2 = -ps^2$$

$$\partial_\eta^2 a(\mu)/\, a(\mu) + [\,\ell(\ell+1)\,\mu^2 - ps^2] = 0$$
$$\partial_\zeta^2 b(\nu) / b(\nu) - [\ell(\ell+1)\,\nu^2 - ps^2] = 0$$

$$\partial_\eta^2 a(\mu) + [\,\ell(\ell+1)\,\mu^2 - ps^2]\, a(\mu) = 0$$
$$\partial_\zeta^2 b(\nu) - [\ell(\ell+1)\,\nu^2 - ps^2]\, b(\nu) = 0$$

I have thus replicated the Lamé equations Hobson showed as page 457 C which, I showed earlier, lead to exactly the same solutions in terms of μ and ν. Therefore, we can continue our discussion this way, showing the angular momentum eigenfunctions to have a factored form which is the product of the same function:

$$-\nabla^2_{\mu\nu} f = \ell(\ell+1)\, f$$

$$f(\mu,\nu) = f_{\ell p}(\mu,\nu) = E_{\ell p}(\mu)\, E_{\ell p}(\nu) = \langle\mu\nu|\ell p\rangle$$

where p takes $(2\ell+1)$ values to span the subspace of this operator which is our same angular momentum operator just expressed in new coordinates.

(d) Resume the eigenket discussion: how Y and E are related

Presumably there will be some connection

$$|\ell m\rangle = \Sigma_p \langle \ell p|\ell m\rangle |\ell p\rangle \qquad \Sigma_p |\ell p\rangle\langle \ell p| = 1^\ell \qquad \Sigma_m |\ell m\rangle\langle \ell m| = 1^\ell$$

Now $|\theta\varphi\rangle$ and $|\mu\nu\rangle$ can both be written as $|x\rangle$ where x is a point on our sphere. The same point will of course have different numbers for the parameter, so when we now say $|x\rangle = |\theta\varphi\rangle = |\mu\nu\rangle$ we don't mean to imply that $\theta\varphi = \mu\nu$. We could enhance the notation to make this clearer, but I think it is OK.

Then let's try this:

$$E_{\ell p}(\mu)\, E_{\ell p}(\nu) = \langle \mu\nu|\ell p\rangle = \langle \mu|\ell p\rangle\langle \nu|\ell p\rangle$$

$$= \langle \theta\varphi|\ell p\rangle = \Sigma_m \langle \theta\varphi\, |\ell m\rangle\langle \ell m|\ell p\rangle = \Sigma_m \langle \ell m|\ell p\rangle\, Y_{\ell m}(\theta,\varphi)$$

which tells us that we can write the E function products as sums of spherical harmonics. Going the other way we then have

$$Y_{\ell m}(\theta,\varphi) = \langle \theta\varphi|\ell m\rangle = \langle \theta\varphi|\, \Sigma_p\, |\ell p\rangle\langle \ell p|\, \ell m\rangle = \Sigma_p \langle \ell p|\, \ell m\rangle \langle \theta\varphi|\, \ell p\rangle = \Sigma_p \langle \ell p|\, \ell m\rangle \langle \mu\nu|\, \ell p\rangle$$

$$= \Sigma_p \langle \ell p|\, \ell m\rangle\, E_{\ell p}(\mu)\, E_{\ell p}(\nu)$$

(e) What can we learn from the connection between Y and E?

The actual connection between $\theta\varphi$ and $\mu\nu$ is pretty simple, namely (I swap x and z now)

$$x = r\sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\,/(ks)$$
$$y = r\sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\,/(hs) \qquad s = \sqrt{k^2-h^2} \qquad 0 < \nu < h < \mu < k$$
$$z = r\,\mu\nu/(hk)$$

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$\Rightarrow \cos\theta = \mu\nu/(kh)$$
$$\sin\theta\cos\varphi = \sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\,/\,(hs)$$
$$\sin\theta\sin\varphi = \sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\,/(ks)$$

which are the equations Hobson mentioned way back on page 455 with no comment. So consider

$$E_{\ell p}(\mu)\, E_{\ell p}(\nu) = \Sigma_m \langle \ell m|\ell p\rangle\, Y_{\ell m}(\theta,\varphi)$$

where for the moment we think symbolically that [ie, real and imaginary parts of Y]

$$Y_{\ell m}(\theta,\varphi) = K_{\ell m} P_{\ell}^{m}(\cos\theta) [\cos(m\varphi), \sin(m\varphi)]$$

and we know that

$$P_n^m(\cos\theta) = (\sin\theta)^m \partial_x^m P_n(x) = \text{poly of degree n in variables } \sin\theta \text{ and } \cos\theta.$$

Therefore we can claim that

$$Y_{\ell m}(\theta,\varphi) = \text{poly of degree n in } \sin\theta, \cos\theta \ * \ \text{poly of degree m in } \sin\varphi,\cos\varphi$$

Now suppose we then replace:

$\cos\theta = \mu\nu/(kh)$ = " a function of $\mu\nu$"

$\sin\theta = \sqrt{1 - [\mu\nu/(kh)]^2}$ = "an algebraic function of $\mu\nu$ "

$\cos\varphi = [\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2} / (hs)]/\sin\theta$ = " a rational function of $\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}$ and $\sin\theta$"

$\sin\varphi = [\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2} / (ks)]/\sin\theta$ = " a rational function of $\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}$ and $\sin\theta$"

At this point in my original notes I went a bit astray and was unhappy. The last two lines above are not really what one wants to do, you have to see the picture at a slightly more detailed level. I then wrote "Using Conical and Spherical Coordinates to explore Lamé functions.doc" where I figured it all out, and there was a lot to figure out actually.

"Using Conical and Spherical Coordinates to explore Lamé functions.doc" goes here!

I obtained the four forms that Hobson gives as p 459, where the various U_k functions end up being $\text{poly}_k(\mu^2,\nu^2,\mu\nu)$ which he should have just called polynomials instead of the confusing "rational algebraical functions". I then went on to consider the equation above, now written as

$$P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)] = \Sigma_p f_{n,p}(m) E_{n,p}(\mu) E_{n,p}(\nu)$$

so show that there really are the four solution "classes" he mentions called K,L,M,N and that each has exactly the form he shows including the degree and the missing alternate terms. I then confirmed Hobson's claim p 460 D and found a typo in that he has LL and MM swapped. I looked just now for some collected Hobson errata on the 1931 book but could find none.

One should take note that the four classes of solutions pretty much agree with what you would expect based on simple Frobenius theory where you learn what the exponents are at each singular point of the Lamé equation (namely, 0 and 1/2 at $z = \pm h$ and $\pm k$). The missing alternate terms have an interpretation I give elsewhere. This Frobenius theory does not, however, deliver as much information as Hobson extracts from the method above which I now like a lot.

273. Elaboration of the four Frobenius solutions K, L, N, M.

The above separate doc explains the general forms we get for these four classes of Lamé functions. Here we want to lean exactly how we figure out the p values for the $E_{n,p}$ with E in the four classes.

[See MUCH better text here: Understanding the Hobson Lamé Recursion Relation Solutions.doc !!]

(1) The K case : have to set detM=0 to find the r+1 distinct p values.

I now understand and agree with the assumed form shown for K(μ). What happens when you insert this form into the Lamé equation? For a given power μ^k I show that the first term gives a mix of μ^{k+2}, μ and μ^{k-2} which is what I mean by my pencil notation +2,0,-2. What is a little confusing to the reader is how Hobson defines the indices on his a_i coefficients. Although the powers skip two, these indices only skip one. This means that the recursors will link a_r , a_{r+1}, a_{r-1}. I understand then how his list of recursors starts off on page 461 (though I have not verified the details). But then the three items showing a_r confuse me. What is r? If r is out in the middle somewhere, why do these three equations have different forms? Well OK. He writes these equations:

r'=1
r'=2
r'= 3
...
r' = r
r' = r+1
r' = r+2

However, in the r+2 equation he *wants* to have the β term vanish, and I agree that will happen if either factor multiplying β in the r+2 term were to vanish. These factors would be

					"special r"	
n-2(r+2)+4	=	n-2r	=0	=>	r = n/2	n even
n-2(r+2)+3	=	n-2r-1	=0	=>	r = (n-1)/2	n odd

Thus, the last equation he writes is for a **"special r"**, so is not "out in the middle". It is because the r in this equation can have two different values that he calls it r. So the symbol r has two meanings: (1) it marks a generic recursion relation out in the middle; (2) it marks the "special r" shown on the last line. So he overloads the symbol r and this confuses the reader!

Now remember that as r increases, we are going *down* in power, not up (which is the more usual case). Look at the form of the series

$$K(\mu) = a_1\mu^n + a_2\mu^{n-2} + ... a_k\mu^{n-2k} + ... + a_N\mu^{n-2N}$$

The series is supposed to *end* when n-2k reaches either 0 or 1 which means we have

n-2N = 0	=>	N = n/2	n even	=>	$a_{n/2+1} = 0$
n-2N = 1	=>	N = (n-1)/2	n odd	=>	$a_{n/2+1/2} = 0$

where on the right I show the first coefficient that must vanish (and all after). Using the second meaning of the symbol r noted above, we are then claiming that we need $a_{r+1} = 0$ and also all following.

So what has to happen such that the coefficient shown and all those following vanish? From the general r' = r recursor, we can see that if we can get two in a row to vanish, we are home free.

Let's encode the recursors as follows (r' is the highest index k of a_k appearing)

$$M_{00}a_0 + M_{01}a_1 = 0 \qquad r' = 1$$
$$M_{10}a_0 + M_{11}a_1 + M_{12}a_2 = 0 \qquad r' = 2$$
$$M_{21}a_1 + M_{22}a_2 + M_{23}a_3 = 0 \qquad r' = 3$$
$$\ldots\ldots$$

Now let's look at the last three rows of this list of equations

$$M_{r-1,r-2}a_{r-2} + M_{r-1,r-1}a_{r-1} + M_{r-1,r}a_r = 0 \qquad r' = r$$
$$M_{r,r-1}a_{r-1} + M_{r,r}a_r + M_{r,r+1}a_{r+1} = 0 \qquad r' = r+1$$
$$M_{r+1,r}a_r + M_{r+1,r+1}a_{r+1} + M_{r+1,r+2}a_{r+2} = 0 \qquad r' = r+2$$

Now if we take the r in this last row to be our "special r", then the red M coefficient vanishes. Notice that this same coefficient appears in the second last row. Now suppose we could solve this matrix problem, where our matrix goes only through the r' = r+1 row and so is now a finite matrix:

$$M_{00}a_0 + M_{01}a_1 = 0 \qquad r' = 1$$
$$M_{10}a_0 + M_{11}a_1 + M_{12}a_2 = 0 \qquad r' = 2$$
$$M_{21}a_1 + M_{22}a_2 + M_{23}a_3 = 0 \qquad r' = 3$$
$$\ldots\ldots$$
$$\ldots\ M_{r-1,r-2}a_{r-2} + M_{r-1,r-1}a_{r-1} + M_{r-1,r}a_r = 0 \qquad r' = r$$
$$M_{r,r-1}a_{r-1} + M_{r,r}a_r = 0 \qquad r' = r+1$$

where I have set $M_{r,r+1} = 0$. The number of rows in this matrix is r+1 where r is the special r. And the number of coefficients is a_0, a_1..... a_r which is r+1 coefficients. Thus we have an (r+1)x(r+1) matrix M and this matrix equation

$$M\mathbf{a} = 0$$

If detM ≠ 0, this equation has no non-trivial solution for vector **a**. But if we can tune the value of p such that detM=0, then we can have a solution for vector **a**. Now this is the tricky part: we have this solution for a_0, a_1..... a_r which satisfies the first r+1 recursion relations. The coefficient a_{r+1} is decoupled from this system since $M_{r+1,r} = 0$. So I think we just arbitrarily *declare* that $a_{r+1} = 0$. That is, we seek a solution with $a_{r+1} = 0$. Given this fact, the last recursor listed above says

$$M_{r+1,r}a_r + M_{r+1,r+1}a_{r+1} + M_{r+1,r+2}a_{r+2} \quad = M_{r+1,r+1}a_{r+1} + M_{r+1,r+2}a_{r+2} = 0$$

so then $a_{r+2} = 0$. But now you have two in a row 0, so the rest must vanish as well. You then have a self consistent solution that meets our earlier-stated requirement that the down-going series end with the term μ^1 or μ^0 which required a_{r+1} and all following to vanish where r is the special r.

Now something note yet noted: in each recursor, the "middle M" has a term linear in p. There are r+1 of these middle M's in our matrix. Here I mark them in red

$$\begin{pmatrix} M_{00} & M_{01} & 0 & 0 \\ M_{10} & M_{11} & M_{12} & 0 \\ 0 & M_{21} & M_{22} & M_{23} \\ 0 & 0 & M_{32} & M_{33} \end{pmatrix} \mathbf{a} = 0 \qquad \begin{pmatrix} r'=0 \\ r'=1 \\ r'=2 \\ r'=3 \end{pmatrix}$$

We know that the determinant is given by

$$\det M = \varepsilon_{ijk\ell}\, M_{0i}M_{1j}M_{2k}M_{3\ell}$$

and this includes the product of the red elements shown. Therefore, the determinant will be a polynomial in p of degree r+1 ! Here are some sample n cases

n=2 r=1 r+1 = 2
n=3 r=1 r+1 = 2

So call the roots $p_1, p_2, \ldots p_{r+1}$. In either the case n = 2 or n = 3, we have 2 p solutions p_1, p_2.

Question: How do we actually *solve* the equation M**a** = 0 given that detM ≠ 0 ? The answer is that you just construct the a_i one at a time from the recursion relations top of page 461starting with a_0, then a_1 and so on until you get to a_n ! For each value of p, you will get a set of coefficients **a**. Really quite simple.

Let's hold on further questions and just see what we learn about the L,M and N solutions.

(2) The L case: set $a_{n-r} = 0$ to truncate poly, solve this for the n-r p values that work.

Here we get a similar set of recursors, but things are done differently. Here you imagine building up your coefficients one a time in a more traditional fashion:

$a_0 = 1$
$a_1 = (Ap+B)a_0$
$a_2 = (Cp+D)a_1 + ka_0$ = quadratic in p
$a_3 = \ldots.$

We again want the series to stop at the μ^0 or μ^1 term so we want $a_{r+1} = 0$ and all following. In this case we just build up and then require that $a_{n-r} = 0$ where r is our same "special value". If we can achieve this, then the next equation says $a_{n-r-1} = 0$ and we are "home free". So the equation for p is not a determinant

in this case, it is whatever you get when you do the above buildup. We will have a_{n-r} being of degree n-r in p, so there will be p solutions p_1, p_2...p_{n-r} and I suspect each p value gives one L function.

(3) The M case: same idea as L case, there are n-r values of p.

Just like the L case, this will also have n-r solutions.

(4) The N case. this time there are r solutions.

Doing the same motions, we find r solutions here.

(5) Summary, add them up to find 2n+1 solutions $E_{n,p}(x)$

So let's add them up

K	r+1	r = n/2 if n even, else (n-1)/2
L	n-r	
M	n-r	
N	r	
	2n+1 solutions!!	

Here then is a little table just showing counts of solutions

n=	0	1	2	3	4	5	
r=	0	0	1	1	2	2	
K	1	1	2	2	3	3	
L	0	1	1	2	2	3	
M	0	1	1	2	2	3	
N	0	0	1	1	2	2	
	1	3	5	7	9	11	= (2n+1)

Now suddenly Byerly's table for n = 0,1,2,3 makes sense and I quote it again here

$E_0(x)$

$$K_0(x) = 1$$
$$L_0(x) = 0$$
$$M_0(x) = 0$$
$$N_0(x) = 0$$

$E_1(x)$

$$K_1(x) = x$$
$$L_1(x) = \sqrt{x^2 - b^2}$$
$$M_1(x) = \sqrt{x^2 - c^2}$$
$$N_1(x) = 0$$

$E_2(x)$

$$K_2^{p_1}(x) = x^2 - \tfrac{1}{3}[b^2 + c^2 - \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$$
$$K_2^{p_2}(x) = x^2 - \tfrac{1}{3}[b^2 + c^2 + \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$$
$$L_2(x) = x\sqrt{x^2 - b^2}$$
$$M_2(x) = x\sqrt{x^2 - c^2}$$
$$N_2(x) = \sqrt{(x^2 - b^2)(x^2 - c^2)}$$

$E_3(x)$

$$K_3^{p_1}(x) = x^3 - \frac{x}{5}[2(b^2 + c^2) - \sqrt{4(b^2 + c^2)^2 - 15b^2c^2}]$$

$$K_3^{p_2}(x) = x^3 - \frac{x}{5}[2(b^2 + c^2) + \sqrt{4(b^2 + c^2)^2 - 15b^2c^2}]$$

$$L_3^{q_1}(x) = \sqrt{x^2 - b^2}[x^2 - \tfrac{1}{5}(b^2 + 2c^2 - \sqrt{(b^2 + 2c^2)^2 - 5b^2c^2})]$$
$$L_3^{q_2}(x) = \sqrt{x^2 - b^2}[x^2 - \tfrac{1}{5}(b^2 + 2c^2 + \sqrt{(b^2 + 2c^2)^2 - 5b^2c^2})]$$
$$M_3^{r_1}(x) = \sqrt{x^2 - c^2}[x^2 - \tfrac{1}{5}(2b^2 + c^2 - \sqrt{(2b^2 + c^2)^2 - 5b^2c^2})]$$
$$M_3^{r_2}(x) = \sqrt{x^2 - c^2}[x^2 - \tfrac{1}{5}(2b^2 + c^2 + \sqrt{(2b^2 + c^2)^2 - 5b^2c^2})]$$
$$N_3(x) = x\sqrt{(x^2 - b^2)(x^2 - c^2)}$$

Now finally, lets write out the four cases which we see in Hobson p 459-460. Here we use a,b... as generic dimensionless coefficients.

$K_{n,p} = (a\mu^n + b\mu^{n-2} + \ldots)$

$L_{n,p} = \sqrt{\mu^2 - h^2}\ (a\mu^{n-1} + b\mu^{n-3} + \ldots)$

$M_{n,p} = \sqrt{\mu^2 - k^2}\ (a\mu^{n-1} + b\mu^{n-3} + \ldots)$

$N_{n,p} = \sqrt{\mu^2 - h^2}\sqrt{\mu^2 - k^2}\ (a\mu^{n-2} + b\mu^{n-4} + \ldots)$

Notice that in all four cases, the $E_{n,p}$ function is the dimension μ^n which is L^n. You of course see this in Byerly's list above where the variable is x instead of μ.

274. How things reduce in the spheroidal limits of ellipsoidal.

Recall first for ellipsoidal coordinates ρ,μ,ν that,

$A^2 = \rho^2-k^2$		z		$0 < \nu < h < \mu < k < \rho$
$B^2 = \rho^2-h^2$		y		
$C^2 = \rho^2-0^2$	$= \rho^2$	x	=>	$C \geq B \geq A$

(a) oblate. If h = 0, then the two larger semi's are the same B = C and you have **oblate**. This seems to crush out the ν coordinate and force it to 0. The idea is that both h and ν → 0 but the ratio ν/h → cosφ'. Meanwhile, claim is that μ/k = sinθ', and then ρ,θ',φ' will end up being the oblate spheroidal coordinates.

He examines the four different sets of p-determining equations for the K,L,M,N functions in this limit and they are all basically the same and I think the distinction fades. It turns out that $\mu' = \sqrt{k^2-\mu^2}$ becomes the variable in the limit of the Lamé equation which is then the Legendre equation and you find $E(\mu) \rightarrow P_n^m(\mu')$ where $m = \sqrt{p}$ = integer. In similar fashion, when you replace μ→ρ, you get $\mu' \rightarrow \sqrt{k^2-\rho^2} \equiv i\rho'$, and then $E(\rho) \rightarrow P_n^m(i\rho')$. Finally, E(ν) → sin/cosφ'. This is not an obvious limit to take, but to his credit Hobson does not leave this little stone unturned. So:

$E(\mu) \rightarrow P_n^m(\mu')$ $E(\rho) \rightarrow P_n^m(i\rho')$ $E(\nu) \rightarrow \sin/\cos\varphi'$

(b) prolate. If h = k, then the two smaller semis are the same A = B and you have **prolate**. In this case the μ coordinate is squeezed out and we end up with μ = h = k. We end up with

$E(\nu) \rightarrow P_n^m(\cos\theta)$ $E(\rho) \rightarrow P_n^m(\rho)$ $E(\nu) \rightarrow \sin/\cos\varphi'$

I have not studied this stuff, but I could do it all if I wanted. The key thing is that the Lame equation becomes the Legendre equation for 2 of the 3 coordinates in these oblate and prolate limits. We saw this happen directly in M&F.

275. The Lamé functions of degree n= 0,1,2,3.

(a) Derivation of all the first-kind Lamé functions with n = 0,1,2 and 3.

For n=0 we get E = K = 1 as the only solution, and then EEE = 111 = 1.

For n=1 we get one each of K, L and M. He then notes that EEE = KKK, LLL and MMM are three possible combinations and that they happen to be proportional to the Cartesian x,y and z solutions. He refers to the triples with all the same as "**normal" solutions.** Remember that it is just these triplets that appear as solutions to the Laplace equation, all three E's have the same n and p !!!

For n = 2 we get two K, and one each L and M and N. The two K ones require you to compute the p values as roots of something, but the L,M,N functions are trivial. He then writes out the five normal functions

KKK K'K'K' LLL MMM NNN

The last three turn out to be proportional to the Cartesian solutions xz, xy, yz and first two y^2-z^2 and z^2-x^2. This is just the Cartesian form of the n = 2 spherical harmonics (modulo some linear combination). There are only 5 such solutions. I computed these in my " Using Conical and Spherical Coordinates to explore Lamé functions.doc". We associate these things with the word "quadrupole". I quote from the other doc

$P_2^2(\cos\theta)\cos(2\varphi) \sim x^2-y^2$ $\quad P_2^2(\cos\theta)\sin(2\varphi) \sim xy$
$P_2^1(\cos\theta)\cos(1\varphi) \sim xz$ $\quad P_2^1(\cos\theta)\sin(1\varphi) \sim zy$
$P_2^0(\cos\theta)\cos(0\varphi) \sim 2z^2-x^2-y^2$ $\quad P_2^0(\cos\theta)\sin(0\varphi) = 0$

Aside: Then how can we have solution = F 1 1 for charged ellipsoid? Well that really is KKK but the first K is replaced by its second kind function, but we have not mentioned these yet. If we included them you could say

$KKK \rightarrow [K_1+K_2]\,[K_1+K_2]\,[K_1+K_2]$

and this would be your set of normal solutions where K_2 are the second kind functions.

For n = 3 we expect 2 each of K,L,M and one N. This time he does not write down the Cartesian solutions.

(b) Proof that the (2n+1) p values are real and distinct.

Hobson then gives a summary of **Lamé's proof** that the (2n+1) p values you get are always real and distinct. This proof starts at the bottom of page 465 and ends at the pencil line page 467. Along the way, Hobson shows that if you take two E functions in the *same class and of the same n*, call them E_1 and E_2, then if p_1 and p_2 are the corresponding p values for these E functions, then

$(p_1-p_2)\int_0^{\omega} d\eta\, E_{n,p1}(\eta)E_{n,p2}(\eta) = 0$ assumes E's in same class

where he has snuck in a new Greek letter variable ω. Recall that earlier we defined

$$\eta(\mu) = \int_h^{\mu} d\mu / \sqrt{\mu^2-h^2}\,\sqrt{k^2-\mu^2}$$

It turns out that $\omega = \eta(k)$ where μ takes its max value. So $\eta = \omega$ is like $\mu = k$. The above says that if the roots are different for the p_i, then the E functions are *orthogonal* in the sense shown

$\int_0^{\omega} d\eta\, E_1(\eta)E_2(\eta) = 0$ involves μ in the range (h,k) I think

Hobson does not talk about S-L problems. All of our first-kind functions E(μ) are polynomials (with a possible radical). Thus all the E(ν) on interval (0,h), say, probably form a complete set since the L operator is probably self-adjoint. So we should be able to expand some f(ν) as he shows, normal **Fourier analysis**. This is probably not of much practical use since it is so hard to compute the higher order E functions. Maybe the E(μ) are also a complete set on (h,k). I admit being a little fuzzy on the endpoints. We cannot take (0,∞) as the interval. Our "two oscillatory" functions would likely be E(ν) and E(μ) for an ellipsoidal situation, and then E(ρ) would be expo. Obviously this S-L fact would imply orthogonality relations for the E functions, and completeness as well. We have just seen one form of orthogonality above where the two E functions are in the same n manifold and have the same class, which is a very special case. [As noted earlier, we do NOT have $E_{n,p}(\mu)$ $E_{n,p}(\nu)$ being two 1D SL problems, so everything is indeed "fuzzy" here.]

276. The Evaluation of a Certain Integral (the 4-E orthogonality)

First he looks at a double integral dη dζ which contains four E functions, has weight $(\mu^2-\nu^2)$. The upper end point of the dη integration is η(k) called ω, and of the dζ integration is ζ(h) called ω'. Big typo noted here. There are really only two specific E functions involved, one is n' and s, the other n and t, and they must be of the same class. He then arrives at p 468 A which is of the form "factors times his integral of interest". Basically it says this

$(n-n') * (p_n^s - p_{n'}^t) *$ integral = 0

So if n≠n' and we have K functions say in different n-spaces, you will get orthogonality. We already know from the previous section that you get orthogonality within the same n space. So basically he is working up the orthogonality situation, I am not going into the details here.

[This four-E integral must be orthogonality for the 2D multi-parameter S-L problem of $E_{n,p}(\mu)$ $E_{n,p}(\nu)$.]

277. Double Fourier in the μν world.

Obviously this is going to use the orthogonality integrals derived in section 276. Since we are going to expand f(μ,ν) on the product E(μ)E(ν), we can see why maybe he needed the four-E integral. Roughly if you assume an expansion as shown in p 470A, the coefficients are as given in p 470 B. He tacks on a little extra fact needed in the next section.

I think of this as similar to the orthogonality of the spherical harmonics. In that case we do have orthogonality separately in the θ and φ variables. You would think that same separation would happen here with the E functions and with μ and ν, but he does not make that claim, he has the two linked together. I will just let this ride. [see later]

278. The Zeros of the E(μ) functions.

Claims the zeros are real and never larger than k, and that you never get degenerate roots. I am still unclear about the S-L problem situation with the two possible intervals (0,h) and (h,k). Certainly in either

case we are not going to get zeros for $\mu > k$ since this is outside our SL problem range. As in earlier work, we have a sort of hybrid world with μ/η and ν/ζ kicking around.

Although I did not realize it on first reading, the zeros of E are significant when you write the polynomial part of E(u) in factored form $\Pi_i\ (\mu-\alpha_i)$. It turns out roots are in ± pairs so $\Pi_i\ (\mu^2-\alpha_i^2)$. This factored form plays a role below.

279. The Second Kind Lamé Functions

He arrives at the usual formula. I don't follow his method of finding it, but I think I can follow MF I page 524-525 where they talk about doing this in general for separated ODE equations Then for the simplest E functions (those with n=0 and n=1) he writes down specific elliptic style integrals for these second kind functions. It seems a bit strange that Hobson never uses the standard F or E functions, but that is fine, he just writes the integral itself. [Byerly does use the F notation I think]. He claims that the second-kind functions are always first and second kind elliptic integrals, never third kind.

280. Potential Problems for an Ellipsoid (473).

This section is quite clear. If on an ellipsoid you happen to have a Dirichlet potential that is the product $E(\mu)E(\nu)$ (indices suppressed), then you *at once* know the exact solution of both the interior and exterior Dirichlet problems! If your Dirichlet potential is some weighted sum of this EE, then of course your two Dirichlet problems are as shown in p 473 B. If the Dirichlet BV potential is written $f(\mu,\nu)$, this weighted sum has the form p 473 C. Hobson then rewrites the two Dirichlet problem solutions as p 474A where now he shows my $E_{n,p}(\rho)$ as $E_n^{(p)}(\rho)$. But what is going on here regarding "class" ? Well, remember that the 2n+1 values of p (his m) include the functions of all four classes, so class is handled correctly. Finally, he states the Fourier coefficient which appears in p 473C in p 474B. Notice that this is a two-variable Fourier transform gizmo. One might wonder where the weight function $(\mu^2-\nu^2)$ comes from. It must be that the Haar measure is $dV = (\mu^2-\nu^2)d\mu d\nu$, but I don't really have any "group theory" information at this point. The main issue is that things don't seem to "factorize".

Comment on SL separation. There *does* seem to be something new going on here in the μ,ν world which I am not used to. In sphericals, we think of $Y(\theta,\varphi)$ as the product of two separate 1D SL problems in θ and φ, and the Y factors into $A(\theta)B(\varphi)$. In each 1D world we have a self-adjoint ODE and there is some weight function Stak calls s(x), and this is what appears in the orthogonality integral of the eigenfunctions of the ODE. But in μ,ν coordinates, we don't get this same factorization idea, and the weight function seems to tangle both μ and ν together. We don't have separate 1D SL problems in μ and in ν! I cannot think of another situation where this arises. Is the Lamé equation self-adjoint? When we solve Lamé for $E_{n,p}(\mu)$, what causes n and p to be quantized? It is the requirement to truncate the Frobenius power series so you get actual solution functions I guess on (h,k) for variable μ. So you would think there would be a complete set of eigenfunctions on this interval without regard to ν.

I did a quick web scan and learned that I am right, there IS something new going on here. Eg,

http://jlms.oxfordjournals.org/content/s2-6/4/705.extract

The author of this 1973 paper says that the Lamé problem and other problems are actually not simple SL problems where you have one eigenvalue parameter λ, but are in fact "two parameter" SL problems with two eigenvalues λ_1 and λ_2. This author describes such a system this way,

1. Introduction

Consider the two-parameter eigenvalue problem for ordinary differential equations defined by the system

$$\frac{d^2 y_r}{dx_r^2} + \{\lambda_1 p_{r1}(x_r) + \lambda_2 p_{r2}(x_r) + q_r(x_r)\} y_r = 0, \qquad (1.1)$$

$$\cos\alpha_r\, y_r(0) - \sin\alpha_r \frac{dy_r(0)}{dx_r} = 0,$$

$$\cos\beta_r\, y_r(1) - \sin\beta_r \frac{dy_r(1)}{dx_r} = 0, \qquad r = 1, 2, \qquad (1.2)$$

where

$$p_{ij}(x_i) \in C[0, 1], \qquad i, j = 1, 2,$$

$$q_i(x_i) \in C[0, 1], \qquad i = 1, 2,$$

and

$$\det\{p_{rs}\}_{r,s=1}^{2} > 0 \qquad \text{for all } x = (x_1, x_2) \in [0, 1] \times [0, 1]. \qquad (1.3)$$

The first is the two ODE's (like our Lamé ones for μ and ν) where you see the two eigenvalues appearing, and p and q functions of the usual SL world. Then next lines are the usual homo BC's. The interval is (0,1).

The problem defined by (1.1) and (1.2) has a history almost as long as the now well established one-parameter Sturm–Liouville problem. Special forms of the problem have been well studied, e.g. boundary value problems for the Mathieu equation, Lamé equation, spheroidal wave equation etc.,† but the general problem has received little attention. We may point out that the system (1.1), (1.2) is readily

The purpose of this paper is two-fold; firstly to establish completeness properties of the eigenfunction products $y_1(x_1)\, y_2(x)$ and to deduce an expansion theorem, and

The general subject here is **"multi-parameter Sturm Liouville problems"**. So perhaps the two "parameters" are in fact n(n+1) and p in the Lamé case. The endpoints of interval (h,k) or (0,h) are singular points, so you do have to call upon the Stak *singular* SL theory, and somehow it must be that you cannot come up with a complete set of eigenfunctions that factorizes. Something different must happen, and this is beyond the scope of my Stak reading.

So OK, this is a "different world", and Hobson is showing how things actually work. That is why we keep seeing integrals of FOUR E functions instead of the expected TWO.

281. Expanding a function on Lamé products: the example of $P_n(\cos\gamma)$

In this section Hobson does an example. He takes the usual $P_n(\cos\gamma)$ [for which we know the Legendre expansion in terms of $Y_{nm}(\theta,\varphi)$ and $Y_{nm}(\theta',\varphi')$ see Stak A.11 p 398] and he comes up with a 4-E expansion of the thing bottom page 475. The result is quite simple. He assumes the expansion p 475A and wants to find the coefficients. We know this thing is right because we are within the n manifold of the

Hilbert Space of functions. I am not clear where equation B is coming from, it seems unfamiliar (does not match my SO(3) stuff), so I am not following his steps, but OK, we have a simple example worked out.

282. Ellipsoidal harmonics expressed in Cartesian coordinates

Before starting, how would I express the E functions in terms of x,y,z? One trick would be to use the inversion of the conicals (which is similar that for the ellipsoidals)

Inverse conical:

$r = \sqrt{x^2+y^2+z^2}$ $\qquad\qquad$ $s^2 = k^2 - h^2$

$\nu = \sqrt{W - \sqrt{W^2 - 4h^2k^2(x/r)^2}} \;/\sqrt{2}$ $\qquad\qquad$ $W = k^2(x/r)^2 + s^2(y/r)^2 + h^2$

$\mu = h\,k(x/r)\,(1/\nu)$

but this is pretty messy. The forward ellipsoidals are probably more helpful:

$x = \rho\mu\nu/(hk)$ $\qquad\qquad$ // $\rho, \mu, \nu = \xi_1, \xi_2, \xi_3$

$y = \sqrt{\rho^2-h^2}\sqrt{\mu^2-h^2}\sqrt{h^2-\nu^2}/(hs)$ $\qquad\qquad$ $s^2 = k^2-h^2$

$z = \sqrt{\rho^2-k^2}\sqrt{k^2-\mu^2}\sqrt{k^2-\nu^2}/(ks)$

Then we can see at once that

$x = K_0(\rho)\,K_1(\mu)\,K_1(\nu)$

$y = iM_1(\rho)\,iM_1(\mu)\,M_1(\nu)/(hs)$

$z = iM_1(\rho)\,M_1(\mu)\,iL_1(\nu)/(ks)$

$xy = \rho\mu\nu/(hk) * \sqrt{\rho^2-h^2}\sqrt{\mu^2-h^2}\sqrt{h^2-\nu^2}/(hs)$

$= \rho\sqrt{\rho^2-h^2} * \mu\sqrt{\mu^2-h^2} * \nu\sqrt{h^2-\nu^2} * 1/(h^2ks)$

$= M_2(\rho) * M_2(\mu) * iM_2(\nu) * 1/(h^2ks)$

We know from the n-manifold with Y's what the Y's are in Cartesians, so the EE have to be linear combinations of these.

So OK, a preliminary look says it is not obvious how to expand EE into x,y,z.

Hobson then starts into **the Niven theory**. I was quite confused at first (just as I was trying to read this in W&W), but then I grokked it. This is explained in "Using Conical and Spherical Coordinates to explore Lamé functions.doc". I will here give a little review of what I do there.

(a) a new way to write $E_{n,p}(\mu)$ and $E_{n,p}(\mu)\,E_{n,p}(\nu)$

This is a little review of stuff in "Using Conical and Spherical Coordinates to explore Lamé functions".

General form for E. First, I show in the doc in full detail that a single E function can be written in this manner

$$E_{n,p}(\mu) = \{1, \sqrt{k^2 - \mu^2}, \sqrt{h^2 - \mu^2}, \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2}\ \}\ [\ 1, \mu]\ \text{poly}_J(\mu^2, \text{by } 1) \qquad (*)$$

I=0 I=1 I=1 I=2
K L M N

where the polynomial degree J is determined in a somewhat complicated manner as follows:

n-I = even	=>	J = (n-I)/2	and	select 1 in [1, μ]
n-I = odd	=>	J = (n-I-1)/2	and	select μ in [1, μ]

where I = 0,1,1,2 label the classes K,L,M,N (see radical factor labels above). For a given E function, you have to make two choices in the form above. First, you select the radical factor appropriate for the class of your E function. Second, you select a factor from [1, μ] according to whether n-I is even or odd. As examples, recall these two Lamé functions

$K_1(\mu) = \mu$ I = 0 n = 1 => n-I = 1 = odd => J = (n-I-1)/2 = 0 so no "poly" part.

$M_2(\mu) = \mu\sqrt{k^2 - \mu^2}$ I= 1 n = 2 => n-I = 1 = odd => J = (n-I-1)/2 = 0 so no "poly" part.

[This second example shows why it would be wrong to replace {...}[1, μ] with this single bracket

$$\{1, \mu, \sqrt{k^2 - \mu^2}, \sqrt{h^2 - \mu^2}, \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2}\ \}$$

which strangely is how things appear in p 476 A, see below.]

Roots occur in ± pairs. Next, I show that the roots of $\text{poly}_J(\mu^2, \text{by } 1)$ must occur in $\pm\alpha_i$ pairs, hence we have

$$\text{poly}_J(\mu^2, \text{by } 1) = \kappa \prod_{i=1}^{J} (\mu^2 - \alpha_i^2)$$

so that our E function form above then becomes

$$E_{n,p}(\mu) = \{\ 1, \sqrt{k^2 - \mu^2}, \sqrt{h^2 - \mu^2}, \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2}\ \}\ [\ 1, \mu]\ \kappa \prod_{i=1}^{J} (\mu^2 - \alpha_i^2)$$

with of course the same two bracket choices and J value as shown above. Here is a little table showing the J values in the various cases

		J	2J
n even	K	n/2	n
	N	(n-2)/2	n-2
	L,M	(n-2)/2	n-2
n odd	K	(n-1)/2	n-1
	N	(n-3)/2	n-3
	L,M	(n-1)/2	n-1

Below p 476 A Hobson neglects to mention the (n-3)/2 case, but he does mention it in mid page 477.

General form for EE . Given the above, we can of course write out a product of two E functions this way:

$$E_{n,p}(\mu)\, E_{n,p}(\nu) \;=\; \kappa^2$$
$$\{\, 1 \,,\, \sqrt{k^2-\mu^2} \,,\, \sqrt{h^2-\mu^2} \,,\, \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2} \,\}\ [\, 1, \mu]$$
$$\{\, 1 \,,\, \sqrt{k^2-\nu^2} \,,\, \sqrt{h^2-\nu^2} \,,\, \sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2} \,\}\ [\, 1, \nu]$$
$$\Pi_{i=1}^{J}\ (\mu^2-\alpha_i^2)\,(\nu^2-\alpha_i^2)$$

where we select one of the four columns in {...} for K,L,M,N and one of the columns in [...] based on the value of n as noted above. Remember that conical atoms have the form $r^n\, E_{n,p}(\mu)\, E_{n,p}(\nu)$.

Comment on Dimensions: (long overdue)

The ellipsoidal x,y,z equations are

$$x = \rho\mu\nu/(hk) \qquad // \ \rho, \mu, \nu = \xi_1, \xi_2, \xi_3$$
$$y = \sqrt{\rho^2-h^2}\,\sqrt{\mu^2-h^2}\,\sqrt{h^2-\nu^2}\,/\,(hs) \qquad s^2 = k^2-h^2$$
$$z = \sqrt{\rho^2-k^2}\,\sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2}\,/(ks)$$

Since we think of h and k as distances (focal distances of an ellipsoid), it follows that we must think of μ and ν and ρ all having dimensions of length. This is obvious for ρ which is the semi-major axis of an ellipsoid, but less obvious for μ and ν. [I just went back and figured out what these were. For the $\mu = \xi_2$ single sheet bloid, μ is the semi-major axis of the elliptical neck. For the $\nu = \xi_3$ double-sheet bloid, ν is the distance from origin to the tip of either of these sheets.]

h,k,ρ,μ,ν,x,y,z all have dimensions L^1, they are all true distances

If we look back at our general forms for the K,L,M,N functions as series in "Using Conical and Spherical Coordinates to explore Lamé functions.doc", we find that all of the $E_{n,p}(\mu)$ functions (assuming no added out front constants) are of dimension L^n . This is also shown in a comment above. A conical solution of the form $r^n\, E_{n,p}(\mu)\, E_{n,p}(\nu)$ must have dimensions L^{3n} (and the same for the ellipsoidal EEE).

(b) a new way to write $E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$: { 1 x y z xy xz yz xyz } Π (roots)

Here we develop this "half way to Cartesians" expression for EEE.

Let's just tack the above E formula together three times and summarize by saying

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \;=\; \kappa^3$$
$$\{\, 1 \,,\, \sqrt{k^2-\rho^2} \,,\, \sqrt{h^2-\rho^2} \,,\, \sqrt{h^2-\rho^2}\ \sqrt{k^2-\rho^2} \,\}\ [\, 1, \rho]$$
$$\{\, 1 \,,\, \sqrt{k^2-\mu^2} \,,\, \sqrt{h^2-\mu^2} \,,\, \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2} \,\}\ [\, 1, \mu]$$
$$\{\, 1 \,,\, \sqrt{k^2-\nu^2} \,,\, \sqrt{h^2-\nu^2} \,,\, \sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2} \,\}\ [\, 1, \nu]$$

$$\Pi_{i=1}^{J}\ (\rho^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)\,(\nu^2-\alpha_i^2) \qquad (*)$$

where J is as noted above, and where you will select the same term in each {...} and [..]. There are thus only 4x2 = 8 possible selections you can make. Recall now the ellipsoidal x,y,z equations

$x = \rho\mu\nu/(hk)$ // $\rho, \mu, \nu = \xi_1, \xi_2, \xi_3$

$y = \sqrt{\rho^2-h^2}\,\sqrt{\mu^2-h^2}\,\sqrt{h^2-\nu^2}\,/\,(hs)$ $s^2 = k^2-h^2$

$z = \sqrt{\rho^2-k^2}\,\sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2}\,/(ks)$

From these we may compute up some of our triple products EEE:

$K_0\ K_0\ K_0$	1
$L_1\ L_1\ L_1$	$\sqrt{h^2-\rho^2}\,\sqrt{h^2-\mu^2}\,\sqrt{h^2-\nu^2} = y * ihs$
$M_1\ M_1\ M_1$	$\sqrt{k^2-\rho^2}\,\sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2} = z * iks$
$N_2\ N_2\ N_2$	$\sqrt{h^2-\rho^2}\ \sqrt{k^2-\rho^2}\,\sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\,\sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2} = (y * ihs)(z * iks)$ $= yz * (-hks^2)$

$K_1\ K_1\ K_1$	$\rho\mu\nu = x * hk$
$L_2\ L_2\ L_2$	$\rho\mu\nu\,\sqrt{k^2-\rho^2}\,\sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2} = (x * hk)(z * iks) = xz * (ihk^2s)$
$M_2\ M_2\ M_2$	$\rho\mu\nu\,\sqrt{h^2-\rho^2}\,\sqrt{h^2-\mu^2}\ \sqrt{h^2-\nu^2} = (x * hk)(y * ihs) = xy * (ih^2ks)$
$N_3\ N_3\ N_3$	$\rho\mu\nu\,\sqrt{h^2-\rho^2}\ \sqrt{k^2-\rho^2}\ \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\ \sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2}$ $= (x * hk)(y * ihs)(z * iks) = xyz * (-h^2k^2s^2)$

where the second four come from choosing μ in $[1,\mu]$, etc. Notice that the dimensions are always L^{3n}. This allows us to rewrite the triple E product this way:

This allows us to rewrite the triple E product this way: (I will call this a "half- Niven Formula")

$E_{n,p}(\rho)\ E_{n,p}(\mu)\ E_{n,p}(\nu)$

$= \{\ 1\ \ x\ y\ z\ xy\ xz\ yz\ xyz\ \}\ \kappa' \Pi_{i=1}^{J}\,[(\rho^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)]$

K K L M M L N N

e o o o e e e o // values of n for which this form applies (n even or odd)

// for a given value of n, there are only 4 types of solutions, not 8 types

where I show the class associated with each of the 8 possibilities and κ' is some norm constant. This **constant κ'** is different of course for each {...} case and for each case has different dimensions! It is a function of h and k and we could figure it out for each case if we wanted.

(c) a second way to write $E_{n,p}(\rho)\ E_{n,p}(\mu)\ E_{n,p}(\nu)$: full Cartesian coordinates

I then show that you can write the root product in this manner

$$(\rho^2-\rho_s^2)(\mu^2-\rho_s^2)(\mu^2-\rho_s^2) = D(\rho_s^2)\,[\,x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1]$$

$$D(\rho_s^2) = \rho_s^2\ (\rho_s^2-h^2)(\rho_s^2-k^2)$$

where ρ_s is some (non-zero) root of E(x) (called α_i above). These roots are just numbers, so the quantity D is just a constant. This then leads to the famous result (I will call this a "full-Niven formula")

$E_{n,p}(\rho)\,E_{n,p}(\mu)\,E_{n,p}(\nu)$ // $\kappa''=\kappa'\,(\Pi_{s=1}^{J}\,D(\rho_s^2))$

$= \{\ 1\ \ x\ y\ z\ xy\ xz\ yz\ xyz\ \}\ \kappa''\ \{\ \Pi_{s=1}^{J}\ [\,x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1]\ \}$

K K L M M L N N

e o o o e e e o // values of n for which this form applies (n even or odd)

which is Hobson p 477 A and which shows how to write ALL triple E products *in Cartesian coordinates*, which is in fact the title of this entire Hobson section, see page 476 top.

Aside on degree of EE: The atoms for sphericals and conicals are $r^nY_{n,m}(\theta,\varphi)$ and $r^n\,E_{n,p}(\mu)\,E_{n,p}(\nu)$. It is because they share the same coordinate r (with the same meaning), that we were able to say that the functions $E_{n,p}(\mu)\,E_{n,p}(\nu)$ must be linear combinations of the $Y_{n,m}(\theta,\varphi)$ with the same n. Of course we know that dim(EE) = L^{2n} whereas dim(Y) = L^0, so the coefficients must have dim(coeff) = L^{2n}. These coefficients are going to be functions of h and k.

Lengthy Aside on degree of EEE:

(1) The atoms for sphericals and conicals are $r^nY_{n,m}(\theta,\varphi)$ and $E_{n,p}(\rho)E_{n,p}(\mu)E_{n,p}(\nu)$. Since we don't have the same shared coordinate ($r \neq \rho$), we *cannot* conclude that $E_{n,p}(\rho)E_{n,p}(\mu)E_{n,p}(\nu)$ is a linear combination of the $r^n\,Y_{n,m}(\theta,\varphi)$ with the same n.

(2)We know from work elsewhere that (homogeneous, and coefficients depend on n and m)

$r^nY_{n,m}(\theta,\varphi) = \mathrm{Poly}_n(x,y,z;\,n,m)$.

(3) We can examine the "order" of the above EEE product with this little table

		J	2J	{}	dim{}	2J+dim{}
n even	K	n/2	n	1	0	n
	N	(n-2)/2	n-2	yz	2	n
	L,M	(n-2)/2	n-2	xy,xz	2	n
n odd	K	(n-1)/2	n-1	x	1	n
	N	(n-3)/2	n-3	xyz	3	n
	L,M	(n-1)/2	n-1	y,z	1	n

The 2J column gives the order in (x,y,z) of the $\Pi_{s=1}^{J}$ factor above, while the dim{} column gives the order of the (legal for given n parity) primitive selected from {...} . We see that the total order is always n, so we conclude that

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu) = \text{poly}_n(x,y,z)$$

This shows that $E_n E_n E_n$ is in fact $\text{poly}_n(x,y,z)$, meaning the highest degree term has degree n . This, the "degree" indicated by the n subscript on the E functions matches the "degree" of the x,y,z polynomial which that EEE is equal to. [From the Cartesian form of EEE, with its -1 term, we can see that the terms will have degrees that jump down 2 at a time, so this is a "by 2" poly.]

At the same time, we know the dimension of $E_n E_n E_n$ is 3n. Dimension and degree are not the same thing, and all we can really say is degree ≤ dimension.

(4) We can certain do the following expansion since we have a complete set on the RHS,

$$E_{n1,p}(\rho)\, E_{n1,p}(\mu)\, E_{n1,p}(\nu) = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} a_{n,m}\, r^n\, Y_{n,m}(\theta,\varphi)$$

Since the $r^n Y_{n,m}(\theta,\varphi) = \text{Poly}_n(x,y,z)$, we cannot expect to have terms in the n sum with n' > n1 since this would introduce terms of degree higher than $\text{Poly}_n(x,y,z)$. So we may tentatively conclude that

$$E_{n1,p}(\rho)\, E_{n1,p}(\mu)\, E_{n1,p}(\nu) = \Sigma_{n=0}^{n1} \Sigma_{m=-n}^{n} a_{n,m}\, r^n\, Y_{n,m}(\theta,\varphi)$$

(5) However, in all the *lowest* cases listed above,

{ 1 x y z xy xz yz xyz }

we DO find that in fact EEE is a linear combination of the $r^n\, Y_{n,m}(\theta,\varphi)$ with the same n. These are the cases which have no extra factor $\Pi_{s=1}^{J}$ [...] (ie, the factor is just 1) . We have already seen these lowest cases in our lists above.

(6) From the Cartesian form of EEE, with its -1 term, we can see that the terms will have degrees that jump down 2 at a time, so this is a "by 2" poly. We claim in (3) above that EEE = $\text{poly}_n(x,y,z)$, so we might write this as EEE = $\text{poly}_n(x,y,z$; by 2). This is what Hobson later says at the start of Section 286 on page 481. For this same reason, in the Y sum mentioned in 4 above, the Y sum also steps down by 2.

Introduction of the θ and Θ and a,b,c symbols

Hobson now introduces some new symbols. First, recall our basic ellipsoid:

$$x^2/\rho^2 + y^2/(\rho^2-h^2) + z^2/(\rho^2-k^2) = 1$$

Suppose we make this replacement of the ellipsoid *label* ρ^2 with θ,

$$\rho^2 = a^2 + \theta$$

where for the moment a is just a constant. Then our ellipsoid looks like this

$$x^2/(a^2+\theta) + y^2/(a^2+\theta-h^2) + z^2/(a^2+\theta-k^2) = 1$$

Notice that $\theta = -a^2 + \rho^2$ ranges from $(0,\infty)$ when ρ is indeed an ellipsoid label. But when we think of ρ as a generic ellipsoidal coordinate, we know that ρ lies in the range $(0,\infty)$ so θ lies in $(-a^2,\infty)$.

Next, we define new symbols b and c according to

$$b^2 = a^2-h^2 \qquad h^2 = a^2-b^2$$
$$c^2 = a^2-k^2 \qquad k^2 = a^2-c^2$$

Note that $a^2 > b^2 > c^2$. Then our ellipsoid becomes

$$x^2/(a^2+\theta) + y^2/(b^2+\theta) + z^2/(c^2+\theta) = 1$$

In this manner we have just reparameterized our ellipse from label ρ^2 to label θ. Remember this is a family of confocal ellipses, where now θ is the label. When $\theta = 0$ we get a "**fundamental ellipsoid**" :

$$x^2/(a^2) + y^2/(b^2) + z^2/(c^2) = 1$$

and we now see the significance of the constants a,b,c: they are the semi-major axes of our fundamental ellipsoid, where a is the largest and c is the smallest. [I think we are **free to set a** to any value we want, and then b and c are determined by h and k above. But **another way** to think of it is that we start off from scratch and specify three numbers $a^2 > b^2 > c^2$ and this in turn determines h and k. If we double the size of a^2, it is not true that b^2 and c^2 double. But if we double all three a^2, b^2, c^2 then h^2 and k^2 double.]

If we set $\rho = \rho_s$, where ρ_s is a non-zero positive root of some E function, then this is just some value of ρ, which implies some value of θ according to $\rho_s^2 = a^2 + \theta_s$. So we can say

$$[\, x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1] \;=\; [\, x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s) - 1 \,]$$

and so our EEE Cartesian formula becomes // full-Niven formula for ellipsoidals.

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$$

$$= \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\}\ \kappa''\ \{\, \Pi_{s=1}^{J} \;\; [\, x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s) - 1 \,] \,\}$$
$$K \;\; K \;\; L \;\; M \;\; M \;\; L \;\; N \;\; N$$

// Hobson p 477 B

where $\theta_s = \rho_s^2 - a^2$ and ρ_s is a positive root of $E(\rho)$. We now define

$$\Theta_s \;\equiv\; [\, x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s) - 1 \,] = \Theta_s(x,y,z;\, \theta_s;\, a,b,c)$$

and then EEE becomes a more compact thing

$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{ \underset{K}{1} \ \underset{K}{x} \ \underset{L}{y} \ \underset{M}{z} \ \underset{M}{xy} \ \underset{L}{xz} \ \underset{N}{yz} \ \underset{N}{xyz} \} \ \kappa'' \ \{ \Pi_{s=1}^{J} \Theta_s(x,y,z;\theta_s;a,b,c) \}$ // Hobson p 477 C

$\sim \ \{ 1 \ x \ y \ z \ xy \ xz \ yz \ xyz \} \Theta_1\Theta_2....\Theta_J$

and he refers to my J as his "m".

Quote from W&W concerning Niven and "species". So finally I understand this formula which appears so mysteriously in W&W, and I quote from there:

23·2. *The four species of ellipsoidal harmonics.*

A consideration of the expressions for spherical harmonics in factors ndicates that there are four possible species of ellipsoidal harmonics to be nvestigated. These are included in the scheme

$$\left\{ \begin{matrix} & x, & yz, & \\ 1, & y, & zx, & xyz \\ & z, & xy, & \end{matrix} \right\} \Theta_1\Theta_2 \dots \Theta_m,$$

here one or other of the expressions in { } is to multiply the product $_1\Theta_2 \dots \Theta_m$.

I remember looking long and had to find what "m" was, W&W are silent, and I see now why that is, because it is a little complicated.

W&W then claim that the four columns correspond to the four "species"

If we write for brevity

$$\Theta_1\Theta_2 \dots \Theta_m = \Pi(\Theta),$$

ıy harmonic of the form $\Pi(\Theta)$ will be called *an ellipsoidal harmonic of the rst species.* A harmonic of any of the three forms* $x\Pi(\Theta)$, $y\Pi(\Theta)$, $z\Pi(\Theta)$ ill be called *an ellipsoidal harmonic of the second species.* A harmonic of ıy of the three forms* $yz\Pi(\Theta)$, $zx\Pi(\Theta)$, $xy\Pi(\Theta)$ will be called *an ellipsoidal rmonic of the third species.* And a harmonic of the form $xyz\Pi(\Theta)$ will be lled *an ellipsoidal harmonic of the fourth species.*

We conclude that W&W's **species** are not the same as Hobson's **classes**. In fact we have

	1	x	y	z	xy	xz	yz	xyz
species	1	2	2	2	3	3	3	4
class	K	K	M	L	L	M	N	N

OK, no big deal. Byerly carefully avoids using the words species, classes or even types. He just says K,L,M,N !

The G and $\mathcal{G}$ functions

Next. We know there are 2n+1 functions in the group $E_{n,p}(\mu)$. Thus, we can define 2n+1 functions which are indeed the "ellipsoidal harmonics", which is to say, what we earlier called "normal functions",

$$G_n^s(\rho,\mu,\nu) = E_{n,s}(\rho)E_{n,s}(\mu)E_{n,s}(\nu) \qquad s = 1,2...2n+1$$

He calls these "internal" since they are first kind functions. The corresponding second kind solutions he refers to as $\mathcal{G}_n^s(\rho,\mu,\nu)$. He says they are "of degree n", which is the in first sense noted above.

Hobson claims a formula for getting $\mathcal{G}_n^s$ as a multiple of G_n^s, I will not attempt to derive this formula on this first pass. I presume he means this:

$$\mathcal{G}_n^s(\rho,\mu,\nu) = F_{n,s}(\rho)F_{n,s}(\mu)F_{n,s}(\nu) \qquad s = 1,2...2n+1$$

but he is not writing things out at this point.

This Section 282 comes to a sudden and unexpected end at this point!

283. Conicals in Cartesian coordinates

This section opens with a huge typo. He writes the x,y,z conical equations completely wrong. Ignoring this, he claims that you go through the same sequence of steps that we did with the ellipsoidals, and we end up with the same general form for the "normal functions" except there is no -1 term in the product factor, which follows since there is no -1 in the conicoid equations for the cones. So I have no trouble at all accepting the general form of his result. The J number comes out being the same. I think I could derive all this stuff if I had to.

(a) a new way to write $E_{n,p}(\mu)$ $E_{n,p}(\nu)$: { 1 x/r y/r z/r xy/r^2 xz/r^2 yz/r^2 xyz/r^3 } Π (roots)

But I will do it right now. We start with the x,y,z equations for conicals (using Hobson's x and z)

$$z = r\sqrt{k^2-\mu^2}\sqrt{k^2-\nu^2}/(ks)$$
$$y = r\sqrt{\mu^2-h^2}\sqrt{h^2-\nu^2}/(hs) \qquad s = \sqrt{k^2-h^2} \qquad 0 < \nu < h < \mu < k$$
$$x = r\,\mu\nu/(hk)$$

We then start with our EEE table above and edit it down to an EE table

$K_0\ K_0$	1
$L_1\ L_1$	$\sqrt{h^2-\mu^2}\sqrt{h^2-\nu^2} = y * ihs/r$
$M_1\ M_1$	$\sqrt{k^2-\mu^2}\sqrt{k^2-\nu^2} = z * ks/r$
$N_2\ N_2$	$\sqrt{h^2-\mu^2}\sqrt{k^2-\mu^2}\sqrt{h^2-\nu^2}\sqrt{k^2-\nu^2} = (y * ihs/r)(z * ks/r)$
	$= yz * (-hks^2/r^2)$

$K_1\ K_1$ $\quad \mu\nu = x * hk/r$

$L_2\ L_2$ $\quad \mu\nu \sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2} = (x * hk/r)(z * ks/r) = xz * (hk^2s/r^2)$

$M_2\ M_2$ $\quad \mu\nu \sqrt{h^2 - \mu^2}\sqrt{h^2 - \nu^2} = (x * hk/r)(y * ihs/r) = xy * (ih^2ks/r^2)$

$N_3\ N_3$ $\quad \mu\nu \sqrt{h^2 - \mu^2}\sqrt{h^2 - \nu^2}\sqrt{k^2 - \nu^2}\sqrt{k^2 - \mu^2}$

$= (x * hk/r)(y * ihs/r)(z * ks/r) = xyz * (ih^2k^2s^2/r^3)$

The location of the various red functions is exactly the same as in the EEE table, so that means our little underlying letters below will be the same. Except we now have factors or r as shown.

We then quote this result from above

$$E_{n,p}(\mu)\, E_{n,p}(\nu) = \kappa^2$$
$$\{ 1, \sqrt{k^2 - \mu^2}, \sqrt{h^2 - \mu^2}, \sqrt{h^2 - \mu^2}\sqrt{k^2 - \mu^2} \}\ [1, \mu]$$
$$\{ 1, \sqrt{k^2 - \nu^2}, \sqrt{h^2 - \nu^2}, \sqrt{h^2 - \nu^2}\sqrt{k^2 - \nu^2} \}\ [1, \nu]$$
$$\Pi_{i=1}^{J}\ (\mu^2 - \alpha_i^2)(\nu^2 - \alpha_i^2)$$

We then install our EE combinations from above, absorbing stuff into the constants, so we get

$$E_{n,p}(\mu)\, E_{n,p}(\nu)$$

$$= \{ 1 \quad x/r \quad y/r \quad z/r \quad xy/r^2 \quad xz/r^2 \quad yz/r^2 \quad xyz/r^3 \} \quad \kappa' \Pi_{i=1}^{J} [(\mu^2 - \alpha_i^2)(\nu^2 - \alpha_i^2)]$$

K	K	L	M	M	L	N	N	
e	o	o	o	e	e	e	o	// n parity for which term applies

Now, the equations which describe either of the elliptic cone families is the same as that of the ellipsoid but you replace the RHS 1 with a 0. See "conical coordinates.doc" for details.

Note: unlike the ellipsoidal case for EEE, here for EE the bracket {....} contains factors or r.

We now wander off and take a look at Section (c) of " Using Conical and Spherical Coordinates to explore Lamé functions.doc" to see what happens next. Let's just copy paste and edit:

(b) a second way to write $E_{n,p}(\mu)\, E_{n,p}(\nu)$

If we take our ellipsoid equation and move the 1 to the left, we get Hobson p 476 being 0,

$$[] \equiv [\ x^2/\rho_s^2 + y^2/(\rho_s^2 - h^2) + z^2/(\rho_s^2 - k^2) - 0\ ] = 0 \qquad \text{// 1 removed}$$

We start here now with the equation for a family of elliptic cones. If x,y,z lies at the intersection of a μ cone and a ν cone, then $\rho_s = \mu$ and $\rho_s = \nu$ both satisfy the above equation.

Consider a point (x,y,z) in Cartesian space. It lies on three orthogonal surfaces whose labels are ρ,μ,ν -- the ellipsoid and two bloids. We know from previous work that ρ^2,μ^2,ν^2 are the three roots of the cubic equation that []=0 implies for variable ρ_s^2 (which MF called ξ^2). We can do it right here. Rewrite [] as follows, by combining the three terms over D

$$[] = N/D = [x^2(\rho_s^2-h^2)(\rho_s^2-k^2) + y^2\rho_s^2(\rho_s^2-k^2) + z^2\rho_s^2(\rho_s^2-h^2) - 0]/[\rho_s^2(\rho_s^2-h^2)(\rho_s^2-k^2)]$$

// last term in N was removed (it had ρ^6)

In N, the coefficient of the leading term ρ_s^4 is just $x^2 + y^2 + z^2 = r^2$, so we have

$$N = r^2[\rho_s^4 + A\rho_s^2 + B] = r^2[(\rho_s^2-\mu^2)(\rho_s^2-\nu^2)]$$

where we have factored the quadratic numerator. The roots must be as shown because we just said 4" up that these root values cause [] = 0. So we have just shown that

$$[x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 0] = r^2[(\rho_s^2-\mu^2)(\rho_s^2-\nu^2)]/D_s$$

=>

$$(\rho_s^2-\mu^2)(\rho_s^2-\nu^2) = (D_s/r^2)[x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2)] \qquad (*)$$

Now recall from our previous section that we had

$$E_{n,p}(\mu)\,E_{n,p}(\nu)$$

$$= \{\underset{K}{1}\ \underset{K}{x/r}\ \underset{L}{y/r}\ \underset{M}{z/r}\ \underset{M}{xy/r^2}\ \underset{L}{xz/r^2}\ \underset{N}{yz/r^2}\ \underset{N}{xyz/r^3}\}\ \kappa' \Pi_{i=1}^{J}[(\mu^2-\alpha_i^2)(\nu^2-\alpha_i^2)]$$

So let's "rename" our roots from α_i to ρ_s and we have

$$E_{n,p}(\mu)\,E_{n,p}(\nu)$$

$$= \{\underset{K}{1}\ \underset{K}{x/r}\ \underset{L}{y/r}\ \underset{M}{z/r}\ \underset{M}{xy/r^2}\ \underset{L}{xz/r^2}\ \underset{N}{yz/r^2}\ \underset{N}{xyz/r^3}\}\ \kappa' \Pi_{s=1}^{J}[(\mu^2-\rho_s^2)(\nu^2-\rho_s^2)]$$

where $\pm\rho_s$ for s = 1..J are the roots of $E_{n,p}(x)$ and are just functions of h and k and the quantized p value. We now replace using (*) to get

$$E_{n,p}(\mu)\,E_{n,p}(\nu)$$

$$= \{\underset{K}{1}\ \underset{K}{x/r}\ \underset{L}{y/r}\ \underset{M}{z/r}\ \underset{M}{xy/r^2}\ \underset{L}{xz/r^2}\ \underset{N}{yz/r^2}\ \underset{N}{xyz/r^3}\}\ \kappa' \Pi_{s=1}^{J}[(D_s/r^2)[x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2)]]$$

We can at this point write

$$\Pi_{s=1}^{J}[(D_s/r^2)[\ldots]] = (\Pi_{s=1}^{J}D_s)(\Pi_{s=1}^{J}(1/r^2))(\Pi_{s=1}^{J}[\ldots]) = r^{-2J}(\Pi_{s=1}^{J}D_s)(\Pi_{s=1}^{J}[\ldots])$$

Now D_s is a function of the root values and h and k, so it is a constant and we absorb it into κ'. Then

$E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{\ 1 \quad x/r \quad y/r \quad z/r \quad xy/r^2 \quad xz/r^2 \quad yz/r^2 \quad xyz/r^3 \ \}\ r^{-2J}\ \kappa''\ \Pi_{s=1}^{J}\ [\ x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2)]$

K K L M M L N N

(c) how to write r^n $E_{n,p}(\mu)$ $E_{n,p}(\nu)$: full Cartesian form

Finally, we can add r^n to get our actual conical harmonics:

$r^n\ E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{\ 1 \quad x/r \quad y/r \quad z/r \quad xy/r^2 \quad xz/r^2 \quad yz/r^2 \quad xyz/r^3 \ \}\ r^{n-2J}\ \kappa''\ \Pi_{s=1}^{J}\ [\ x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2)]$

K K L M M L N N

e o o o e e e o // n is even or odd, only 4 apply in each case

1 2 2 2 3 3 3 4 // W&W "species"

Now we want to carefully study the powers or r which appear in the above equation. Consider this table:

		I	n-I	J	2J	(s-1) {}	ord{}	2J+ord{}	n-2J	r^p	r^{tot}RHS
n even	K	0	even	(n-0)/2	n	1	0	n	0	0	0
	L,M	1	odd	(n-1-1)/2	n-2	xy,xz	2	n	2	-2	0
	N	2	even	(n-2)/2	n-2	yz	2	n	2	-2	0
n odd	K	0	odd	(n-0-1)/2	n-1	x	1	n	1	-1	0
	L,M	1	even	(n-1)/2	n-1	y,z	1	n	1	-1	0
	N	2	odd	(n-2-1)/2	n-3	xyz	3	n	3	-3	0

In every case, we find that if we combine r^{n-2J} with the attached power r^p inside {...}, we get r^0. This is what the last three columns of the above table are saying. Therefore, we could slide r^{n-2J} into the {...} and in each item replace $r^{n-2J}\ r^p = 1$. We then get this much simpler equation:

$r^n\ E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{\ 1 \quad x \quad y \quad z \quad xy \quad xz \quad yz \quad xyz \ \}\ \kappa''\ \Pi_{s=1}^{J}\ [\ x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2)]$

K K L M M L N N

e o o o e e e o // Hobson p 478 A

We can of course replace ρ_s^2 by θ_s as we did before in the ellipsoidal case to get

$r^n\ E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{\ 1 \quad x \quad y \quad z \quad xy \quad xz \quad yz \quad xyz \ \}\ \kappa''\ \Pi_{s=1}^{J}\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\theta_s) + z^2/(\ c^2+\theta_s)\ ]$

K K L M M L N N

e o o o e e e o // Hobson p 478 B

n-I = even => J = (n-I)/2 and select 1 in [1, μ]
n-I = odd => J = (n-I-1)/2 and select μ in [1, μ]

Comment: Let's look again at the last columns in our table above

		I	n-I	J	2J	(s-1) {}	ord{}	2J+ord{}	n-2J	r^p	r^{tot}RHS
n even	K	0	even	(n-0)/2	n	1	0	n	0	0	0
	L,M	1	odd	(n-1-1)/2	n-2	xy,xz	2	n	2	-2	0
	N	2	even	(n-2)/2	n-2	yz	2	n	2	-2	0
n odd	K	0	odd	(n-0-1)/2	n-1	x	1	n	1	-1	0
	L,M	1	even	(n-1)/2	n-1	y,z	1	n	1	-1	0
	N	2	odd	(n-2-1)/2	n-3	xyz	3	n	3	-3	0

The column ord{} in our table above can be identified with **"species"** s-1 where s = 1,2,3,4, exactly the species defined by W&W in our clip above. What we see is that (s-1) + (n-2J) = 0 which says that

$2J = n + s - 1$ $J = (n + s - 1)/2$ $n-2J = -(s-1)$

Comment: If either variable μ or ν equals one of the root values ρ_s , the LHS vanishes, and so does $\Pi_{s=1}^{J}[]$ because one of the factors [] = 0, the one with that ρ_s value. This follows from our result above

$$(\rho_s^2 - \mu^2)(\rho_s^2 - \nu^2) = (D_s/r^2)\,[\, x^2/\rho_s^2 + y^2/(\rho_s^2 - h^2) + z^2/(\rho_s^2 - k^2)] \qquad (*)$$

We don't yet know how to find what these ρ_s values are.

Notice that without the -1 present, our " $\Pi_{s=1}^{J}$ object is $Poly_{2J}(x,y,z)$. Therefore we conclude

$r^n E_{n,p}(\mu)\, E_{n,p}(\nu) = Poly_{2J+s}(x,y,z)$ where s is the *species* s = 1,2,3,4

So, if you want to test to see if something satisfies Laplace's equation, these are the forms you have to deal with. If we make the simplest choice {1}, then we would apply ∇^2 to $\Pi_{s=1}^{J}[...]$, for example.

---

284. Finding the roots.

I have noted above that finding the roots I first called α_i, and later ρ_s following Hobson, is a problem that must be dealt with. We are really talking about those eigenvalues of the separation constant originally called p. We now have it in another form called θ_s according to $\rho_s^2 = a^2 + \theta_s$, so we now regard our problem of finding the eigenvalues of θ_s. [Recall these are really the eigenvalues of p (indirectly), that are determined by the 2D non-separable μ,ν Sturm Liouville problem in conicals. For a given integer n,

only certain very specific p values yield Frobenius solutions which are finite because they truncate. The infinite series solutions all diverge.]

Hobson suggests we take our Cartesian solution form (the conicals one) and stuff that into the Laplace equation and see what happens. This what he does on page 479 where he selects just the "1" out of the {...} bracket as the simplest case. I think he is doing all this with the conicals form rather than the ellipsoidal form, since there is no -1. What he finds is that you end up with a set of m (my J) equations all equaling 0 in which the m unknowns θ_s appear. [Note that a particular "harmonic" has "m" factors each of which has one θ_s where s = 1,2...m, so a given "harmonic", regardless of which leading factor you take, has some set of θ_s going 1 to m. I think he is claiming that for the lowest harmonic, there is only one factor and you get a quadratic equation for θ_s.] Notice in this system of equations that the last column has coefficient 4, all other columns are 1.

If you take instead the "x" from the {...}, he says you get the same set of equations but the first column all has 3 instead of 1 in the numerator. Similar 3's appear in a systematic fashion as you take other {...} entries. I have not verified this, it seems reasonable. This reminds me of that determinant thing he first did back for the K functions.

The m equations are called the "characteristic equations".

Let's now try to verify Hobson's work on page 479. But first, we do some special cases to get the flavor.

The cases n = 0 and n = 1.

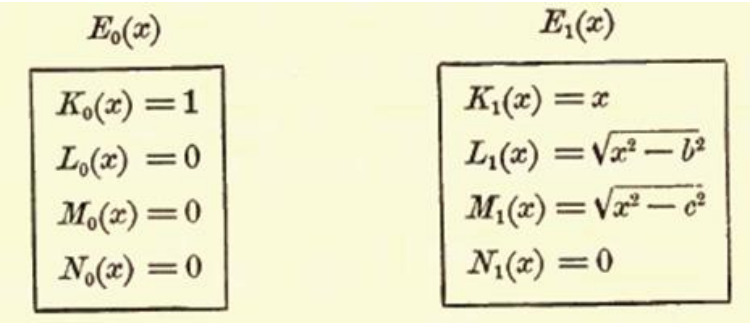
Recall our J rule, and recall that for a given J, we get J+1 Lamé functions since $a_{J+1}(p) = 0$:

n-I	= even	=>	J = (n-I)/2	and	select 1 in [1, μ]
n-I	= odd	=>	J = (n-I-1)/2	and	select μ in [1, μ]

The negative J cases don't exist, they are just bookkeeping artifacts.

For the case n = 0, the rule gives for K,L,M,N that J = 0,-1,-1,-2., so we get only the K_0 function.

For the case n = 1, the rule gives for K,LM,N that J = 0,0,0,-1 , so we get one function each of K,L,M,



The case n = 2 . // a familiar phrase !

We start by setting n = 2 in our conical form (that is, we look at harmonics of degree 2)

$r^n\ E_{n,p}(\mu)\ E_{n,p}(\nu)$

$= \{\ 1\ \ x\ \ y\ \ z\ \ xy\ \ xz\ \ yz\ \ xyz\ \}\ \kappa''\ \Pi_{s=1}^{J}\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\theta_s) + z^2/(\ c^2+\theta_s)\ ]$

K K L M M L N N

e o o o e e e o

Since n is even, our only four forms here are { 1 xy xz yz }. This is species 1 and 3. From our table above

		I	n-I	J	2J	{}	ord{}	2J+ord{}	n-2J	r^p	r^{tot}RHS
n even	K	0	even	(n-0)/2	2	1	0	2	0	0	0
	L,M	1	odd	(n-1-1)/2	0	xy,xz	2	2	2	-2	0
	N	2	even	(n-2)/2	0	yz	2	2	2	-2	0

For our three primitive functions xy,xz,yz there is no $\Pi_{s=1}^{J}$ factor and for these we have just

$r^2\ E_{2,p}(\mu)\ E_{2,p}(\nu)\ =\ \{\ xy\ \ xz\ yz\ \}\ \kappa''$

The one other function will have J = 1 so we will have

$r^2\ E_{2,p}(\mu)\ E_{2,p}(\nu)\ =\ \{\ 1\}\ \kappa''\ \ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\theta_s) + z^2/(\ c^2+\theta_s)\ ]$

This n = 2 harmonic has to satisfy Laplace. If we apply ∇^2 to this thing, we get

$\nabla^2[r^2\ E_{2,p}(\mu)\ E_{2,p}(\nu)]\ =\ \{\ 1\}\ \kappa''\ 2\ \ [\ 1/(a^2+\theta_s) + 1/\ (b^2+\theta_s) + 1/(\ c^2+\theta_s)\ ]\ = 0$

and this we require that

$[\ 1/(a^2+\theta_s) + 1/\ (b^2+\theta_s) + 1/(\ c^2+\theta_s)\ ] = 0$ // Hobson p 479 A

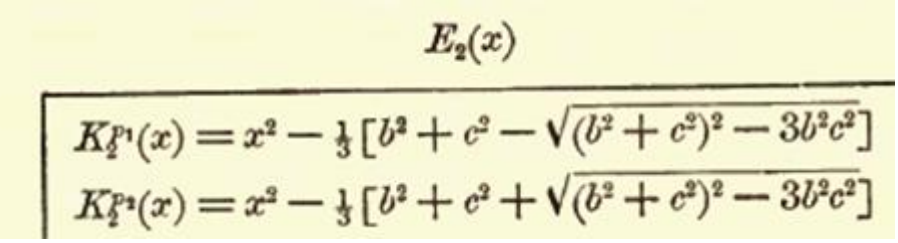
which we can solve for the two roots θ_s. Suppose the roots were θ_1 and θ_2. These are going with K functions and we would conclude that

$K_{2,p} = (\ a\mu^2 + b)\ = a(\mu^2 - \theta^2)$ set a = 1

$K_{2,p1} = (\ a\mu^2 + b)\ = (\mu^2 - \theta_1^2)$

$K_{2,p2} = (\ a\mu^2 + b)\ = (\mu^2 - \theta_2^2)$

and there are the guys you see in the Byerly table:



$$E_2(x)$$

$$K_2^{p_1}(x) = x^2 - \tfrac{1}{3}[b^2 + c^2 - \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$$

$$K_2^{p_2}(x) = x^2 - \tfrac{1}{3}[b^2 + c^2 + \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$$

So here is what our 2n+1 = 5 harmonics look like for n = 2:

$r^2\ E_{2,p}(\mu)\ E_{2,p}(\nu)\ =\ \{\ xy/r^2\ \ xz/r^2\ \ yz/r^2\ \}\ \kappa''$ 3

$r^2\ E_{2,p}(\mu)\ E_{2,p}(\nu)\ =\ \{\ 1\}\ \kappa''\ \ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ]$ 2

You don't need to know the roots to know the first 3, but you do for the second 2. Below we revisit the n = 2 case after learning a few more things.

The case n = 3 and also some general results.

I do this in **Appendix A** below. For n = 3 there are going to be 7 conical harmonics. One of them is this

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu)\ =\ \{\ xyz\ \}\ \kappa''$ N class

which we can see by inspection satisfies $\nabla^2 u = 0$. Another harmonic will have this form

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{x\}\ \kappa''\ \ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ]$ (*) K class

If we require that $\nabla^2 u = 0$, we show in Appendix A that we then require that

$3/(a_1^2+\theta_s)\ + 1/(a_2^2+\theta_s)\ + 1/(a_3^2+\theta_s)\ = 0$

which agrees with Hobson's p 479 D modified by his comments. This is also a quadratic in θ_s so there will be two roots θ_1 and θ_2 and we will then get 2 K class solutions of the form shown in (*) above.

We can see what will happen in the other cases. Another harmonic has this form

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{y\}\ \kappa''\ \ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ]$ (*) L class

and we can see from symmetry that our condition will come out being

$1/(a_1^2+\theta_s)\ +3/(a_2^2+\theta_s)\ + 1/(a_3^2+\theta_s)\ = 0$

Remember that the a_i = a,b,c are numbers we select and which determine h, k. If h and k are pre-specified, we can pick a out of the blue, and then b and c are determined by h and k, as observed in a note

above where a,b,c were introduced. So, this is a different quadratic equation with different roots θ_1 and θ_2 and this gives two L class solutions.

The last case will have {z} and will give two M class solutions. So here is a summary then of the n = 3 harmonics (you can see that we get 2n+1 = 7 of them)

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{ xyz \}\ \kappa''$ N class

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{x\}\ \kappa''\ [\ x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)\ ]$ K class, θ_1 and θ_2

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{y\}\ \kappa''\ [\ x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)\ ]$ L class, θ_1' and θ_2'

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{z\}\ \kappa''\ [\ x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)\ ]$ M class, θ_1'' and θ_2''

The main point is this: using this "method", we are finding the solutions θ_s for each class of solutions. These then tell us the "roots" for each class according to $\rho_s^2 = a^2 + \theta_s$. These roots are the zeros of the corresponding E function. We already know the general form of the K,J,M,N functions, and the only missing ingredient was to find the coefficients of the series factors of these functions, and that is what the roots tell us! Recall from above our single E formula

$$E_{n,p}(\mu) = \{ 1\ ,\ \sqrt{k^2 - \mu^2}\ ,\ \sqrt{h^2 - \mu^2}\ ,\ \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2}\ \}\ [\ 1, \mu]\ \kappa\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$$

I=0 I=1 I=1 I=2

K L M N

n-I = even => J = (n-I)/2 and select 1 in [1, μ]

n-I = odd => J = (n-I-1)/2 and select μ in [1, μ]

We can arbitrarily scale our functions so κ = 1, and then we get this

$K_{n,p} = [\ 1, \mu]\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$

$L_{n,p} = \sqrt{\mu^2 - h^2}\ [\ 1, \mu]\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$

$M_{n,p} = \sqrt{\mu^2 - k^2}\ [\ 1, \mu]\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$

$N_{n,p} = \sqrt{\mu^2 - h^2}\ \sqrt{\mu^2 - k^2}\ [\ 1, \mu]\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$

where each function has its distinct roots. We can break this list down as follows

n even:

$K_{n,p} = \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-0)/2 = n/2

$L_{n,p} = \sqrt{\mu^2 - h^2}\ \mu\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-1-1)/2 = (n-1)/2

$M_{n,p} = \sqrt{\mu^2 - k^2}\ \mu\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-1-1)/2 = (n-1)/2

$N_{n,p} = \sqrt{\mu^2 - h^2}\ \sqrt{\mu^2 - k^2}\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-2)/2 = (n-2)/2

n odd:

$K_{n,p} = \mu\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-0-1)/2 = (n-1)/2

$L_{n,p} = \sqrt{\mu^2 - h^2}\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-1)/2 = (n-1)/2

$M_{n,p} = \sqrt{\mu^2 - k^2}\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-1)/2 = (n-1)/2

$N_{n,p} = \sqrt{\mu^2 - h^2}\ \sqrt{\mu^2 - k^2}\ \mu\Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (n-2-1)/2 = (n-3)/2

Important point not to be missed: We now know the exact form of ALL Lamé functions for any order we want in terms of the root values. What the above table does NOT tell us is how many Lamé functions we will have in each class. But we figured that out earlier and it was this:

K	r+1	r = n/2 if n even, else (n-1)/2
L	n-r	
M	n-r	
N	r	
	2n+1 solutions!!	

So for n = 3 we have r = 1 and the four counts are 2, 2, 2, 1, agreeing with what we see above. So once we have a way to determine the root values ρ_s^2 in each case, we know EVERYTHING.

So, we have just done the case n = 3 which is in the second group. For the N function J = 0 and we find simply that $N_{n,p} = \sqrt{\mu^2 - h^2}\sqrt{\mu^2 - k^2}\ \mu$. For all the other functions we have J = 1. So here are our expected n = 3 solutions

$K_{3,p} = \mu(\mu^2 - \rho_s^2) = \mu^3 - \mu\rho_s^2$
$L_{3,p} = \sqrt{\mu^2 - h^2}\ (\mu^2 - \rho_s^2)$
$M_{3,p} = \sqrt{\mu^2 - k^2}\ (\mu^2 - \rho_s^2)$
$N_{3,p} = \sqrt{\mu^2 - h^2}\sqrt{\mu^2 - k^2}\ \mu$

where in the first three lines there are two distinct roots ρ_s^2 for each line. Here is Byerly, where we need to replace his x,b,c with our μ,h,k . You can see the full agreement and of course Byerly shows the pair of coefficients in each case.

$E_3(x)$

$$K_3^{p_1}(x) = x^3 - \frac{x}{5}[2(b^2+c^2) - \sqrt{4(b^2+c^2)^2 - 15b^2c^2}]$$

$$K_3^{p_2}(x) = x^3 - \frac{x}{5}[2(b^2+c^2) + \sqrt{4(b^2+c^2)^2 - 15b^2c^2}]$$

$$L_3^{q_1}(x) = \sqrt{x^2-b^2}[x^2 - \tfrac{1}{5}(b^2+2c^2 - \sqrt{(b^2+2c^2)^2 - 5b^2c^2})]$$

$$L_3^{q_2}(x) = \sqrt{x^2-b^2}[x^2 - \tfrac{1}{5}(b^2+2c^2 + \sqrt{(b^2+2c^2)^2 - 5b^2c^2})]$$

$$M_3^{r_1}(x) = \sqrt{x^2-c^2}[x^2 - \tfrac{1}{5}(2b^2+c^2 - \sqrt{(2b^2+c^2)^2 - 5b^2c^2})]$$

$$M_3^{r_2}(x) = \sqrt{x^2-c^2}[x^2 - \tfrac{1}{5}(2b^2+c^2 + \sqrt{(2b^2+c^2)^2 - 5b^2c^2})]$$

$$N_3(x) = x\sqrt{(x^2-b^2)(x^2-c^2)}$$

Sample θ_s computation. We can verify one of these cases just for fun. For the K class we got this requirement

$$3/(a^2+\theta_s) + 1/(b^2+\theta_s) + 1/(c^2+\theta_s) = 0$$

where these our "our abc", not Byerly's. Writing this as N/D we find that

$$N = 3(b^2+\theta_s)(c^2+\theta_s) + (a^2+\theta_s)(c^2+\theta_s) + (a^2+\theta_s)(b^2+\theta_s) = 0$$

$$\theta_s^2 [3+1+1] + \theta_s[3(c^2+b^2) + (a^2+c^2) + (a^2+b^2)] + [3b^2c^2 + a^2c^2 + a^2b^2] = 0$$

$$\theta_s^2 [5] + \theta_s[2a^2 + 4b^2 + 4c^2] + [3b^2c^2 + a^2c^2 + a^2b^2] = 0$$

At this point we use

$b^2 = a^2-h^2$ $\quad$ $h^2 = a^2-b^2$ $\quad$ $b^2+c^2 = 2a^2 - h^2-k^2$

$c^2 = a^2-k^2$ $\quad$ $k^2 = a^2-c^2$

We then find these coefficients:

$$[2a^2 + 4b^2 + 4c^2] = [2a^2 + 4(b^2) + 4*(c^2)] = [2a^2 + 4(a^2-h^2) + 4*(a^2-k^2)]$$

$$= [10a^2-4h^2-4k^2] = 2 [5a^2-2h^2-2k^2] = 2 [5a^2-2s^2] \qquad s^2 = h^2+k^2 \text{ as earlier}$$

$$[3b^2c^2 + a^2c^2 + a^2b^2] = [3b^2c^2 + a^2(c^2 + b^2)]$$

$$= [3(b^2)(c^2) + a^2(c^2 + b^2)] = [3(a^2-h^2)(a^2-k^2) + a^2(2a^2 - h^2-k^2)]$$

$$= [3a^4 - 3a^2(h^2+k^2) + 3h^2k^2 + 2a^4 - a^2(h^2+k^2)] = [5a^4 - 4a^2s^2 + 3h^2k^2]$$

So our quadratic now reads (note that θ_2 has dimensions L^2)

$$\theta_s^2 [5] + \theta_s 2 [5a^2-2s^2] + [5a^4 - 4a^2s^2 + 3h^2k^2] = 0$$

$$\theta_s = [-2 [5a^2-2s^2] \pm \{ 2^2 [5a^2-2s^2]^2 - 4*5* [5a^4 - 4a^2s^2 + 3h^2k^2] \}^{1/2} / 2*5$$

$$= [- [5a^2-2s^2] \pm \{ [5a^2-2s^2]^2 - 5* [5a^4 - 4a^2s^2 + 3h^2k^2] \}^{1/2} / 5$$

Now we compute

$$\{\} = [5a^2-2s^2]^2 - 5* [5a^4 - 4a^2s^2 + 3h^2k^2]$$

$$= 25a^4 + 4s^4 - 20a^2s^2 - 25a^4 + 20a^2s^2 - 15h^2k^2$$

$$= 4s^4 - 15h^2k^2$$

so we then end up with

$$\theta_s = \{ \ [2s^2-5a^2] \pm \sqrt{4s^4 - 15h^2k^2} \ \} / 5$$

But now we have to find $\rho_s^2 = a^2 + \theta_s$ and that gives

$$5\rho_s^2 = 5a^2 + 5\theta_s = 5a^2 + \{ \ [2s^2-5a^2] \pm \sqrt{4s^4 - 15h^2k^2} \ \}$$

$$= 2s^2 \pm \sqrt{4s^4 - 15h^2k^2}$$

and we note approvingly that the free parameter a^2 no longer appears! Now, according to above we should have

$$K_{3,p} = \mu(\mu^2 - \rho_s^2) = \mu^3 - \mu\rho_s^2$$

$$= \mu^3 - \mu(1/5)[2s^2 \pm \sqrt{4s^4 - 15h^2k^2}]$$

and this agrees exactly with Byerly where we take our μ,h,k →x,b,c:

$$K_3^{p_1}(x) = x^3 - \frac{x}{5}[2(b^2+c^2) - \sqrt{4(b^2+c^2)^2 - 15b^2c^2}]$$

$$K_3^{p_2}(x) = x^3 - \frac{x}{5}[2(b^2+c^2) + \sqrt{4(b^2+c^2)^2 - 15b^2c^2}]$$

The n = 2 case revisited

According to our "theory" of the last section, we expect

n even:

$K_{n,p} = \Pi_{s=1}^{J}(\mu^2 - \rho_s^2)$ J = (2-0)/2 = 1

$L_{n,p} = \sqrt{\mu^2 - h^2}\ \mu\ \Pi_{s=1}^{J}(\mu^2 - \rho_s^2)$ J = (2-1-1)/2 = 0

$M_{n,p} = \sqrt{\mu^2 - k^2}\ \mu\ \Pi_{s=1}^{J}(\mu^2 - \rho_s^2)$ J = (2-1-1)/2 = 0

$N_{n,p} = \sqrt{\mu^2 - h^2}\sqrt{\mu^2 - k^2}\ \Pi_{s=1}^{J}(\mu^2 - \rho_s^2)$ J = (2-2)/2 = 0

K	r+1 = 2	r = n/2 if n even, else (n-1)/2 = 1
L	n-r = 1	
M	n-r = 1	
N	r = 1	
	2n+1 solutions!! = 5 solutions	

Here are our solutions from the above list

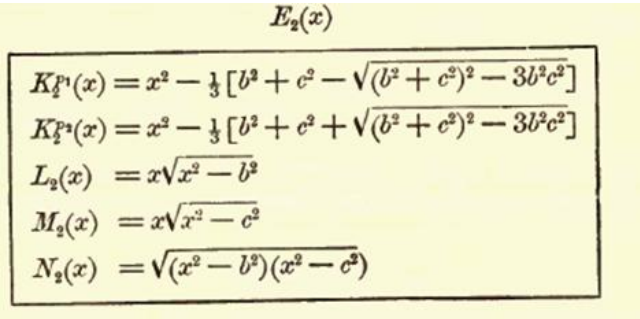
$K_{2,p} = (\mu^2 - \rho_s^2)$ with θ_1 and θ_2

$L_{2,p} = \sqrt{\mu^2 - h^2}\ \mu \qquad 1$

$M_{2,p} = \sqrt{\mu^2 - k^2}\ \mu \qquad 1$

$N_{2,p} = \sqrt{\mu^2 - h^2}\ \sqrt{\mu^2 - k^2} \qquad 1$

and here is the Byerly claim, which is in agreement:



$E_2(x)$

$K_2^{p_1}(x) = x^2 - \frac{1}{3}[b^2 + c^2 - \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$

$K_2^{p_2}(x) = x^2 - \frac{1}{3}[b^2 + c^2 + \sqrt{(b^2 + c^2)^2 - 3b^2c^2}]$

$L_2(x) = x\sqrt{x^2 - b^2}$

$M_2(x) = x\sqrt{x^2 - c^2}$

$N_2(x) = \sqrt{(x^2 - b^2)(x^2 - c^2)}$

I will not take the time to compute the two θ_1, θ_2 values except to say they are the solutions to this quadratic equation, as we found above

$[\ 1/(a^2+\theta_s) + 1/\ (b^2+\ \theta_s) + 1/(\ c^2+\ \theta_s)\ ] = 0$

The n = 4 case.

I won't compute the roots here, but I will figure out what I expect the Byerly yellow box would look like if he had that box. I am now stepping out into new territory.

n even:

$K_{4,p} = \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (4-0)/2 = 2

$L_{4,p} = \sqrt{\mu^2 - h^2}\ \mu\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (4-1-1)/2 = 1

$M_{4,p} = \sqrt{\mu^2 - k^2}\ \mu\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (4-1-1)/2 = 1

$N_{4,p} = \sqrt{\mu^2 - h^2}\ \sqrt{\mu^2 - k^2}\ \Pi_{s=1}^{J}\ (\mu^2 - \rho_s^2)$ J = (4-2)/2 = 1

K	r+1 = 3	r = n/2 if n even, else (n-1)/2 = 2
L	n-r = 2	
M	n-r = 2	
N	r = 2	
	2n+1 solutions!! = 9 solutions	

$K_{4,p} = (\mu^2 - \rho_1^2)\ (\mu^2 - \rho_2^2)$ 3 // there will be 3 distinct sets of 2 roots

$L_{4,p} = \sqrt{\mu^2 - h^2}\ \mu\ \ (\mu^2 - \rho_1^2)$ 2

$M_{4,p} = \sqrt{\mu^2 - k^2}\ \mu(\mu^2 - \rho_1^2)$ 2

$N_{4,p} = \sqrt{\mu^2 - h^2}\ \sqrt{\mu^2 - k^2}\ (\mu^2 - \rho_1^2)$ 2

How would we find those three distinct sets of 2 roots? Presumably we have 479 D which would look like this

$$1/(a^2+\theta_1)+ 1/(b^2+\theta_1) + 1/(c^2+\theta_1) + 4/(\theta_1-\theta_2) = 0$$
$$1/(a^2+\theta_2)+ 1/(b^2+\theta_2) + 1/(c^2+\theta_2) + 4/(\theta_2-\theta_1) = 0$$

There are only two equations since m = J = 2. Let's rewrite this in more compact notation

$$1/(a+x)+ 1/(b+x) + 1/(c+x) + 4/(x-y) = 0$$
$$1/(a+y)+ 1/(b+y) + 1/(c+y) - 4/(x-y) = 0$$

How do you solve equations like this? Rationalize each and show the numerators:

$$(b+x)\,(c+x)\,(x-y) + (a+x)\,(c+x)\,(x-y) + (a+x)\,(b+x)\,(x-y) + 4(a+x)\,(b+x)\,(c+x) = 0$$
$$(b+y)\,(c+y)\,(x-y) + (a+y)\,(c+y)\,(x-y) + (a+y)\,(b+y)\,(x-y) - 4(a+y)\,(b+y)\,(c+y) = 0$$

$$p_3(x,y) = 0$$
$$q_3(x,y) = 0$$

We could plot each of these polynomials as a surface over the x,y plane. Each will intersect the z = 0 plane in some set of curves. We want then to know the points (x,y) at which these curves might intersect. This is a subject I know nothing about! Solving simultaneous polynomials in multiple variables.

Suppose we think of these two polynomials in this manner

$p_3(x; y) = 0$ A cubic in x, with y = constant
$q_3(y; x) = 0$ A cubic in y, with x = constant

We could solve $p_3(x; y) = 0$ for the three cubic solutions which we might then call $x_{pi}(y)$, i = 1,2,3. We can take each of these values and put it into $q_3(y; x)$ and for each $x_{pi}(y)$, we would get three roots for y (assuming things are very general in terms of the cubic curves).

y_{1j} = three roots of $q_3(y; x_{p1}(y)) = 0$
y_{2j} = three roots of $q_3(y; x_{p2}(y)) = 0$
y_{3j} = three roots of $q_3(y; x_{p3}(y)) = 0$

But each of these situations is not just a polynomial in y, because the fixed parameter is a function of y as well. So what do we do? I happen to know the general form for a cubic solution

$$x_{p1}(y) = \text{poly}_2 + \sqrt{\text{poly}_4} * \cos\{\,[1/3]\cos^{-1}(\text{poly}_6/\sqrt{\text{poly}_{12}}\,)\,\}$$

so each of the above equations is then a huge mess indeed. This is not the right way to go.

I guess the answer is that there will be three distinct solutions of the form (x,y) and these then give our three K functions. But I don't know how to prove this off hand.

Let's look back again at our equations, maybe they are simpler than I think:

$$(b+x)(c+x)(x-y) + (a+x)(c+x)(x-y) + (a+x)(b+x)(x-y) + 4(a+x)(b+x)(c+x) = 0$$
$$(b+y)(c+y)(x-y) + (a+y)(c+y)(x-y) + (a+y)(b+y)(x-y) - 4(a+y)(b+y)(c+y) = 0$$

Well each equation is linear in y. We can solve the first equation for y to get

$$y = \frac{4abc + 6ax^2 + 5abx + 6bx^2 + 6x^2c + 5acx + 7x^3 + 5bcx}{ab + 2ax + ac + 2xc + 2bx + bc + 3x^2}$$

This think has the form $y = p_3(x)/p_2(x)$. What happens if we insert this solution for y into the second equation? The LHS of the second equation is this:

$$2bxy - 5bcy + bcx - 4abc + abx + 2axy - 5acy + acx + 2xcy - 5aby - 6by^2 - 6cy^2 + 3y^2x - 6ay^2 - 7y^3$$

$$= Ay^3 + By^2 + Cy + D$$

$$= A\,[p_3(x)/p_2(x)]^3 + B\,[p_3(x)/p_2(x)]^2 + C\,[p_3(x)/p_2(x)]^1 + D$$

$$= \{ A\,[p_3(x)]^3 + B\,[p_3(x)^2 p_2(x)] + C\,[p_3(x)p_2(x)^2]^1 + D\,p_2(x)^3 \} / p_2(x)^3$$
$$\qquad 9 \qquad\qquad 8 \qquad\qquad\qquad 7 \qquad\qquad 6 \qquad 6$$

The numerator here is of the form $poly_9(x)$ so we expect 9 solutions in general for x. I had Maple do this, and here is what Maple says:

```
> solve(h=0,x);
```

$$-a, -b, -c, \text{RootOf}(245\_Z^6 + (490b + 490c + 490a)\_Z^5 + (929ac + 929bc + 296b^2 + 929ab + 296c^2 + 296a^2)\_Z^4 + (48b^3 + 520b^2c + 520a^2b + 1636abc + 48a^3 + 520a^2c + 520ab^2 + 520bc^2 + 520ac^2 + 48c^3)\_Z^3 + (263a^2b^2 + 72a^3b + 818ab^2c + 818abc^2 + 72ca^3 + 263c^2a^2 + 818a^2bc + 72b^3c + 72c^3a + 72bc^3 + 72ab^3 + 263b^2c^2)\_Z^2 + (346a^2b^2c + 30a^3c^2 + 30b^3c^2 + 30a^3b^2 + 346a^2bc^2 + 84a^3bc + 30b^2c^3 + 346ab^2c^2 + 30a^2c^3 + 84ab^3c + 84bc^3a + 30a^2b^3)\_Z + 21a^2b^3c + 110a^2b^2c^2 + 21a^3b^2c + 21a^3bc^2 + 21ab^2c^3 + 3b^3c^3 + 21ab^3c^2 + 21a^2bc^3 + 3a^3c^3 + 3a^3b^3)$$

The first three roots all cause both of our equations to diverge

$$1/(a+x) + 1/(b+x) + 1/(c+x) + 4/(x-y) = 0$$
$$1/(a+y) + 1/(b+y) + 1/(c+y) - 4/(x-y) = 0$$

so I think we can throw them out. We are left with 6 roots which are the roots of the 6th order polynomial shown. Assuming the roots are all distinct, we then get x_i for i = 1,2...6. For each root, we will get a corresponding y_i value from our y = equation. So it seems that we might get six (θ_1,θ_2) solutions to our pair of equations.

$$1/(a^2+\theta_1)+ 1/(b^2+\theta_1) + 1/(c^2+\theta_1) + 4/(\theta_1-\theta_2) = 0$$
$$1/(a^2+\theta_2)+ 1/(b^2+\theta_2) + 1/(c^2+\theta_2) + 4/(\theta_2-\theta_1) = 0$$

so this will correspond to six pairs (ρ_1,ρ_2) for our solution

$K_{4,p} = (\mu^2- \rho_1^2) (\mu^2- \rho_2^2)$ 3 // there will be 3 distinct sets of 2 roots

When I plug in some Maple numbers for a,b,c I seem to get ρ_s^2 = 6 distinct positive value for ρ_1. But perhaps the solutions will be pairwise equal and there will only be 3 pairs the are distinct, notice the symmetry of the K function. OK, enough on this digression!

One question I do have: can the roots be expressed in closed form? I guess that is a question about finding the roots of a polynomial of order n. Here from a paper

" Any polynomial equation of degree ≤ 4 can be solved in closed form; for higher degrees a general closed form solution is impossible [Kur80, MP65] "

[Kur80] A. Kurosh. Higher Algebra. Mir Publishers, Moscow, 3rd edition, 1980.

[MP65] A.P. Mishina and I.V. Proskuryakov. Higher Algebra: Linear Algebra, Polynomials, General Algebra. Pergamon Press Ltd., Oxford, 1st English edition, 1965.

So I guess the lesson here is that starting with n = 4, you cannot just write the roots as we do for lower n values. You have to compute the actual numbers numerically!

I just spent some time looking for a "table" of Lame functions, they are not to be had! Proprietary papers claim to have data, but Marriott is not working today, or Jim's account has expired. Byerly says nothing about n > 4.

The case of general n.

First I tried doing this in line here, but it got too complicated, so it got shipped off to " Derive Hobson p 479 B,C and D.doc". Here I will review the main points derived in that doc.

(0) The general idea is this: We have in p 478 B a general Cartesian form for the $r^n E_n E_n$ "harmonics", which are the "conical harmonics", not the ellipsoidal ones. As was the case earlier, it is easier to get useful results from the conical form than from the ellipsoidal form. Our plan will be to apply ∇^2 to certain selected harmonics, and since they are harmonics, we require that ∇^2(harmonic) = 0. It turns out this creates a system of equations we can use to find the roots θ_s for any harmonic we want.

Problem 1: What happens if we apply ∇^2 to $\{1\}\ \{\ \Pi_{s=1}^{J}\ [x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)]\ \}$?

By selecting {1}, we are talking only the first solution type (K) of the Niven form. Our goal is to find a way to compute the roots θ_s for the K class {1} solutions for an arbitrary value of the (even) order n. Once we solve this problem, we will go onto **Problem 2** which is how to do the other cases in { 1 x y z xy xz yz xyz }. It turns out that Problem 1 is where most of the work resides. There is so much work, that I broke it into about 12 different subsections in the supporting doc, and even just *reviewing* the results of those sections takes a lot of space below:

(1,2) My first act is to replace the series above with a better notation

$$x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s) = \Sigma_i\ x_i^2/(a_i^2+\theta_s) \equiv f_s \qquad // = \Theta_s \text{ of Hobson}$$

(3) I show that

$$\partial_j\ (\Pi_{s=1}^{J}\ f_s) = \Sigma_{k=1}^{J}\ (\partial j\ f_k)\ (\Pi_{s\neq k}\ f_s)$$

(4) Our Problem 1 is to work with $\nabla^2\ [\ \{1\}\ \{\ \Pi_{s=1}^{J}\ f_s\ \}\ ] = 0$ and see what we learn. In this subsection I show that

$$\nabla^2\ \Pi_{s=1}^{J}\ [f_s] = \Sigma_j\ \partial_j\ \{\ \partial_j\ (\Pi_{s=1}^{J}\ [f_s])\ \} = \Sigma_{k=1}^{J}\ [\nabla^2 f_k\ (\Pi_{s\neq k}\ f_s) + \Sigma_j\ (\partial_j\ f_k)\ \partial_j\ (\Pi_{s\neq k}\ f_s)\ ]$$

(5) Here I derive an expression for the object appearing in the second term above

$$\partial_j\ (\Pi_{s\neq k}\ f_s) = \Sigma_{n\neq k}\ (\partial_j f_n)\ (\Pi_{s\neq n,\, s\neq k}\ f_s)$$

(6) I plug this fact into the result of (4) to get our result as a sum of **first and second terms**:

$$\nabla^2\ \Pi_{s=1}^{J}\ [f_s] = \Sigma_k\ (\nabla^2 f_k)\ (\Pi_{s\neq k}\ f_s) + \Sigma_k\ \Sigma_{n\neq k}\ \Sigma_j\ (\partial_j\ f_k)(\partial_j f_n)\ (\Pi_{s\neq n,\, s\neq k}\ f_s)$$

where we could it we wanted write $(\partial_j\ f_k)(\partial_j f_n) = \nabla f_k \bullet \nabla f_n$ so all would be in vector notation.

(7) Here I compute the **first term** and I show that

$$1st = \Sigma_k\ (\nabla^2 f_k)\ (\Pi_{s\neq k}\ f_s) = 2\ \Sigma_k\ \Sigma_i\ [1/(a_i^2+\theta_k)]\ \ (\Pi_{s\neq k}\ f_s)$$

and this is the final form that the first term takes. We then worry about the messier second term.

(8) I then install the f_i expressions into the **second term** and it becomes

$$2nd = 4\ \Sigma_k\ \Sigma_{n\neq k}\ \Sigma_j\ \ [x_j^2 / [(a_j^2+\theta_k)\ (a_j^2+\theta_n)\ ]\ (\Pi_{s\neq n,\, s\neq k}\ f_s)$$

$$= 4\ \Sigma_k\ \Sigma_{n\neq k}\ F_{kn}\ (\Pi_{s\neq n,\, s\neq k}\ f_s) \qquad \text{where } F_{kn} \equiv \Sigma_j\ \ [x_j^2 / [(a_j^2+\theta_k)\ (a_j^2+\theta_n)\ ].$$

(9) You can think of this as a double sum $\Sigma_k \Sigma_{n\neq k}$ of a symmetric summand. I show here with the typical double summation triangles picture that you can write

$$\Sigma_k \Sigma_{n\neq k} S_{nk} = 2 \Sigma_k \Sigma_{n<k} S_{kn} = 2 \Sigma_n \Sigma_{k<n} S_{kn}$$

(10) So we use this fact to rewrite our second term

$$2nd = 8 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s)$$

Each sum goes up to J (Hobson's m) so this 2nd sum has J(J-1)/2 terms. Our total result is then

$$\nabla^2 \Pi_{s=1}^{J} [f_s] = 2 \Sigma_k \Sigma_i [1/(a_i^2 + \theta_k)] (\Pi_{s\neq k} f_s) + 8 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s)$$

We require this to = 0, so this says

$$\Sigma_k \Sigma_i [1/(a_i^2 + \theta_k)] (\Pi_{s\neq k} f_s) + 4 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s) \} = 0$$

J terms J(J-1)/2 terms

where $F_{kn} \equiv \Sigma_j [x_j^2 / [(a_j^2 + \theta_k) (a_j^2 + \theta_n)]$

and this is exactly what Hobson p 479B says.

(11) I then derive in short order his next result which is

$$F_{nk} = \{ \Sigma_i x_i^2 / [(a_i^2 + \theta_n) (a_i^2 + \theta_k)] \} = (f_n - f_k)/(\theta_k - \theta_n) \quad // n \leftrightarrow k \text{ symmetric}$$

and this is Hobson p 479C. This then causes our second term to have this form

$$4 \Sigma_k \Sigma_{n<k} F_{kn} (\Pi_{s\neq n, s\neq k} f_s) = 4 \Sigma_k \Sigma_{n<k} (f_n - f_k)/(\theta_k - \theta_n) (\Pi_{s\neq n, s\neq k} f_s)$$

(12) This is a very tricky step. I write the above out as the two obvious terms f_n and f_k and then only in the *second* term I go back to that triangle summation picture and use another result it implies,

$$\Sigma_k \Sigma_{n<k} = \Sigma_n \Sigma_{k>n}$$

and this leads to a final and very simple form for the second term

$$= 4\Sigma_k \Sigma_{n\neq k} [1/(\theta_k - \theta_n)] (\Pi_{s\neq k} f_s)$$

so our final result $\nabla^2 u = 0$ now says

$$\Sigma_k \Sigma_i [1/(a_i^2 + \theta_k)] (\Pi_{s\neq k} f_s) + 4\Sigma_k \Sigma_{n\neq k} [1/(\theta_k - \theta_n)] (\Pi_{s\neq k} f_s) = 0$$

But now both terms have the same factor $(\Pi_{s\neq k} f_s)$ so we rewrite as

$$\Sigma_k \ (\Pi_{s\neq k} f_s) \ \{ \Sigma_i [1/(a_i^2 + \theta_k)] \ + \ 4\Sigma_k \ \Sigma_{n\neq k} [1/(\theta_k-\theta_n)] \ \} \ = 0$$

We then say that we satisfy the Laplace if we can make all the {...} vanish, that is

$$\Sigma_i [1/(a_i^2 + \theta_k)] \ + \ 4 \Sigma_{n\neq k} [1/(\theta_k-\theta_n)] \ = 0 \qquad\qquad k = 1...J$$

where I have slightly changed our meaning of $\Sigma_{n\neq k}$ so it now means a double sum excluding the diagonal elements. This last result is Hobson p 479 D . We have J equations in J unknowns θ_i , so we at least have a chance of solving this system of equations for one or more solutions $\{ \theta_1, \theta_2.... \theta_J \}_i$ where i lists off however many solutions there are. For any root solution set, we know that ∇^2(harmonic) = 0.

Problem 2: We want here to look at all the other cases $\{ x \ y \ z \ xy \ xz \ yz \ xyz \} \Pi_{s=1}^{J} [f_s]$.

In the same doc we are reviewing, I show that we can generalize the Problem 1 result to be this

$$\Sigma_k \ \Sigma_i [k_i/(a_i^2 + \theta_k)] \ \ (\Pi_{s\neq k} f_s) + \ 4\Sigma_k \ \Sigma_{n\neq k} [1/(\theta_k-\theta_n)] \ (\Pi_{s\neq k} f_s) \ = 0$$

where k_i = 3 if x_i appears in {...}, otherwise k_i = 1. Then the Problem 1 case has all k_i = 1 and is just a special case of Problem 2. Of course we have x_1= x, x_2=y and x_3=z.

285. More on finding the roots.

He now rewrites our system of equations in a more compact notation, and allows that we might have those 3's in the first columns. He has also divided by 4, so k_i = 1/4 or 3/4. The subject now is not to find the exact θ_s, but to learn general things about where they lie. In θ language, the roots lie in range ($-a^2$,$-c^2$) where a,b were introduced above as the semi's of the basic ellipsoid. We can break this range into two regions by putting $-b^2$ in the middle. He states something about how many roots show up in each of these two ranges, and this work is credited to Klein. Klein uses a method that reminds me of morphing between the conical case and a simpler cases where I think both cones are circular.

I suspect a modern computer program could find the solutions to the characteristic equation for any order, maybe even Maple could do that. By the way, here is something from the Maple help:

coords - coordinate systems supported in Maple

Description:

- At present, Maple supports the following coordinate systems:

  In three dimensions - bipolarcylindrical, bispherical, cardioidal, cardioidcylindrical, casscylindrical, confocalellip, confocalparab, conical, cylindrical, ellcylindrical, ellipsoidal, hypercylindrical, invcasscylindrical, invellcylindrical, invoblspheroidal, invprospheroidal, logcoshcylindrical, logcylindrical, maxwellcylindrical, oblatespheroidal, paraboloidal, paraboloidal2, paracylindrical, prolatespheroidal, rectangular, rosecylindrical, sixsphere, spherical, tangentcylindrical, tangentsphere, and toroidal.

  In two dimensions - bipolar, cardioid, cassinian, cartesian, elliptic, hyperbolic, invcassinian, invelliptic, logarithmic, logcosh, maxwell, parabolic, polar, rose, and tangent.

- REFERENCES:

  Moon, P. and D.E.Spencer. "Field Theory Handbook, 2nd Ed." Berlin: Springer-Verlag, 1971.

  Spiegel, Murray R. "Mathematical Handbook of Formulas and Tables." New York: McGraw Hill Book Company, 1968. 126-130.

Notice my little Schaum book and Moon and Spencer as references. Maple does not know about Lamé functions, and I think this is true even for Maple 14. Web does not mention any third party Lamé code. It would have the tough job of having to compute the roots for whatever degree the user specified. I suppose I could do it in a month of work.

286. How are the ellipsoidal harmonics related to the spherical harmonics?

This is a very obscure and thankfully short section. Hobson opens with my comment already noted above about how you might write. He then claims that his equation A "is a spherical harmonic". I would say it was an ellipsoidal harmonic. So I think he just means it is a sum of spherical harmonics. Very confusing (as usual).

He then talks about G_n as ellipsoidal harmonics and H_n as spherical ones and H_n "is related to" G_n.

He then makes a wild claim of orthogonality of the G_n functions over the ellipsoid if they are of the same degree but are different ones. The weight is called $\overline{p}$ but his explanation does not explain what it is, perhaps it is that Kepler distance thing. It is odd that he feels no need to justify p 481 B. Well, he gives a verbal proof in the last paragraph that I don't follow, what are these "octants"? If in x,y,z space, why do you get the sign situation he mentions? Well, I guess if you put in forms as shown in p 481A, there will be definite sign situations in the quadrants and you would get cancellations maybe, but it is all pretty vague because among other things the dS and weight are vague.

He then does the strange thing of showing that orthogonality of the G_n as just stated implies the orthogonality of the H_n which we already know. This proof is on page 482 and I have not tried to follow it. He is setting x,y,z = a,b,c for some reason. I don't think that is a point on any ellipsoid.

He wraps up with a very obscure theorem about a "ternary quintic of degree n" which I am happy to ignore.

Notes: On page 482, the words and equations make no sense to me, I don't understand the intention of the author. I now have to parse the words which consumes my time.

- Yes, B is the factor we are used to which appears in the Niven formulas.
- Yes, if you set θ = 0, setting Θ = 0 is the same as equation C and yes, this is called the fundy ellipsoid. '

But WHAT is he saying is equal to expression D ? Well, consider the expression on the RHS of B. This is a function of x,y,z and θ. θ can be any ≥ 0 value, and x,y,z can be any point in 3D space. If we select θ = 0, the equation C will be true for x,y,z points on the fundy ellipsoid.

Now, suppose we insist that x,y,z be on the fundy ellipsoid. They must then satisfy C. What happens if we select a particular x,y,z on the fundy and insert those x,y,z values into the RHS of equation B? One way to do that would be to solve C say for z^2 and then put that into RHS B. Let's go try that in Maple. In Maple I use the symbols x,y,z,a,b,c to stand for the squared values to reduce clutter. I first enter RHSB and expression D which I call expD:

```
restart;
t := theta;
```

$$t := \theta$$

```
expD := -t*(x/(a*(a+t)) + y/(b*(b+t)) + z/(c*(c+t)) );
```

$$expD := -\theta\left(\frac{x}{a\,(a+\theta)}+\frac{y}{b\,(b+\theta)}+\frac{z}{c\,(c+\theta)}\right)$$

$$t := \theta$$

```
RHSB := x/(a+t) +y/(b+t)+z/(c+t) - 1;
```

$$RHSB := \frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{z}{c+\theta}-1$$

I then manually solve equation C for z and enter that:

```
z := c*(1 - x/a - y/b);
```

$$z := c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

This now replaces z everywhere in Maple. So I have Maple first print out RHSB:

```
RHSB:
```

$$\frac{x}{a+\theta}+\frac{y}{b+\theta}+\frac{c\left(1-\frac{x}{a}-\frac{y}{b}\right)}{c+\theta}-1$$

```
RHSB1 := rationalize(RHSB);
```

$$RHSB1 := \frac{\theta\,(x\,a\,b^2+x\,a\,b\,\theta+y\,a^2\,b+y\,a\,\theta\,b-c\,x\,b^2-c\,x\,b\,\theta-c\,y\,a^2-c\,y\,a\,\theta-a^2\,b^2-a^2\,b\,\theta-a\,b^2\,\theta-a\,b\,\theta^2)}{c\,a^2\,b^2+a^2\,b^2\,\theta+c\,a^2\,b\,\theta+a^2\,b\,\theta^2+c\,a\,b^2\,\theta+a\,b^2\,\theta^2+c\,a\,b\,\theta^2+a\,b\,\theta^3}$$

And then I have it print out expression D

```
expD;
```

$$-\theta\left(\frac{x}{a\,(a+\theta)}+\frac{y}{b\,(b+\theta)}+\frac{1-\frac{x}{a}-\frac{y}{b}}{c+\theta}\right)$$

```
expD1 := rationalize(expD);
```

$$expD1 := \frac{\theta\,(x\,a\,b^2+x\,a\,b\,\theta+y\,a^2\,b+y\,a\,\theta\,b-c\,x\,b^2-c\,x\,b\,\theta-c\,y\,a^2-c\,y\,a\,\theta-a^2\,b^2-a^2\,b\,\theta-a\,b^2\,\theta-a\,b\,\theta^2)}{c\,a^2\,b^2+a^2\,b^2\,\theta+c\,a^2\,b\,\theta+a^2\,b\,\theta^2+c\,a\,b^2\,\theta+a\,b^2\,\theta^2+c\,a\,b\,\theta^2+a\,b\,\theta^3}$$

And sure enough, these two things are equal

```
RHSB1-expD1;
```

$$0$$

So now I can put in my own words to clarify Hobson's words:

" If you select any point x,y,z located on the fundy ellipsoid and you insert that point into the expression that is the RHS of equation B, you get expression D. "

I have no idea why this fact is useful or interesting. Now, suppose in D we define

$x_1 = x/a$ $\quad y_1 = y/b$ $\quad z_1 = z/c$

Then when x,y,z lie on the fundy ellipsoid, you find that x_1,y_1,z_1 lie on a unit sphere, I agree. So if we have two different spaces S and S_1, then the fundy ellipsoid in space S "corresponds" to the unit sphere in space S_1, I agree.

And I agree that if you make this change of variables, then D becomes E. But I still have no idea why either D or E is interesting.

Now where on earth does F come from? This claim just comes out of the blue with no support. It looks like either the ellipsoidal or conical Cartesian forms we have seen before. Since the G functions were defined to be the ellipsoidal harmonics in Cartesians, I will assume this mysterious F is coming from page 477 B. The only way I can see to make the {...} match is to set x=a, y=b, c=z . This would then give us

$$EEE^{\texttt{eval at xyz=abc}} = \{1\ a\ b\ c\ bc\ ac\ ab\ abc\}\ \{\Pi_{s=1}^{J}\,[\,a^2/(a^2+\theta_s)+b^2/(b^2+\theta_s)+c^2/(c^2+\theta_s)-1\,]\,\}$$

Now it is true that if we set x=a, y=b, c=z, then we are setting $x_1 = y_1 = z_1 = 1$ which is at the corner of a cube in S_1 space. In S space, this point lies at a corner of a rectangular box enclosing the fundy ellipsoid. But all we are doing above is evaluating $EEE_{\texttt{Cartesian}}$ at this strange point in space. I don't see where to go from the above expression. So I am completely blocked at this point, I have no moves left. But maybe if I skip down to the next section where more of this stuff occurs, the logjam can be broken. Maybe *Hobson is just summarizing his plans for the next section.*

287. Expanding Cartesian EEE = G_n on the Cartesian spherical harmonics = H_n.

He is writing $H_n(x,y,z)$ here I think to stand for $r^n Y_{nm}(\theta,\varphi)$ when converted to Cartesians. Luckily, I am now an "expert" in this subject. I show elsewhere that

$$r^n P_n^m(\cos\theta)\cos(m\varphi) = r^n \text{poly}_n(x/r,y/r,z/r;\, n,m) = \text{Poly}_n(x,y,z;\, n,m)$$
$$r^n P_n^m(\cos\theta)\sin(m\varphi) = r^n \text{poly}_n(x/r,y/r,z/r;\, n,m) = \text{Poly}_n(x,y,z;\, n,m)$$

so I will assume then that $H_n(x,y,z)$ stands for one of these homogeneous polynomials we see on the RHS of my two equations above. Fine.

I have already agreed to the fact stated by Hobson at the start of section 286 that we can write "G = $E_n E_n E_n$ as a sum of $r^n Y_{nm}$, and only n, n-2 and so on Y's are involved" which he requotes at the start of this section. So in our new language, we would say

$$G_{n1}(x,y,z) = \text{sum}[\, n = n_1 \text{ down by 2 to 0 or 1, } H_n(x,y,z)\,]$$

Now expression C I recognize as a conical harmonic of degree n, and I know this is a linear combination of the tesseral harmonics of degree n, so I guess any such linear combination would be called "a spherical harmonic". So here is how I will write B and C:

[Warning: Hobson has shifted notation. Whereas I am using s as my summation index, he has suppressed this index completely. And what I call J, which he previously called m, is now being called s. So I will just change my summation index to i and replace my J with s.]

$$G_n(x,y,z) = \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\} \{ \Pi_{i=1}^{s} [\, x^2/(a^2+\theta_i) + y^2/(b^2+\theta_i) + z^2/(c^2+\theta_i) - 1 \,] \}$$

$$H_n(x,y,z) = \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\} \{ \Pi_{i=1}^{s} [\, x^2/(a^2+\theta_i) + y^2/(b^2+\theta_i) + z^2/(c^2+\theta_i) \,] \}$$

So his question is: can we write the G_n as a sum of H_n ?

First, notice in passing this key fact which is true just because we are listing off cases:

$$\{\, 1 \;\; x/a \;\; y/b \;\; z/c \;\; x/a\,y/b \;\; x/a\,z/c \;\; x/a\,z/c \;\; x/a\,y/b\,z/c \,\} * \{\, 1 \;\; a \;\; b \;\; z \;\; ab \;\; ac \;\; bc \;\; abc \,\}$$
$$= \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\}$$

or

$$\{\, 1 \;\; x_1 \;\; y_1 z_1 \;\; x_1 y_1 \;\; x_1 z_1 \;\; y_1 z_1 \;\; x_1 y_1 z_1 \,\} * \{\, 1 \;\; a \;\; b \;\; z \;\; ab \;\; ac \;\; bc \;\; abc \,\}$$
$$= \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\}$$

or

$$\{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\} / \{\, 1 \;\; x_1 \;\; y_1 z_1 \;\; x_1 y_1 \;\; x_1 z_1 \;\; y_1 z_1 \;\; x_1 y_1 z_1 \,\}$$
$$= \{\, 1 \;\; a \;\; b \;\; z \;\; ab \;\; ac \;\; bc \;\; abc \,\} \qquad (**)$$

So, from the G definition, in new notation, we have

$$G_n(x,y,z) = \{\, 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \,\} \{ \Pi_{i=1}^{s} [\, x^2/(a^2+\theta_i) + y^2/(b^2+\theta_i) + z^2/(c^2+\theta_i) - 1 \,] \}$$

Now let's *assume* x,y,z on the fundy ellipsoid so we can replace [...] as in p 482 E. Then we have

$$G_n(x,y,z) = \{\ 1\ \ x\ y\ z\ xy\ xz\ yz\ xyz\ \}$$
$$* \ \ \Pi_{i=1}^{s} \{(-\theta_i)\ [\ x_1^2/(a^2+\theta_i) + y_1^2/\ (b^2+\theta_i) + z_1^2/(\ c^2+\theta_i)\ ]\ \}$$

$$= \{\ 1\ \ x\ y\ z\ xy\ xz\ yz\ xyz\ \}\ \ (-1)^s\ \Pi_{i=1}^{s}\theta_i$$
$$* \ \{\ \Pi_{i=1}^{s}\ [\ x_1^2/(a^2+\theta_i) + y_1^2/\ (b^2+\theta_i) + z_1^2/(\ c^2+\theta_i)\ ]\ \} \qquad \alpha$$

From our definition of H applied to sub 1 coordinates we know that

$$H_n(x_1,y_1,z_1) = \{\ 1\ \ x_1\ \ y_1\ z_1\ \ x_1y_1\ \ x_1z_1\ \ y_1z_1\ \ x_1y_1z_1\ \}$$
$$* \ \Pi_{i=1}^{s}\ [\ x_1^2/(a^2+\theta_i) + y_1^2/\ (b^2+\theta_i) + z_1^2/(\ c^2+\theta_i)\ ] \qquad \beta$$

If we divide equations α/β we see that the big Π factors cancel, and we use (**) for the {}/{} ratio and the result is

$$G_n(x,y,z)/\ H_n(x_1,y_1,z_1)\ =\ \ (-1)^s\ \Pi_{i=1}^{s}\theta_i\ \{\ 1\ \ a\ \ b\ \ z\ \ ab\ \ ac\ \ bc\ \ abc\ \}$$

from which we obtain

$$G_n(x,y,z) = \ (-1)^s\ \Pi_{i=1}^{s}\theta_i\ \{\ 1\ \ a\ \ b\ \ z\ \ ab\ \ ac\ \ bc\ \ abc\ \}\ H_n(x_1,y_1,z_1) \qquad \text{Hobson p 483 F}$$

and I think the ice is starting to thaw here. This last equation is true only if x,y,z lie on the fundy ellipsoid in S space, which means only if x_1,y_1,z_1 lie on the unit sphere in S_1 space.

Page 484: We are now entered a high-algebra area! Danger! The opening equation A I first tried to prove in-line here, then I tried to prove in a separate doc, and I could not do it. He refers to an earlier section of his book which I don't have. So let's just accept A as true and move on.

The waypoints given on the page are not unreasonable, and we move down step by step. He then calls upon another fancy theorem near the top of page 485 (also shown earlier in the book) and applies it to his situation. The endgame result is then page 486 A. Basically this is of the form

$$G_n(x,y,z) = \ \text{(fancy differential operator acting on)}\ H_n(x,y,z)$$

This thing is a tool that Hobson will use in subsequent sections. I think we are getting way off track now relative to my interests, but at least I will see where he is going. He then explains that this result above was *derived by himself* in an 1892 paper which improved Niven's proof of 1891. Hobson's claim to fame is that he did all the "cases" at once with the {.....} notation. So Hobson is showing all the algebra he did in that former paper just to get it into the historical record, and that is why we have all the detail level on pages 484-485.

287. Expanding Cartesian EEE = G_n on the Cartesian spherical harmonics = H_n, continued.

We are continuing in the "high algebra" mode. His opener here is B. In this monster thing, the integral is over the surface of a sphere or radius R. The function f(x,y,z) is some "reasonable function" in my words.

He does not state what the function $Y_n(x,y,z)$ is, perhaps it is a spherical harmonic of degree n. He then does a Phil-pondered "morph thing" where he replaces x,y,z on this sphere with x_1, x_2, x_3 which are on an ellipsoid. He then figures out how to convert the area thing dS to what it is on the ellipsoid. Here he makes it a little clearer that $\bar{p}$ is in fact the Kelvin distance. I think abc comes from the Kelvin surface area of an ellipsoid.

So we make a replacement in 486 B to get 487 A. In the morphing process, the ∇^2 appearing in B have become the fancier elliptic D^2 operator which he used in his previous section. He next changes f to another function which he also calls f (just morphing it) and we are then at p 487 B. He then takes the point (ξ,η,ζ) to be an "external point", I guess meaning outside our new ellipsoid (I don't think these have anything to do with ellipsoidal coordinates). Then f is redefined into F as shown in C, and we end up with p 487 D, where now the D operator is now in the new coordinates. This is getting pretty wild. We are still integrating over our ellipsoid surface, these new coordinates are bystanders.

Pages 488-490: He next picks a very special form for his function F, calling it φ, and a new symbol ρ as shown in B. This ρ is the distance between an integration point and the external point. Our big mess integral is now C. He then sets $\varphi(\rho) = 1/\rho$ to get D, and he then interprets the integral as having the form $\int dA\ \sigma/R$ $(R=\rho)$ which is the potential a certain surface charge σ (he call this charge "matter").

Now we pause to quote the result E which I have in fact derived. He then sets $Y_n = H_n$ (perhaps confirming my guess that Y_n was some arbitrary spherical harmonic). He then uses E in D to get F where we are now integrating of $G_n(x,y,z)/\rho$. This integral is thereby expressed as an explicit function, if you know what H_n is. He then interprets the integral as a potential of a weird charge distribution $\bar{p}\ G(x,y,z)$. Somehow he shows that this surface integral is the external ellipsoidal harmonic $\mathcal{G}$ and so in J we have an expression for this thing in terms of H_n . His point is that this result J expresses $\mathcal{G}$ as a sum of spherical harmonics. On page 490 he wraps up this section by putting a condition on our "external point" which makes sure the series in fact converges, something Niven also worried about.

289. The ellipsoidal Dirichlet problem all done in Cartesian coordinates.

Suppose we prescribe some f(x,y,z) on an ellipsoid. We would like to write that as an integral of ellipsoidal harmonics. On page 490 A he formally writes this expansion where he shows the coefficients as integrals and has a normalizer downstairs, I think he mentions this earlier in the chapter. Having such an expansion hinges on the completeness of the G functions which implies their orthogonality as shown in B. We know from earlier that if we can get our Dirichlet prescription expressed in harmonics, then our external potential has the form shown in C. This is all "in general". He then assumes that the Dirichlet potential is a polynomial of a certain kind, and ends up with page 492A which expresses f(x,y,z) over the H_n of ratio coordinates. But then we can in effect replace H_n with G_n as in our by now familiar result B and we end up with an unstated expansion of f(x,y,z) on the G_n. This is what we want in doing a Dirichlet problem in ellipsoidals (in Cartesian coordinates), but you can see that the general expansion of f is a mess. Part of writing this expansion is that you apply that operator D multiple times to f.

All this stuff seems to be an alternative approach to doing problems in ellipsoidal coordinates which is much easier at least for simple boundary conditions. Here we are doing brute force electrostatics using the Cartesian coordinate versions of the ellipsoidal harmonics. I wonder it this really has any use?

290. Reduction of Ellipsoidal harmonics to Spheroidal ones.

The approach here is to look at those "characteristic equations" for the roots of the Lamé functions which determine the θ_i. He says a = b is our limit. I think maybe he has changed gears to c > b > a so that a=b is then the prolate limit. But then he does not really take this limit at this point, and a and b are kept separate. I guess he will take the limit later. So he is taking the prolate limit in this Section.

What happens to those characteristic equations when we go to this limit? He claims that some number σ of the roots become just $\theta_i = -a^2$ and he then works on finding the rest. He defines q_i in terms of the θ_i, and then f(q) as a certain function top p 493. He goes that this f(q) solves a certain ODE for which the solution is a certain sine and cosine thing in variable 2σχ with $q = \sin^2 \chi$. He mentions "κ factors" a this point, with no reference to where they are defined. I think this last ODE applies only for the f(q) involving the set of roots $\theta_i = -a^2$. The rest of the roots create another ODE which is of course the Legendre equation on page 494, and we then get those functions but in a strange variable $\mu = p^{1/2}$ with p top page 494. We finally end up with functions (page 494 arrow) which *look like* spherical harmonics, but the variables are μ and φ. He never cleanly defines φ, so unclear. He calls this "**spherical** harmonics" so somehow he must have a = b = c but I don't see where c is set equal to the other two. A strange ending then to this section.

291. Reduction of Ellipsoidal harmonics to Spheroidal ones continued.

We are back to a = b now and he writes his D operator and in this case, G = (operator) H seems to simplify a lot. Somehow then he gets the result middle of page 495. Let's recall my *prolate* spheroidal atoms

$$(1) \qquad \overset{\text{osc}}{[\, P_n^m(\eta), Q_n^m(\eta)\,]}\ \overset{\text{expo}}{[P_n^m(\xi), Q_n^m(\xi)\,]}\ \overset{\text{osc}}{[\,\sin(m\varphi),\cos(m\varphi)]}$$

$$(2) \qquad \overset{\text{expo}}{[\, P_{i\tau-1/2}^m(\eta), Q_{i\tau-1/2}^m(\eta)\,]}\ \overset{\text{osc}}{[P_{i\tau-1/2}^m(\xi), Q_{i\tau-1/2}^m(\xi)\,]}\ \overset{\text{expo}}{[\,\sin(m\varphi),\cos(m\varphi)]}$$

If we compare my first line to his page 495 result, it seems that

his r = η
his cosθ = ξ
his φ = φ
his σ = m

so p 495 is claiming to show the prolate harmonics emerging from the ellipsoidal ones.

Then he says after doing all this work "a similar reduction applies to the oblates".

His last act is to use the "trick" formula in its θ form to get the second kind harmonic which again he calls an "external harmonic.

These last two sections really are strange. He claims to be "reducing ellipsoidal to the prolate" but we never see the thing we start with. It is all done in terms of the roots θ_i. We thought maybe he would start with G somehow and reduce this. He does show G bottom of page 494 in terms of H. I guess he puts H =

P_n somehow. I think this section was a "last gasp" of Hobson, a footnote detail that was not worth explaining in more detail.

292. Lamé relation to other functions.

We earlier showed that the helper coordinates ξ,η,ζ were given as elliptic integrals of the ρ,μ,ν coordinates. Here he suggests that the Lamé functions might be written themselves as elliptic integrals of the Jacobi and Weierstrass type. [recall that the Weierstrass elliptic function is that funny **ρ**(u) thing that you see in various places like W&W] He says W&W talk about this a little (4th Ed was 1927 which predates Hobson's book by 4 years). This is a version of the Lamé equation using this funny function

$$\frac{d^2\Lambda}{du^2} = \{n(n+1)\wp(u) + B\}\Lambda,$$

Hobson is just saying he knows this stuff exists, but he won't be dealing with it.

His final comments are that someone has fiddled with Lamé for n being half integers, and he thinks that using n = iτ-1/2 might be useful in elliptic cone boundary stuff, which I could believe. Recall the closely related conical, toroidal and Mehler functions that I comment on somewhere.

THE END!

Appendix A. Finding the θ_s system of equations for the case n = 3

We start by setting n = 3 in our conical form (that is, we look at harmonics of degree 2)

$$r^n\ E_{n,p}(\mu)\ E_{n,p}(\nu)$$

$$= \{\ 1\ \ x\ \ y\ \ z\ \ xy\ \ xz\ \ yz\ \ xyz\ \}\ \kappa''\ \Pi_{s=1}^{J}\ [\ x^2/(a^2+\theta_s) + y^2/(b^2+\theta_s) + z^2/(c^2+\theta_s)\ ]$$

	K	K	L	M	M	L	N	N
	e	o	o	o	e	e	e	o

Since n is odd, our only four forms here are { x y z xyz }. From our table above

		I	n-I	J	2J	{}	ord{}	2J+ord{}	n-2J	r^p	r^{tot}RHS
n odd	K	0	odd	(3-0-1)/2	2	x/r	1	n	1	-1	0
	L,M	1	even	(3-1)/2	2	y/r,z/r	1	n	1	-1	0
	N	2	odd	(3-2-1)/2	0	xyz/r^3	3	n	3	-3	0

For our primitive function xyz there is no $\Pi_{s=1}^{J}$ factor and for this we have just

$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{ xyz \}\ \kappa''$ // these E functions are N functions.

Let's look next at the {x} item, where we have

$$r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu) = \{ x\}\ \kappa''\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ]$$

Our task is now to apply ∇^2 to this thing. First let's refer to x as x_1 so we have

$$\partial_i\ [r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu)] = \partial_i(x_1)\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ] + (x_1)\ \partial_i\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ]$$

Meanwhile

$$\partial_i\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ] = \partial_i\ [\ x_1^2/(a_1^2+\theta_s) + x_2y^2/\ (a_2^2+\ \theta_s) + x_3^2/(\ a_3^2+\ \theta_s)\ ]$$

$$= 2\ x_i/(a_i^2+\ \theta_s)$$

Therefore we have shown that

$$\partial_i\ [r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu)] = \partial_i(x_1)\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ] + (x_1)\ \partial_i\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ]$$

$$= \delta_{i1}\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ] + (x_1)\ 2\ x_i/(a_i^2+\ \theta_s)$$

Then our next step is this

$$\partial_i^2\ [r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu)] = \partial_1\ [\ x^2/(a^2+\theta_s) + y^2/\ (b^2+\ \theta_s) + z^2/(\ c^2+\ \theta_s)\ ] + 2\ \partial_i\ [x_1\ x_i/(a_i^2+\ \theta_s)]$$

$$= 2x_1/(a^2+\theta_s)\ + [2/(a_i^2+\ \theta_s)]\ \partial_i\ [x_1\ x_i]$$

$$= 2x_1/(a^2+\theta_s)\ + \Sigma_i\ [2/(a_i^2+\ \theta_s)]\ [\ x_1(\partial_i x_i) + (\partial_i x_1)x_i]$$

$$= 2x_1/(a^2+\theta_s)\ + \Sigma_i\ [2/(a_i^2+\ \theta_s)]\ [\ x_1\ 1 + \delta_{i1}x_i]$$

$$= 2x_1/(a^2+\theta_s)\ + 2x_1\Sigma_i\ [1\ /(a_i^2+\ \theta_s)]\ +\ 2x_1/(a_1^2+\ \theta_s)$$

$$= 4x_1/(a_1^2+\theta_s)\ + 2x_1\Sigma_i\ [1\ /(a_i^2+\ \theta_s)]$$

$$= 2x_1\{\ 2/(a_1^2+\theta_s)\ + \Sigma_i\ [1\ /(a_i^2+\ \theta_s)]\ \ \ \}$$

If we require now that $\nabla^2\ [r^3\ E_{3,p}(\mu)\ E_{3,p}(\nu)] = 0$, we have

$\Sigma_i\ [1\ /(a_i^2 + \theta_s)]\ + 2/(a_1^2+\theta_s)\ = 0$

$3/(a_1^2+\theta_s)\ + 1/(a_2^2+\theta_s)\ + 1/(a_3^2+\theta_s)\ = 0$

and this agrees with Hobson p 479 D where m = 1 so we have only one equation and we have none of the second type of term. [Note