This is the Title PhL 3.26.05

Motivation: While reading Hobson, we encounter the comparison of solving the Laplace equation in two different coordinate systems: r,θ,φ, and r,μ,ν, which is the conical system. We have a connection between θ,φ and μ,ν. The atoms in conicals are $r^n\ E_{n,p}(\mu)\ E_{n,p}(\nu)$ and we think these are related by a matrix transformation to the atoms in sphericals which are taken as the tessorals $r^n P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$. On a sphere at r=1 we think either the set $E_{n,p}(\mu)\ E_{n,p}(\nu)$ or the set $P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$ forms a complete 2D basis for the expansion of any function on the sphere. Thus, we expect there to be a matrix relation between these functions within the angular momentum n manifold of the full 3D Hilbert Space. This is the subject discussed by Hobson.

Overview.

1. I first **compute the 2D Cartesian atoms** of $\nabla^2 f = 0$ and find them to be the following 28 items:

[sin(kx), cos(kx), x , 1] * [sh(ky), ch(ky), y, 1], $\nabla^2 = \partial_x^2+\partial_y^2$
[sin(ky), cos(ky), y , 1] * [sh(kx), ch(kx), x, 1]

When we separate the Laplace equation using u(x,y) = a(x)b(y) we get certain little eigenvalue problems for the two separated functions: $-\partial_x^2\ a(x) = k^2\ a(x)$ and $-\partial_y^2\ b(y) = -k^2\ b(y)$ where k^2 must of course be a real number. If we take $k^2 > 0$, we get the first row of atoms above, and $k^2 < 0$ gives the second row. The simpler solutions x,1 and y,1 arise from the nullspaces of $-\partial_x^2\ a(x) = 0$ and $-\partial_y^2\ b(y) = 0$. If we have no boundary conditions, we have no restrictions on k, so k can be any real or any imaginary number, in which case k^2 is a real number in (-∞,∞). When we talk about linearly combining the above atoms, I think we are completely general if we just restrict to k in (0,∞) . The negative k axis would give nothing new in terms of our general form, nor would the (0,i∞) range or its negative.

2. I then **pose two questions**:
Do the above "atoms" form a complete basis for an expansion of any function f(x,y) ?
Do the above "atoms" form a complete basis for an expansion of any Laplace solution f(x,y) ?

For the first equation, if we take f(x,y) = some F(x) which has a Fourier integral, then our atoms are sufficient to expand the function, and the same for F(y): we just take the trig functions and the 1 in the appropriate variable. If we take yF(x) or xF(y)we are still good. But what about x^2y ? Since x^2 has no Fourier integral (it diverges), so we cannot fit x^2y with $\int_0^\infty dk\ A_k \cos(kx)\ y$. So I think this provides a simple counterexample to the first question, and the answer there is NO.

The second question is not quite so clear and I try some examples later. You would think the answer would be NO.

3. At this point, I construct the **atoms for the r,θ polar coordinates** and find them to be:

$\nabla^2_{r\theta} f = 0 \quad => \quad r\,\partial_r[\, r\,\partial_r f] + \partial_\theta^2 f = 0$

atoms = $[\sin(n\theta), \cos(n\theta)] * [r^n, r^{-n}]$, $n = 1,2,3....$
E ln(r) + F

I take note in passing of some atoms converted to Cartesians:

$r^2\cos(2\theta) = 2xy$
$r^3\cos(3\theta) = x^3 - 3xy^2$
$r^{-1}\cos\theta = r\cos\theta/r^2 = x/(x^2+y^2)$
$\ln r = \ln(\sqrt{x^2+y^2})$

4. I then **tried to find a Laplace solution which I could not fit with Cartesian atoms**. My idea was to generate a weird solution in polar coordinates, then convert it to Cartesians. I can see that the 1st item above is clearly a Cartesian atom fit. The 2nd item looked like a good candidate. I tried a fit of this form

$$x^3 - 3xy^2 = \int_0^\infty dk \{ A_k \sin(kx) + C_k x \cos(ky) \}$$

$$= f_{odd,arb}(x) + C x f_{even,arb}(y)$$

We would have a good fit here if x^3 had a Fourier Sine integral $\int_0^\infty dk \sin(kx) A(k) = x^3$. But if there were such an integral for x^3 it would have $A(k) = c \int_0^\infty dx \sin(kx) x^3$ but this diverges. So I think in retrospect that $x^3 - 3xy^2$ is a Laplace solution which cannot be fit by the Cartesian atoms.

I looked next at the 3rd item above, and I was a bit surprised to find that I could write it as

$$x/(x^2+y^2) = \int_0^\infty dk\, e^{-k|y|}\sin(kx)$$

so this one is not a counterexample.

Finally, the 4th item certainly looks like another counterexample, but I did not play with it.

5. I next tried to **find a Laplace solution which I could not fit with Polar atoms**. I tried e^{kx} . I was able to write (after much fiddling)

$$e^{kx} = e^{kr\cos\theta} = a_0/2 + \Sigma_{n=1}^\infty a_n(r)\cos(n\theta) \qquad a_n(r) = 2 I_n(kr)$$

Since we cannot write $I_n(kr)$ as a linear combination of r^n and r^{-n}, we at once have a counterexample: a function e^{kx} which is a Laplace solution for sure (it is an atom in fact), but it cannot be expanded in terms of Polar atoms.

6. **Conclusions**. When we talk about a "complete set", we refer to a set of eigenfunctions of some self-adjoint ODE operator with some homogeneous boundary conditions. For example, the Helmholtz equation $-\nabla^2 f = \lambda f$, with the BC that f vanishes on the 1st quadrant π edge square, has eigenfunctions

sin(nx)sin(my) and eigenvalues $\lambda_{n,m} = (n^2+m^2)$ and these would serve as a complete set for expanding a function f(x,y) *which vanishes on this square perimeter*. Such a function is not a solution of the Laplace equation (except when n=m=f=0), by the way. We might in some sense regard $\sin(k_x x)\sin(k_y y)$ as a complete set for functions in the entire x,y plane which vanish at infinity. But in our list of Laplace Cartesian atoms, the closest thing we find to this general form is sin(kx) sh(ky). There is just no way in the world that anyone could claim that the Laplace Cartesian atoms form a complete set on the plane. These atoms are solutions of the Laplace equation, which is the Helmholtz eigenvalue equation with $\lambda = 0$. The set of eigenfunctions with $\lambda = 0$ is not a complete set because you need ALL the eigenvalues of $-\nabla^2 f$. So we certainly expect to find functions which cannot be expanded in Cartesian atoms, and some such functions could be solutions to the Laplace equation.

7. The Hobson situation is something quite different. We feel that on a sphere at r=1 either the set $E_{n,p}(\mu)$ $E_{n,p}(\nu)$ or the set $P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$ forms a complete 2D basis for the expansion of any function on the sphere. At least in the spherical case, we know that the form shown represents a product of two well-defined S-L problems each with a complete set of eigenfunctions, so the combination is than a complete 2D basis on θ,φ. I think this is true as well with the E functions, and in that case each S-L problem involves the same Lamé equation. This suggests that any reasonable function on the sphere can be expanded in either complete set. Roughly we would claim that

$$f(\text{sphere}) = \Sigma_{n=0}^{\infty} \Sigma_p f_{n,p} E_{n,p}(\mu) E_{n,p}(\nu) = \Sigma_{n=0}^{\infty} \Sigma_m g_{n,m} P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$$

This would seem to imply that, for each n, we have this balance:

$$\Sigma_p f_{n,p} E_{n,p}(\mu) E_{n,p}(\nu) = \Sigma_m g_{n,m} P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$$

If we take $f_{n,p} = \delta_{p,p1}$ then the following is some function on the sphere:

$$f(\text{sphere}) = \Sigma_{n=0}^{\infty} \Sigma_p \delta_{p,p1} E_{n,p}(\mu) E_{n,p}(\nu) = \Sigma_{n=0}^{\infty} E_{n,p1}(\mu) E_{n,p1}(\nu)$$

The balance above would then argue that we can write

$$E_{n,p1}(\mu) E_{n,p1}(\nu) = \Sigma_m g_{n,m}(p_1) P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$$

and some coefficient $g_{n,m}(p_1)$ exists to make this work. Similarly, we ought be have

$$P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)] = \Sigma_p f_{n,p}(m) E_{n,p}(\mu) E_{n,p}(\nu)$$

and some coefficient $f_{n,p}(m_1)$ exists to make this work.

This situation superficially seems similar to our Polar/Cartesian situation with x,y and r,θ except here it is μ,ν and θ,φ. In the first case we can represent any point in the plane as x,y or r,θ and in the second case we can represent any point on the sphere as μ,ν or θ,φ. The big difference is an old friend: in 3D, you can arrange to have two of the three coordinates be oscillatory and have two SL problems each with complete sets of associated functions. In 2D you cannot have this! One variable will be oscillatory and the other must then be expo. Such was the situation of this entire doc.

I thought *here* I would ponder for a while the simpler situation of Laplace atoms in 2D on a disk. The coordinates for comparison are x,y and r,θ. The atoms in the x,y system can be taken as [we shall elaborate on these lists of atoms below, these are incomplete!]

x,y atoms:	[cos(kx),sin(kx)] [ch(ky),sh(ky)]	k = sep constant
r,θ atoms:	$[r^n, r^{-n}]$ [sin(θ),cos(nθ)]	n = integers

Question #1: where are the functions like "x" or "xy" in the x,y coordinate system?

Answer. First, let's verify:

$\nabla^2\cos(kx)\text{ch}(ky) = [\partial_x^2+\partial_y^2]\cos(kx)\text{ch}(ky)$

$= \text{ch}(ky)\,\partial_x^2\cos(kx) + \cos(kx)\,\partial_y^2\,\text{ch}(ky)$

$= -k^2\,\text{ch}(ky)\cos(kx) + k^2\cos(kx)\,\text{ch}(ky) = 0$ QED

$[\partial_x^2+\partial_y^2]x = 0$ QED
$[\partial_x^2+\partial_y^2]xy = 0+0 = 0$ QED

Put this question on hold for a moment. [It gets answered below]

Question #2: Do the x,y atoms form a complete set on the plane? [no]

The notion of complete set requires boundary conditions. In 1D we know how an ODE plus homo BC's leads to a SL problem and if the ODE operator is self-adjoint, we get the eigenfunctions forming a complete set. The notion of eigenfunctions involves the eigenvalue equation $-\nabla^2 u = \lambda u$, which is not the Laplace equation, so eigenfunctions are not Laplace solutions (unless $\lambda = 0$ is an eigenvalue). [$L = -\nabla^2$]

In the case of x,y we could consider a BC which makes u = 0 on the perimeter of a square of edge π whose lower left corner lies at the origin. We suspect that the eigenfunctions are sin(nx)sin(my). We find that $-\nabla^2[\sin(nx)\sin(my)] = (n^2+m^2)[\sin(nx)\sin(my)]$ so we have $\lambda_{n,m} = (n^2+m^2)$ where n and m are integers. We could then regard { sin(nx)sin(my) } as a complete set for the expansion of any f(x,y) *which vanishes on this perimeter*. The functions x and xy don't meet this condition, so we don't expect to be able to expand x in terms of the atoms sin(nx)sin(my).

If we have no boundary conditions, then there is no notion of a complete set, and we conclude that the x,y atoms do not form a complete set on the plane.

Dialog: But I thought that with a set of Laplace equation atoms, you could express *any* solution to the Laplace equation as a linear combination of the atoms? [this is wrong, see response]

Response: We know that each Laplace atom satisfies Laplace with no BC's, so any linear combination of Laplace atoms does solve Laplace. But we don't know that an arbitrary function can be written as a linear combination of Laplace atoms. Nor do we know that an arbitrary Laplace solution can be written as a linear combination of Laplace atoms. [We will show a counterexample below.]

Question #3: What *are* the Laplace atoms in 2D in x,y coordinates?

Answer: Let's do the separation

$$[\partial_x^2+\partial_y^2]\, a(x)b(y) = 0$$

$$(\partial_x^2\, a(x))\, b(y) + a(x)\, (\partial_y^2\, b(y)) = 0$$

$$(\partial_x^2\, a(x))/a(x) = -\, (\partial_y^2\, b(y))/b(y) = -k^2 \qquad \text{separation constant } k$$

$$-\partial_x^2\, a(x) = k^2\, a(x)$$

Think of this separated equation as an eigenvalue equation where $L = -\partial_x^2$ and k^2 is then the eigenvalue. We have no boundary conditions so k^2 is just a free real value. The eigenfunctions are then sin(kx) and cos(kx) if $k^2 > 0$, and they are sh(kx) and ch(kx) if $k^2 < 0$ and k imaginary. In the first case, we see that Asin(kx)+Bcos(kx) is a solution for a(x). But this is a particular solution, and we are allowed to add to it any solution of the homo equation $-\partial_x^2\, f(x) = 0$. This operator $L = -\partial_x^2$ has a non-vanishing nullspace and the homo solutions have the form f(x) = Cx+D. Thus our most general solution to our separated equation in x is this, assuming $k^2 > 0$:

$$a(x) = A\sin(kx)+B\cos(kx) + Cx + D$$

Meanwhile, the other separated equation is this

$$-\partial_y^2\, b(y) = -k^2\, b(y)$$

If $k^2 > 0$, then most general solution to this equation is

$$b(y) = A'\text{sh}(ky)+B'\text{ch}(ky) + C'y + D'$$

Therefore, our separated solution can have this form

$$u(x,y) = [A\sin(kx)+B\cos(kx) + Cx + D][\, A'\text{sh}(ky)+B'\text{ch}(ky) + C'y + D']$$

and of course we swap trigs and bolics if $k^2 < 0$, which is the same as doing $x \leftrightarrow y$. Without further ado, then, we can see that our complete list of atoms is this, where you take one from each bracket

[sin(kx), cos(kx), x , 1] [sh(ky), ch(ky), y, 1],
[sh(kx), ch(kx), x, 1] [sin(ky), cos(ky), y , 1]

It would seem that there are then 32 atoms, but the ones xy,x,y,1 are counted twice, so there are in fact 28 atoms. You cannot represent x^2 with this list of atoms, but that is a legal function f(x,y) on the plane. So we certainly cannot say that the set of Cartesian atoms of the Laplace equation is a complete set on the plane. Below we come up with $f(x,y) = x^3 - 3xy^2$ which also cannot be represented by the atoms, and it too is a function on the plane, and moreover, it satisfies the Laplace equation!

Fact: If we define f(x,y) as an arbitrary linear combination of the above many atoms, then f(x,y) will solve the Laplace equation.

Question: Can we think of some Laplace solution which *cannot* be written as a sum of said atoms?

Answer: I think $r^2\cos(2\theta)$ is a Laplace solution in polar coordinates. We then have

$$r^2\cos(2\theta) = r^2\, 2\sin(\theta)\cos(\theta) = 2xy \qquad \text{// so this is NOT a counterexample}$$

$$r^3\cos(3\theta) = r^3[\,4\cos^3(\theta) - 3\cos(\theta)] = 4x^3 - 3r^2x = 4x^3 - 3(x^2+y^2)x = x^3 - 3xy^2$$

Let's verify that this is a Laplace solution

$$[\,\partial_x^2+\partial_y^2][\,x^3 - 3xy^2] = 6x - 6x = 0$$

So here is our counterexample! The Laplace solution $x^3 - 3xy^2$ cannot be written as a linear combination of the atoms listed above! For example, x^3 is not an atom, nor is xy^2. In our counterexample, it is the *interaction* between the two terms which satisfies Laplace, each term does not separately satisfy.

Comment: By working in polar coordinates, we have revealed Laplace solutions which are functions of x and y but which we never "thought of" looking at our list of Cartesian atoms.

Question: What are the 2D Laplace atoms in polar coordinates?

Answer: Let's do the separation. First, we need the 2D Laplace equation in polar coordinates.

(a) What is the 2D Laplacian

Rather than do lots of derivatives, we can use our general theory which says this

$$\text{lap}(f) = \text{div grad}(f) = (1/\sqrt{g'})\,\partial_i\,[\sqrt{g'}\,g'^{ij}\,(\partial_j f)\,]$$

For orthogonal coordinates this becomes

$$\nabla^2 f = (1/\sqrt{g'})\,\Sigma_{i=1,2}\,\partial_i\,[\sqrt{g'}\,g'^{ii}\,(\partial_i f)\,]$$

But we know that

$$\sqrt{g'} = \sqrt{\det(g')} = Q_1 Q_2 \qquad g'^{ij} = Q_i^{-2}\,\delta^{ij}$$

=>

$$\nabla^2 f = (1/Q_1Q_2)\ \Sigma_{i=1,2}\ \partial_i\,[Q_1Q_2/Q_i^2\,(\partial_i f)\,]$$

$$= (1/Q_1Q_2)\{\ \partial_1[\,Q_2/Q_1\,\partial_1 f]\ + \partial_2[\,Q_1/Q_2\,\partial_2 f]\ \}$$

where we can take 1,2 = r,θ. Maple tells us the metric tensor:

```
restart;
with(linalg):
arning, new definition for norm
arning, new definition for trace
R[1] := r*cos(theta);
```

ordering for r is 1,2 = x,y

$$R_1 := r\cos(\theta)$$

```
R[2] := r*sin(theta);
```

$$R_2 := r\sin(\theta)$$

```
q[1] := r; q[2] := theta;
```

ordering for q is 1,2 = r,theta

$$q_1 := r$$

$$q_2 := \theta$$

```
T := (i,j) -> diff(R[j],q[i]);
```

Note that (1) i is the row index. (2) This is the "tilt-up" matrix.

$$T := (i,j) \to \frac{\partial}{\partial q_i} R_j$$

```
Tup := matrix(2,2,T);
```

$$Tup := \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}$$

```
Tupt := transpose(Tup);
```

$$Tupt := \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

```
gdn := simplify(evalm(Tup &* Tupt));
```

see page 17 of tensors and curvilinear coordinates.doc

$$gdn := \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

So we have then $Q_1 = 1$ and $Q_2 = r$. Therefore

$$\nabla^2 f = (1/Q_1Q_2)\{\ \partial_1[\,Q_2/Q_1\,\partial_1 f]\ + \partial_2[\,Q_1/Q_2\,\partial_2 f]\ \}$$

$$= (1/Q_2)\{\ \partial_1[\,Q_2\,\partial_1 f]\ + \partial_2[\,1/Q_2\,\partial_2 f]\ \}$$

$$= (1/r)\{\ \partial_1[\,r\,\partial_1 f]\ + \partial_2[(1/r)\,\partial_2 f]\ \}$$

$= (1/r)\{\ \partial_r[\ r\ \partial_r f]\ + \partial_\theta[(1/r)\ \partial_\theta f]\ \}$

$= (1/r)\{\ \partial_r[\ r\ \partial_r f]\ + (1/r)\ \partial_\theta[\partial_\theta f]\ \}$

$= (1/r)\{\ \partial_r[\ r\ \partial_r f]\ + (1/r)\ \partial_\theta^2 f]\ \}$

Therefore we have found that

$\nabla^2_{r\theta}\ f\ = (1/r)\ \partial_r[\ r\ \partial_r f]\ +\ (1/r)^2\ \partial_\theta^2 f$

$= (1/r)\ \{\ \partial_r f + r\partial_r^2 f\ \} +\ (1/r)^2\ \partial_\theta^2 f$

$=\ \partial_r^2 f\ + (1/r)\partial_r f +\ (1/r)^2\ \partial_\theta^2 f$

Note: compare this to the spherical coordinates Laplacian, there is a huge difference:

$\nabla^2\ f =\ (1/r^2)\ \partial_r[\ r^2\ \partial_r f]\ +\ (1/r)^2\ (1/\sin\theta)\partial_\theta(\sin\theta\partial_\theta f) +\ (1/r)^2\ (1/\sin^2\theta)\partial_\varphi^2 f$

(b) do the separation

Let's then use the first form to do our separation

$\nabla^2_{r\theta}\ f\ = 0\ \Rightarrow\quad r\ \partial_r[\ r\ \partial_r f]\ +\ \partial_\theta^2 f\ = 0$

Let $f(r,\theta) = a(r)b(\theta)$ and we get

$b(\theta) r\ \partial_r[\ r\ \partial_r\ a(r)]\ +\ a(r)\ \partial_\theta^2\ b(\theta)\ = 0$

$r\ \partial_r[\ r\ \partial_r\ a(r)]/a(r)\ +\ \partial_\theta^2\ b(\theta)/b(\theta)\ = 0$

Separation is then

$\partial_\theta^2\ b(\theta) = -k^2 b(\theta)$

$r\ \partial_r[\ r\ \partial_r\ a(r)] = k^2 a(r)$

(c) solve the separated equations

Solutions of the first equation for a full circle are

$b(\theta) = A\sin(k\theta) + B\cos(k\theta)$

$b(\theta+2\pi) = b(\theta)\quad \Rightarrow\quad A\sin(k\theta) + B\cos(k\theta) = A\sin(k\theta+2\pi k) + B\cos(k\theta+ 2\pi k)$

$\Rightarrow$ k = integer, call it n.

Are there homogeneous solutions to add?

$\partial_\theta^2\, b(\theta) = 0 \qquad b(\theta) = A + B\theta$

The Bθ term is not periodic so B = 0, and the A term is already included with n = 0 and cos. So nothing new comes from the homo solutions.

The radial equation is then

$r\, \partial_r[\, r\, \partial_r\, a(r)] = n^2 a(r)$

We just try $r^{\pm n}$ and find

$r\, \partial_r[\, r\, \partial_r\, r^{\pm n}] = r\, \partial_r[\, r\, (\pm n)\, r^{\pm n-1}] = (\pm n) r\, \partial_r[\, r^{\pm n}] = (\pm n) r(\, \pm n)\, r^{\pm n-1}$

$= (\pm n)^2\, r^{\pm n} = n^2\, r^{\pm n}$

so we have found our particular solution to be

$a(r) = Ar^n + Br^{-n}$

What about homo solutions?

$r\, \partial_r[\, r\, \partial_r\, a(r)] = 0$

$r\partial_r^2\, a(r) + \partial_r\, a(r) = 0$

$\partial_r^2 u + (1/r)\, \partial_r u = 0$

Let $g = \partial_r u$ so this says

$\partial_r g + (1/r) g = 0$

$\partial_r g = -\,(1/r) g$

$dg/g = -dr/r$

$\ln(g/g_0) = -\ln(r/r_0) = \ln(r_0/r)$

$g = A/r$ // verifies that $\partial_r g + (1/r) g = 0$

Now we need to solve

$\partial_r u = A/r$

$du = Adr/r$

$u-u_0 = A\ln(r/r_0)$

$u = A\ln(r) + B$

This is the famous result, and it is the same as the non-homo solution with n = 0, so to speak.

(d) construct the atoms list

So here are the atoms for this problem:

$b(\theta) = A\sin(n\theta) + B\cos(kn)$ n = 0,±1,±2.... // includes $b(\theta)$ = const

$a(r) = Cr^n + Dr^{-n}$ n = ±1,±2....
$= E\ln(r) + F$ n = 0

But the two signs add nothing, and we can just ignore the negative n values

So to get our atoms, we do this

atoms = $[\sin(n\theta), \cos(n\theta)]\ [r^n, r^{-n}]$, n = 1,2,3....
$E\ln(r) + F$

Any linear combination of these 6 atoms will be a solution of the Laplace equation. So this brings to mind some interesting Cartesian solutions one does not see in the Cartesian atoms list, such as

$r^{-1}\cos\theta = r\cos\theta/r^2 = x/(x^2+y^2)$ $\ln(\sqrt{x^2+y^2})$ etc

Comment: When I started this doc, I was wondering if I could write the Cartesian atoms as linear combinations of the Polar ones and vice versa. But neither set of atoms is a complete set, so this is simply not always possible. We saw an example above

Polar Atom = $r^3\cos(3\theta) = x^3 - 3xy^2$

Can this be expressed as a linear combination of Cartesian atoms which are these : ?

$[\sin(kx), \cos(kx), x, 1]\ [\mathrm{sh}(ky), \mathrm{ch}(ky), y, 1]$, k = real
$[\mathrm{sh}(kx), \mathrm{ch}(kx), x, 1]\ [\sin(ky), \cos(ky), y, 1]$

Let's try it. We would have to select atoms that do not have expo behavior because $x^3 - 3xy^2$ does not have such behavior. Based on even/odd symmetry, we are then left with

$$x^3 - 3xy^2 = \int dk \{ A_k \sin(kx) + B_k x + C_k x \cos(ky) \}$$

$$= f_{odd,arb}(x) + B x + C x f_{even,arb}(y)$$

So we could select A_k such that $f_{odd,rb}(x) = x^3$ and B = 0 and C_k such that C $f_{even,arb}(y) = -y^2$. Wow, so you CAN do it after all!

Well let's take another case

Polar Atom = $r^{-1}\cos\theta = r\cos\theta/r^2 = x/(x^2+y^2)$

This has the same even/odd symmetry (odd in x, even in y) as our last example, so we have the same limited solution form

$$x/(x^2+y^2) = f_{odd,rb}(x) + B x + C x f_{even,arb}(y)$$

$$= f(x) + Bx + xg(y)$$

$$\Rightarrow x = [f(x) + Bx + xg(y)] (x^2+y^2)$$

$$x = f(x)x^2 + Bx^3 + x^3g(y) + f(x)y^2 + Bxy^2 + xy^2g(y)$$

$$x - f(x)x^2 - Bx^3 = x^3g(y) + f(x)y^2 + Bxy^2 + xy^2g(y)$$

The RHS cannot depend on y. We could try $g(y) = y^2$ and $f(x) = -x^3$ in an effort to get rid of the x^3 term on the RHS. Then we are left with

$$x + x^5 - Bx^3 = Bxy^2 + xy^4$$

which does not fly. The other choice is g(y) = 0 so RHS is $f(x)y^2 + Bxy^2$, so need B = 0 and then we are left with RHS = $f(x)y^2$ and then we need f = 0 so RHS = 0 and that does not fly either.

But what if we make our second kind solutions more explicit using these atoms

$$[\sin(kx), \cos(kx), x, 1] [e^{ky}, e^{-ky}, y, 1], \qquad k = \text{real}$$
$$[e^{kx}, e^{-kx}, x, 1] [\sin(ky), \cos(ky), y, 1]$$

What about $e^{-k|y|}$? This let's us add a new term of this form

$$x/(x^2+y^2) = \int dk \{ A_k \sin(kx) + B_k x + C_k x \cos(ky) + D_k e^{-k|y|}\sin(kx) + E_k e^{-k|y|}x \}$$

$$= f_{odd,arb}(x) + B x + C x f_{even,arb}(y) + \int dk\, D_k e^{-k|y|}\sin(kx) + x \int dk\, E_k e^{-k|y|}$$

$= f_{odd,arb}(x) + B\,x + C\,x\,f_{even,arb}(y) + \int dk\, D_k\, e^{-k|y|}\sin(kx) + x f_{even}(y)$

We know that $\int e^{-ky} dk = 1/y$. so the integral can change a decaying behavior. So maybe it is possible that you could find D_k such that

$$x/(x^2+y^2) = \int_0^\infty dk\, D_k\, e^{-k|y|}\sin(kx)$$

Apply $\int_0^\infty dx \sin(k'x)$ to both sides:

$$\int_0^\infty dx \sin(k'x)\, x/(x^2+y^2) = \int_0^\infty dk\, D_k\, e^{-k|y|} \int_0^\infty dx \sin(k'x)\sin(kx)$$

3. $$\int_0^\infty \frac{x\sin(ax)}{\beta^2+x^2}\,dx = \frac{\pi}{2}e^{-a\beta}$$

$$\text{LHS} = \int_0^\infty dx \sin(k'x)\, x/(x^2+y^2) = (\pi/2)\, e^{-k'|y|}$$

Then we have also from transforms.doc that

$$\int_0^\infty dz \sin(kz)\sin(k'z) = (\pi/2)\delta(k-k')$$

$$\text{RHS} = \int_0^\infty dk\, D_k\, e^{-k|y|}\ (\pi/2)\delta(k-k') = (\pi/2)\, D_{k'} e^{-k'|y|}$$

Then we have

$$(\pi/2)\, e^{-k'|y|} = (\pi/2)\, D_{k'} e^{-k'|y|} \quad \Rightarrow \quad D_{k'} = 1 \quad !!!$$

So once again we are able to write this function as a superposition of Cartesian atoms!

$$x/(x^2+y^2) = \int_0^\infty dk\, e^{-k|y|}\sin(kx)$$

Who'd a thunk it? Here is some verification

$$\int_0^\infty e^{-px}\sin(qx+\lambda)\,dx = \frac{1}{p^2+q^2}(q\cos\lambda + p\sin\lambda)$$

Set $\lambda = 0$ and $x = k$ to get

$$\int_0^\infty dk\, e^{-pk} \sin(qk) = q/(p^2+q^2)$$

$$\int_0^\infty dk\, e^{-|y|k} \sin(xk) = x/(y^2+x^2) \qquad \text{QED!}$$

Going the other way, consider

Cartesian atom $= e^{kx} = e^{krcos\theta}$

Can you expand this in terms of the Polar atoms which are these : ?

$[\sin(n\theta), \cos(n\theta),1]\ [r^n, r^{-n}, \ln(r), 1] \qquad n = 1,2,3....$

Well consider

$$e^{krcos\theta} = \Sigma_{n=0}^{\infty} (krcos\theta)^n/n! = \Sigma_{n=0}^{\infty} k^n r^n (cos\theta)^n/n!$$

This does not work, but maybe something else will. Consider

$$e^{krcos\theta} = K + \Sigma_{n=1}^{\infty} \cos(n\theta)[\, A_{n,k}\, r^n + B_{n,k}\, r^{-n} + C_{n,k} \ln(r) + D_{n,k}]$$

To see if this is possible, we can write $e^{krcos\theta}$ as a Fourier Series, and then just see if the coefficient comes out having the form allowed in the square bracket. So consider

$$e^{krcos\theta} = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta)$$

This is a Fourier series period $L = \pi$ Schaum p 131.

$$a_n = (1/\pi) \int_0^{2\pi} d\theta\, e^{krcos\theta} \cos(n\theta) = (2/\pi) \int_0^{\pi} d\theta\, e^{krcos\theta} \cos(n\theta)$$

Trashing a bunch of work with GR7, I now know this integral is Bateman II p 81:

$$\pi J_n(z) = i^{-n} \int_0^{\pi} d\theta\, e^{izcos\theta} \cos(n\theta)$$

So set $iz = k$ and we find

$$\pi J_n(-ik) = i^{-n} \int_0^{\pi} d\theta\, e^{kcos\theta} \cos(n\theta)$$

But we know that

$I_n(k) = i^{-n} J_n(ik) = i^n J_n(-ik) \quad \Rightarrow J_n(-ik) = i^{-n} I_n(k)$

so we really have this result

$$\pi I_n(k) = \int_0^{\pi} d\theta\, e^{k\cos\theta} \cos(n\theta)$$

$$\pi I_n(kr) = \int_0^{\pi} d\theta\, e^{kr\cos\theta} \cos(n\theta)$$

So our Fourier coefficients are simply

$a_n = (2/\pi)\, \pi I_n(kr) = 2 I_n(kr)$

Thus we have been able to express our Cartesian atom e^{kx} as a linear combination of polar atoms!

$e^{kr\cos\theta} = a_0/2 + \Sigma_{n=1}^{\infty} a_n \cos(n\theta) \qquad a_n = 2 I_n(kr)$

Now does this match our allowed form from above?

$e^{kr\cos\theta} = K + \Sigma_{n=1}^{\infty} \cos(n\theta)[\, A_{n,k}\, r^n + B_{n,k}\, r^{-n} + C_{n,k} \ln(r) + D_{n,k}]$

To make this work we need

$K = 2 I_0(kr) = 2$ this part is OK

$[\, A_{n,k}\, r^n + B_{n,k}\, r^{-n} \,] = 2 I_n(kr)$ this part is not OK

But it is not possible to write the special function $I_n(kr)$ as a linear combination of r^n, r^{-n}, lnr and 1.

So FINALLY I have found an example where a Laplace solution in one world canNOT be written as a linear combination of atoms in the other world. It took me all day to find this example.