Using Conical and Spherical Coordinates to explore Lamé functions PhL 10.30.10

This subject is discussed in Hobson's ellipsoidal harmonics chapter. As he explains there, the conical harmonics are $r^n\ E_{n,p}(\mu)\ E_{n,p}(\nu)$ where these are exactly the same E functions which appear in the ellipsoidal harmonics $E_{n,p}(\rho)E_{n,p}(\mu)E_{n,p}(\nu)$ (although the coordinates μ,ν are not the same in the two cases). These E's are the first-kind Lamé functions. Since the conical system is simpler than the ellipsoidal one, and has a very close connection to the spherical system, it is useful to glean certain facts about the E functions in the conicals context. We do that below. As a reminder, the conical system and the ellipsoidal system both have two constants which Hobson calls k and h, with k > h.

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Overview (2 pages)

In **Section 1** I give the x,y,z equations for conicals. Since both conicals and sphericals use labeled spheres, it is easy to relate the conical μ,ν to spherical θ,φ, and that connection is made here. The harmonics for sphericals are taken to be the tesseral harmonics rather than the official Y functions, since the former are real and we shall be interested in real things.

In **Section 2**, I then take the Cartesian forms of the tesserals (obtained from work done in another doc) and convert these to μ,ν conical variables. Since we have $\text{poly}_n(x/r,y/r,z/r;\ n,m)$ for any tesseral, all we have to do is replace things like x/r using the conical x,y,z formulas. The results can be summarized as follows (which does take considerable effort to derive, all done below)

$P_n^m(\cos\theta)\cos(m\varphi)$:

$= \text{poly}_n\ (\ \mu^2,\nu^2,\ \mu\nu;\ a'_m;\ \text{by 1})$ // m even Case I

$= \sqrt{k^2 - \mu^2}\ \sqrt{k^2 - \nu^2}\ \ \text{poly}_{n-1}\ (\ \mu^2,\nu^2,\ \mu\nu;\ a'_m;\ \text{by 1})$ // m odd Case II

$P_n^m(\cos\theta)\sin(m\varphi)$:

$= \sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\;\; \text{poly}_{n-1}\,(\,\mu^2,\nu^2,\,\mu\nu;\, a'_m;\, \text{by } 1)$ // m odd Case IV

$= \sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\;\;\sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\;\; \text{poly}_{n-2}\,(\,\mu^2,\nu^2,\,\mu\nu;\, a'_m;\, \text{by } 1)$ // m even Case III

These results are extremely important and will soon lead to an understanding of the E functions.

At this point I have a comment on Hobson's phrase "rational algebraical functions" and on the general modern definition of an "algebraic function".

In **Section 3** I first quote from "Polar and Cartesian atoms.doc" which somehow gets off into the conical coordinates despite its title:

$E_{n,p}(\mu)\,E_{n,p}(\nu) = \Sigma_m\, g_{n,m}(p)\, P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$

$P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)] = \Sigma_p\, f_{n,p}(m)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$

The first line claims that the product EE as shown (both factors have the same n and the same p) is simply a linear combination of tesseral harmonics of degree n. But in Section 2 we showed how we can write out the tesserals in terms of μ and ν. I then consider the four cases shown, I,II,III and IV, and in each case I deduce that form which the function E which makes up EE must have. Here are the conclusions:

$P_n^m(\cos\theta)\cos(m\varphi)$ with m = even:

$$E_{n,p}(\mu) = K_{n,p}(\mu) = [\, a\mu^n + b\mu^{n-2} + c\mu^{n-4} + \ldots \,]$$

$P_n^m(\cos\theta)\cos(m\varphi)$ with m = odd:

$$E_{n,p}(\mu) = M_{n,p}(\mu) = \sqrt{\mu^2 - k^2}\,[\, a\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$

$P_n^m(\cos\theta)\sin(m\varphi)$ with m = odd:

$$E_{n,p}(\mu) = L_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\,[\, a\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$

$P_n^m(\cos\theta)\sin(m\varphi)$ with m = even:

$$E_{n,p}(\mu) = N_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\,\sqrt{\mu^2 - k^2}\;\; [\, a\mu^{n-2} + b\mu^{n-4} + c\mu^{n-6} + ..]$$

A few comments are in order:

(1) the coefficients a,b,c... are generic, they are not the same in the above four series. They are functions of n,p,h,k. Recall that n and p are the quantized parameters which appear in the Lamé ODE which cause the Frobenius series solutions to truncate and thus be convergent. We know n = integers, but it is the p values that take work to compute.

(2) In each series, every other term is missing. I show how this arises from the fact that $E_{n,p}(\mu)\,E_{n,p}(\nu)$ is symmetric in μ and ν, so that RHS aggregate terms which are not symmetric get ruled out.

(3) I show all the square roots with "the variable first". For $E_{n,p}(\mu)$ we know $k > \mu > h$, and for $E_{n,p}(\nu)$ we know $k > h > \nu > 0$. So in practical application, if one root comes out imaginary, we know the coefficients a,b,c must all be imaginary, because we know that the E functions are real.

(4) I have shown the class (species) names K,L,M,N exactly as shown in Hobson page 459-460.

(5) The E functions are really quite simple in form, but obtaining the coefficients for larger values of n takes an ever increasing amount of work.

(6) The root structure is consistent with the Frobenius fact that exponents are 0 and 1/2 about the singular points of the Lamé ODE which occur at z = ±h and ± k.

(7) These are all "first kind" Lamé functions because they blow up as $\mu \to \infty$.

(8) Notice that the descending power series don't all start with μ^n. We can say that the first term in one of the series is μ^{n-I} where I=0,1,1,2 for K,L,M,N.

I then show the following facts (noting that Hobson has L↔M as a typo)

$P_n^m(\cos\theta)\cos(m\varphi) = \Sigma\ \alpha\ K(\mu)K(\nu)$ m = even
$P_n^m(\cos\theta)\cos(m\varphi) = \Sigma\ \alpha\ M(\mu)M(\nu)$ m =odd
$P_n^m(\cos\theta)\sin(m\varphi) = \Sigma\ \alpha\ N(\mu)N(\nu)$ m = even
$P_n^m(\cos\theta)\sin(m\varphi) = \Sigma\ \alpha\ L(\mu)L(\nu)$ m = odd

or

$P_n^{2m}(\cos\theta)\cos(2m\varphi) = \Sigma\ \alpha\ K(\mu)K(\nu)$
$P_n^{2m+1}(\cos\theta)\cos([2m+1]\varphi) = \Sigma\ \alpha\ M(\mu)M(\nu)$
$P_n^{2m}(\cos\theta)\sin(2m\varphi) = \Sigma\ \alpha\ N(\mu)N(\nu)$
$P_n^{2m+1}(\cos\theta)\sin([2m+1]\varphi) = \Sigma\ \alpha\ L(\mu)L(\nu)$

I verify that Hobson and Byerly agree on their definitions of K,L,M,N.

Section 4 is entitled "New ways to write E and EE and EEE."

In **part (a)** I first "rewrite" the above E functions in this manner:

$$E_{n,p}(\mu) = \{\ 1\ ,\ \sqrt{k^2-\mu^2}\ ,\ \sqrt{h^2-\mu^2}\ ,\ \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\ \}\ [\ 1,\ \mu]\ \kappa\ \Pi_{i=1}^{J}\ (\mu^2-\alpha_i^2)$$

K M L N
I = 0 I=1 I=1 I=2 // class dependent

J = (n-I)/2 n-I = even => select 1 in [1, μ]
J = (n-I-1)/2 n-I = odd => select μ in [1, μ]

For any given value of n, for each of the four classes we compute J from n and I, and follow the rule shown for selecting either 1 or μ from the square bracket. Notice that J is the number of $(\mu^2-\alpha_i^2)$ factors, and α_i^2 are in effect a recoding of the coefficient information a,b,c which appeared above in the series.
It is seen that for each constant α_i, the E function has zeros located at $\pm\alpha_i$. There may be a zero at μ=0 depending on the case shown. The rule for J applies to all equations below and won't be repeated.

In **part (b)** I combine three of these E functions to get the fairly obvious result

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \;=\; \kappa^3$$
$$\{\, 1\,,\ \sqrt{k^2-\rho^2}\,,\ \sqrt{h^2-\rho^2}\,,\ \sqrt{h^2-\rho^2}\ \sqrt{k^2-\rho^2}\,\}\ [\,1,\ \rho]$$
$$\{\, 1\,,\ \sqrt{k^2-\mu^2}\,,\ \sqrt{h^2-\mu^2}\,,\ \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\,\}\ [\,1,\ \mu]$$
$$\{\, 1\,,\ \sqrt{k^2-\nu^2}\,,\ \sqrt{h^2-\nu^2}\,,\ \sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2}\,\}\ [\,1,\ \nu]$$
$$\Pi_{i=1}^{J}\ (\rho^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)\,(\nu^2-\alpha_i^2) \qquad (*)$$

There are only four different EEE expressions here, not 8, because for each n, and each column in {}, you select a s specific column in [] as explained in the single-E rules above. Next, for certain low order cases, we can express the EEE functions in terms of x,y,z by just using the x,y,z equations for ellipsoidal coordinates. Here is the list of interesting EEE products:

e	$K_0\, K_0\, K_0$	1
o	$L_1\, L_1\, L_1$	$\sqrt{h^2-\rho^2}\,\sqrt{h^2-\mu^2}\,\sqrt{h^2-\nu^2} \;=\; y * ihs$
o	$M_1\, M_1\, M_1$	$\sqrt{k^2-\rho^2}\,\sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2} \;=\; z * iks$
e	$N_2\ N_2\, N_2$	$\sqrt{h^2-\rho^2}\ \sqrt{k^2-\rho^2}\,\sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\,\sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2} = (y * ihs)(z * iks)$ $= yz * (-hks^2)$
o	$K_1\, K_1\, K_1$	$\rho\mu\nu \;=\; x * hk$
e	$L_2\ \ L_2\, L_2$	$\rho\mu\nu\,\sqrt{k^2-\rho^2}\,\sqrt{k^2-\mu^2}\,\sqrt{k^2-\nu^2} = (x * hk)(z * iks) = xz * (ihk^2 s)$
e	$M_2\, M_2\, M_2$	$\rho\mu\nu\,\sqrt{h^2-\rho^2}\,\sqrt{h^2-\mu^2}\ \sqrt{h^2-\nu^2} = (x * hk)(y * ihs) = xy * (ih^2 ks)$
o	$N_3\, N_3\, N_3$	$\rho\mu\nu\,\sqrt{h^2-\rho^2}\ \sqrt{k^2-\rho^2}\ \sqrt{h^2-\mu^2}\ \sqrt{k^2-\mu^2}\ \sqrt{h^2-\nu^2}\ \sqrt{k^2-\nu^2}$ $= (x * hk)(y * ihs)(z * iks) = xyz * (-h^2k^2s^2)$

This allows us to write our EEE product in the following manner

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$$

$$= \{\ 1\ \ x\ y\ z\ xy\ xz\ yz\ xyz\ \}\ \ \kappa'\ \Pi_{i=1}^{J}\,[(\rho^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)]$$
$$\quad K\ K\ L\ M\ M\ L\ \ N\ \ N$$
$$\quad e\ \ o\ o\ o\ e\ \ e\ \ e\ \ o \qquad \text{// values of n for which this form applies (n even or odd)}$$

where now κ' has absorbed the various functions of h,k that arise in each case. So basically we have replaced all that square root stuff and the [1 μ] type brackets with simple Cartesian functions. It is important to understand that for a given n, only four of the eight items in the {...} list apply, one "form" for each class. In general there will be multiple functions in each class that have the form shown, a fact not obvious just staring at the above notation. But we know there will be 2n+1 total functions for a given n. Hobson prefers to write the root product above in this alternate fashion

$$\Pi_{i=1}^{J}\,[(\rho^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)\,(\mu^2-\alpha_i^2)] \;=\; \Pi_{i=1}^{J}\,[(\rho^2-\rho_i^2)\,(\mu^2-\rho_i^2)\,(\mu^2-\rho_i^2)]$$

$$= \Pi_{s=1}^{J}\,[(\rho^2-\rho_s^2)\,(\mu^2-\rho_s^2)\,(\mu^2-\rho_s^2)]$$

so $\rho_i^2 = \alpha_i^2$, just a new name, and the product index is changed from i to s.

In **part (c)** I show how the product of roots on the right above may be replaced with a product of simple Cartesian functions, and we get our next form, the "full Niven Cartesian form" for the ellipsoidal harmonics,

$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$ // $\kappa'' = \kappa' (\Pi_{s=1}^{J} D(\rho_s^2))$

$= \{ 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \}$ κ'' $\{ \Pi_{s=1}^{J} [x^2/\rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1] \}$

K	K	L	M	M	L	N	N	
e	o	o	o	e	e	e	o	n = even or odd
1	2	2	2	3	3	3	4	species s

Once we have this form, we can define a new number called the "species" s which I have written as the last line above. This is one plus the number of Cartesian coordinates appearing in the {...} terms. Although I don't show it here, it turns out that we can write J = n -s. **********

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1. Basic facts.

Here are the basic facts: (I use the usual x,y,z meanings, not Hobson p 456; h goes with y)

$x = r \sqrt{k^2 - \mu^2} \sqrt{k^2 - \nu^2} /(ks)$
$y = r \sqrt{\mu^2 - h^2} \sqrt{h^2 - \nu^2} /(hs)$ $\quad s = \sqrt{k^2-h^2}$ $\quad 0 < \nu < h < \mu < k$
$z = r\, \mu\nu/(hk)$

$x = r \sin\theta \cos\varphi$
$y = r \sin\theta \sin\varphi$
$z = r \cos\theta$

$\Rightarrow \cos\theta = \mu\nu/(kh)$
$\sin\theta \cos\varphi = \sqrt{\mu^2 - h^2} \sqrt{h^2 - \nu^2} / (hs)$
$\sin\theta \sin\varphi = \sqrt{k^2 - \mu^2} \sqrt{k^2 - \nu^2} /(ks)$

$P_n^m(\cos\theta)\, [\cos(m\varphi), \sin(m\varphi)]$ // tesseral harmonics

$P_n^m(\cos\theta) = (\sin\theta)^m \, \partial_x^m \, P_n(x)$

2. Expressing the tesseral harmonics in terms of μ and ν.

First, we need to quote some results from "spherical harmonics in Cartesian coordinates.doc" we shall be using. In our study there, we find that

$P_n^m(\cos\theta)\cos(m\varphi) = \{ \partial_x^m \, P_n(x) \} * \{ (\sin\theta)^m \cos(m\varphi) \} = \{\text{first piece}\} * \{\text{second piece cos}\}$
$P_n^m(\cos\theta)\sin(m\varphi) = \{ \partial_x^m \, P_n(x) \} * \{ (\sin\theta)^m \sin(m\varphi) \} = \{\text{first piece}\} * \{\text{second piece sin}\}$

We find these facts:

$$\{\text{first piece}\} = \{ \partial_x^m \, P_n(x) \} = (1/r)^{n-m} [\text{Poly}_{n-m}(x,y,z; n,m)] = \text{Poly}_{n-m}(x/r,y/r,z/r; n,m)$$
$$\{\text{second piece cos}\} = \{ (\sin\theta)^m \cos(m\varphi) \} = (1/r)^m [\text{Poly}_m(x,y; m)] = \text{Poly}_m(x/r,y/r; m)$$
$$\{\text{second piece sin}\} = \{ (\sin\theta)^m \sin(m\varphi) \} = (1/r)^m [\text{Poly}_m(x,y; m)] = \text{Poly}_m(x/r,y/r; m)$$

Here are some notational details that one should understand:

• In the first piece, the coefficients are just numbers (ie, dimensionless) and depend on both n and m. This is the meaning of the n,m labels after the semicolon.

• In the second piece, he coefficients are just numbers (ie, dimensionless) and depend just on m.

• The two second pieces are not the same polynomial, we are just showing the "form". That is to say, the coefficients are not the same.

• Poly(x_1, x_2, x_3, ...x_n) means that every term in the polynomial has the same degree. This is usually called a homogeneous polynomial.

• In the poly forms shown above, where the arguments are dimensionless ratios like x/r, everything in the polynomial is dimensionless: the ratios and the coefficients. All terms have the same degree.

We can combine the two pieces if we want (but we don't want to do this below) to get

$$P_n^m(\cos\theta)\cos(m\varphi) = (1/r)^n \, \text{Poly}_n(x,y,z; n,m) = \text{Poly}_n(x/r,y/r,z/r; n,m)$$
$$P_n^m(\cos\theta)\sin(m\varphi) = (1/r)^n \, \text{Poly}_n(x,y,z; n,m) = \text{Poly}_n(x/r,y/r,z/r; n,m)$$

which tells us that all the spherical harmonics have a certain form when expressed in Cartesians. I have written a Maple program to spit out these polynomials in the doc just mentioned above.

But there is more to say about the doc referred to above. Since we are about to think about the r,μ,ν conical system, it is helpful to express our various "pieces" in a different way. We show in the doc that

$$\{\text{first piece}\} = \{ a\,(z/r)^{n-m} + b\,(z/r)^{n-m-2} + c\,(z/r)^{n-m-4} + \ldots \}$$
$$\{\text{second piece cos}\} = \{ a\,(x/r)^m + b\,[1-(z/r)^2]\,(x/r)^{m-2} + c\,[1-(z/r)^2]^2\,(x/r)^{m-4} \ldots \}$$
$$\{\text{second piece sin}\} = (y/r)\,\{ a\,(x/r)^{m-1} + b\,[1-(z/r)]^2\,(x/r)^{m-3} + c\,[1-(z/r)]^4\,(x/r)^{m-5} + \ldots \}$$

so this shows a little more information that our Poly forms given above.

As a reminder, in the first piece series, the coefficients are just "numbers" and all terms are dimensionless, but these numbers do depend on both n and m. In the second piece series, the same is true, but the coefficient numbers depend only on m. The coefficients are "generic" (not same in different series)

What we want to do is substitute for x/r etc from the x,y,z equations which are used to define conical coordinates.

$x/r = \sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\,/(ks)$
$y/r = \sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\,/(hs)$ $\qquad s = \sqrt{k^2-h^2}$ $\qquad 0 < \nu < h < \mu < k$
$z/r = \mu\nu/(hk)$

so we "like" seeing "z/r" since it is the simplest of these three expression: any expression $(z/r)^I$ is just going go be a product of $\mu^I\nu^I$ so there are never any square roots. Thus, the entire "first piece" above has the following form (constants have changed a little)

$\{ a\,(\mu\nu)^{n-m} + b\,(\mu\nu)^{n-m-2} + c\,(\mu\nu)^{n-m-4} + ... \}$ // first piece

which is a polynomial in $(\mu\nu)$ which is missing every other power:

first piece = $\text{poly}_{n-m}(\mu\nu;\, a_{mn};\, \text{by } 2)$

Now, μ and ν have dimensions of distance, so our new a,b,c coefficients now have dimensions (they are functions of h and k) such that each *term* in the series is dimensionless. We are no longer "homogeneous".

The **second piece cos** has a slightly different nature depending on whether m is even or odd.

<u>If m is even</u>, then all ratios $(x/r)^I$ shown there have even I. Since

$(x/r) = (\mu^2-h^2)\,(h^2-\nu^2)/(ks)^2 = \alpha\,(\mu^2-h^2)^{1/2}\,(h^2-\nu^2)^{1/2}$

$[1-(z/r)^2] = [1-(\mu\nu/(hk))^2] = [\,1 - \mu^2\nu^2/(hk)^2] = 1-\beta\mu^2\nu^2$

we can examine our second piece cos,

$\{ a'\,(x/r)^m + b'\,[1-(z/r)^2]\,(x/r)^{m-2} + c'\,[1-(z/r)^2]^2(x/r)^{m-4} .... \}$ =

and install factors like these

$(x/r)^m = \alpha^m\,(\mu^2-h^2)^{m/2}\,(h^2-\nu^2)^{m/2}$

$(x/r)^{m-2} = \alpha^{m-2}\,(\mu^2-h^2)^{m/2-1}\,(h^2-\nu^2)^{m/2-1}$ etc

to get [we are doing m = even here, so m/2 = integer]

$\{ a'\,(x/r)^m + b'\,[1-(z/r)^2]\,(x/r)^{m-2} + c'\,[1-(z/r)^2]^2(x/r)^{m-4} .... \}$ =

$\{ a'\,\alpha^m(\mu^2-h^2)^{m/2}\,(h^2-\nu^2)^{m/2} + b'\,[1-\beta\mu^2\nu^2]\,\alpha^{m-2}(\mu^2-h^2)^{m/2-1}\,(h^2-\nu^2)^{m/2-1} + ...$

This form is a polynomial in μ^2 and ν^2 of degree m. That is, highest term is $(\mu^2)^{m/2}\,(\nu^2)^{m/2}$. The poly is not symmetric under $\mu \leftrightarrow \nu$. The powers step down by 1 , so next lower is $(\mu^2)^{m/2-1}\,(\nu^2)^{m/2}$. So let's write

second piece cos = poly_m (μ^2,ν^2; a'_m; by 1) // m even

Note: If you have p(x,y) having a term x^3y^2, the degree of that term is 5. Note that this polynomial is NOT homogeneous in its arguments because we just showed two terms of different degree.

Now suppose m is odd, so we write m = M+1 with M even. Then we have for our second piece,

$\{ a' (x/r)^m + b' [1- (z/r)^2] (x/r)^{m-2} + c' [1-(z/r)^2]^2 (x/r)^{m-4} .... \}$

$= (x/r) \{ a' (x/r)^M + b' [1- (z/r)^2] (x/r)^{M-2} + c' [1-(z/r)^2]^2 (x/r)^{M-4} .... \}$

$= (x/r)\ \text{poly}_M (\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd

$= \sqrt{k^2 - \mu^2}\ \sqrt{k^2 - \nu^2}\ \ \text{poly}_{m-1} (\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd

As usual we are just talking "forms", not specific coefficients. So here is a summary of what we have so far:

$P_n^m(\cos\theta)\cos(m\varphi)$ = first piece * second piece :

first piece = $\text{poly}_{n-m}(\mu\nu; a_{mn}; \text{by } 2)$

second piece = $\text{poly}_m (\mu^2,\nu^2; a'_m; \text{by } 1)$ // m even
second piece = $\sqrt{k^2 - \mu^2}\ \sqrt{k^2 - \nu^2}\ \ \text{poly}_{m-1} (\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd

Now let's look at the $\sin(m\varphi)$ harmonic. The first piece is the same. The second piece is this:

$= (y/r) [d (x/r)^{m-1} + e\ [1- (z/r)^2] (x/r)^{m-3} + f\ [1- (z/r)^2]^2 (x/r)^{m-5} + ...]$

Again we detect two different cases depending on the nature of m. This time things are simpler if m is odd. Then we have m = M+1 with M even and the above becomes

$= (y/r)\ \text{poly}_M (\mu^2,\nu^2; a'_m; \text{by } 1)$

$= (y/r)\ \text{poly}_{m-1} (\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd

$= \sqrt{\mu^2 - h^2}\ \sqrt{h^2 - \nu^2}\ \ \text{poly}_{m-1} (\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd

Now suppose m = even. We can factor out one (x/r) in the above to get

$= (y/r)(x/r) [d (x/r)^{m-2} + e\ [1- (z/r)^2] (x/r)^{m-4} + f\ [1- (z/r)^2]^2 (x/r)^{m-6} + ...]$

$= (y/r)(x/r)\ \text{poly}_{m-2}(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m even

$= \sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}\ \text{poly}_{m-2}(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m even

where we simply note the similar form to our previous case. We summarize this case now:

$P_n^m(\cos\theta)\sin(m\varphi)$ = first piece * second piece :

first piece = $\text{poly}_{n-m}(\mu\nu; a_{mn}; \text{by } 2)$

second piece = $\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}\ \text{poly}_{m-1}(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd
second piece = $\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}\ \text{poly}_{m-2}(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m even

Our world is so complicated now that I feel the need for very frequent recapitulations, so let's have one right here

Summary to this point:

$P_n^m(\cos\theta)\cos(m\varphi)$ = first piece * second piece :

first piece = $\text{poly}_{n-m}(\mu\nu; a_{mn}; \text{by } 2)$

second piece = $\text{poly}_m(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m even
second piece = $\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}\ \text{poly}_{m-1}(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd

---

$P_n^m(\cos\theta)\sin(m\varphi)$ = first piece * second piece :

first piece = $\text{poly}_{n-m}(\mu\nu; a_{mn}; \text{by } 2)$

second piece = $\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}\ \text{poly}_{m-1}(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m odd
second piece = $\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}\ \text{poly}_{m-2}(\mu^2,\nu^2; a'_m; \text{by } 1)$ // m even

---

We descry four distinct cases here, two for each tesseral starting form.

Aside: It is completely clear that it is a cos(mφ) case that involves square roots with k's and these will be the M functions later on.

Continue Processing

We now want to combine our polynomial factors, perhaps giving up some of our delicate information. First, if someone hands you $p_4(x,y,z)$, you take that to mean a highest term might be x^4 or xy^2z etc. So if you combined $p_2(z)p_4(x,y)$, you would get $p_6(x,y,z)$. Just we, we claim that

cos(mφ) harmonic, m = even:
$poly_{n-m}(\mu\nu;\ a_{mn};\ by\ 2)\ poly_{m}(\ \mu^2,\nu^2;\ a'_{m};\ by\ 1)$ $= poly_{n}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$

When we combine by 2 with by 1, we are doubtless going to end up with by 1, as indicated. the other situation here would be

cos(mφ) harmonic, m = odd
$poly_{n-m}(\mu\nu;\ a_{mn};\ by\ 2)\ poly_{m-1}(\ \mu^2,\nu^2;\ a'_{m};\ by\ 1)$ $= poly_{n-1}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$

sin(mφ) harmonic, m = odd
$poly_{n-m}(\mu\nu;\ a_{mn};\ by\ 2)\ poly_{m-1}(\ \mu^2,\nu^2;\ a'_{m};\ by\ 1)$ $= poly_{n-1}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$

sin(mφ) harmonic, m = even
$poly_{n-m}(\mu\nu;\ a_{mn};\ by\ 2)\ poly_{m-2}(\ \mu^2,\nu^2;\ a'_{m};\ by\ 1)$ $= poly_{n-2}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$

We can now redo our summary with this combined information, where full now means we have combined the first and second pieces together into a single piece:

$P_n^m(\cos\theta)\cos(m\varphi)$ = first piece * second piece :

full = $poly_{n}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$ // m even
full = $\sqrt{k^2 - \mu^2}\ \sqrt{k^2 - \nu^2}\ poly_{n-1}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$ // m odd

---

$P_n^m(\cos\theta)\sin(m\varphi)$ = first piece * second piece :

full = $\sqrt{\mu^2 - h^2}\ \sqrt{h^2 - \nu^2}\ poly_{n-1}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$ // m odd
full= $\sqrt{\mu^2 - h^2}\ \sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \mu^2}\ \sqrt{k^2 - \nu^2}\ poly_{n-2}(\ \mu^2,\nu^2,\ \mu\nu;\ a'_{m};\ by\ 1)$ // m even

---

Comment on Hobson p 459 B

I get his results as quoted if I associate my $poly_n$ with his U_n and similarly for n-1 and n-2. But I take issue with his characterization of these functions. I have shown above that what he calls U_n and the other two cases are in fact *polynomials* in ($\mu^2,\nu^2,\ \mu\nu$). This means a polynomial of three variables, which I think is still called a polynomial. He calls these "rational algebraical functions". His language suggests somehow "rational functions" are involved with a numerator and denominator. Sure, a polynomial is rational with denominator 1. The term algebraic suggests radicals, but there are none in these polys. I think the implication of radicals of the word algebraic is a later usage.

Here is one web definition

Sci-Tech Dictionary: rational algebraic expression

Home > Library > Science > Sci-Tech Dictionary

(ˈrash·ən·əl ˈal·jəˈbrā·ik ikˈspresh·ən)

(*mathematics*) An algebraic expression that equals a quotient of polynomials.

I will assume the word algebraical as used in older times just meant related to algebra, such as a polynomial is algebraic.

I just found a little PDF that discusses the meaning of "what is an algebraic function". People do agree with the wiki definition that I am used to. Consider

$$X^1 - (ax^2+bx+c)X^0 = 0$$

This is a polynomial in X whose coefficients are polynomials in x. The root is

$$X(x) = (ax^2+bx+c)$$

so sure, a polynomial falls into the class of algebraic functions. But I think in GR7 and the like, we are more interested in algebraic functions which are not polynomials, as in

$$X^3 - (ax^2+bx+c)X^0 = 0 \qquad X(x) = \sqrt[3]{ax^2+bx+c}$$

3. Finding the general form of the functions $E_{n,p}(\mu)$: the famous K,L,M,N forms.

In doc "Polar and Cartesian" we show the P and E functions are both complete sets on the unit sphere and therefore we are going to be able to express one in terms of the other this way

$$E_{n,p}(\mu)\, E_{n,p}(\nu) = \Sigma_m\, g_{n,m}(p)\, P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$$

$$P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)] = \Sigma_p\, f_{n,p}(m)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \qquad (*)$$

where we are looking at a specific value of n here. We now want to use our knowledge gleaned in the previous Section of our current doc to see if we can somehow "deduce" the nature of the E functions from the second equation above. Each term in equation (*) must FACTOR as shown into the product of two functions.

Regarding dimensions: We shall later find that $\dim[E_{n,p}(\mu)] = n$. Thus, $\dim(EE) = 2n$ and we have

$$\dim[g_{n,m}(p)] = 2n \qquad \dim[f_{n,m}(p)] = -2n$$

So let's consider again our four cases.

Case I: $\mathbf{P_n^m(\cos\theta)\cos(m\varphi)}$ **with m = even:**

$\text{poly}_n(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1) = \Sigma_p f_{n,p}(m) E_{n,p}(\mu) E_{n,p}(\nu) \equiv F_n(\mu,\nu)$ which is symmetric $\mu \leftrightarrow \nu$

All the non-vanishing terms on the LHS must therefore be symmetric under $\mu \leftrightarrow \nu$, or they must occur in pairs like this

$[(\mu^2)^a(\nu^2)^b + (\mu^2)^b(\nu^2)^a] (\mu\nu)^c$ where a+b+c = n and a,b,c = integers

$[\mu^{2a}\nu^{2b} + \mu^{2b}\nu^{2a}] (\mu\nu)^c$

Let's just consider *a priori* this candidate form for E

$E_{n,p}(\mu) = a\mu^n + b\mu^{n-1} + c\mu^{n-3} + ...$

Then

$E_{n,p}(\mu) E_{n,p}(\nu) = (a\mu^n + b\mu^{n-1} + c\mu^{n-3} + ...)(a\nu^n + b\nu^{n-1} + c\nu^{n-3} + ...)$

$= a^2 \mu^n\nu^n + ab \mu^n\nu^{n-1} + ba \mu^{n-1}\nu^n + b^2 \mu^{n-1}\nu^{n-1} + ...$

Notice that this produces the symmetric term

$\mu^n\nu^{n-1} + \mu^{n-1}\nu^n$

Can we write this in the form shown above,

$[\mu^{2a}\nu^{2b} + \mu^{2b}\nu^{2a}] (\mu\nu)^c = \mu^{2a+c}\nu^{2b+c} + \nu^{2a+c}\mu^{2b+c}$

We would need

$2a+c = n$
$2b+c = n-1 \quad \Rightarrow \quad 2(a-b) = n-(n-1) = 1 \quad \Rightarrow \quad a-b = 1/2$

But this is impossible because a and b are supposed to be integers and cannot differ by 1/2. The only way to eliminate these illegal symmetric pairs is to adjust the candidate series to read

$E_{n,p}(\mu) = a\mu^n + b\mu^{n-2} + c\mu^{n-4} + ..$

Then we have

$E_{n,p}(\mu) E_{n,p}(\nu) = (a\mu^n + b\mu^{n-2} + c\mu^{n-4} + ...)(a\nu^n + b\nu^{n-2} + c\nu^{n-4} + ...)$

$= a^2 (\mu\nu)^n + ab [\mu^{n-2}\nu^n + \nu^{n-2}\mu^n] + ...$

Now let's verify that this symmetric pair term is legal. We write it again as

$$\mu^{2a+c}\nu^{2b+c} + \nu^{2a+c}\mu^{2b+c}$$

We would need

$$2a+c = n-2$$
$$2b+c = n \quad \Rightarrow \quad 2(a-b) = n-(n-2) = 1 \quad \Rightarrow \quad a-b = 1$$

and this is no problem regardless of whether n is even or odd.

Our conclusion at this point is this:

The E functions will have this form:

$$E_{n,p}(\mu) = a\mu^n + b\mu^{n-2} + c\mu^{n-4} + ..$$

Case II: For $P_n^m(\cos\theta)\cos(m\varphi)$ with m = odd.

Now what happens for m = odd? In this case we are going to have

$$\sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\ \text{poly}_{n-1}(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$$

$$= \Sigma_p\, f_{n,p}(m)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \equiv F_n(\mu,\nu) \text{ which is symmetric } \mu \leftrightarrow \nu$$

An obvious solution in this case seems to be

$$E_{n,p}(\mu) = \sqrt{k^2 - \mu^2}\,[\, a\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$

We basically just repeat all of our discussion above, but here we have this radical out front, and we have n → n-1. So our conclusions are now

$P_n^m(\cos\theta)\cos(m\varphi)$ m = even:

$$E_{n,p}(\mu) = a\mu^n + b\mu^{n-2} + c\mu^{n-4} + .. \qquad \text{// called } K_{n,p}(\mu)$$

$P_n^m(\cos\theta)\cos(m\varphi)$ m = odd:

$$E_{n,p}(\mu) = \sqrt{k^2 - \mu^2}\,[\, a\mu^{n-1} + b\mu^{n-3} + c\mu^{n-5} + ..]$$

Hobson negates the radical so we can absorb the i into the coefficients which then all become imaginary when μ is in our usual "physical range". So

$$E_{n,p}(\mu) = \sqrt{\mu^2 - k^2}\,[\, ia\mu^{n-1} + ib\mu^{n-3} + ic\mu^{n-5} + ..] \qquad \text{// called } M_{n,p}(\mu)$$

Now let's continue onto our other two cases.

Case III: $P_n^m(\cos\theta)\sin(m\varphi)$ for m = even

$$\sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}\ \ \text{poly}_{n-2}(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$$

$$= \left(\sqrt{\mu^2 - h^2}\sqrt{k^2 - \mu^2}\right)\left(\sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \nu^2}\right)\text{poly}_{n-2}(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$$

$$= \Sigma_p\, f_{n,p}(m)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \quad \equiv F_n(\mu,\nu) \text{ which is symmetric } \mu \leftrightarrow \nu$$

We need to flip the first radical and add an i so we have

$$= \left(\sqrt{h^2 - \mu^2}\sqrt{k^2 - \mu^2}\right)\left(\sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \nu^2}\right) i\ \text{poly}_{n-2}(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$$

$$= \Sigma_p\, f_{n,p}(m)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \quad \equiv F_n(\mu,\nu) \text{ which is symmetric } \mu \leftrightarrow \nu$$

Everything is really "real", but we are just trying to get the μ and ν functional forms to be the same. This suggests that we try

$$E_{n,p}(\mu) = \sqrt{h^2 - \mu^2}\sqrt{k^2 - \mu^2}\ [\, a\mu^{n-2} + b\mu^{n-4} + c\mu^{n-6} + ..]$$

and then all of our previous work applies, but we now have n → n-2 and we have these two radicals out front. Hobson flips both radicals, which creates a -1 which we can absorb into coefficients and we then write

$$E_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\sqrt{\mu^2 - k^2}\ [\, a\mu^{n-2} + b\mu^{n-4} + c\mu^{n-6} + ..] \qquad \text{// called } N_{n,p}(\mu)$$

Case IV: $P_n^m(\cos\theta)\sin(m\varphi)$ we have for m = odd

We get

$$= \sqrt{\mu^2 - h^2}\sqrt{h^2 - \nu^2}\ \ \text{poly}_{n-1}(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$$

$$= \Sigma_p\, f_{n,p}(m)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \quad \equiv F_n(\mu,\nu) \text{ which is symmetric } \mu \leftrightarrow \nu$$

I will first flip the second radical argument to rephrase this as

$$= \sqrt{\mu^2 - h^2}\sqrt{\nu^2 - h^2}\ \ i\,\text{poly}_{n-1}(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$$

$$= \Sigma_p\, f_{n,p}(m)\, E_{n,p}(\mu)\, E_{n,p}(\nu) \quad \equiv F_n(\mu,\nu) \text{ which is symmetric } \mu \leftrightarrow \nu$$

The obvious choice here is this:

$$E_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\ [\, ia\mu^{n-1} + ib\mu^{n-3} + ic\mu^{n-5} + ..] \qquad \text{// called } L_{n,p}(\mu)$$

Don't worry about the i's, everything is real if you put the radicals in the proper coordinate order.

Summary of conclusions to this point:

$P_n^m(\cos\theta)\cos(m\varphi)$ with m = even:

Need $E_{n,p}(\mu) = K_{n,p}(\mu) = a\mu^n + b\mu^{n-2} + c\mu^{n-4} + ...$
$E_{n,p}(\nu) = K_{n,p}(\nu) = a\nu^n + b\nu^{n-2} + c\nu^{n-4} + ...$

$P_n^m(\cos\theta)\cos(m\varphi)$ with m = odd:

Need $E_{n,p}(\mu) = M_{n,p}(\mu) = \sqrt{\mu^2 - k^2}\,[\, ia\mu^{n-1} + ib\mu^{n-3} + ic\mu^{n-5} + ..]$
$E_{n,p}(\nu) = M_{n,p}(\nu) = \sqrt{\nu^2 - k^2}\,[\, ia\nu^{n-1} + ib\nu^{n-3} + ic\nu^{n-5} + ..]$

$P_n^m(\cos\theta)\sin(m\varphi)$ with m = even:

Need $E_{n,p}(\mu) = N_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\sqrt{\mu^2 - k^2}\,[\, a\mu^{n-2} + b\mu^{n-4} + c\mu^{n-6} + ..]$
$E_{n,p}(\nu) = N_{n,p}(\nu) = \sqrt{\nu^2 - h^2}\sqrt{\nu^2 - k^2}\,[\, a\nu^{n-2} + b\nu^{n-4} + c\nu^{n-6} + ..]$

$P_n^m(\cos\theta)\sin(m\varphi)$ with m = odd:

Need $E_{n,p}(\mu) = L_{n,p}(\mu) = \sqrt{\mu^2 - h^2}\,[\, ia\mu^{n-1} + ib\mu^{n-3} + ic\mu^{n-5} + ..]$
$E_{n,p}(\nu) = L_{n,p}(\nu) = \sqrt{\nu^2 - h^2}\,[\, ia\nu^{n-1} + ib\nu^{n-3} + ic\nu^{n-5} + ..]$

Regarding dimensions: We will usually take the leading coefficient as unity, or at worst, some dimensionless number. So just think of a as dimensionless and you see that all four cases of the E function have $\dim(E_n) = n$. The terms in the series part are thus all of dimension n, which means that the coefficients b,c,d... must have compensating dimensions.

I would then summarize these results as follows:

$P_n^m(\cos\theta)\cos(m\varphi) = \Sigma\,\alpha\,K(\mu)K(\nu)$ m = even

$P_n^m(\cos\theta)\cos(m\varphi) = \Sigma\,\alpha\,M(\mu)M(\nu)$ m =odd

$P_n^m(\cos\theta)\sin(m\varphi) = \Sigma\,\alpha\,N(\mu)N(\nu)$ m = even

$P_n^m(\cos\theta)\sin(m\varphi) = \Sigma\,\alpha\,L(\mu)L(\nu)$ m = odd

which we can rewrite as follows, where in each case m is an integer.

$P_n^{2m}(\cos\theta)\cos(2m\varphi) = \Sigma\,\alpha\,K(\mu)K(\nu)$

$P_n^{2m+1}(\cos\theta)\cos([2m+1]\varphi) = \Sigma\, \alpha\, M(\mu)M(\nu)$

$P_n^{2m}(\cos\theta)\sin(2m\varphi) = \Sigma\, \alpha\, N(\mu)N(\nu)$

$P_n^{2m+1}(\cos\theta)\sin([2m+1]\varphi) = \Sigma\, \alpha\, L(\mu)L(\nu)$

Interestingly, this does not agree with Hobson page 460, he has L and M reversed, and this must be a typo in his book! We know (earlier aside) that the M function involves root with k, and that appears in our cos(mφ) case.

Check With Byerly on definition of L and M

His situation is this

136. If we are dealing with an *ellipsoid* instead of a spheroid, we can take as our orthogonal system of surfaces a set of *confocal quadrics*;

$$\left.\begin{aligned} \frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2}-1=0 \\ \frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}+\frac{z^2}{\mu^2-c^2}-1=0 \\ \frac{x^2}{\nu^2}+\frac{y^2}{\nu^2-b^2}+\frac{z^2}{\nu^2-c^2}-1=0 \end{aligned}\right\} \quad (1)$$

where $\lambda^2 > c^2 > \mu^2 > b^2 > \nu^2$. Here the first surface is an ellipsoid, the

so since $c^2 > b^2$ we identify k = c and h = b and c > b. A little later we see these statements:

$$L_m^p(x) = \sqrt{x^2-b^2}\,[x^{m-1} + a_{m-3}x^{m-3} + a_{m-5}x^{m-5} + \cdots], \quad (10)$$

$$M_m^p(x) = \sqrt{x^2-c^2}\,[x^{m-1} + a_{m-3}x^{m-3} + a_{m-5}x^{m-5} + \cdots] \quad (11)$$

The one with the larger constant in it is called M and this *agrees with Hobson* p 460 top.

4. New ways to write E and EE and EEE.

(a) a new way to write $E_{n,p}(\mu)$ $E_{n,p}(\nu)$ and $E_{n,p}(\mu)$

In the above we claimed that we can expand an EE product within the n manifold onto the tesseral harmonics this way:

$E_{n,p}(\mu)\, E_{n,p}(\nu) = \Sigma_m\, g_{n,m}(p)\, P_n^m(\cos\theta)[\sin(m\varphi),\cos(m\varphi)]$

But we also claimed that

$P_n^m(\cos\theta)\cos(m\varphi)$ = first piece * second piece :

full = $\text{poly}_n\,(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$ // m even

full = $\sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\;\text{poly}_{n-1}\,(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$ // m odd

---

$P_n^m(\cos\theta)\sin(m\varphi)$ = first piece * second piece : ----9+87

full = $\sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\;\text{poly}_{n-1}\,(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$ // m odd

full= $\sqrt{\mu^2 - h^2}\,\sqrt{h^2 - \nu^2}\;\sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\;\text{poly}_{n-2}\,(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$ // m even

---

Each of the tesseral harmonics has the form $\text{poly}(\mu^2,\nu^2, \mu\nu)$, and therefore we must conclude that (here I have adjusted the poly coefficients so that all the radical factors have the variable second)

$$E_{n,p}(\mu)\, E_{n,p}(\nu) = \{ 1, \sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\,, \sqrt{h^2 - \mu^2}\,\sqrt{h^2 - \nu^2}\,, \sqrt{h^2 - \mu^2}\,\sqrt{h^2 - \nu^2}\;\sqrt{k^2 - \mu^2}\,\sqrt{k^2 - \nu^2}\,\} * \Sigma_m\, g_{n,m}(p)\text{poly}_{n-I}\,(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$$

where I = 0,1 or 2 depending on which factor you select from the bracket (class = K, LM, N). Think of this

$E_{n,p}(\mu)\, E_{n,p}(\nu)$ = [radical factors piece] * [polynomial piece]

where [polynomial piece] = $\text{poly}_{n-I}\,(\mu^2,\nu^2, \mu\nu; a'_m; \text{by } 1)$. It is obvious that the RHS is symmetric in $\mu\leftrightarrow\nu$ because the LHS is manifestly so. But we have more information that just that fact. We know that the polynomial piece is $f(\mu^2,\nu^2,\mu\nu)$. Thus, you could never have something like

$E_{n,p}(\mu)\, E_{n,p}(\nu)_{\text{polynomial piece}} = (A\mu+B)\,(A\nu+B)$

Although symmetric, this has terms linear in μ and ν, which are not allowed!

Now, I think there is a nice way to characterize this poly piece. Consider

$f(\mu^2,\nu^2, \mu\nu) = \Sigma_{abc}\, f_{abc}(\mu^2)^a(\nu^2)^b(\mu\nu)^c$

We can write

$(\mu\nu)^c = (\mu^2)^{c/2}\,(\nu^2)^{c/2}$ if c is even

$$(\mu\nu)^c = \mu\nu\,(\mu^2)^{(c-1)/2}\,(\nu^2)^{(c-1)/2} \qquad \text{if c is odd}$$

So split the series f into its two pieces based on even or odd

$$\begin{aligned} f(\mu^2,\nu^2,\mu\nu) &= \Sigma_{ab}\Sigma_{c_even}\, f_{abc}(\mu^2)^a(\nu^2)^b(\mu\nu)^c \\ &\quad + \Sigma_{ab}\Sigma_{c_odd}\; f_{abc}(\mu^2)^a(\nu^2)^b(\mu\nu)^c \\ &= \Sigma_{ab}\Sigma_{c_even}\, f_{abc}(\mu^2)^a(\nu^2)^b(\mu^2)^{c/2}\,(\nu^2)^{c/2} \\ &\quad + \Sigma_{ab}\Sigma_{c_odd}\, f_{abc}(\mu^2)^a(\nu^2)^b\,\mu\nu\,(\mu^2)^{(c-1)/2}\,(\nu^2)^{(c-1)/2} \\ &= \Sigma_{ab}\Sigma_{c_even}\, f_{abc}(\mu^2)^{a+c/2}(\nu^2)^{b+c/2} \\ &\quad + \mu\nu\;\Sigma_{ab}\Sigma_{c_odd}\, f_{abc}(\mu^2)^{a+(c-1)/2}\,(\nu^2)^{b+(c-1)/2} \\ &= \text{poly}_1(\mu^2,\nu^2) + \mu\nu\,\text{poly}_2(\mu^2,\nu^2) \end{aligned}$$

So we now have

$$\begin{aligned} & E_{n,p}(\mu)\,E_{n,p}(\nu) \\ &= * \{ 1, \sqrt{k^2-\mu^2}\sqrt{k^2-\nu^2}, \sqrt{h^2-\mu^2}\sqrt{h^2-\nu^2}, \sqrt{h^2-\mu^2}\sqrt{h^2-\nu^2}\;\sqrt{k^2-\mu^2}\sqrt{k^2-\nu^2} \} \\ &\quad * \Sigma_m\, g_{n,m}(p)\,\text{poly}_{n-I}(\mu^2,\nu^2,\mu\nu;\, a'_m;\, \text{by } 1) \\ &= * \{ 1, \sqrt{k^2-\mu^2}\sqrt{k^2-\nu^2}, \sqrt{h^2-\mu^2}\sqrt{h^2-\nu^2}, \sqrt{h^2-\mu^2}\sqrt{h^2-\nu^2}\;\sqrt{k^2-\mu^2}\sqrt{k^2-\nu^2} \} \\ &\quad * \Sigma_m\, g_{n,m}(p)\,[\text{poly}_1(\mu^2,\nu^2;\, a_m';\, \text{by } 1) + \mu\nu\,\text{poly}_2(\mu^2,\nu^2\, a_m';\, \text{by } 1)] \end{aligned}$$

Remember that we are just picking ONE of the bracket factors and this depends on what p is.

Now suppose α is a non-vanishing root of $E_{n,p}(\mu)$. Then, we have

$$0 = E_{n,p}(\alpha)\,E_{n,p}(\alpha) = \{..\, f(\alpha^2...\} * \Sigma_m\, g_{n,m}(p)\,[\text{poly}_1(\alpha^2,\alpha^2;\, a_m';\, \text{by } 1) + \alpha^2\,\text{poly}_2(\alpha^2,\alpha^2\, a_m';\, \text{by } 1)]$$

Hobson shows in his zeros section why such a zero cannot be at h or k, and therefore the radical factor is non-zero. So we conclude that

$$\Sigma_m\, g_{n,m}(p)\,[\text{poly}_1(\alpha^2,\alpha^2;\, a_m';\, \text{by } 1) + \alpha^2\,\text{poly}_2(\alpha^2,\alpha^2\, a_m';\, \text{by } 1)] = 0$$

If this equation is true for some α, then it must also be true for -α ! Thus, <u>the roots of $E_{n,p}(\mu)$ must always occur in pairs.</u>

Now we bring in more knowledge. We know that $E_{n,p}(\mu)$ itself (not a product of two of them) has this form

$$E_{n,p}(\mu) = \{ 1, \sqrt{k^2-\mu^2}, \sqrt{h^2-\mu^2}, \sqrt{h^2-\mu^2}\;\sqrt{k^2-\mu^2} \}\,\text{poly}_{n-I}(\mu, \text{by } 2) \qquad (*)$$

// Note that dim[$\text{poly}_{n-I}(\mu, \text{by } 2)$] = n-I so dim E = I + (n-I) = n

where again I depends on which class (bracket factor) you have. If n-I is odd, we can factor out μ from the poly (the lowest power is μ). So we have

$$\text{poly}_{n-I}(\mu, \text{by } 2) = \text{poly}_{(n-I)/2}(\mu^2, \text{by } 1) \qquad n\text{-}I = \text{even}$$

$$\text{poly}_{n-I}(\mu, \text{by } 2) = \mu\, \text{poly}_{(n-I-1)/2}(\mu^2, \text{by } 1) \qquad n\text{-}I = \text{odd}$$

Let's define the following J symbol,

J = (n-I)/2	n-I = even	<=	n even AND K or N, or, n odd AND L or M
J = (n-I-1)/2	n-I = odd	<=	n even AND L or M, or, n odd AND K or N

So what exactly is J?

		J
n even	K	n/2
	N	(n-2)/2
	L,M	(n-2)/2
n odd	K	(n-1)/2
	N	(n-3)/2
	L,M	(n-1)/2

We can then write our (*) above in this new way

$$E_{n,p}(\mu) = \{1, \sqrt{k^2 - \mu^2}, \sqrt{h^2 - \mu^2}, \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2}\ \}\ [1, \mu]\ \text{poly}_J(\mu^2, \text{by } 1) \qquad (*)$$

where in [1, μ] we select the 1 *or* the μ depending on n-I even or odd! So finally, since the roots occur in pairs, we can write (κ is some constant controlling norm of the E functions)

$$\text{poly}_J(\mu^2, \text{by } 1) = \kappa \prod_{i=1}^{J} (\mu^2 - \alpha_i^2)$$

Yes, if we have the factor μ, it represents a single root at μ = 0, and then this is a root in addition to the other roots shown in the product.

Dimension Check:

dim {.....} = 0, 1, 1, 2 = I

dim [..] = 0 if n-I even
1 if n-I odd

dim poly_J = 2J = (n-I) n-I even // think of just the leading term
= (n-I-1) n-I odd

dim(E) = dim {.....} + dim [..] + dim poly_J

$= I + 0 + (n-I) = n$ n-I even

$= I + 1 + (n-I-1) = n$ n-I odd so dimension checks, we get n

Summary of this result:

$$E_{n,p}(\mu) = \{ 1 , \sqrt{k^2 - \mu^2} , \sqrt{h^2 - \mu^2} , \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2} \} [1, \mu] \kappa \Pi_{i=1}^{J} (\mu^2 - \alpha_i^2)$$

I = 0 I=1 I=1 I=2 // class dependent

J = (n-I)/2 n-I = even => select 1 in [1, μ]

J = (n-I-1)/2 n-I = odd => select μ in [1, μ]

// Note that $\dim[\Pi_{i=1}^{J} (\mu^2 - \alpha_i^2)] = 2J$. J =int[(n-I)/2]

// Note that, for a given n, there are only 4 types of solutions, not 8 types

Let's write out these 4 types for each n more explicitly:

n even:

$$E_{n,p}(\mu) = \{ 1 , \mu\sqrt{k^2 - \mu^2} , \mu\sqrt{h^2 - \mu^2} , \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2} \} \kappa \Pi_{i=1}^{J} (\mu^2 - \alpha_i^2)$$ J =int[(n-I)/2]

n odd:

$$E_{n,p}(\mu) = \{ \mu , \sqrt{k^2 - \mu^2} , \sqrt{h^2 - \mu^2} , \mu \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2} \} \kappa \Pi_{i=1}^{J} (\mu^2 - \alpha_i^2)$$ J =int[(n-I)/2]

Here is a table explicitly showing what J is in all possible cases

		J
n even	K	n/2
	N	(n-2)/2
	L,M	(n-2)/2
n odd	K	(n-1)/2
	N	(n-3)/2
	L,M	(n-1)/2

and now we have learned a bit more about what one of these $E_{n,p}(\mu)$ functions looks like in terms of the number of root pairs in the polynomial part.

As an example, suppose n = 1. We have

$K_1 = \mu$ n = 1 I = 0, n-I = 1 = odd, so J = (n-I-1)/2 = 0, so no roots Π = 1.

$L_1 = i\sqrt{k^2 - \mu^2}$ n = 1 I = 1, n-I = 0 = even, so J = (n-1)/2 = 0, so no roots Π = 1.

$K_{2\pm} = (\mu^2 - [c \pm d])$ n=2 I= 0 , n-I even, J = (n-I)/2 = 1, one pair roots, select 1 (no μ)

(b) a new way to write $E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$

Now what happens if we multiple three E functions together using our formula above for each:

$$E_{n,p}(\rho) = \{ 1 , \sqrt{k^2 - \rho^2} , \sqrt{h^2 - \rho^2} , \sqrt{h^2 - \rho^2}\ \sqrt{k^2 - \rho^2} \}\ [1, \rho]\ \kappa\ \Pi_{i=1}^{J} (\rho^2 - \alpha_i^2)$$
$$E_{n,p}(\mu) = \{ 1 , \sqrt{k^2 - \mu^2} , \sqrt{h^2 - \mu^2} , \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2} \}\ [1, \mu]\ \kappa\ \Pi_{i=1}^{J} (\mu^2 - \alpha_i^2)$$
$$E_{n,p}(\nu) = \{ 1 , \sqrt{k^2 - \nu^2} , \sqrt{h^2 - \nu^2} , \sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \nu^2} \}\ [1, \nu]\ \kappa\ \Pi_{i=1}^{J} (\nu^2 - \alpha_i^2)$$

Here all three are the same E function, so we are talking a "normal" solution. The E's have the same n and the same class and the same p within that class if several. It is a bit clumsy trying to write the triple product EEE but here it is:

$$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu) = \kappa^3$$
$$\{ 1 , \sqrt{k^2 - \rho^2} , \sqrt{h^2 - \rho^2} , \sqrt{h^2 - \rho^2}\ \sqrt{k^2 - \rho^2} \}\ [1, \rho]$$
$$\{ 1 , \sqrt{k^2 - \mu^2} , \sqrt{h^2 - \mu^2} , \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2} \}\ [1, \mu]$$
$$\{ 1 , \sqrt{k^2 - \nu^2} , \sqrt{h^2 - \nu^2} , \sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \nu^2} \}\ [1, \nu]$$
$$\Pi_{i=1}^{J} (\rho^2 - \alpha_i^2)\ (\mu^2 - \alpha_i^2)\ (\nu^2 - \alpha_i^2) \qquad (*)$$

// Note that for a given n, there are 4 types of solutions, not 8 types.

where it must be understood that, given a specific n and p, you pick the same factor in {...} in each variable, and you pick the same factor in [...] as well. I think the above is a precision statement of what Hobson has written in **p 476 A**.

For example, if all are K and n odd, then n-I is odd, J = (n-I-1)/2, and we get

$$K_{n,p}(\rho)\, K_{n,p}(\rho)\, K_{n,p}(\rho) = \kappa^3\ 1*1*1\ \text{x}\ \rho*\mu*\nu\ \Pi_{i=1}^{J} (\rho^2 - \alpha_i^2)\ \Pi_{i=1}^{J} (\mu^2 - \alpha_i^2)\ \Pi_{i=1}^{J} (\mu^2 - \alpha_i^2)$$

$$= \rho\mu\nu\ \kappa^3\ \Pi_{i=1}^{J} [(\rho^2 - \alpha_i^2)\ (\mu^2 - \alpha_i^2)\ (\mu^2 - \alpha_i^2)]$$

The number of [...] factors is just J as given by the table above. This is then the number of α_i root pairs.

Let's write out the four cases for each n:

$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu) = \kappa^3\ \ \Pi_{i=1}^{J} (\rho^2 - \alpha_i^2)\ (\mu^2 - \alpha_i^2)\ (\nu^2 - \alpha_i^2)$ <u>times the entry below:</u>

n even:

K { 1,

M $\rho\mu\nu \sqrt{k^2 - \rho^2}\ \sqrt{k^2 - \mu^2}\ \sqrt{k^2 - \nu^2}$,

L $\rho\mu\nu \sqrt{h^2 - \rho^2}\ \sqrt{h^2 - \mu^2}\ \sqrt{h^2 - \nu^2}$,

N $\sqrt{h^2 - \rho^2}\ \sqrt{k^2 - \rho^2}\ \sqrt{h^2 - \mu^2}\ \sqrt{k^2 - \mu^2}\ \sqrt{h^2 - \nu^2}\ \sqrt{k^2 - \nu^2}$ }

n odd:

K { $\rho\mu\nu$,

M $\sqrt{k^2 - \rho^2}\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2}$,

L $\sqrt{h^2 - \rho^2}\sqrt{h^2 - \mu^2}\sqrt{h^2 - \nu^2}$,

N $\rho\mu\nu\sqrt{h^2 - \rho^2}\sqrt{k^2 - \rho^2}\sqrt{h^2 - \mu^2}\sqrt{k^2 - \mu^2}\sqrt{h^2 - \nu^2}\sqrt{k^2 - \nu^2}$ }

Recall now the ellipsoidal x,y,z equations

$x = \rho\mu\nu/(hk)$ // $\rho, \mu, \nu = \xi_1, \xi_2, \xi_3$

$y = \sqrt{\rho^2-h^2}\sqrt{\mu^2- h^2}\sqrt{h^2- \nu^2} / (hs)$ $s^2 = k^2-h^2$

$z = \sqrt{\rho^2-k^2}\sqrt{k^2- \mu^2}\sqrt{k^2- \nu^2} /(ks)$

From these we may compute up some of our triple products EEE:

e $K_0\ K_0\ K_0$ 1

o $L_1\ L_1\ L_1$ $\sqrt{h^2 - \rho^2}\sqrt{h^2 - \mu^2}\sqrt{h^2 - \nu^2} = y * ihs$

o $M_1\ M_1\ M_1$ $\sqrt{k^2 - \rho^2}\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2} = z * iks$

e $N_2\ N_2\ N_2$ $\sqrt{h^2 - \rho^2}\sqrt{k^2 - \rho^2}\sqrt{h^2 - \mu^2}\sqrt{k^2 - \mu^2}\sqrt{h^2 - \nu^2}\sqrt{k^2 - \nu^2} = (y * ihs)(z * iks)$
$= yz * (-hks^2)$

o $K_1\ K_1\ K_1$ $\rho\mu\nu = x * hk$

e $L_2\ L_2\ L_2$ $\rho\mu\nu\sqrt{k^2 - \rho^2}\sqrt{k^2 - \mu^2}\sqrt{k^2 - \nu^2} = (x * hk)(z * iks) = xz * (ihk^2s)$

e $M_2\ M_2\ M_2$ $\rho\mu\nu\sqrt{h^2 - \rho^2}\sqrt{h^2 - \mu^2}\sqrt{h^2 - \nu^2} = (x * hk)(y * ihs) = xy * (ih^2ks)$

o $N_3\ N_3\ N_3$ $\rho\mu\nu\sqrt{h^2 - \rho^2}\sqrt{k^2 - \rho^2}\sqrt{h^2 - \mu^2}\sqrt{k^2 - \mu^2}\sqrt{h^2 - \nu^2}\sqrt{k^2 - \nu^2}$
$= (x * hk)(y * ihs)(z * iks) = xyz * (-h^2k^2s^2)$

where the second four come from choosing μ in [1,μ], etc. Notice that the dimensions are always L^{3n}. In the leftmost column, I indicate whether the solution type applies for even or odd values of n based on the listing just done above. We see that this e and o label matches the nature of the index on the triple product shown, in all cases.

This allows us to rewrite the triple E product this way:

$E_{n,p}(\rho)\ E_{n,p}(\mu)\ E_{n,p}(\nu)$

$= \{ 1\ \ x\ y\ z\ xy\ xz\ yz\ xyz \}\ \kappa' \Pi_{i=1}^{J} [(\rho^2- \alpha_i^2)(\mu^2- \alpha_i^2)(\mu^2- \alpha_i^2)]$
K K L M M L N N
e o o o e e e o // values of n for which this form applies (n even or odd)

// for a given value of n, there are only 4 types of solutions, not 8 types

where I show the class associated with each of the 8 possibilities and κ' is some norm constant. This **constant κ'** is different of course for each {...} case and for each case has different dimensions! It is a

function of h and k and we could figure it out for each case if we wanted. Let's show this in a little more detail:

	{ 1	x	y	z	xy	xz	yz	xyz }
κ' =	1	hk	ihs	iks	ih^2ks	ihk^2s	$-hks^2$	$-h^2k^2s^2$
class =	K	K	L	M	M	L	N	N
dimκ'	0	2	2	2	4	4	4	6

The above form is implied in Hobson but not written down as such.

(c) a second way to write $E_{n,p}(\rho)$ $E_{n,p}(\mu)$ $E_{n,p}(\nu)$

If we take our ellipsoid equation and move the 1 to the left, we get Hobson p 476 being 0,

$$[] \equiv [x^2/ \rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1] = 0$$

Consider a point (x,y,z) in Cartesian space. It lies on three orthogonal surfaces whose labels are ρ,μ,ν -- the ellipsoid and two bloids. We know from previous work that ρ^2,μ^2,ν^2 are the three roots of the cubic equation that []=0 implies for variable ρ_s^2 (which MF called ξ^2). We can do it right here. Rewrite [] as follows, by combining the three terms over D

$$[x^2(\rho_s^2-h^2)(\rho_s^2-k^2) + y^2\rho_s^2(\rho_s^2-k^2) + z^2 \rho_s^2(\rho_s^2-h^2) - \rho_s^2(\rho_s^2-k^2)(\rho_s^2-h^2)]/D = [] = 0$$

where D is the product of the three denominator factors,

$$D(\rho_s^2) = \rho_s^2 (\rho_s^2-h^2)(\rho_s^2-k^2)$$

Change all the signs

$$[\rho_s^2(\rho_s^2-k^2)(\rho_s^2-h^2) - x^2(\rho_s^2-h^2)(\rho_s^2-k^2) - y^2\rho_s^2(\rho_s^2-k^2) - z^2 \rho_s^2(\rho_s^2-h^2)/D = - [] = 0$$

But we know that the three roots are ρ^2,μ^2,ν^2 so we can rewrite this as

$$[(\rho_s^2- \rho^2)(\rho_s^2- \mu^2)(\rho_s^2- \nu^2)]/D = - [] = 0$$

So the main idea is that we can write

$$[x^2/ \rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1] = - [(\rho_s^2- \rho^2)(\rho_s^2- \mu^2)(\rho_s^2- \nu^2)]/D$$

which we rewrite as follows

$$(\rho^2- \rho_s^2)(\mu^2- \rho_s^2)(\mu^2- \rho_s^2) = D(\rho_s^2) [x^2/ \rho_s^2 + y^2/(\rho_s^2-h^2) + z^2/(\rho_s^2-k^2) - 1] \qquad (*)$$

where ρ_s is just a dummy variable. Now recall from our previous section that we had

$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{ 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \;\} \;\; \kappa' \prod_{i=1}^{J} [(\rho^2 - \alpha_i^2)(\mu^2 - \alpha_i^2)(\mu^2 - \alpha_i^2)]$

So let's "rename" our roots from α_i to ρ_s and we have

$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{ 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \;\} \;\; \kappa' \prod_{s=1}^{J} [(\rho^2 - \rho_s^2)(\mu^2 - \rho_s^2)(\mu^2 - \rho_s^2)]$

where $\pm \rho_s$ for s = 1..J are the roots of $E_{n,p}(x)$ and are just functions of h and k and the quantized p value. We now replace using (*) to get

$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$

$= \{ 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \;\} \;\; \kappa' (\prod_{s=1}^{J} D(\rho_s^2)) \; \{ \prod_{s=1}^{J} [\, x^2/\rho_s^2 + y^2/(\rho_s^2 - h^2) + z^2/(\rho_s^2 - k^2) - 1] \}$

But $(\prod_{s=1}^{J} D_s)$ is just a constant (a function of h,k and the root values) then which we can absorb into κ', so we then have

$E_{n,p}(\rho)\, E_{n,p}(\mu)\, E_{n,p}(\nu)$ // $\kappa'' = \kappa' (\prod_{s=1}^{J} D(\rho_s^2))$

$= \{ 1 \;\; x \;\; y \;\; z \;\; xy \;\; xz \;\; yz \;\; xyz \;\} \;\; \kappa'' \; \{ \prod_{s=1}^{J} [\, x^2/\rho_s^2 + y^2/(\rho_s^2 - h^2) + z^2/(\rho_s^2 - k^2) - 1] \}$

K K L M M L N N
e o o o e e e o

and here again is the table explicitly showing what J is in all possible cases

		J
n even	K	n/2
	N	(n-2)/2
	L,M	(n-2)/2
n odd	K	(n-1)/2
	N	(n-3)/2
	L,M	(n-1)/2

I think it would be good to do a dimensional check at this point since things are quite messy. Copy down from above

	e	o	o	o	e	e	e	o
	{ 1	x	y	z	xy	xz	yz	xyz }
κ' =	1	hk	ihs	iks	ih^2ks	ihk^2s	$-hks^2$	$-h^2k^2s^2$
class =	K	K	L	M	M	L	N	N
dimκ'	0	2	2	2	4	4	4	6
dim term	0	3	3	3	6	6	6	9

$D(\rho_s^2) = \rho_s^2 \; (\rho_s^2 - h^2)(\rho_s^2 - k^2) \quad => \quad \dim D = 6$

dim $[(\Pi_{s=1}^{J} D(\rho_s^2))] = 6J$

This seems to say that each of the 8 terms has a different dimension.

1 term:	dim = 0 + 6J	I=0	J = n/2 since n even =>	6J = 3n = as expected
x term:	dim = 3 + 6J	I=1	J = (n-1)/2 since n odd =>	3+6(n-1)/2 = 3+3(n-1) = 3n
etc				

Note that $\pm \rho_s$ for s = 1..J are the roots (other than the possible 0 root) of $E_{n,p}(x)$ and are just functions of h and k and the quantized p value. Look now what we have done! We have written the EEE "normal" product entirely in *Cartesian coordinates*. I think this is rather amazing. This is Hobson **p 477 A.** There are four possible classes into which E can fall, and for each one there are two choices you can make from {.....}, and then those choices combined with the last product of factors is the actual result in Cartesian coordinates that EEE takes, and the only unknown is the constant κ" which is normalization dependent. Of course to really know the last product of factors in detail, you have to know all those roots.

At this point, we are talking EEE functions which are *ellipsoidal* harmonics, so we have moved away from the topic of this document which was mainly how *conical* coordinates provide information about the individual E functions. So the flow now resumes back into my Hobson notes doc where I summarize the above results, and then move onward.