Proof of a Theorem? PhL 11.12.10

Here I try (and fail) to prove Hobson p 484 A.

Assume that we have a homogeneous polynomial of this form

$$H_n(x,y,z) = \text{Poly}_n(x,y,z;\, n,m) = a\, x^{n-3}y^2z + \text{similar terms}$$

$$= \Sigma_{abc}\, F_{abc}\, x^a y^b z^c \qquad a+b+c = n$$

The claim of the theorem is this:

$$H_n(\partial_x, \partial_y, \partial_z)\,(1/r) = (-1)^n\,(2n)!\,/\,(2^n n!) * H_n(x,y,z) / r^{2n+1}$$

We start off restating the theorem as

$$\Sigma_{abc}\, F_{abc}\, \partial_x^a \partial_y^b \partial_z^c\,(1/r) = (-1)^n\,(2n)!\,/\,(2^n n!) * \Sigma_{abc}\, F_{abc}\, x^a y^b z^c / r^{2n+1}$$

Obviously [wrong!] this will have to be true term by term, so we will have to show that

$$\partial_x^a \partial_y^b \partial_z^c\,(1/r) = (-1)^n\,(2n)!\,/\,(2^n n!)\; x^a y^b z^c / r^{2n+1} \qquad a+b+c = n$$

I suspect an induction proof is in order here. Suppose a+b+c = 1. Then we want to show

$$\partial_x^a \partial_y^b \partial_z^c\,(1/r) = -\,x^a y^b z^c / r^3 \qquad a+b+c = 1$$

We call upon some known supporting facts:

$$(\partial_i r) = (x_i/r)$$

$$\partial_i(r^n) = n\, r^{n-1}(x_i/r) = n\, x_i\, r^{n-2}$$

$$\Rightarrow\ \partial_i(r^{-n}) = (-n)\, x_i\, r^{-n-2}$$

There are only three cases to consider where one of a,b,c is 1. For the first case we have

abc = 100:	$\partial_x^a \partial_y^b \partial_z^c\,(1/r) = \partial_x(1/r) = (-1)\, x\, r^{-3}$	agrees with claim
abc = 010:	$\partial_x^a \partial_y^b \partial_z^c\,(1/r) = \partial_y(1/r) = (-1)\, y\, r^{-3}$	agrees with claim
abc = 001:	$\partial_x^a \partial_y^b \partial_z^c\,(1/r) = \partial_z(1/r) = (-1)\, z\, r^{-3}$	agrees with claim

Hmmm, OK. Let's assume this is true for some specific integer k > 0:

$$\partial_x^a \partial_y^b \partial_z^c\,(1/r) = (-1)^k\,(2k)!\,/\,(2^k k!)\; x^a y^b z^c / r^{2k+1} \qquad a+b+c = k$$

We want to show it is true for k→k+1, which means we want to show this is true:

$$\partial_x^a \partial_y^b \partial_z^c (1/r) = (-1)^{k+1} (2k+2)! / (2^{k+1}(k+1!)\, x^a y^b z^c / r^{2k+3} \qquad a+b+c = k+1$$

We can simply redefine the symbol a in the last line above to be a+1 so we get

$$\partial_x^{a+1} \partial_y^b \partial_z^c (1/r) = (-1)^{k+1} (2k+2)! / (2^{k+1}(k+1!)\, x^{a+1} y^b z^c / r^{2k+3} \qquad a+b+c = k \quad (1)$$

So equation (1) is the thing we want to prove. There are two other ways to get to k+1. Instead of incrementing a by 1, we could do this to b or c. But I think the symmetry of the situation tells us that we only have to prove the case where we increment a. Therefore, here is what we need to prove:

$$\partial_x [\, \partial_x^a \partial_y^b \partial_z^c (1/r)] \;=\; (-1)^{k+1} (2k+2)! / (2^{k+1}(k+1!)\, x^{a+1} y^b z^c / r^{2k+3} \qquad a+b+c = k$$

But we know from our starting position that

$$\partial_x^a \partial_y^b \partial_z^c (1/r) = (-1)^k (2k)! / (2^k k!)\, x^a y^b z^c / r^{2k+1} \qquad a+b+c = k$$

Therefore, what we need to prove is this:

$$\partial_x [(-1)^k (2k)! / (2^k k!) * x^a y^b z^c / r^{2k+1}]$$

$$= (-1)^{k+1} (2k+2)! / (2^{k+1}(k+1!) * x^{a+1} y^b z^c / r^{2k+3} \qquad a+b+c = k$$

which we can rewrite as

$$(-1)^k (2k)! / (2^k k!) * y^b z^c \, \partial_x [x^a / r^{2k+1}]$$

$$= (-1)^{k+1} (2k+2)! / (2^{k+1}(k+1!) * x^{a+1} y^b z^c / r^{2k+3} \qquad a+b+c = k \qquad \text{// need to prove}$$

$$= (-1)\,(-1)^k (2k+2)\,(2k+1)\,(2k)! / (2^k\, 2\,(k+1)k!) * x^{a+1} y^b z^c / r^{2k+3}$$

We can cancel various factors on both sides, so this then is what we need to prove

$$\partial_x [x^a / r^{2k+1}]$$

$$= (-1)\,(2k+2)\,(2k+1) / (\,2\,(k+1)) * x^{a+1} / r^{2k+3}$$

$$= (-1)\,(2k+1)\, x^{a+1} / r^{2k+3} \qquad \text{// need to prove}$$

So is this true? We can compute the LHS like this

$$\partial_x [x^a \; r^{-2k-1}] \;= a x^{a-1} r^{-2k-1} \; + x^a \, \partial_x (r^{-2k-1})$$

$$= a x^{a-1} r^{-2k-1} \; + x^a \,(-2k-1)\, x\, r^{-2k-3}$$

$$= a x^{a-1} r^{-2k-1} + (-1)(2k+1)\, x^{a+1} / r^{2k+3}$$

Unfortunately, we are getting here an extra first term that we don't want. **Something has gone wrong.** In order to track down this bug, let's as usual try the case k = 2. Then we have this assumed starting position

$$\partial_x^a \partial_y^b \partial_z^c (1/r) = (-1)^2 (4)! / (2^2 2!)\, x^a y^b z^c / r^5 \qquad a+b+c = 2$$

And specifically, let's take the case abc = 200 so we have

$$\partial_x^2 (1/r) = (-1)^2 (4)! / (2^2 2!)\, x^2 / r^5 = 24/8\, x^2 / r^5 = 3\, x^2/r^5 \qquad 2+0+0 = 2$$

First of all, is this even true for a starting position? We have

$$\partial_x^2 (1/r) = \partial_x \partial_x (1/r) = \partial_x [(-1)\, x\, r^{-3}] = (-1)\, \partial_x [x\, r^{-3}]$$

$$= (-1)\, [\, r^{-3} + x\ (-3)\, x\, r^{-3-2}\,] = -1/r^3 + 3\, x^2/r^5$$

So we have just found a **counterexample to our supposed theorem!**

Well, in my theorem here I have not really used the fact that $\Sigma_{abc} F_{abc} x^a y^b z^c$ is a harmonic! I was hoping that fact would not be needed, but I guess it is needed. So the extra information we have is this

$$\nabla^2 [\Sigma_{abc} F_{abc} x^a y^b z^c] = 0$$

My idea of term by term on the theorem is then no go. Now consider

$$\nabla^2 (x^a y^b z^c) = \partial_x^2 (x^a y^b z^c) + \text{other terms}$$

$$= a(a-1) x^{a-2} y^b z^c + .. = a(a-1)\, (x^a y^b z^c)/x^2 + ... \qquad \text{assuming } a \geq 2, \text{ else term is } 0$$

$$= (x^a y^b z^c)[\, a(a-1)/x^2 + b(b-1)/y^2 + c(c-1)/z^2 \ ]$$

Our Laplace condition is then is this

$$\Sigma_{abc} F_{abc} (x^a y^b z^c)[\, a(a-1)/x^2 + b(b-1)/y^2 + c(c-1)/z^2 \ ] = 0 \qquad a+b+c = n$$

so this restricts F_{abc} .

So let's start over on our theorem proof keeping the sum and F_{abc}. The theorem claims that

$$\Sigma_{abc} F_{abc} \partial_x^a \partial_y^b \partial_z^c (1/r) = (-1)^n (2n)! / (2^n n!) * \Sigma_{abc} F_{abc} x^a y^b z^c / r^{2n+1} \qquad a+b+c = n$$

Our induction starting point would be this

$$\Sigma_{abc}\, F_{abc}\, \partial_x^a \partial_y^b \partial_z^c\, (1/r) = (-1)^k\, (2k)!\, /\, (2^k k!) * \Sigma_{abc}\, F_{abc}\, x^a y^b z^c\, /\, r^{2k+1} \quad a+b+c = k$$

What we want to show is this

$$\Sigma_{abc}\, F_{abc}\, \partial_x^a \partial_y^b \partial_z^c\, (1/r)$$

$$= (-1)^{k+1}\, (2k+2)!\, /\, (2^{k+1}(k+1)!) * \Sigma_{abc}\, F_{abc}\, x^a y^b z^c\, /\, r^{2k+2} \qquad a+b+c = k+1$$

If we let a→a+1 this can be restated (this being one of three possible cases)

$$\Sigma_{abc}\, F_{a+1bc}\, \partial_x^{a+1} \partial_y^b \partial_z^c\, (1/r) \qquad \text{// want to show}$$

$$= (-1)^{k+1}\, (2k+2)!\, /\, (2^{k+1}(k+1)!) * \Sigma_{abc}\, F_{a+1bc}\, x^{a+1} y^b z^c\, /\, r^{2k+2} \qquad a+b+c = k$$

Now let's go compute

$$\Sigma_{abc}\, F_{a+1bc}\, \partial_x^{a+1} \partial_y^b \partial_z^c\, (1/r)$$

$$= \partial_x \left[\Sigma_{abc}\, F_{a+1bc}\, \partial_x^a \partial_y^b \partial_z^c\, (1/r) \right] = \text{I don't know}$$

Stop. Unless I have missed some obvious fact, this theorem is non-trivial to prove. I can't prove every single thing I see in the world, so I will just assume it is true and try to see where Hobson is headed. I'm sure the theorem is correct. The proof may involve fancy Lagrange multipliers or some such mess.