

LAMÉ FUNCTIONS OF THE FIRST KIND GENERATED BY COMPUTER

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Abstract. The definitions of the Lamé functions and the ellipsoidal coordinates inherent to them are introduced. Proceeding from Lamé's differential equation the four classes of Lamé functions of the first kind are generated by computer with the aid of formula manipulation techniques. For this purpose algebraic expressions for the coefficients in the Lamé polynomials are constructed by virtue of recurrence formulae and presented in tabular as well as machine readable form for further processing. The determination of a parameter appearing in the polynomial coefficients leads to an algebraic eigenvalue problem. It is shown how this eigenvalue problem is amenable to numerical treatment. The polynomial coefficients together with the eigenvalues are characteristic for the Lamé functions and define them completely.

1. Introduction

In connection with theoretical investigations, the Lamé functions have been repeatedly employed for the formulation of properties emerging from bodies of ellipsoidal shape. Historically, Lamé had introduced these functions in conjunction with the problem of determining the steady temperature in an ellipsoidal conductor when the temperature is prescribed on the boundary of the conductor. In geophysics and particularly in the area of equilibrium figures, occasionally the Lamé functions have been used when a high degree of precision is imposed on the mathematical formulation of structures (Oppenheim, 1922; Darwin, 1910). More recently they have become amenable to the representation of the potential of the Earth as well as other celestial bodies (Doubochine, 1966). A systematic account of Lamé's investigation was given by Heine (1878) and subsequently by Hobson (1931). Considerable use of this account has been made in parts of this paper.

It is a requirement of the practical applications that the Lamé functions are made available in a machine readable form for further processing by means of a computer rather than in a tabular compilation. Therefore the approach pursued in this report aims at the generation of the characteristic coefficients of the Lamé polynomials, which are the principal constituents of the four classes of Lamé functions of the first kind. Once these coefficients have been established, it is only a matter of formality to build up the pertinent Lamé functions.

The line of thought governing the generation of Lamé functions starts out from a differential equation of the second order usually referred to as Lamé's equation. In terms of indeterminate coefficients, the structure of the polynomial expressions which form part of the solutions of this differential equation are known from the theory. By inserting them into the differential equation, recurrence formulae for the deter-

mination of the polynomial coefficients are obtained through a set of conditional equations. In a second step, the recurrence formulae are applied to establish the complete expressions of the individual coefficients up to any specified order.

Because of the formal character of the approach, it exposes itself to the utilization of formula manipulation techniques. Indeed both the formation of the conditional equations and the generation of the polynomial coefficients for the four classes of Lamé functions have been accomplished by means of FORMAC (FORMula MANipulation Compiler). More precisely the computer programs have been written in PL/I-FORMAC (IBM, 1967) and allowance has been made to punch the polynomial coefficients as PL/I statements for processing in the framework of related investigations. With only little modification, the coefficients can be punched in a format compatible with the definition of FORTRAN statements.

This paper is comprised of three chapters. In the course of the first one the definition of the Lamé functions is given jointly with the adopted notations and only as much of it is presented as is essential for an understanding of the ensuing paragraphs. The second chapter deals with the formal solution of Lamé's differential equation resulting in the conditional equations and recurrence formulae for the polynomial coefficients. Finally, the third chapter is dedicated to the determination of the polynomial coefficients, which leads to an algebraic eigenvalue problem. The method for its solution is briefly described. For illustration and checking purposes, a selection of polynomial coefficients pertaining to lower order Lamé functions is presented in the Appendix.

2. The Definition of the Lamé Functions

Linked to the Lamé functions are the ellipsoidal coordinates which are the independent variables of this type of function. The relationship between rectangular coordinates x, y, z and ellipsoidal coordinates ξ_1, ξ_2, ξ_3 is described in the subsequent paragraph followed by a paragraph on the various classes of Lamé functions. Wherever no misunderstanding can arise the symbol ξ stands for each of the three ellipsoidal coordinates.

2.1 ELLIPSOIDAL COORDINATES

For given rectangular coordinates x, y, z the cubic equation in θ^2 ,

$$\frac{x^2}{\theta^2} + \frac{y^2}{\theta^2 - h^2} + \frac{z^2}{\theta^2 - k^2} = 1, \quad (1)$$

with $0 \leq h < k$, has three real roots $\xi_1^2, \xi_2^2, \xi_3^2$, which fulfill the inequality

$$0 \leq \xi_3^2 \leq h^2 \leq \xi_2^2 \leq k^2 \leq \xi_1^2 < \infty. \quad (2)$$

On substituting these three roots into (1), the equations of three confocal surfaces are defined in the rectangular coordinates x, y, z : an ellipsoid, a hyperboloid of one sheet and a hyperboloid of two sheets. If the three equations resulting from (1) by inserting $\xi_1^2, \xi_2^2, \xi_3^2$ are successively solved for x^2, y^2, z^2 , one obtains for the coor-

ordinates of the point of intersection the following relations:

$$x^2 = \frac{\xi_1^2 \xi_2^2 \xi_3^2}{h^2 k^2} \quad (3)$$

$$y^2 = \frac{(\xi_1^2 - h^2)(\xi_2^2 - h^2)(h^2 - \xi_3^2)}{h^2(k^2 - h^2)} \quad (4)$$

$$z^2 = \frac{(\xi_1^2 - k^2)(k^2 - \xi_2^2)(k^2 - \xi_3^2)}{k^2(k^2 - h^2)}. \quad (5)$$

It is in this sense that Heine (1878) and Hobson (1931) have formulated the definition of the ellipsoidal coordinates ξ_1, ξ_2, ξ_3 of a point in terms of its rectangular coordinates x, y, z .

By virtue of (3), (4), and (5) a unique relationship between the rectangular coordinates and the ellipsoidal coordinates is established, provided that an appropriate convention on the selection of the sign has been adopted when the square root is taken on the right hand side of the preceding equations.

2.2 THE FOUR CLASSES OF LAMÉ FUNCTIONS OF THE FIRST KIND

In the framework of this report the symbol $E(\xi)$ is representative of any Lamé function of the first kind. Especially $E_n^s(\xi)$, $s=0, 1, \dots, 2n$, denotes the set of Lamé functions of the order n . They are subdivided into four classes which for a given n encompass $2n+1$ linear independent functions. Let

$$\sigma = \begin{cases} \frac{1}{2}n & \text{for } n \text{ even} \\ \frac{1}{2}(n-1) & \text{for } n \text{ odd.} \end{cases} \quad (6)$$

Then the classes are represented by

$$\begin{aligned} \sigma + 1 & \text{ functions } K_n^s(\xi) \\ n - \sigma & \text{ functions } L_n^s(\xi) \\ n - \sigma & \text{ functions } M_n^s(\xi) \\ \sigma & \text{ functions } N_n^s(\xi), \end{aligned}$$

and K, L, M, N are abbreviations for the expressions defining the Lamé functions of the first, second, third and fourth class respectively:

$$K_n^s(\xi) = a_{0s}\xi^n + a_{1s}\xi^{n-2} + \dots + \begin{cases} a_{\sigma s} & \text{for } n \text{ even} \\ a_{\sigma s}\xi & \text{for } n \text{ odd,} \end{cases} \quad (7)$$

$$\begin{aligned} L_n^s(\xi) = \sqrt{\xi^2 - h^2} & \left[b_{0s}\xi^{n-1} + b_{1s}\xi^{n-3} + \dots \right. \\ & \left. \dots + \begin{cases} b_{n-\sigma-1,s}\xi & \text{for } n \text{ even} \\ b_{n-\sigma-1,s} & \text{for } n \text{ odd} \end{cases} \right], \end{aligned} \quad (8)$$

$$M_n^s(\xi) = \sqrt{\xi^2 - k^2} \left[c_{0s} \xi^{n-1} + c_{1s} \xi^{n-3} + \dots \right. \\ \left. \dots + \begin{cases} c_{n-\sigma-1,s} \xi & \text{for } n \text{ even} \\ c_{n-\sigma-1,s} & \text{for } n \text{ odd} \end{cases} \right], \quad (9)$$

$$N_n^s(\xi) = \sqrt{\xi^2 - h^2} \sqrt{\xi^2 - k^2} \left[d_{0s} \xi^{n-2} + d_{1s} \xi^{n-4} + \dots \right. \\ \left. \dots + \begin{cases} d_{\sigma-1,s} & \text{for } n \text{ even} \\ d_{\sigma-1,s} \xi & \text{for } n \text{ odd} \end{cases} \right]. \quad (10)$$

As soon as the polynomial coefficients a_{rs} , b_{rs} , c_{rs} , d_{rs} in (7) to (10) are determined, the Lamé functions of the four classes will be completely defined. In the ensuing sections the algebraic procedures for the determination of these coefficients are supplied.

3. The Generation of the Lamé Functions

3.1 THE CONDITIONAL EQUATIONS FOR THE POLYNOMIAL COEFFICIENTS

In the theory of the Lamé functions (Hobson, 1931) it has been proven that $E(\xi)$ is the solution of the differential equation

$$(\xi^2 - h^2)(\xi^2 - k^2) \frac{d^2 E}{d\xi^2} + \xi(2\xi^2 - h^2 - k^2) \frac{dE}{d\xi} \\ + \{p(h^2 + k^2) - n(n+1)\xi^2\} E = 0, \quad (11)$$

where p is a parameter, which will be defined later. For the sake of convenience and better visualization the subscripts n and superscripts s have been deleted here. They will also be omitted in subsequent occurrences of Lamé functions wherever misunderstandings cannot possibly arise.

From the preceding section it is inferred that the polynomial coefficients in the expressions

$$K(\xi) = P_K(\xi), \quad (12)$$

$$L(\xi) = \sqrt{\xi^2 - h^2} P_L(\xi), \quad (13)$$

$$M(\xi) = \sqrt{\xi^2 - k^2} P_M(\xi), \quad (14)$$

$$N(\xi) = \sqrt{\xi^2 - h^2} \sqrt{\xi^2 - k^2} P_N(\xi) \quad (15)$$

must be determined so that the differential Equation (11) is satisfied by (12) to (15). P_K , P_L , P_M , and P_N , are abbreviations for the polynomial parts in Equations (7) to (10) respectively, and are sometimes called Lamé polynomials.

Substituting successively the right hand sides of (12) to (15) and their first and second derivatives into (11) yields the following differential equations for P_K , P_L , P_M and P_N .

$$(\xi^2 - h^2)(\xi^2 - k^2) \frac{d^2 P_K}{d\xi^2} + \xi(2\xi^2 - h^2 - k^2) \frac{dP_K}{d\xi} \\ + \{p(h^2 + k^2) - n(n+1)\xi^2\} P_K = 0 \quad (16)$$

which is evidently identical with the differential Equation (11) for the Lamé functions. Furthermore, one obtains

$$(\xi^2 - h^2)(\xi^2 - k^2) \frac{d^2 P_L}{d\xi^2} + \xi(4\xi^2 - h^2 - 3k^2) \frac{dP_L}{d\xi} + \{p(h^2 + k^2) - k^2 - (n-1)(n+2)\xi^2\} P_L = 0, \quad (17)$$

$$(\xi^2 - h^2)(\xi^2 - k^2) \frac{d^2 P_M}{d\xi^2} + \xi(4\xi^2 - k^2 - 3h^2) \frac{dP_M}{d\xi} + \{p(h^2 + k^2) - h^2 - (n-1)(n+2)\xi^2\} P_M = 0, \quad (18)$$

which results from (17) when h is replaced by k ,

$$(\xi^2 - h^2)(\xi^2 - k^2) \frac{d^2 P_N}{d\xi^2} + 3\xi(2\xi^2 - h^2 - k^2) \frac{dP_N}{d\xi} + \{(p-1)(h^2 + k^2) - (n(n+1) - 6)\xi^2\} P_N = 0. \quad (19)$$

The next objective is the determination of the coefficients in the polynomials P_K , P_L , P_M and P_N . Let

$$\alpha = h^2 + k^2, \quad \beta = h^2 k^2. \quad (20)$$

On substituting the polynomial P_K and its first and second derivatives in (16) one finds the conditional equations for the polynomial coefficients by equating to zero the coefficients of the various powers of ξ , namely

$$\begin{aligned} 2(2n-1)a_1 - \alpha(p - (n-0)^2)a_0 &= 0 \\ 4(2n-3)a_2 - \alpha(p - (n-1)^2)a_1 - \beta n(n-1)a_0 &= 0 \\ 6(2n-5)a_3 - \alpha(p - (n-4)^2)a_2 - \beta(n-2)(n-3)a_1 &= 0 \\ 2\sigma(2n+1-2\sigma)a_\sigma - \alpha(p - (n+2-2\sigma)^2)a_{\sigma-1} - \\ &\quad - \beta(n+4-2\sigma)(n+3-2\sigma)a_{\sigma-2} = 0 \\ (2\sigma+2)(2n-1-2\sigma)a_{\sigma+1} - \alpha(p - (n-2\sigma)^2)a_\sigma - \\ &\quad - \beta(n+2-2\sigma)(n+1-2\sigma)a_{\sigma-1} = 0. \end{aligned} \quad (21)$$

Here and in the following paragraphs the second index s has been deleted wherever the clarity of the representation is not affected.

Likewise the conditional equations for the polynomial coefficients in P_L , P_M , P_N are deduced.

3.2 THE RECURRENCE FORMULAE FOR THE POLYNOMIAL COEFFICIENTS

It becomes evident from the conditional equations that the polynomial coefficients a_i of P_K can be determined by means of a recurrence formula, which is of the form

$$2i(2n+1-2i)a_i = \alpha\{p - (n+2-2i)^2\}a_{i-1} + \beta(n+4-2i)(n+3-2i)a_{i-2}, \quad (22)$$

$a_{-1}=0$, $a_0=C_1=\text{constant}$, $i=1, 2, \dots, \sigma+1$.

Analogously by substitution of P_L, P_M, P_N into the corresponding differential Equations (17), (18), (19) recurrence formulae for the determination of the polynomial coefficients b_i of P_L , c_i of P_M and d_i of P_N are obtained in turn:

$$2i(2n+1-2i)b_i = [\alpha\{p-(n+1-2i)^2\} - (2n+3-4i)k^2]b_{i-1} + \beta(n+3-2i)(n+2-2i)b_{i-2}, \quad (23)$$

$b_{-1}=0, b_0=C_2=\text{constant}; i=1, 2, \dots, (n-\sigma).$

On replacing k by h in (23) the recurrence formula for the coefficients c_i of P_M takes the form

$$2i(2n+1-2i)c_i = [\alpha\{p-(n+1-2i)^2\} - (2n+3-4i)h^2]c_{i-1} + \beta(n+3-2i)(n+2-2i)c_{i-2}, \quad (24)$$

$c_{-1}=0, c_0=C_3=\text{constant}; i=1, 2, \dots, (n-\sigma),$

and finally one finds for the coefficients d_i of P_N :

$$2i(2n+1-2i)d_i = \alpha\{p-(n+1-2i)^2\}d_{i-1} + \beta(n+2-2i)(n+1-2i)d_{i-2}, \quad (25)$$

$d_{-1}=0, d_0=C_4=\text{constant}; i=1, 2, \dots, \sigma.$

The constants $C_j (j=1, 2, 3, 4)$ may be chosen arbitrarily, e.g. $C_j=1$. Each of the recurrence formulae contains the parameter p . In order to complete the construction of the polynomial coefficients the parameter p must be chosen according to the constraints to which it is exposed. This will be the subject of the next chapter.

3.3 FORMAC APPLICATIONS

The conditional equations for the polynomial coefficients in P_K, P_L, P_M, P_N have been constructed by using the algebraic manipulation features of the programming language PL/I-FORMAC (IBM, 1967). Moreover, validity checks have been performed confirming the formation of the recurrence formulae (22) to (25). After the verification of the recurrence formulae they were transcribed into the FORMAC language resulting in programs which basically allow the algebraic formulation of the polynomial coefficients up to any arbitrary order. Precautions were taken in the programs so that the generated coefficients can subsequently be utilized as input to further studies.

A few sets of polynomial coefficients of lower order are listed for reference in the Appendix. In fact, the coefficients up to the order $n=20$ could easily be generated on an IBM computer 360 Model 50. The increasing length of the resulting expressions made it impractical to generate the coefficients of still higher orders than 20 besides the fact that limitations of the computer configuration prevented it.

4. Determination of the Polynomial Coefficients

4.1 THE PARAMETER CONSTRAINTS

In order to arrive at integer rational functions for P_K, P_L, P_M, P_N one must make the

recurrence formulae (22) to (25) subject to constraint which demands that the parameter p in the polynomial coefficients must be determined so that

$$a_{\sigma+1} = a_{\sigma+1,s} = 0 \quad \text{in} \quad P_K(\xi) = P_K \quad (26)$$

$$b_{n-\sigma} = b_{n-\sigma,s} = 0 \quad \text{in} \quad P_L(\xi) = P_L \quad (27)$$

$$c_{n-\sigma} = c_{n-\sigma,s} = 0 \quad \text{in} \quad P_M(\xi) = P_M \quad (28)$$

$$d_{\sigma} = d_{\sigma,s} = 0 \quad \text{in} \quad P_N(\xi) = P_N. \quad (29)$$

If the condition (26) is introduced in the conditional Equations (21) for the polynomial coefficients in P_K a system of $\sigma+1$ linear homogeneous equations in the unknowns $a_0, a_1, \dots, a_{\sigma}$ is obtained. Putting

$$\lambda = \alpha p \quad (30)$$

the matrix

$$U_K = (u_{lm}) \quad (31)$$

of $(\sigma+1)$ -th order pertaining to the system is composed of the elements listed below:

$$\begin{aligned} u_{11} &= u_{12} = \dots u_{1,\sigma-1} = 0, \\ u_{1,\sigma} &= 2(2n-1), \\ u_{1,\sigma+1} &= \alpha n^2 - \lambda; \\ u_{21} &= u_{22} = \dots u_{2,\sigma-2} = 0, \\ u_{2,\sigma-1} &= 4(2n-3), \\ u_{2,\sigma} &= \alpha(n-2)^2 - \lambda, \\ u_{2,\sigma+1} &= -\beta n(n-1); \\ u_{31} &= u_{32} = \dots u_{3,\sigma-3} = 0, \\ u_{3,\sigma-2} &= 6(2n-5), \\ u_{3,\sigma-1} &= \alpha(n-4)^2 - \lambda, \\ u_{3,\sigma} &= -\beta(n-2)(n-3), \\ u_{3,\sigma+1} &= 0; \\ \vdots & \quad \quad \quad \vdots \quad \quad \quad \vdots \\ u_{\sigma+1,1} &= \alpha(n-2\sigma)^2 - \lambda, \\ u_{\sigma+1,2} &= -\beta(n+2-2\sigma)(n+1-2\sigma), \\ u_{\sigma+1,3} &= u_{\sigma+1,4} = \dots = u_{\sigma+1,\sigma+1} = 0. \end{aligned}$$

Essentially U_K is a function of p , h , and k . A necessary and sufficient condition for a non trivial solution $a_0, a_1, \dots, a_{\sigma}$ of the aforementioned system of linear homogeneous equations is

$$\det U_K = 0. \quad (32)$$

In an analogous manner the conditional equations for the parameters p associated with P_L , P_M and P_N can be established; i.e.

$$\det U_L = 0, \quad U_L \text{ of } (n - \sigma)\text{-th order} \quad (33)$$

$$\det U_M = 0, \quad U_M \text{ of } (n - \sigma)\text{-th order} \quad (34)$$

$$\det U_N = 0, \quad U_N \text{ of } \sigma\text{-th order.} \quad (35)$$

It has been proven in the theoretical treatment of the Lamé functions (Heine, 1878) that for each value $n=0, 1, 2, \dots$ there exist $\sigma+1$ real and unequal values for p associated with P_K ; $n-\sigma$ real and unequal values for p associated with P_L ; $n-\sigma$ real and unequal values for p associated with P_M ; σ real and unequal values for p associated with P_N , adding up to a total of $2n+1$ values corresponding to the $2n+1$ linear independent Lamé functions of the order n .

In order to find these distinct values of p the conditions (32) to (35) are subjected to appropriate matrix transformations which lead to algebraic eigenvalue problems.

4.2 THE EIGENVALUE PROBLEM

If the m -th column of the matrix U_K is interchanged with the $(\sigma+2-m)$ -th one, where m assumes the values $m=1, 2, \dots, \sigma+1$, then one obtains

$$\det (T_K - \lambda I) = (-1)^{1/2\sigma(\sigma+1)} \det U_K. \quad (36)$$

Herein, I stands for the unity matrix and T_K for a triple diagonal matrix whose elements are related to those of U_K through

$$t_{l, \sigma+2-m} = u_{lm} + \lambda \cdot \varepsilon, \quad (37)$$

$$l=1, 2, \dots, \sigma+1; m=1, 2, \dots, \sigma+1;$$

$$\varepsilon = \begin{cases} 0 & \text{for } l \neq \sigma+2-m \\ 1 & \text{for } l = \sigma+2-m. \end{cases}$$

As can be inferred from (36) the fulfilment of the condition $\det U_K=0$ is equivalent with the determination of the eigenvalues of the matrix T_K which is of triple diagonal form:

$$T_K = \begin{bmatrix} t_{11} & t_{12} & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ t_{21} & t_{22} & t_{23} & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & t_{32} & t_{33} & t_{34} & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & \cdot & \cdot & \dots & 0 & t_{\sigma, \sigma-1} & t_{\sigma, \sigma} & t_{\sigma, \sigma+1} \\ 0 & 0 & \cdot & \cdot & \dots & \dots & 0 & t_{\sigma+1, \sigma} & t_{\sigma+1, \sigma+1} \end{bmatrix}. \quad (38)$$

According to the properties of the Lamé functions the eigenvalues under consideration are all real and unequal. Consequently there exist $\sigma+1$ eigenvalues which by virtue of Equation (30) yield $\sigma+1$ different values for the parameters p . Thus with reference to (12) the $\sigma+1$ Lamé functions of class K are established as soon as the eigenvalues

have been calculated. Since in the past numerical analysis combined with computer usage have made the calculation of eigenvalues a manageable task, the aforementioned eigenvalue problem can be considered as solved. In this context it may be referenced that the left-right-algorithm developed by Rutishauser (1958) was used in several test cases for the numerical calculation of the eigenvalues and has proven to be an efficient method.

In an analogous manner the parameters pertaining to the other classes of Lamé functions can be found by solving the eigenvalue problems described below.

The functions of the class L are amenable to the eigenvalue problem

$$\det(T_L - \lambda I) = (-1)^{1/2(n-\sigma-1)(n-\sigma)} \det U_L = 0, \quad (39)$$

$$t_{l, n-\sigma+1-m} = u_{lm} + \lambda \cdot \varepsilon, \quad (40)$$

$$l = 1, 2, \dots, n - \sigma; m = 1, 2, \dots, n - \sigma;$$

$$\varepsilon = \begin{cases} 0 & \text{for } l \neq n - \sigma + 1 - m \\ 1 & \text{for } l = n - \sigma + 1 - m. \end{cases}$$

An eigenvalue problem formally identical with the one for the class L functions holds for the class M functions except that k is replaced by h wherever it appears explicitly.

The functions of class N are associated with the eigenvalue problem

$$\det(T_N - \lambda I) = (-1)^{1/2(\sigma-1)\sigma} \det U_N = 0, \quad (41)$$

$$t_{l, \sigma+1-m} = u_{lm} + \lambda \cdot \varepsilon, \quad (42)$$

$$l = 1, 2, \dots, \sigma; m = 1, 2, \dots, \sigma;$$

$$\varepsilon = \begin{cases} 0 & \text{for } l \neq \sigma + 1 - m \\ 1 & \text{for } l = \sigma + 1 - m. \end{cases}$$

The preceding four eigenvalue problems give rise to the determination of

$$(\sigma + 1) + (n - \sigma) + (n - \sigma) + \sigma = 2n + 1 \quad (43)$$

values of the parameters p . Because of (12) to (15) the definition of $2n+1$ linear independent Lamé functions of order n can thus be considered accomplished.

5. Conclusion

From the actual generation of the Lamé functions emerged the distinct impression that much of their complexity has been removed by the utilization of computerized formula manipulation techniques. Hence, the lengthy and cumbersome expressions for the Lamé functions should no longer restrain their applications to celestial mechanics, geophysics and other appropriate domains. The simplifications are mainly ascribed to the algebraic formulation of the Lamé functions in a machine readable form for further algebraic and numerical treatment. Also, the usual apprehension connected with the algebraic eigenvalue problem embedded in the theory of the Lamé

functions has experienced remarkable attenuation by powerful procedures for eigenvalue calculations being practiced nowadays. In view of the capabilities of modern computer systems, the eigenvalue problem and the mathematically involved expressions for the Lamé functions do not any longer impair their use in analytical and numerical investigations.

By way of illustration some samples of lower order Lamé polynomial coefficients are compiled in the Appendix. Shortage of space prevents the inclusion of the higher order coefficients because of the way they increase in length, and, besides, they are of no fundamental importance in this context. However, the author will be pleased to reproduce these and higher order coefficients in machine readable form should it be desired.

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Appendix

ALGEBRAIC EXPRESSIONS FOR THE LAMÉ POLYNOMIAL COEFFICIENTS

Below, the polynomial coefficients of the four classes of Lamé functions of the first

kind are compiled up to the order $n=6$. The notations used comply with those introduced in Section 3 with the understanding that

ALPHA stands for α ,

BETA stands for β ,

P stands for p ,

subscripts are placed between parentheses.

The computer output format has lent itself directly to the display of the generated expressions.

1. Polynomial Coefficients of the Lamé Functions of Type K

N = 0

I = 0

A(0) = C

N = 1

I = 0

A(0) = C

N = 2

I = 0

A(0) = C

I = 1

A(1) = 1/6 P ALPHA A(0) - 2/3 ALPHA A(0)

N = 3

I = 0

A(0) = C

I = 1

A(1) = 1/10 P ALPHA A(0) - 9/10 ALPHA A(0)

N = 4

I = 0

A(0) = C

$$I = 1$$

$$A(1) = 1/14 P \text{ ALPHA } A(0) - 8/7 \text{ ALPHA } A(0)$$

$$I = 2$$

$$A(2) = 3/5 \text{ BETA } A(0) - 1/14 P \text{ ALPHA}^2 A(0) + \\ + 1/280 P^2 \text{ ALPHA}^2 A(0) + 8/35 \text{ ALPHA}^2 A(0)$$

$$N = 5$$

$$I = 0$$

$$A(0) = C$$

$$I = 1$$

$$A(1) = 1/18 P \text{ ALPHA } A(0) - 25/18 \text{ ALPHA } A(0)$$

$$I = 2$$

$$A(2) = 5/7 \text{ BETA } A(0) - 17/252 P \text{ ALPHA}^2 A(0) + \\ + 1/504 P^2 \text{ ALPHA}^2 A(0) + 25/56 \text{ ALPHA}^2 A(0)$$

$$N = 6$$

$$I = 0$$

$$A(0) = C$$

$$I = 1$$

$$A(1) = 1/22 P \text{ ALPHA } A(0) - 18/11 \text{ ALPHA } A(0)$$

$$I = 2$$

$$A(2) = 5/6 \text{ BETA } A(0) - 13/198 P \text{ ALPHA}^2 A(0) + \\ + 1/792 P^2 \text{ ALPHA}^2 A(0) + 8/11 \text{ ALPHA}^2 A(0)$$

$$I = 3$$

$$A(3) = 13/396 P \text{ BETA ALPHA } A(0) - 379/693 \text{ BETA ALPHA } A(0) + \\ + 7/297 P \text{ ALPHA}^3 A(0) - 1/594 P^2 \text{ ALPHA}^3 A(0) + \\ + 1/33264 P^3 \text{ ALPHA}^3 A(0) - 16/231 \text{ ALPHA}^3 A(0)$$

2. Polynomial Coefficients of the Lamé Functions of Type L

$$N = 1$$

$$I = 0$$

$$B(0) = C$$

$$N = 2$$

$$I = 0$$

$$B(0) = C$$

$$N = 3$$

$$I = 0$$

$$B(0) = C$$

$$I = 1$$

$$B(1) = 1/10 P \text{ ALPHA } B(0) - 2/5 \text{ ALPHA } B(0) - 1/2 K^2 B(0)$$

$$N = 4$$

$$I = 0$$

$$B(0) = C$$

$$I = 1$$

$$B(1) = 1/14 P \text{ ALPHA } B(0) - 9/14 \text{ ALPHA } B(0) - 1/2 K^2 B(0)$$

$$N = 5$$

$$I = 0$$

$$B(0) = C$$

$$I = 1$$

$$B(1) = 1/18 P \text{ ALPHA } B(0) - 8/9 \text{ ALPHA } B(0) - 1/2 K^2 B(0)$$

$$I = 2$$

$$B(2) = -1/36 P K^2 \text{ ALPHA } B(0) + 29/126 K^2 \text{ ALPHA } B(0) + \\ + 3/7 \text{ BETA } B(0) - 5/126 P \text{ ALPHA}^2 B(0) + 1/504 P^2 \text{ ALPHA}^2 B(0) + \\ + 8/63 \text{ ALPHA}^2 B(0) + 5/56 K^4 B(0)$$

$$N = 6$$

$$I = 0$$

$$B(0) = C$$

$$I = 1$$

$$B(1) = 1/22 P \text{ ALPHA } B(0) - 25/22 \text{ ALPHA } B(0) - 1/2 K^2 B(0)$$

$$I = 2$$

$$B(2) = -1/44 P K^2 \text{ ALPHA } B(0) + 137/396 K^2 \text{ ALPHA } B(0) + \\ + 5/9 \text{ BETA } B(0) - 17/396 P \text{ ALPHA}^2 B(0) + 1/792 P^2 \text{ ALPHA}^2 B(0) + \\ + 25/88 \text{ ALPHA}^2 B(0) + 7/72 K^4 B(0)$$

3. Polynomial Coefficients of the Lamé Functions of Type M

$$N = 1$$

$$I = 0$$

$$C(0) = C$$

$$N = 2$$

$$I = 0$$

$$C(0) = C$$

$$N = 3$$

$$I = 0$$

$$C(0) = C$$

$$I = 1$$

$$C(1) = 1/10 P \text{ ALPHA } C(0) - 2/5 \text{ ALPHA } C(0) - 1/2 H^2 C(0)$$

$$N = 4$$

$$I = 0$$

$$C(0) = C$$

$$I = 1$$

$$C(1) = 1/14 P \text{ ALPHA } C(0) - 9/14 \text{ ALPHA } C(0) - 1/2 H^2 C(0)$$

$$N = 5$$

$$I = 0$$

$$C(0) = C$$

$$I = 1$$

$$C(1) = 1/18 P \text{ ALPHA } C(0) - 8/9 \text{ ALPHA } C(0) - 1/2 H^2 C(0)$$

$$I = 2$$

$$C(2) = -1/36 P H^2 \text{ ALPHA } C(0) + 29/126 H^2 \text{ ALPHA } C(0) + \\ + 3/7 \text{ BETA } C(0) - 5/126 P \text{ ALPHA}^2 C(0) + 1/504 P^2 \text{ ALPHA}^2 C(0) + \\ + 8/63 \text{ ALPHA}^2 C(0) + 5/56 H^4 C(0)$$

$$N = 6$$

$$I = 0$$

$$C(0) = C$$

$$I = 1$$

$$C(1) = 1/22 P \text{ ALPHA } C(0) - 25/22 \text{ ALPHA } C(0) - 1/2 H^2 C(0)$$

$$I = 2$$

$$C(2) = -1/44 P H^2 \text{ ALPHA } C(0) + 137/396 H^2 \text{ ALPHA } C(0) + \\ + 5/9 \text{ BETA } C(0) - 17/396 P \text{ ALPHA}^2 C(0) + 1/792 P^2 \text{ ALPHA}^2 C(0) + \\ + 25/88 \text{ ALPHA}^2 C(0) + 7/72 H^4 C(0)$$

4. Polynomial Coefficients of the Lamé Functions of Type N

$$N = 2$$

$$I = 0$$

$$D(0) = C$$

$$N = 3$$

$$I = 0$$

$$D(0) = C$$

$$N = 4$$

$$I = 0$$

$$D(0) = C$$

$$I = 1$$

$$D(1) = 1/14 P \text{ ALPHA } D(0) - 9/14 \text{ ALPHA } D(0)$$

$$N = 5$$

$$I = 0$$

$$D(0) = C$$

$$I = 1$$

$$D(1) = 1/18 P \text{ ALPHA } D(0) - 8/9 \text{ ALPHA } D(0)$$

$$N = 6$$

$$I = 0$$

$$D(0) = C$$

$$I = 1$$

$$D(1) = 1/22 P \text{ ALPHA } D(0) - 25/22 \text{ ALPHA } D(0)$$

$$I = 2$$

$$D(2) = 1/3 \text{ BETA } D(0) - 17/396 P \text{ ALPHA}^2 D(0) + \\ + 1/792 P^2 \text{ ALPHA}^2 D(0) + 25/88 \text{ ALPHA}^2 D(0)$$