Ellipse with ρ,θ coordinates PhL 3.26.05

Motivation: Here we discuss a strange coordinate system ρ,θ for the x,y plane that has no name and which nobody uses. My interest is this: in a way, this is a simplified version of Hobson's discussion of ellipsoidal coordinates ρ,μ,ν versus spherical coordinates r,θ,φ. I am confused by the tangent vector situation, and want to see what that looks like here.

Maple work is done in "rho theta ellipse system.mws".

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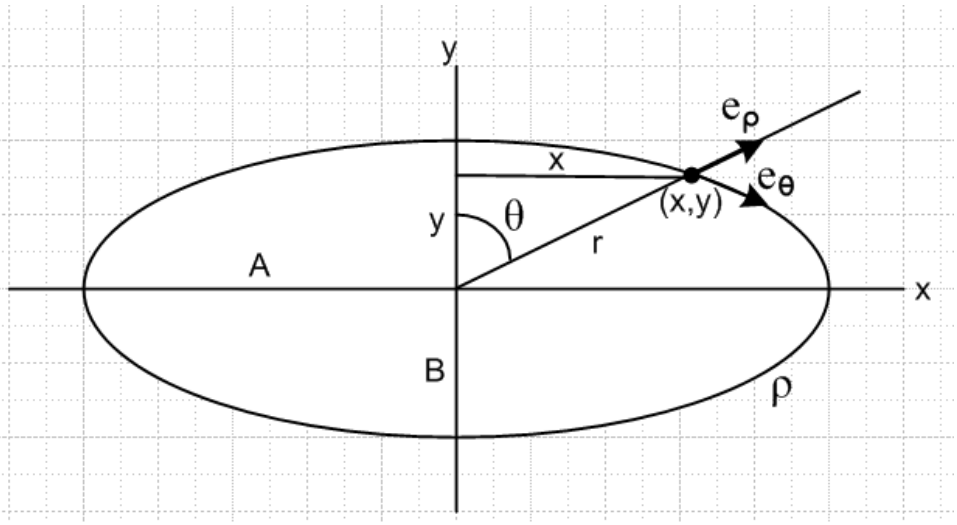
1. Description of our strange coordinate system

I want a 2D coordinate system where θ is a normal polar coordinate, but ρ is an "ellipse label" . Consider the following equation for an ellipse: [θ will be the spherical polar angle in Hobson]

$x^2/\rho^2 + y^2/(\rho^2-f^2) = 1$ $\quad A = \rho$ = semi-major $\quad B = \sqrt{\rho^2-f^2}$ = semi-minor

$A^2 - B^2 = f^2$ $\quad f$ = focal distance $\quad \rho \geq f$

As we vary ρ, we get a family of confocal ellipses with focal distance f. Thus, ρ is our ellipse label, as desired. Here is our opening picture:

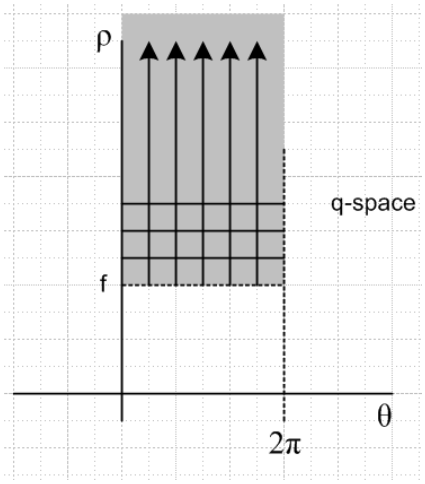


If ρ is very large, our ellipse becomes a large circle of radius ρ. As ρ → f, the ellipse becomes a tight string capturing the two foci.

2. Question about the level curves

In the transformation from x,y to ρ,θ, what are the "level curves"?

The q-space region which corresponds to the entire x-y plane is shown below in gray (it goes up forever). The set of line segments at various ρ values (ρ = constant) map into the confocal ellipses in the x,y plane, so those ellipses are level curves for ρ. The vertical half rays θ = constant in q space map into origin based rays in x,y space, so these rays are the level curves for θ. So the drawing above shows one level curve for ρ, and one level curve for θ.



Now, since we are in a 2D world, a "level curve" is the same thing as a "coordinate line" (see tensor doc). Therefore, the tangent base vectors are going to go along the level curves, and this is shown in the first picture above. We expect these two tangent vectors to be non-orthogonal as the picture requires. Since $g'_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j$, we expect the metric tensor to have no 0 elements (see below).

3. What is the relationship between x,y and ρ,θ ?

First off, from our ellipse equation above we obtain this quadratic equation situation for ρ^2

$$\rho^4 - (x^2+y^2+f^2)\rho^2 + x^2f^2 = 0$$

$$\Rightarrow \quad \rho^2 = [(x^2+y^2+f^2) \pm \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2}\,]/2$$

Both roots are clearly ≥ 0. I think I could show that one of these solutions has ρ < f and the other ρ > f, so only the + sign root is meaningful. Well, let's actually show it:

(a) Show that $\rho_\pm \gtrless f$

$$(x^2+y^2+f^2) \pm \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2} \quad \gtrless \quad 2f^2 \quad ?$$

$$\pm \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2} \quad \gtrless \quad 2f^2 - (x^2+y^2+f^2) \quad ?$$

$$\pm \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2} \quad \gtrless \quad f^2 - (x^2+y^2) \quad ?$$

I am unable to come up with a clean inequality theorem and notation that handles both inequality cases at the same time, so I am forced now to consider the two cases separately.

First the + case:

$$\sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2} \quad > \quad f^2 - (x^2+y^2) \quad ?$$

$$(x^2+y^2+f^2)^2 - 4x^2f^2 \quad > \quad [f^2 - (x^2+y^2)]^2 \quad ?$$

$$(r^2+f^2)^2 - 4x^2f^2 \quad > \quad (f^2 - r^2)^2 \quad ?$$

$$(r^2+f^2)^2 - (r^2-f^2)^2 \quad > \quad 4x^2f^2 \quad ?$$

$$4r^2f^2 \quad > \quad 4x^2f^2 \quad ? \qquad \text{yes since } y^2 > 0$$

Second the - case:

$$-\sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2} \quad < \quad f^2 - (x^2+y^2) \quad ?$$

$$\sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2} \quad > \quad (x^2+y^2) - f^2 \quad ?$$

$$(x^2+y^2+f^2)^2 - 4x^2f^2 \quad > \quad [f^2 - (x^2+y^2)]^2 \quad ?$$

From here on this case is the same as the previous case so the answer is again yes. Therefore, we have shown that

$\rho_\pm^2 \qquad \rho_\pm^2 \gtrless f^2 \qquad \Rightarrow \qquad \rho_\pm \gtrless f$

so only ρ_+ is meaningful and the other solution is spurious (it might have something to do with hyperbolas, but I don't care about that now).

(b) Express ρ,θ in terms of x, y

So, given a point (x,y) we can determine which ellipse it lies on as follows:

$$\rho^2 = [(x^2+y^2+f^2) + \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2}\,]/2$$

We also know that

$\tan\theta = x/y$ // for our unusual polar angle definition

So here then we have $\rho(x,y)$ and $\theta(x,y)$

$$\rho = \sqrt{[(x^2+y^2+f^2) + \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2}\,]/2}$$

$\theta = \tan^{-1}(x/y)$

(c) Express x,y in terms of ρ,θ

What about the other direction? One's first "thought" is this

$x = r\sin\theta$
$y = r\cos\theta$

$r^2 = x^2 + y^2$

but we don't know r as $r(\rho,\theta)$. We shall learn it soon! So let's instead invert our two equations above:

$$2\rho^2 = (x^2+y^2+f^2) + \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2}$$

$x^2 = y^2\tan^2\theta \qquad \Rightarrow \qquad x^2 + y^2 = y^2(1+\tan^2\theta) = y^2\sec^2\theta$

$y^2 = x^2\cot^2\theta \qquad \Rightarrow \qquad x^2 + y^2 = x^2(1+\cot^2\theta) = x^2\csc^2\theta$

The first equation then becomes

$$2\rho^2 = (x^2+y^2+f^2) + \sqrt{(x^2+y^2+f^2)^2 - 4x^2f^2}$$

$$= x^2\csc^2\theta + f^2 + \sqrt{(x^2\csc^2\theta + f^2)^2 - 4x^2f^2}$$

We can in theory solve this equation for x^2

$$2\rho^2 - x^2\csc^2\theta - f^2 = \sqrt{(x^2\csc^2\theta + f^2)^2 - 4x^2f^2}$$

$$(2\rho^2 - x^2\csc^2\theta - f^2)^2 = (x^2\csc^2\theta + f^2)^2 - 4x^2f^2$$

$$4\rho^4 + x^4\csc^4\theta + f^4 - 4\rho^2x^2\csc^2\theta - 4\rho^2f^2 + 2x^2f^2\csc^2\theta = x^4\csc^4\theta + f^4 + 2x^2\csc^2\theta f^2 - 4x^2f^2$$

$$4\rho^4 - 4\rho^2x^2\csc^2\theta - 4\rho^2f^2 = -4x^2f^2$$

$$\rho^4 - \rho^2x^2\csc^2\theta - \rho^2f^2 = -x^2f^2$$

$$\rho^4 - \rho^2x^2\csc^2\theta - \rho^2f^2 + x^2f^2 = 0$$

$$(-\rho^2x^2\csc^2\theta + x^2f^2) + \rho^4 - \rho^2f^2 = 0$$

$$x^2(-\rho^2\csc^2\theta + f^2) + \rho^2(\rho^2 - f^2) = 0$$

$$x^2(\rho^2\csc^2\theta - f^2) - \rho^2(\rho^2 - f^2) = 0$$

$$x^2(\rho^2\csc^2\theta - f^2) = \rho^2(\rho^2 - f^2)$$

$$x^2 = \rho^2(\rho^2 - f^2)/(\rho^2\csc^2\theta - f^2)$$

and we can see since $\csc^2\theta > 1$ and $\rho > f$ that both numerator and denominator are positive. Keep processing a bit

$$x^2 = \sin^2\theta\,\rho^2(\rho^2 - f^2)/(\rho^2 - f^2\sin^2\theta)$$

We can then compute y^2 this way

$$y^2 = x^2\cot^2\theta = \cos^2\theta\,\rho^2(\rho^2 - f^2)/(\rho^2 - f^2\sin^2\theta)$$

Now let's define

$$R^2 = \rho^2(\rho^2 - f^2)/(\rho^2 - f^2\sin^2\theta) \qquad R = \rho\sqrt{(\rho^2 - f^2)/(\rho^2 - f^2\sin^2\theta)}$$

We have then showed that

$$x = R\sin\theta$$

$y = R\cos\theta$

$r^2 = R^2$ => $r = R$

(d) Verify the above result for x,y as functions of ρ,θ

If we insert the above x,y into the ρ^2 equation, should just get $\rho^2 = \rho^2$. I tried to make Maple show this, but it could not figure it out, but I was then above to verify the above with some random numbers:

2. We want to verify that when our solved-for x and y expressions are inserted into rho2, we get rho2 = rho^2.

First, here are the basic expressions:

```
> restart;
> x := R*sin(theta);
```

$$x := R\sin(\theta)$$

```
> y := R*cos(theta);
```

$$y := R\cos(\theta)$$

```
> R := rho* sqrt((rho^2-f^2)/(rho^2-f^2*sin(theta)^2));
```

$$R := \rho\sqrt{\frac{\rho^2 - f^2}{\rho^2 - f^2\sin(\theta)^2}}$$

This should come out saying rho2 = rho^2, independent of f and theta, but I cannot make Maple say it is so , trying everything I know how to do for several hours.

```
> rho2 := ((x^2+y^2+f^2)+sqrt((x^2+y^2+f^2)^2 - (2*x*f)^2))/2;
```

$$\rho 2 := \frac{1}{2}\frac{\rho^2(\rho^2 - f^2)\sin(\theta)^2}{\%1} + \frac{1}{2}\frac{\rho^2(\rho^2 - f^2)\cos(\theta)^2}{\%1} + \frac{1}{2}f^2 + \frac{1}{2}\sqrt{\left(\frac{\rho^2(\rho^2 - f^2)\sin(\theta)^2}{\%1} + \frac{\rho^2(\rho^2 - f^2)\cos(\theta)^2}{\%1} + f^2\right)^2 - 4\frac{\rho^2(\rho^2 - f^2)\sin(\theta)^2 f^2}{\%1}}$$

$$\%1 := \rho^2 - f^2\sin(\theta)^2$$

So I will go for the consolation prize and pick oddball numbers and see if it works right:

```
> f := 4.245213;
```

$$f := 4.245213$$

```
> rho := 5.3456234;
```

$$\rho := 5.3456234$$

```
> theta := 2.32453;
```

$$\theta := 2.32453$$

```
> sqrt(rho2);
```

$$5.345623400$$

QED.

4. Compute the Tup matrix

Here is the Maple code which does this

```
> restart;
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> R := rho* sqrt((rho^2-f^2)/(rho^2-f^2*sin(theta)^2));
```

$$R := \rho \sqrt{\frac{\rho^2 - f^2}{\rho^2 - f^2 \sin(\theta)^2}}$$

```
> r[1] := R*sin(theta);
```

$$r_1 := \rho \sqrt{\frac{\rho^2 - f^2}{\rho^2 - f^2 \sin(\theta)^2}} \sin(\theta)$$

```
> r[2] := R*cos(theta);
```

$$r_2 := \rho \sqrt{\frac{\rho^2 - f^2}{\rho^2 - f^2 \sin(\theta)^2}} \cos(\theta)$$

```
> q[1] := rho; q[2] := theta;
```

ordering for q is 1,2 = rho,theta

$$q_1 := \rho$$

$$q_2 := \theta$$

```
> T := (i,j) -> diff(r[j],q[i]);
```

Note that (1) i is the row index. (2) This is the "tilt-up" matrix.

$$T := (i,j) \to \frac{\partial}{\partial q_i} r_j$$

```
> Tup := matrix(2,2,T);
```

$$Tup := \begin{bmatrix} \%2 \sin(\theta) + \frac{1}{2} \frac{\rho \sin(\theta) \left(2 \frac{\rho}{\%1} - 2 \frac{(\rho^2 - f^2)\rho}{\%1^2} \right)}{\sqrt{\frac{\rho^2 - f^2}{\%1}}} & \%2 \cos(\theta) + \frac{1}{2} \frac{\rho \cos(\theta) \left(2 \frac{\rho}{\%1} - 2 \frac{(\rho^2 - f^2)\rho}{\%1^2} \right)}{\sqrt{\frac{\rho^2 - f^2}{\%1}}} \\ \frac{\rho \sin(\theta)^2 (\rho^2 - f^2) f^2 \cos(\theta)}{\sqrt{\frac{\rho^2 - f^2}{\%1}}\, \%1^2} + \rho\, \%2 \cos(\theta) & \frac{\rho \cos(\theta)^2 (\rho^2 - f^2) f^2 \sin(\theta)}{\sqrt{\frac{\rho^2 - f^2}{\%1}}\, \%1^2} - \rho\, \%2 \sin(\theta) \end{bmatrix}$$

$$\%1 := \rho^2 - f^2 \sin(\theta)^2$$

$$\%2 := \sqrt{\frac{\rho^2 - f^2}{\%1}}$$

The top row is the vector $\mathbf{e}_\rho$. We can see that the components of the vector have this form

$\mathbf{e}_\rho$ = (sinθ * stuff, cosθ * stuff) = stuff (sinθ, cosθ) => e_x/e_y = tanθ

Knowing that ρ > f, we know that %1 > 0 and %2 > 0, so stuff > 0 as well. Thus result then agrees exactly with the picture above.

The second row is the vector $\mathbf{e}_\theta$. For θ in the first quadrant (as in our drawing), the x component is positive. The y component could have either sign. In our picture, the y component is negative, so hopefully this is all correct.

5. Compute the metric tensor.

I did this in Maple, it is a horrendous mess but I was able to simplify the elements, and below I am able to grind it down to something chewable. Maple says (after much work)

```
> g11 := gdn[1,1]:simplify(%):factor(%);
```

$$-\frac{(-f^4+\cos(\theta)^2 f^4+2f^2\rho^2-2\cos(\theta)^2 f^2\rho^2-\rho^4)^2}{(f-\rho)(f+\rho)(\rho^2-f^2+\cos(\theta)^2 f^2)^3}$$

```
> g22 := gdn[2,2]:simplify(%):factor(%);
```

$$\frac{(f-\rho)(f+\rho)(-f^4+\cos(\theta)^2 f^4+2f^2\rho^2-2\cos(\theta)^2 f^2\rho^2-\rho^4)\rho^2}{(\rho^2-f^2+\cos(\theta)^2 f^2)^3}$$

```
> g12 := gdn[1,2]:simplify(%):factor(%);
```

$$-\frac{(-f^4+\cos(\theta)^2 f^4+2f^2\rho^2-2\cos(\theta)^2 f^2\rho^2-\rho^4)\sin(\theta)\rho\cos(\theta)f^2}{(\rho^2-f^2+\cos(\theta)^2 f^2)^3}$$

```
> g21 := gdn[2,1]:simplify(%):factor(%);
```

$$-\frac{(-f^4+\cos(\theta)^2 f^4+2f^2\rho^2-2\cos(\theta)^2 f^2\rho^2-\rho^4)\sin(\theta)\rho\cos(\theta)f^2}{(\rho^2-f^2+\cos(\theta)^2 f^2)^3}$$

We can see that $g_{21} = g_{12}$ as expected. So here are manual results

$g_{11} = (-f^4 + f^4\cos^2\theta + 2f^2\rho^2 - 2\rho^2 f^2\cos^2\theta - \rho^4)^2 / [(\rho^2-f^2)(\rho^2-f^2+ \cos^2\theta\, f^2)^3]$

$= (f^4 [-1 + \cos^2\theta] + 2 f^2\rho^2[1 - \cos^2\theta] - \rho^4)^2 / [(\rho^2-f^2)(\rho^2-f^2[1- \cos^2\theta])^3]$

$= (-f^4\sin^2\theta + 2\rho^2 f^2\sin^2\theta - \rho^4)^2 / [(\rho^2-f^2)(\rho^2-f^2\sin^2\theta)^3]$ dim = L^0

g_{22} has the same long factor in the numerator so the result is this

$g_{22} = \rho^2(\rho^2-f^2) (f^4\sin^2\theta - 2\rho^2 f^2\sin^2\theta + \rho^4) / (\rho^2-f^2\sin^2\theta)^3$ dim = L^2

and $g_{12} = g_{21}$ is similar

$g_{12} = g_{21} = \rho f^2\sin\theta\cos\theta(f^4\sin^2\theta - 2\rho^2 f^2\sin^2\theta + \rho^4) / (\rho^2-f^2\sin^2\theta)^3$ dim = L^1

And we summarize

$g_{\rho\rho} = (-f^4\sin^2\theta + 2\rho^2 f^2\sin^2\theta - \rho^4)^2 / [(\rho^2-f^2)(\rho^2-f^2\sin^2\theta)^3]$ dim = L^0

$g_{\theta\theta} = \rho^2(\rho^2-f^2) (f^4\sin^2\theta - 2\rho^2 f^2\sin^2\theta + \rho^4) / (\rho^2-f^2\sin^2\theta)^3$ dim = L^2

$g_{\rho\theta} = \rho f^2 \sin\theta\cos\theta(f^4\sin^2\theta - 2\rho^2 f^2 \sin^2\theta + \rho^4) / (\rho^2 - f^2\sin^2\theta)^3$ dim = L^1

Here is a more compact presentation:

$g_{\rho\rho} = A^2 B / (\rho^2 - f^2)$ dim = L^0 $A = (f^4\sin^2\theta - 2\rho^2 f^2 \sin^2\theta + \rho^4)$

$g_{\theta\theta} = \rho^2(\rho^2 - f^2)\, AB$ dim = L^2 $B = 1/(\rho^2 - f^2\sin^2\theta)^3$

$g_{\rho\theta} = \rho f^2 \sin\theta\cos\theta\, AB$ dim = L^1

$(ds)^2 = g_{\rho\rho}(d\rho)^2 + g_{\theta\theta}(d\theta)^2 + 2\, g_{\rho\theta}(d\rho)(d\theta)$

Obviously this ρ,θ coordinate system is non-orthogonal. If f goes to zero, $A = \rho^4$, $B = 1/\rho^6$ and we get

$g_{\rho\rho} = \rho^8\rho^{-6} / (\rho^2) = 1$

$g_{\theta\theta} = \rho^2(\rho^2)\,\rho^{-2} = \rho^2$

$g_{\rho\theta} = 0$

$(ds)^2 = (d\rho)^2 + \rho^2\,(d\theta)^2$

and this is the result for a circle with r = ρ.

6. The Hobson connection

3D	2D
ellipsoidal coordinates	bipolar coordinates (ellipses and hyperbolas)
spherical coordinates	polar coordinates r and θ
conical coordinates	polar coordinates r and θ
ellipsoidal ρ,θ,φ system	the ρ,θ system of this doc

In the 3D conical system, we have the usual polar cone pointing up which becomes our polar angle θ cone of polar coordinates. The other sideways 3D conical cone does not exist in the 2D system. So it happens that the 2D analog of both spherical and conical coordinates is the polar coordinates.