Where do the Cartesian ellipsoidal expressions come from PhL 12.24.09
What is the meaning of c ?

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Overview (10.10.10, 1/2 page)

In **Part I** I show how you start with the conic equations for the three conicoids of the ellipsoidal system, and you solve this system of three equations for x,y,z in terms of ξ_1, ξ_2, ξ_3 using **Cramer's rule** and this gives the "usual" set of equations for x,y,z that people normally take as the defining equations for the ellipsoidal coordinate system.

In **Part II** I show that with c > 0, the three zero-plane slice ellipses of a general ellipsoid have these focal distances: (I have added supporting data on the right)

$f_x = \sqrt{b^2 - c^2}$ // that is, this is the focal distance for the x=0 plane ellipse slice
$f_y = \sqrt{a^2 - c^2}$ $c < \xi_3 < b < \xi_2 < a < \xi_1$
$f_z = \sqrt{a^2 - b^2}$ $C \geq B \geq A.$ $A^2 = \xi_1^2 - a^2$ $B^2 = \xi_1^2 - b^2$ $C^2 = \xi_1^2 - c^2$

and

$f_z^2 = f_y^2 - f_x^2$ $x^2/(\xi_1^2 - a^2) + y^2/(\xi_1^2 - b^2) + z^2/(\xi_1^2 - c^2) = 1$ $c < b < a < \xi_1$

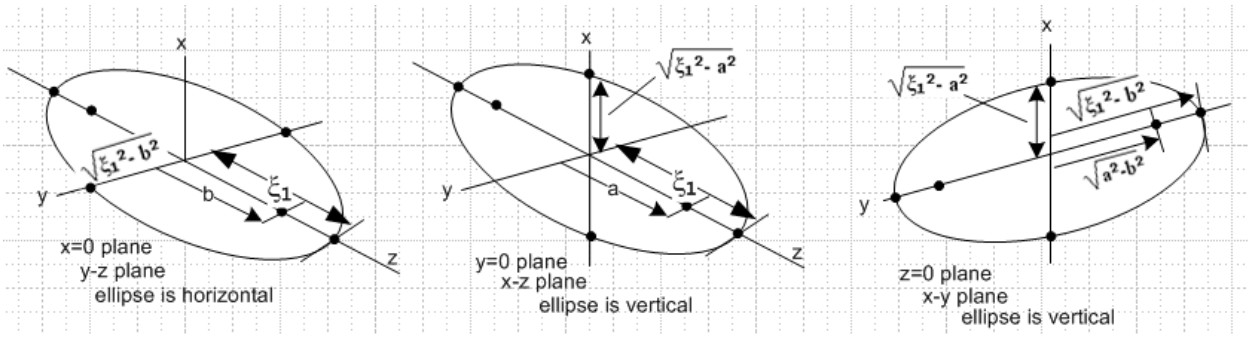
So, the three focal distances cannot be independently set, and there are really only two independent variables which describe the shapes of a confocal family of ellipses (the family members each have a ξ_1 label). So the simplest thing to do is set c = 0 and have a and b be those two independent variables. Then we get

$f_x = b$ // that is, this is the focal distance for the x=0 plane ellipse slice
$f_y = a$ $0 < \xi_3 < b < \xi_2 < a < \xi_1$
$f_z = \sqrt{a^2 - b^2}$ $C \geq B \geq A.$ $A^2 = \xi_1^2 - a^2$ $B^2 = \xi_1^2 - b^2$ $C^2 = \xi_1^2$

and

$f_z^2 = f_y^2 - f_x^2$ $x^2/(\xi_1^2 - a^2) + y^2/(\xi_1^2 - b^2) + z^2/\xi_1^2 = 1$ $0 < b < a < \xi_1$

and here is my picture from the long doc,



Part I: Where do the Cartesian ellipsoidal expressions come from

In my ellipsoidal notes, I often quoted the MF expressions for x,y,z,

512 *Ordinary Differential Equations* [CH. 5

$$x = \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2(a^2 - b^2)}}; \quad \xi_1 > a > \xi_2 > b > \xi_3 > 0$$

$$y = \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2(b^2 - a^2)}}; \quad z = \frac{\xi_1 \xi_2 \xi_3}{ab} \qquad (5.1.34)$$

$$h_1 = \sqrt{\frac{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}}; \qquad h_2 = \sqrt{\frac{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}{(\xi_2^2 - a^2)(\xi_2^2 - b^2)}}; \quad \text{etc.}$$

but I never thought about "where these equations come from". I think this is how it works. We start out with our three intersecting quadric surfaces

$$x^2/(\xi_1^2 - a^2) + y^2/(\xi_1^2 - b^2) + z^2/(\xi_1^2 - c^2) = 1 \qquad c < b < a < \xi_1$$

$$-x^2/(a^2 - \xi_2^2) + y^2/(\xi_2^2 - b^2) + z^2/(\xi_2^2 - c^2) = 1 \qquad c < b < \xi_2 < a$$

$$-x^2/(a^2 - \xi_3^2) - y^2/(b^2 - \xi_3^2) + z^2/(\xi_3^2 - c^2) = 1 \qquad c < \xi_3 < b < a$$

Each surface is labeled by a ξ_i coordinate. I think we can solve the above three equations for x,y,z with a simple Cramer's rule. I seem to recall this from somewhere, but I guess I never wrote it down. I will do it in Maple. Write the above three equations in matrix form:

$$= \begin{bmatrix} \frac{1}{\xi 1^2 - a^2} & \frac{1}{\xi 1^2 - b^2} & \frac{1}{\xi 1^2 - c^2} \\ \frac{1}{\xi 2^2 - a^2} & \frac{1}{\xi 2^2 - b^2} & \frac{1}{\xi 2^2 - c^2} \\ \frac{1}{\xi 3^2 - a^2} & \frac{1}{\xi 3^2 - b^2} & \frac{1}{\xi 3^2 - c^2} \end{bmatrix} \cdot \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then tell Maple to compute the inverse of this matrix and apply to the column of 1's to get x^2 etc. We get

$$\begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} = \begin{bmatrix} -\frac{(-\xi 3^2 + a^2)(-\xi 2^2 + a^2)(-\xi 1^2 + a^2)}{a^4 - b^2 a^2 - c^2 a^2 + c^2 b^2} \\ \frac{(-\xi 1^2 + b^2)(-\xi 3^2 + b^2)(-\xi 2^2 + b^2)}{c^2 b^2 - b^4 + b^2 a^2 - c^2 a^2} \\ -\frac{(-\xi 1^2 + c^2)(-\xi 2^2 + c^2)(-\xi 3^2 + c^2)}{c^4 + b^2 a^2 - c^2 b^2 - c^2 a^2} \end{bmatrix}$$

Manually we can write this as

$$x^2 = (\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2) / [(a^2 - c^2)(a^2 - b^2)]$$
$$y^2 = (\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2) / [(b^2 - a^2)(b^2 - c^2)]$$
$$z^2 = (\xi_1^2 - c^2)(\xi_2^2 - c^2)(\xi_3^2 - c^2) / [(c^2 - b^2)(c^2 - a^2)]$$

Now we usually set $c = 0$ without loss of generality, and this becomes

$$\begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} := \begin{bmatrix} -\dfrac{(-\xi 3^2 + a^2)(-\xi 2^2 + a^2)(-\xi 1^2 + a^2)}{a^4 - b^2 a^2} \\ \dfrac{(-\xi 1^2 + b^2)(-\xi 3^2 + b^2)(-\xi 2^2 + b^2)}{-b^4 + b^2 a^2} \\ \dfrac{\xi 1^2 \xi 2^2 \xi 3^2}{b^2 a^2} \end{bmatrix}$$

which we can compare now to our MF quote:

512 *Ordinary Differential Equations* [CH. 5

$$x = \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2(a^2 - b^2)}}; \quad \xi_1 > a > \xi_2 > b > \xi_3 > 0$$

$$y = \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2(b^2 - a^2)}}; \quad z = \frac{\xi_1 \xi_2 \xi_3}{ab} \qquad (5.1.34)$$

$$h_1 = \sqrt{\frac{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}{(\xi_1^2 - a^2)(\xi_1^2 - b^2)}}; \qquad h_2 = \sqrt{\frac{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}{(\xi_2^2 - a^2)(\xi_2^2 - b^2)}}; \quad \text{etc.}$$

and there you have it! So here I will write out the results

$x^2 = (\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2) / [a^2(a^2 - b^2)]$
$y^2 = (\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2) / [b^2(b^2 - a^2)]$
$z^2 = \xi_1^2 \xi_2^2 \xi_3^2 / a^2 b^2$

Comments: Each ξ_i labels a quadric surface, ie, a conicoid. I know these intersect in 8 places with obvious symmetry, but let's think of the first octant only. The above expressions tell you (x,y,z) in terms of (ξ_1, ξ_2, ξ_3). This is the one Cartesian point in the first octant the lies on all three surfaces.

WW in their Lame chapter write the general x,y,z equations in this way

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2 = -\frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(a^2-b^2)(b^2-c^2)},$$

$$z^2 = \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(a^2-c^2)(b^2-c^2)}.$$

where they refer to their coordinates as (λ,μ,ν) instead of $(\xi_{1,2,3})$. This shows that when $c \neq 0$, the denominators do factor (my solution above did not show things factored).

Part I: What is the significance of parameter c (which we end up setting to 0) 10.10.10

For the ellipse with c showing we have

$$x^2/(\xi_1^2-a^2) + y^2/(\xi_1^2-b^2) + z^2/(\xi_1^2-c^2) = 1 \qquad c < b < a < \xi_1$$

which says

$A^2 = \xi_1^2-a^2$
$B^2 = \xi_1^2-b^2$
$C^2 = \xi_1^2-c^2$ => $C \geq B \geq A$
$\xi_1^2 = A^2+a^2 = B^2+b^2 = C^2+c^2$

If we take a slice in any coordinate zero plane, we get an ellipse. For example, for x = 0 we get

$y^2/(\xi_1^2-b^2) + z^2/(\xi_1^2-c^2) = 1$ $c < b < \xi_1$
$B^2 = \xi_1^2-b^2$
$C^2 = \xi_1^2-c^2$ => $C \geq B$

An ellipse has only one focal distance, and for this ellipse it is

$$f^2 = C^2-B^2 = b^2-c^2$$

Now let's try the y = 0 plane ellipse:

$x^2/(\xi_1^2-a^2) + z^2/(\xi_1^2-c^2) = 1$ $c < a < \xi_1$
$A^2 = \xi_1^2-a^2$
$C^2 = \xi_1^2-c^2$ => $C \geq A$
$f^2 = C^2-A^2 = a^2-c^2$

And finally we do the z = 0 plane ellipse

$x^2/(\xi_1^2-a^2) + b^2/(\xi_1^2- b^2) = 1$ $\quad c < b < \xi_1$

$A^2 = \xi_1^2-a^2$

$B^2 = \xi_1^2-b^2$ $\quad \Rightarrow \quad$ $B \geq A$

$f^2 = B^2- A^2 = a^2- b^2$

Here is a summary, and notice that things are not "cyclic" in nature, the middle item is reverse cyclic:

plane	focal distance squared
x=0	$b^2- c^2$
y=0	$a^2- c^2$
z=0	$a^2- b^2$

$x^2/(\xi_1^2-a^2) + y^2/(\xi_1^2- b^2)+ z^2/(\xi_1^2- c^2) = 1$ $\quad c < b < a < \xi_1$

There are three focal distances and three parameters to describe them. BUT, consider these three equations for the three focal distances,

$f_x = \sqrt{b^2- c^2}$

$f_y = \sqrt{a^2- c^2}$

$f_z = \sqrt{a^2- b^2}$

We find that

$f_z^2 = f_y^2 – f_x^2$

This tells us that for a completely general ellipsoid, you cannot arbitrarily set the three focal distances of the zero coordinate slice ellipses. Only two are independent.

For example, if $f_x = f_y$ you get $f_z= 0$ which means the cross section in the z=0 plane is a circle. We then have $a = b > c$ so $A = B < C$ and we have a prolate spheroid with sym axis in the z direction.

Another example is $f_z = f_y$ which tells us $f_x = 0$. In this case $c = b < a$ so $C = B > A$ and we then have an oblate spheroid with sym axis in the x direction.

But there is more. You can select two of the three focal distances independently, but there are restrictions. If the two you want to select are f_x and f_y, you must have $f_y \geq f_x$! And since $f_x^2 = f_y^2 – f_z^2$, if you want to pick f_y and f_z, then you must make sure $f_y \geq f_z$. But since $f_y^2 = f_x^2 + f_z^2$, you can select f_x and f_z any way you want.

So one conclusion we reach is that there is no need for three parameters a,b,c to define a confocal general ellipsoid family (whose members are described by label ξ_1). Two will do. The convention is to set c = 0 and then we have

$f_x = b$ // Notice that $f_x = b$, not a, though a appears with x in the conic

$f_y = a$

$f_z = \sqrt{a^2 - b^2}$

And we already have stated that $a \geq b$ as part of our system definition, so we are saying $f_y \geq f_x$ which gives a non-negative f_z.

So we can regard c in two ways:

First, it is a redundant parameter in defining a confocal family.

Second, if we look at our last conicoid equation above

$$- x^2/(a^2 - \xi_3^2) - y^2/(b^2 - \xi_3^2) + z^2/(\xi_3^2 - c^2) = 1 \qquad c < \xi_3 < b < a$$

we see that c sets the lower range endpoint for the ξ_3 variable which defines the 2-sheeted bloid. Whatever $c \geq 0$ you select for this endpoint, the range of ξ_3 will be (c,b). Were you to set c close to b, then all the "action" of this ξ_3 variable takes place in a narrow range, but the surfaces span the same set of surfaces regardless of how you decide to "scale" the ξ_3 variable. There is nothing to be gained by having $c \neq 0$ except it gives some "term symmetry" to the conic equations.