This is the Title PhL 3.26.05

In retrospect, the first section A is all one should read here. I don't have anything better than that.

In Section A I use my $\mathbf{n}_i = \nabla(g_i)$ rule on the conicoid equations to compute the normals on the three surfaces, and then I just dot them together and then use the x,y,z equations and find $\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$ etc.

A. GEOMETRIC REASON WHY THE ELLIPSOIDAL LEVEL-SURFACE NORMALS ARE ORTHOGONAL. 2

B. AN EARLIER LESS FOCUSED VERSION OF A. 5

1. The ellipsoidal level surfaces and the x,y,z equations 5

2. Finding the normal vectors at some point x 5

3. Show that $\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$ 7

C. ANOTHER VERSION TRYING TO MAKE BETTER USE OF INDICES. 8

1. The ellipsoidal level surfaces and the x,y,z equations 8

2. Restate the above in a more regular notation 8

3. Finding the normal vectors at some point x 9

3. Show that the three normal vectors are mutually perpendicular. 10

D. REHASH OF THE "METRIC TENSOR METHOD". 13

Overview

In **Section A** I use my $\mathbf{n}_i = \nabla(g_i)$ rule on the conicoid equations to compute the normals on the three surfaces at some point **x** on the surface, and then I just dot them together and then use the x,y,z equations and find $\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$ etc. Here is a summary. The conicoid equations are

$$g_i(x,y,z) \equiv x^2/(\xi_i^2 - a^2) + y^2/(\xi_i^2 - b^2) + z^2/(\xi_i^2 - c^2) - 1 = 0 \qquad i=1,2,3 \qquad c < b < a$$

The Cramer's rule x,y,z equations are the following, and a,b,c cyclic for y^2 and z^2

$$x^2 = (\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2) / [(a^2 - c^2)(a^2 - b^2)]$$

The normals are

$$\mathbf{n}_i = \nabla(g_i) = 2[\mathbf{x}/(\xi_i^2 - a^2) + \mathbf{y}/(\xi_i^2 - b^2) + \mathbf{z}/(\xi_i^2 - c^2)] \qquad i = 1,2,3$$

so the dot product of two normals becomes, when x^2 etc are substituted in,

$$\mathbf{n}_i \bullet \mathbf{n}_j/4 = [x^2] / [(\xi_i^2 - a^2)(\xi_j^2 - a^2)] + \text{cyclic } (x,y,z;\ a,b,c)$$

$$= [(\xi_1^2-a^2)(\xi_2^2-a^2)(\xi_3^2-a^2)/ [(a^2-c^2)(a^2-b^2)(\xi_i^2-a^2)(\xi_j^2-a^2)] + \text{cyclic (a,b,c)}$$

When i and j are different, let k be "the third index". We then get at once

$$\mathbf{n}_i \bullet \mathbf{n}_j/4 = [(\xi_k^2-a^2) / [(a^2-c^2)(a^2-b^2)] + \text{cyclic}$$

$$= [(\xi_k^2-a^2)(b^2-c^2) + \text{cyclic}] /[(a^2-c^2)(a^2-b^2)(b^2-c^2)]$$

But direct calculation shows that

$(b^2-c^2)+ \text{cyclic} = 0$ AND $a^2(b^2-c^2) + \text{cyclic} = 0$

So the numerator is 0 and we then have $\mathbf{n}_i \bullet \mathbf{n}_j = 0$ for $i \neq j$. We know that close to some point the three level-surfaces in general form a parallelogram with tangent base vectors $\mathbf{e}_i$. But if the three parallelogram face normals $\mathbf{n}_i$ are orthogonal, then so are the $\mathbf{e}_i$ and our coordinate system is then orthogonal.

Section B is where I started and I give there a first-cut proof that $n_1 \bullet n_2 = 0$ with various extra steps and non-relevant side calculations. It was not obvious to me at the point that $n_2 \bullet n_3 = 0$ for example.

In **Section C**, then, I replaced a,b,c with a_1, a_2, a_3 and x,y,x with x_1, x_2, x_3 . I hoped this more organized approach would shed light on things, but in fact it made things more obscure.

In **Section D** I rehash the metric tensor method, talking about the meaning of the base vectors and drawing the parallelogram picture. But this really added nothing to an understanding of why the normals are orthogonal. I cannot help but wonder if the first person to discover this fact was surprised. Who might that person have been? Web scan reveals nada, same for internal book scan. Here is a little

> Associated with the history of the confocal ellipsoidal coordinates are some of the most illustrious names in mathematics. Lamé (1837) introduced them to solve Laplace's equation for regions bounded by ellipsoids or by parts of surfaces belonging to a system of confocal quadrics. Independently, Jacobi (1839) also discovered them and with their aid solved the problems of the attraction by two fixed centres and of the construction of geodesics on a triaxial ellipsoid (Arnold 1978).
>
> Stäckel (1891) carried out an important investigation into the general conditions for the Hamilton–Jacobi equation to separate in orthogonal curvilinear coordinate systems. He wrote down the most general metric for which this is possible and deduced the conditions which the potential must satisfy (Goldstein 1980). Weinacht (1924) gave the explicit expression for the general form of the potential in confocal ellipsoidal coordinates. Eisenhart (1934) constructed the separable systems of Stäckel in all the degenerate forms of the confocal quadrics.

I think I will give my prize to Lamé in 1837.

A. Geometric reason why the ellipsoidal level-surface normals are orthogonal.

The equations for the three ellipsoidal level surfaces are all the same, having the form for i = 1,2,3

$$g_i(x,y,z) \equiv x^2/(\xi_i^2-a^2) + y^2/(\xi_i^2- b^2) + z^2/(\xi_i^2- c^2) - 1 = 0 \qquad c < b < a$$

The nature of the surface (ellipsoid, bloid..) is determined by the value of parameter ξ relative to a,b,c:

$$c < \xi_3 < b < \xi_2 < q < \xi_1$$
bloid' bloid ellipsoid

The normal to any of these surfaces is given by (see my geometry doc) $\mathbf{n}_i = \nabla(g_i)$ so consider:

$$\mathbf{n}_i = \nabla(g_i) = 2[\,\mathbf{x}/(\xi_i^2-a^2)+ \mathbf{y}/(\xi_i^2-b^2)+ \mathbf{z}/(\xi_i^2-c^2)]$$

$$\mathbf{n}_1 = \nabla(g_1) = 2[\,\mathbf{x}/(\xi_1^2-a^2)+ \mathbf{y}/(\xi_1^2-b^2)+ \mathbf{z}/(\xi_1^2-c^2)]$$
$$\mathbf{n}_2 = \nabla(g_2) = 2[\,\mathbf{x}/(\xi_2^2-a^2)+ \mathbf{y}/(\xi_2^2-b^2)+ \mathbf{z}/(\xi_2^2-c^2)]$$
$$\mathbf{n}_3 = \nabla(g_3) = 2[\,\mathbf{x}/(\xi_3^2-a^2)+ \mathbf{y}/(\xi_3^2-b^2)+ \mathbf{z}/(\xi_3^2-c^2)]$$

Taking the dot product gives, for example,

$$\mathbf{n}_1 \bullet \mathbf{n}_2/4 = [x^2] / [(\xi_1^2-a^2)(\xi_2^2-a^2)] + [y^2] / [(\xi_1^2-b^2)(\xi_2^2-b^2)] + [z^2] / [(\xi_1^2-c^2)(\xi_2^2-c^2)]$$

We want x,y,z to be not just *any* point, but a point at the intersection of the three level surfaces with labels ξ_1, ξ_2 and ξ_3. That is to say, we want x,y,z to be a point whose ellipsoidal coordinates are ξ_1, ξ_2, ξ_3. If we write out our three g_i equations above, we can solve for x,y,z using a 3x3 Cramer's rule with this result

$$x^2 = (\xi_1^2-a^2)(\xi_2^2-a^2)(\xi_3^2-a^2)/ [(a^2-c^2)(a^2-b^2)]$$
$$y^2 = (\xi_1^2-b^2)(\xi_2^2-b^2)(\xi_3^2-b^2)/ [(b^2-a^2)(b^2-c^2)]$$
$$z^2 = (\xi_1^2-c^2)(\xi_2^2-c^2)(\xi_3^2-c^2)/ [(c^2-b^2)(c^2-a^2)] \qquad \text{// the "xyz equations"}$$

Given any equation, you get the next by doing a cyclic rotation of a,b,c. If we now insert these values into our dot product, we get

$$\mathbf{n}_1 \bullet \mathbf{n}_2/4 = [x^2] / [(\xi_1^2-a^2)(\xi_2^2-a^2)] \qquad \text{// from above}$$
$$+ [y^2]/ [(\xi_1^2-b^2)(\xi_2^2-b^2)]$$
$$+ [z^2]/ [(\xi_1^2-c^2)(\xi_2^2-c^2)]$$

$$= [(\xi_1^2-a^2)(\xi_2^2-a^2)(\xi_3^2-a^2)/ [(a^2-c^2)(a^2-b^2)]] / [(\xi_1^2-a^2)(\xi_2^2-a^2)] \qquad \text{// Cramer's in}$$
$$+ [(\xi_1^2-b^2)(\xi_2^2-b^2)(\xi_3^2-b^2)/ [(b^2-a^2)(b^2-c^2)]]/ [(\xi_1^2-b^2)(\xi_2^2-b^2)]$$
$$+ [(\xi_1^2-c^2)(\xi_2^2-c^2)(\xi_3^2-c^2)/ [(c^2-b^2)(c^2-a^2)]]/ [(\xi_1^2-c^2)(\xi_2^2-c^2)]$$

$$= [(\xi_1^2-a^2)(\xi_2^2-a^2)(\xi_3^2-a^2)/ [(a^2-c^2)(a^2-b^2)(\xi_1^2-a^2)(\xi_2^2-a^2)] \qquad \text{// a rewrite}$$

$$+ [(\xi_1^2-b^2)\,(\xi_2^2-b^2)\,(\xi_3^2-b^2)/\,[\,(b^2-a^2)(b^2-c^2)(\xi_1^2-b^2)\,(\xi_2^2-b^2)]$$
$$+ [(\xi_1^2-c^2)\,(\xi_2^2-c^2)\,(\xi_3^2-c^2)/\,[\,(c^2-b^2)(c^2-a^2)(\xi_1^2-c^2)\,(\xi_2^2-c^2)]$$

$$= [(\xi_3^2-a^2)/\,[\,(a^2-c^2)(a^2-b^2)] \qquad \text{// after cancellations}$$
$$+ [(\xi_3^2-b^2)/\,[\,(b^2-a^2)(b^2-c^2)]$$
$$+ [(\xi_3^2-c^2)/\,[\,(c^2-b^2)(c^2-a^2)]$$

$$\Rightarrow \quad (\mathbf{n}_1 \bullet \mathbf{n}_2/4)\,(a^2-c^2)\,(a^2-b^2)\,(b^2-c^2)$$

$$= (\xi_3^2-a^2)\,(b^2-c^2)$$
$$+ (\xi_3^2-b^2)\,(c^2-a^2)$$
$$+ (\xi_3^2-c^2)\,(a^2-b^2)$$

$$= A\xi_3^2 + B = 0\xi_3^2 + 0 = 0 \qquad \text{// just do it!}$$

More generally we have

$$\mathbf{n}_i \bullet \mathbf{n}_j/4 = [x^2] / [(\xi_i^2-a^2)\,(\xi_j^2-a^2)]$$
$$+ [y^2] / [(\xi_i^2-b^2)\,(\xi_j^2-b^2)]$$
$$+ [z^2] / [(\xi_i^2-c^2)\,(\xi_j^2-c^2)]$$

$$= [(\xi_1^2-a^2)\,(\xi_2^2-a^2)\,(\xi_3^2-a^2)/\,[\,(a^2-c^2)(a^2-b^2)(\xi_i^2-a^2)\,(\xi_j^2-a^2)] \qquad \text{// Cramer's and rewrite}$$
$$+ [(\xi_1^2-b^2)\,(\xi_2^2-b^2)\,(\xi_3^2-b^2)/\,[\,(b^2-a^2)(b^2-c^2)(\xi_i^2-b^2)\,(\xi_j^2-b^2)]$$
$$+ [(\xi_1^2-c^2)\,(\xi_2^2-c^2)\,(\xi_3^2-c^2)/\,[\,(c^2-b^2)(c^2-a^2)(\xi_i^2-c^2)\,(\xi_j^2-c^2)]$$

If $i \neq j$, we always get two cancellations on each line. If t is the letter that i and j are *not*, then we get

$$\mathbf{n}_i \bullet \mathbf{n}_j/4 =$$

$$= [(\xi_t^2-a^2)/\,[\,(a^2-c^2)(a^2-b^2)] \qquad \text{// after cancellations}$$
$$+ [(\xi_t^2-b^2)/\,[\,(b^2-a^2)(b^2-c^2)]$$
$$+ [(\xi_t^2-c^2)/\,[\,(c^2-b^2)(c^2-a^2)]$$

and this is the same form we had above with t = 3, but the value of t does not matter and we get 0. **QED.**

Suppose i and j are the same. In this case, let r and s be "the other two indices" and we get

$$\mathbf{n}_i \bullet \mathbf{n}_i/4 = [(\xi_1^2-a^2)\,(\xi_2^2-a^2)\,(\xi_3^2-a^2)/\,[\,(a^2-c^2)(a^2-b^2)(\xi_i^2-a^2)^2] \qquad \text{// the rewrite}$$
$$+ [(\xi_1^2-b^2)\,(\xi_2^2-b^2)\,(\xi_3^2-b^2)/\,[\,(b^2-a^2)(b^2-c^2)\,(\xi_i^2-b^2)^2]$$
$$+ [(\xi_1^2-c^2)\,(\xi_2^2-c^2)\,(\xi_3^2-c^2)/\,[\,(c^2-b^2)(c^2-a^2)\,(\xi_i^2-c^2)^2]$$

$$= [(\xi_r^2-a^2)\,(\xi_s^2-a^2) / [\,(a^2-c^2)(a^2-b^2)(\xi_i^2-a^2)] \qquad \text{// after cancellation}$$
$$+ [(\xi_r^2-b^2)\,(\xi_s^2-b^2) / [\,(b^2-a^2)(b^2-c^2)(\xi_i^2-b^2)]$$

$$+ [(\xi_r^2-c^2)\,(\xi_s^2-c^2) / [\,(c^2-b^2)(c^2-a^2)(\xi_i^2-c^2)]$$

and this tells us the length^2/4 of our normal vector n_i as defined by $\mathbf{n_i} = \nabla(\,g_i)$. It is not very simple.

B. An earlier less focused version of A.

1. The ellipsoidal level surfaces and the x,y,z equations

Here we have the ellipsoid and the two bloids for the ellipsoidal level surfaces

$x^2/(\,\xi_1^2-a^2) + y^2/(\,\xi_1^2- b^2) + z^2/(\xi_1^2- c^2) = 1$	$c < b < a < \xi_1$	ellipsoid
$- x^2/(a^2- \xi_2^2) + y^2/(\,\xi_2^2- b^2) + z^2/(\xi_2^2- c^2) = 1$	$c < b < \xi_2 < a$	bloid
$- x^2/(a^2- \xi_3^2) - y^2/(b^2- \xi_3^2) + z^2/(\xi_3^2- c^2) = 1$	$c < \xi_3 < b < a$	bloid'

We know that we can obtain the x,y,z equations using Cramer's rule and we find (in "ellipsoidal Cramer's rule...doc") that, for general c,

$$x^2 = (\xi_1^2-a^2)\,(\xi_2^2-a^2)\,(\xi_3^2-a^2)/\ [\,(a^2-c^2)(a^2-b^2)]$$
$$y^2 = (\xi_1^2-b^2)\,(\xi_2^2-b^2)\,(\xi_3^2-b^2)/\ [\,(b^2-a^2)(b^2-c^2)]$$
$$z^2 = (\xi_1^2-c^2)\,(\xi_2^2-c^2)\,(\xi_3^2-c^2)/\ [\,(c^2-b^2)(c^2-a^2)]$$

where things are cyclic in a,b,c.

2. Finding the normal vectors at some point x

Now, the first conicoid equation above (the ellipsoid) can be written as

$$g_1(x,y,z;\,\xi_1)= 0$$

We know from "surface geometry questions.doc" that the normal to this surface is given by

$$\hat{\mathbf{n}}_1 = \nabla(g_1) / |\nabla(g_1)|$$
$$\mathbf{n}_1 = \nabla(g_1) \qquad \text{// unnormalized}$$

where the direction is that of maximum positive change, and where ∇ is the usual 3D gradient. How do we compute this gradient?

$$g_1(x,y,z) = x^2/(\,\xi_1^2-a^2) + y^2/(\,\xi_1^2- b^2)+ z^2/(\xi_1^2- c^2)- 1$$

$$\partial_x\, g_1(x,y,z) = 2x/(\,\xi_1^2-a^2)$$
$$\partial_y\, g_1(x,y,z) = 2y/(\,\xi_1^2-b^2)$$
$$\partial_z\, g_1(x,y,z) = 2z/(\,\xi_1^2-c^2)$$

$$\mathbf{n_1} = \nabla(g_1) = 2[\ \mathbf{x}/(\ \xi_1^2-a^2)+\ \mathbf{y}/(\ \xi_1^2-b^2)+\ \mathbf{z}/(\ \xi_1^2-c^2)]$$

What is the magnitude of this normal?

$$n_1^2 = \mathbf{n_1}\bullet\ \mathbf{n_1} = 4\ [\mathbf{x}/(\ \xi_1^2-a^2)+\ \mathbf{y}/(\ \xi_1^2-b^2)+\ \mathbf{z}/(\ \xi_1^2-c^2)]\bullet\ [\mathbf{x}/(\ \xi_1^2-a^2)+\ \mathbf{y}/(\ \xi_1^2-b^2)+\ \mathbf{z}/(\ \xi_1^2-c^2)]$$

$$= 4\ [x^2/(\ \xi_1^2-a^2)^2\ +\ y^2/(\ \xi_1^2-b^2)^2\ +\ z^2/(\ \xi_1^2-c^2)^2\ ]$$

Now notice this fact

$$\partial_{\xi 1}\ (\ \xi_1^2-a^2)^{-1} = -2\xi_1(\ \xi_1^2-a^2)^{-2}$$

$$\partial_{\xi 1}\ g_1\ = -2\xi_1\ [x^2/(\ \xi_1^2-a^2)^2\ +\ y^2/(\ \xi_1^2-b^2)^2\ +\ z^2/(\ \xi_1^2-c^2)^2\ ]$$

$$\Rightarrow\ 2[x^2/(\ \xi_1^2-a^2)^2\ +\ y^2/(\ \xi_1^2-b^2)^2\ +\ z^2/(\ \xi_1^2-c^2)^2\ ]\ = -\ \partial_{\xi 1}\ g_1/\ \xi_1$$

We thus obtain this simple form for n_1^2 (but we shall not make use of it, at least not here)

$$n_1^2 = -2\ \ (\partial_{\xi 1}\ g_1)/\ \xi_1$$

We can write this *another* way by using our x,y,z equations so that (we now set c = 0)

$$n_1^2/4 = x^2/(\ \xi_1^2-a^2)^2\ +\ y^2/(\ \xi_1^2-b^2)^2\ +\ z^2/(\ \xi_1^2-c^2)^2$$

$$\begin{aligned}
&= \ \ x^2/(\ \xi_1^2-a^2)^2 \\
&\ + y^2/(\ \xi_1^2-b^2)^2 \\
&\ + z^2/(\ \xi_1^2-c^2)^2
\end{aligned}$$

$$\begin{aligned}
&= \ \ [(\xi_1^2-a^2)\ (\xi_2^2-a^2)\ (\xi_3^2-a^2)/\ \ [\ (a^2-c^2)(a^2-b^2)]]/(\ \xi_1^2-a^2)^2 \\
&\ + [(\xi_1^2-b^2)\ (\xi_2^2-b^2)\ (\xi_3^2-b^2)/\ [\ (b^2-a^2)(b^2-c^2)]]/(\ \xi_1^2-b^2)^2 \\
&\ + [(\xi_1^2-c^2)\ (\xi_2^2-c^2)\ (\xi_3^2-c^2)/\ \ [\ (c^2-b^2)(c^2-a^2)]]/(\ \xi_1^2-c^2)^2
\end{aligned}$$

$$\begin{aligned}
&= \ \ [\ (\xi_2^2-a^2)\ (\xi_3^2-a^2)/\ \ [\ (a^2-c^2)(a^2-b^2)(\xi_1^2-a^2)] \\
&\ + [\ (\xi_2^2-b^2)\ (\xi_3^2-b^2)/\ [\ (b^2-a^2)(b^2-c^2)(\xi_1^2-b^2)] \\
&\ + [(\xi_2^2-c^2)\ (\xi_3^2-c^2)/\ \ [\ (c^2-b^2)(c^2-a^2)(\xi_1^2-c^2)\ ]
\end{aligned}$$

which is not very enlightening (we can see that it is still cyclic).

The gradients of the other two surfaces are the same with a change in ξ_i, so here are our three normals at the point (x,y,z):

$$\mathbf{n_1} = \nabla(g_1) = 2\mathbf{x}/(\ \xi_1^2-a^2)+\ 2\mathbf{y}/(\ \xi_1^2-b^2)+\ 2\mathbf{z}/(\ \xi_1^2-c^2)$$
$$\mathbf{n_2} = \nabla(g_2) = 2\mathbf{x}/(\ \xi_2^2-a^2)+\ 2\mathbf{y}/(\ \xi_2^2-b^2)+\ 2\mathbf{z}/(\ \xi_2^2-c^2)$$

$$\mathbf{n_3} = \nabla(g_3) = 2\mathbf{x}/(\xi_3^2-a^2) - 2\mathbf{y}/(\xi_3^2-b^2) + 2\mathbf{z}/(\xi_3^2-c^2)$$

or in general

$$\mathbf{n_i} = \nabla(g_i) = 2\mathbf{x}/(\xi_i^2-a^2) - 2\mathbf{y}/(\xi_i^2-b^2) + 2\mathbf{z}/(\xi_i^2-c^2)$$

3. Show that $\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$

Consider

$$\mathbf{n_1} \bullet \mathbf{n_2} = 4\,[\mathbf{x}/(\xi_1^2-a^2) + \mathbf{y}/(\xi_1^2-b^2) + \mathbf{z}/(\xi_1^2-c^2)] \bullet [\,\mathbf{x}/(\xi_2^2-a^2) + \mathbf{y}/(\xi_2^2-b^2) + \mathbf{z}/(\xi_2^2-c^2)]$$

$$= 4\{\ x^2/[(\xi_1^2-a^2)(\xi_2^2-a^2)] + y^2/[(\xi_1^2-b^2)(\xi_2^2-b^2)] + z^2/[(\xi_1^2-c^2)(\xi_2^2-c^2)]\ \}$$

Now once again let's use the x,y,z equations here:

$$\begin{aligned}
\mathbf{n_1} \bullet \mathbf{n_2}/4 = {} & [x^2]/[(\xi_1^2-a^2)(\xi_2^2-a^2)] \\
& + [y^2]/[(\xi_1^2-b^2)(\xi_2^2-b^2)] \\
& + [z^2]/[(\xi_1^2-c^2)(\xi_2^2-c^2)]
\end{aligned}$$

$$\begin{aligned}
= {} & [(\xi_1^2-a^2)(\xi_2^2-a^2)(\xi_3^2-a^2)/\ [\,(a^2-c^2)(a^2-b^2)]] / [(\xi_1^2-a^2)(\xi_2^2-a^2)] \\
& + [(\xi_1^2-b^2)(\xi_2^2-b^2)(\xi_3^2-b^2)/\ [\,(b^2-a^2)(b^2-c^2)]]/ [(\xi_1^2-b^2)(\xi_2^2-b^2)] \\
& + [(\xi_1^2-c^2)(\xi_2^2-c^2)(\xi_3^2-c^2)/\ [\,(c^2-b^2)(c^2-a^2)]]/ [(\xi_1^2-c^2)(\xi_2^2-c^2)]
\end{aligned}$$

$$\begin{aligned}
= {} & [(\xi_1^2-a^2)(\xi_2^2-a^2)(\xi_3^2-a^2)/\ [\,(a^2-c^2)(a^2-b^2)(\xi_1^2-a^2)(\xi_2^2-a^2)] \\
& + [(\xi_1^2-b^2)(\xi_2^2-b^2)(\xi_3^2-b^2)/\ [\,(b^2-a^2)(b^2-c^2)(\xi_1^2-b^2)(\xi_2^2-b^2)] \\
& + [(\xi_1^2-c^2)(\xi_2^2-c^2)(\xi_3^2-c^2)/\ [\,(c^2-b^2)(c^2-a^2)(\xi_1^2-c^2)(\xi_2^2-c^2)]
\end{aligned}$$

$$\begin{aligned}
= {} & [(\xi_3^2-a^2)/\ [\,(a^2-c^2)(a^2-b^2)] \\
& + [(\xi_3^2-b^2)/\ [\,(b^2-a^2)(b^2-c^2)] \\
& + [(\xi_3^2-c^2)/\ [\,(c^2-b^2)(c^2-a^2)]
\end{aligned}$$

$$(\mathbf{n_1} \bullet \mathbf{n_2}/4)(a^2-c^2)(a^2-b^2)(b^2-c^2)$$

$$\begin{aligned}
= {} & (\xi_3^2-a^2)(b^2-c^2) \\
& - (\xi_3^2-b^2)(a^2-c^2) \\
& + (\xi_3^2-c^2)(a^2-b^2)
\end{aligned}$$

This is a sum of 12 terms, and Maple or by hand tells us the result is 0. So we have shown that

$$\mathbf{n_1} \bullet \mathbf{n_2} = 0$$

Summary: We have computed n_1^2 and found that $\mathbf{n_1} \bullet \mathbf{n_2} = 0$, which means the ellipsoid normal and the first bload normal are in fact orthogonal at the same point **x**. We would like to look at these two results and immediately come up with an expression for n_2^2 and $\mathbf{n_1} \bullet \mathbf{n_3}$, say, but it is not obvious how to do this without reworking all the algebra pretty much from scratch. My feeling is that there should be a better notation which will give us results for n_i^2 and $\mathbf{n_i} \bullet \mathbf{n_j}$. That "better notation" is the subject of the next Part of this doc.

C. Another version trying to make better use of indices.

1. The ellipsoidal level surfaces and the x,y,z equations

Here we have the ellipsoid and the two bloids for the ellipsoidal level surfaces

$$\begin{aligned} x^2/(\xi_1^2-a^2) + y^2/(\xi_1^2-b^2) + z^2/(\xi_1^2-c^2) &= 1 \qquad c < b < a < \xi_1 \\ -x^2/(a^2-\xi_2^2) + y^2/(\xi_2^2-b^2) + z^2/(\xi_2^2-c^2) &= 1 \qquad c < b < \xi_2 < a \\ -x^2/(a^2-\xi_3^2) - y^2/(b^2-\xi_3^2) + z^2/(\xi_3^2-c^2) &= 1 \qquad c < \xi_3 < b < a \end{aligned}$$

which is written here in a form where the denominators are always positive. We can of course write this in the following form

$$\begin{aligned} x^2/(\xi_1^2-a^2) + y^2/(\xi_1^2-b^2) + z^2/(\xi_1^2-c^2) &= 1 \qquad c < b < a < \xi_1 \\ x^2/(\xi_2^2-a^2) + y^2/(\xi_2^2-b^2) + z^2/(\xi_2^2-c^2) &= 1 \qquad c < b < \xi_2 < a \\ x^2/(\xi_3^2-a^2) + y^2/(\xi_3^2-b^2) + z^2/(\xi_3^2-c^2) &= 1 \qquad c < \xi_3 < b < a \end{aligned}$$

so that all three equations look exactly the same apart from the different ξ_i parameters.

We know that we can obtain the x,y,z equations using Cramer's rule and we find (in "ellipsoidal Cramers rule...doc") that, for general c,

$$\begin{aligned} x^2 &= (\xi_1^2-a^2)(\xi_2^2-a^2)(\xi_3^2-a^2)/\ [(a^2-c^2)(a^2-b^2)] \\ y^2 &= (\xi_1^2-b^2)(\xi_2^2-b^2)(\xi_3^2-b^2)/\ [(b^2-a^2)(b^2-c^2)] \\ z^2 &= (\xi_1^2-c^2)(\xi_2^2-c^2)(\xi_3^2-c^2)/\ [(c^2-b^2)(c^2-a^2)] \end{aligned}$$

where things are cyclic in a,b,c.

2. Restate the above in a more regular notation

Suppose we define

$$\begin{aligned} a_1 &= a,\ a_2 = b,\ a_3 = c \\ x_1 &= x,\ x_2 = y,\ x_3 = z \end{aligned}$$

Then any of our level surface equations in the second group become

$$x_1^2/(\xi_i^2-a_1^2) + x_2^2/(\xi_i^2-a_2^2)+ x_3^2/(\xi_i^2-a_3^2) = 1 \qquad i = 1,2,3$$

or

$$\Sigma_{k=1}^{3} x_k^2/(\xi_i^2-a_k^2) = 1 \qquad i = 1,2,3$$

The distinction between the three surface shapes is this

ellipsoid	$a_3 < a_2 < a_1 < \xi_1$
bloid	$a_3 < a_2 < \xi_2 < a_1$
bloid'	$a_3 < \xi_3 < a_2 < a_1$

The Cramer's Rule results above become

$$x_1^2 = (\xi_1^2-a_1^2)(\xi_2^2-a_1^2)(\xi_3^2-a_1^2)/[(a_1^2-a_3^2)(a_1^2-a_2^2)]$$
$$x_2^2 = (\xi_1^2-a_2^2)(\xi_2^2-a_2^2)(\xi_3^2-a_2^2)/[(a_2^2-a_1^2)(a_2^2-a_3^2)]$$
$$x_3^2 = (\xi_1^2-a_3^2)(\xi_2^2-a_3^2)(\xi_3^2-a_3^2)/[(a_3^2-a_2^2)(a_3^2-a_1^2)]$$

which can be combined into one line

$$x_i^2 = (\xi_1^2-a_i^2)(\xi_2^2-a_i^2)(\xi_3^2-a_i^2)/[(a_i^2-a_{i-1}^2)(a_i^2-a_{i+1}^2)]$$

or

$$x_i^2 = (\xi_1^2-a_i^2)(\xi_2^2-a_i^2)(\xi_3^2-a_i^2)/[\Pi_{j\neq i}(a_i^2-a_j^2)]$$

or

$$x_i^2 = \Pi_{k=1}^{3}(\xi_k^2-a_i^2) / [\Pi_{j\neq i}(a_i^2-a_j^2)]$$

So to summarize, we have

$\Sigma_{k=1}^{3} x_k^2/(\xi_i^2-a_k^2) = 1 \quad i = 1,2,3$	surfaces
$x_i^2 = \Pi_{k=1}^{3}(\xi_k^2-a_i^2) / [\Pi_{j\neq i}(a_i^2-a_j^2)]$	x,y,z equations

3. Finding the normal vectors at some point x

If we define

$$g_i(x,y,z) \equiv \Sigma_{k=1}^{3} x_k^2/(\xi_i^2-a_k^2) - 1$$

Then our level surface equations may be written

$$g_i(x,y,z) = 0 \qquad i = 1,2,3$$

We know from "surface geometry questions.doc" that the normal to this surface is given by

$$\hat{\mathbf{n}}_i = \nabla(g_i) / |\nabla(g_i)|$$
$$\mathbf{n}_i = \nabla(g_i) \qquad \text{// unnormalized}$$

where the direction is that of maximum positive change, and where ∇ is the usual 3D gradient. How do we compute this gradient?

$$\nabla(g_i) = \nabla(\Sigma_{k=1}^{3} x_k^2/ (\xi_i^2-a_k^2) - 1) = \Sigma_{n=1}^{3} \hat{\mathbf{x}}_n \partial_n [\Sigma_{k=1}^{3} x_k^2/ (\xi_i^2-a_k^2) - 1]$$

$$= \Sigma_{n=1}^{3} \hat{\mathbf{x}}_n \partial_n [\Sigma_{k=1}^{3} x_k^2/ (\xi_i^2-a_k^2) = \Sigma_{n=1}^{3} \Sigma_{k=1}^{3} \{1/ (\xi_i^2-a_k^2)\} \hat{\mathbf{x}}_n \partial_n (x_k^2)$$

$$= \Sigma_{n=1}^{3} \Sigma_{k=1}^{3} \{1/ (\xi_i^2-a_k^2)\} \hat{\mathbf{x}}_n 2 x_k (\partial_n x_k) = \Sigma_{n=1}^{3} \Sigma_{k=1}^{3} \{1/ (\xi_i^2-a_k^2)\} \hat{\mathbf{x}}_n 2 x_k \delta_{nk}$$

$$= \Sigma_{n=1}^{3} \{1/ (\xi_i^2-a_n^2)\} \hat{\mathbf{x}}_n 2 x_n = 2 \Sigma_{n=1}^{3} \mathbf{x}_n / (\xi_i^2-a_n^2)$$

So we are then claiming that,

$$\mathbf{n}_i = 2 \Sigma_{k=1}^{3} \mathbf{x}_k / (\xi_i^2-a_k^2)$$

where, if we define our usual Cartesian unit vectors to be $\hat{\mathbf{k}}$ for k = 1,2,3, we have

$$\mathbf{x}_k \equiv (\mathbf{x} \bullet \hat{\mathbf{k}}) \hat{\mathbf{k}} \qquad\qquad \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3$$

So $\mathbf{n}_i$ is an unnormalized normal vector which is normal to the surface whose label is ξ_i at point $\mathbf{x}$.

3. Show that the three normal vectors are mutually perpendicular.

We compute the dot product:

$$\mathbf{n}_i \bullet \mathbf{n}_j = [2 \Sigma_{n=1}^{3} \mathbf{x}_n / (\xi_i^2-a_n^2)] \bullet [2 \Sigma_{m=1}^{3} \mathbf{x}_m / (\xi_j^2-a_m^2)]$$

$$= 4 \Sigma_{n=1}^{3} \Sigma_{m=1}^{3} 1/[(\xi_i^2-a_n^2) (\xi_j^2-a_m^2)] \mathbf{x}_n \bullet \mathbf{x}_m$$

$$= 4 \Sigma_{n=1}^{3} \Sigma_{m=1}^{3} 1/[(\xi_i^2-a_n^2) (\xi_j^2-a_m^2)] \delta_{nm} x_n^2$$

$$= 4 \Sigma_{n=1}^{3} x_n^2/[(\xi_i^2-a_n^2) (\xi_j^2-a_n^2)]$$

Now into this expression we want to insert our Cramer's rule result which was

$$x_i^2 = \Pi_{k=1}^{3}(\xi_k^2-a_i^2) / [\Pi_{j \neq i}(a_i^2-a_j^2)] \qquad \text{x,y,z equations}$$

or

$$x_n^2 = \Pi_{k=1}^{3}(\xi_k^2-a_n^2) / [\Pi_{j \neq n}(a_n^2-a_j^2)] \qquad \text{x,y,z equations}$$

or

$$x_n^2 = \Pi_{k=1}^{3}(\xi_k^2-a_n^2) / [\Pi_{s \neq n}(a_n^2-a_s^2)] \qquad \text{x,y,z equations}$$

In other words, we are replacing the x,y,z coordinates of **x** with their values given in ellipsoidal coordinates. These coordinates are, of course, the same ξ_i values appearing in $\mathbf{n}_i \bullet \mathbf{n}_j$ above . We then have

$$\mathbf{n}_i \bullet \mathbf{n}_j = 4 \Sigma_{n=1}^{3} \{x_n^2\}/[(\xi_i^2-a_n^2)(\xi_j^2-a_n^2)]$$

$$= 4 \Sigma_{n=1}^{3} \{ \Pi_{k=1}^{3}(\xi_k^2-a_n^2) / [\Pi_{s\neq n}(a_n^2-a_s^2)]\}/[(\xi_i^2-a_n^2)(\xi_j^2-a_n^2)]$$

$$= 4 \Sigma_{n=1}^{3} \{ \Pi_{k=1}^{3}(\xi_k^2-a_n^2) / [(\xi_i^2-a_n^2)(\xi_j^2-a_n^2)\Pi_{s\neq n}(a_n^2-a_s^2)] \}$$

$$= 4 \Sigma_{n=1}^{3} \{ \Pi_{k=1}^{3}(\xi_k^2-a_n^2) / [(\xi_i^2-a_n^2)(\xi_j^2-a_n^2) D_n \} \qquad D_n = \Pi_{s\neq n}(a_n^2-a_s^2)$$

Now multiply both sides by $(a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2)$

$$(a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2)\, \mathbf{n}_i \bullet \mathbf{n}_j$$

$$= 4 \Sigma_{n=1}^{3} \{ \Pi_{k=1}^{3}(\xi_k^2-a_n^2) / [(\xi_i^2-a_n^2)(\xi_j^2-a_n^2)] * (a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2) / [\Pi_{s\neq n}(a_n^2-a_s^2)] \}$$

Now consider the second ratio. There are three factors up top, and two down. Now we shall look at each case separately

n=1 $\quad (a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2) / [(a_1^2-a_2^2)(a_1^2-a_3^2)] = -(a_2^2-a_3^2)$

n=2 $\quad (a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2) / [(a_2^2-a_1^2)(a_2^2-a_3^2)] = -(a_3^2-a_1^2)$

n=3 $\quad (a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2) / [(a_3^2-a_1^2)(a_3^2-a_2^2)] = -(a_1^2-a_2^2)$

The second two cases are cyclics of the first one. We can then write out the n=1 term only and say

$$(a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2)\, \mathbf{n}_i \bullet \mathbf{n}_j$$

$$= 4\ (a_3^2-a_2^2)\ \{ [\Pi_{k=1}^{3}(\xi_k^2-a_1^2)]\ / [(\xi_i^2-a_1^2)(\xi_j^2-a_1^2)]\} + \text{cyclic}$$

Notice that both sides have cyclic symmetry under cycling of the a_i indices, and that both sides are symmetric under $i \leftrightarrow j$. The LHS turns into itself under any cyclic rotation of the a_i indices, and so does the RHS when all three terms are included.

Now suppose $i \neq j$ and suppose t is the index that is neither i nor j from the set 1,2,3. Then in the {} ratio, the 2 denominator factors cancel 2 of the 3 numerator factors, leaving just $(\xi_t^2-a_1^2)$. We then have

$$(1/4)(a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2)\, \mathbf{n}_i \bullet \mathbf{n}_j = (a_3^2-a_2^2)(\xi_t^2-a_1^2) + \text{cyclic} \qquad i \neq j$$

The ξ_t^2 terms are these

ξ_t^2 [$(a_3^2-a_2^2)$ + cyclic] = ξ_t^2 [$(a_3^2-a_2^2) + (a_1^2-a_3^2) + (a_2^2-a_1^2)$] = $\xi_t^2[0] = 0$

The other terms are these

$(a_2^2-a_3^2)a_1^2$ + cyclic = $(a_2^2-a_3^2)a_1^2 + (a_3^2-a_1^2)a_2^2 + (a_1^2-a_2^2)a_3^2$

$= 21 - 31 + 32 - 12 + 13 - 23 = 0$

Therefore, we have shown that

$\mathbf{n}_i \bullet \mathbf{n}_j = 0$ if $i \neq j$

So we have achieved our goal of generalizing our Part I proof that $\mathbf{n}_1 \bullet \mathbf{n}_2 = 0$.

Now go back and try i = j instead. We then get

$(a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2)\ \mathbf{n}_i \bullet \mathbf{n}_i$

$= 4\ (a_3^2-a_2^2)\ \{ [\Pi_{k=1}^{3}(\xi_k^2-a_1^2)] / [(\xi_i^2-a_1^2)(\xi_i^2-a_1^2)]\}$ + cyclic

This time, only one factor top and bottom cancels and we get a fairly ugly result

$(a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2)\ \mathbf{n}_i \bullet \mathbf{n}_i$

$= 4\ (a_3^2-a_2^2)\ \{ [\Pi_{k\neq i}(\xi_k^2-a_1^2)] / [(\xi_i^2-a_1^2)]\}$ + cyclic

Therefore

$n_i^2 = \mathbf{n}_i \bullet \mathbf{n}_i = 4\ (a_3^2-a_2^2)\ \{ [\Pi_{k\neq i}(\xi_k^2-a_1^2)] / [(\xi_i^2-a_1^2)]\} / [(a_1^2-a_2^2)(a_2^2-a_3^2)(a_3^2-a_1^2)]$

$= -4\ \{ [\Pi_{k\neq i}(\xi_k^2-a_1^2)] / [(\xi_i^2-a_1^2)]\} / [(a_1^2-a_2^2)(a_3^2-a_1^2)]$ + cyclic

$= 4\ \{ [\Pi_{k\neq i}(\xi_k^2-a_1^2)] / [(\xi_i^2-a_1^2)]\} / [(a_1^2-a_2^2)(a_1^2-a_3^2)]$ + cyclic

$= 4\ \{ [\Pi_{k\neq i}(\xi_k^2-a_1^2)] / [(\xi_i^2-a_1^2)(a_1^2-a_2^2)(a_1^2-a_3^2)]\}$ + cyclic

$= 4\ \{ [\Pi_{k\neq i}(\xi_k^2-a_1^2)] / [(\xi_i^2-a_1^2)(a_1^2-a_2^2)(a_1^2-a_3^2)]\}$ + cyclic

Specifically, we have

$n_1^2 = 4\ \{ (\xi_2^2-a_1^2)(\xi_3^2-a_1^2)] / [(\xi_1^2-a_1^2)(a_1^2-a_2^2)(a_1^2-a_3^2)]\}$ + cyclic
$n_2^2 = 4\ \{ (\xi_3^2-a_1^2)(\xi_1^2-a_1^2)] / [(\xi_2^2-a_1^2)(a_1^2-a_2^2)(a_1^2-a_3^2)]\}$ + cyclic
$n_3^2 = 4\ \{ (\xi_1^2-a_1^2)(\xi_2^2-a_1^2)] / [(\xi_3^2-a_1^2)(a_1^2-a_2^2)(a_1^2-a_3^2)]\}$ + cyclic

so we have generalized our earlier expression for n_1^2

D. Rehash of the "metric tensor method".

In our method above, we used our brute force geometry normal expression to obtain the desired result. Here consider the x,y,z equations again

$$x_i^2 = \Pi_{k=1}^{3}(\xi_k^2 - a_i^2) \,/\, [\,\Pi_{j\neq i}(a_i^2 - a_j^2)]$$

Think of the ξ_k as the q^k of our general curvilinear discussion. The transformation matrix T is this

$$T_k^{\,i} = (\mathbf{e}_{\xi k})^i = \partial x^i / \partial \xi_k$$

which is perhaps more intuitively written as

$$\delta x^i = (\mathbf{e}_{\xi k})^i \, \delta\xi_k \quad \Rightarrow \quad \delta \mathbf{x}_k = (\mathbf{e}_{\xi k})\, \delta\xi_k$$

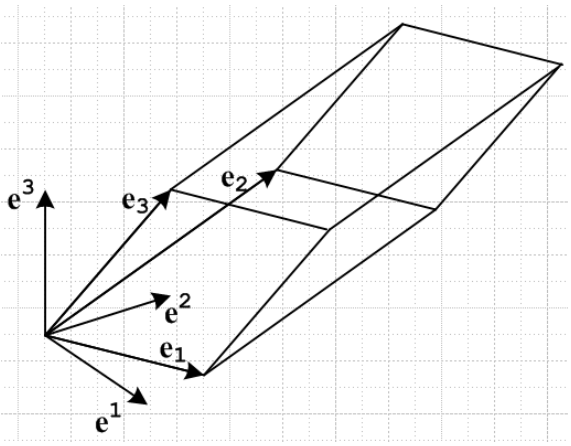
The meaning is what is important. If you vary only curvilinear coordinate ξ_k and keep the other two fixed, then $\delta\mathbf{x}_k$ is how you move in x,y,z space. You move along the ξ_k "coordinate line", and $\mathbf{e}_{\xi k}$ is known as the tangent base vector for ξ_k.

Let's take k = 1 to be more explicit. We want to vary ξ_1 but keep ξ_2 and ξ_3 fixed, and see where we go in Cartesian space from a starting point which lies on the intersection of the three surfaces having these labels. So in this case, $\delta\mathbf{x}_1$ is a little displacement off the ξ_1 ellipsoid, but we remain ON the ξ_2 bloid and we remain on the ξ_3 bloid.

Suppose we could show that $\delta\mathbf{x}_k \bullet \delta\mathbf{x}_j = 0$ for any pair k≠j. What would that mean in terms of the three level surfaces which intersect at our starting point?

vary ξ_1 $\quad \delta\mathbf{x}_1 \sim \mathbf{e}_1$ lies on surface 2 and 3
vary ξ_2 $\quad \delta\mathbf{x}_2 \sim \mathbf{e}_2$ lies on surface 3 and 1
vary ξ_3 $\quad \delta\mathbf{x}_3 \sim \mathbf{e}_3$ lies on surface 1 and 2

Consider this picture:



Since our three level surfaces are presumably smooth, if we look only in a tiny neighborhood of some point of interest, those three surfaces look like the faces of a parallelogram in the most general case. In this picture, the left surface is surface 1 because we stay on it if we vary along $\mathbf{e}_2$ or $\mathbf{e}_3$. The surface number is the index that does not appear on the two vectors spanning the surface. So the front face is surface 2 and the bottom is surface 3.

Basically this says that locally, we have a little parallelogram-shaped coordinate system and the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ point along the three axes. If we find that these three axis vectors are orthogonal, then we know that locally we have a Cartesian coordinate system with these things as axes. It is just that simple, and this is then called an orthogonal coordinate system.

Now back to our x,y,z equations:

$$x_i^2 = \Pi_{k=1}^{3}(\xi_k^2 - a_i^2) \,/\, [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]$$

Let's compute the three vectors

$$(\mathbf{e}_{\xi j})_i = \partial x_i/\partial_{\xi j} = \partial x_i^2/\partial_{\xi j} * \partial x_i/\partial x_i^2 = (1/2x_i)\,\partial x_i^2/\partial_{\xi j}$$

$$= (1/2x_i)\,\partial_{\xi j}\,\{\ \Pi_{k=1}^{3}(\xi_k^2 - a_i^2) \,/\, [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]\ \}$$

$$= (1/2x_i) * 1/\ [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]\ \ * \partial_{\xi j}\,\{\ \Pi_{k=1}(\xi_k^2 - a_i^2)\ \}$$

$$= (1/2x_i) * 1/\ [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]\ \ * 2\,\xi_j\,\Pi_{k\neq j}(\xi_k^2 - a_i^2)$$

$$= \Pi_{k\neq j}(\xi_k^2 - a_i^2)\ \ (\xi_j/x_i) \,/\, [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]$$

Then we can say

$$\mathbf{e}_{\xi j}\bullet\mathbf{e}_{\xi j'} = (\mathbf{e}_{\xi j})_i(\mathbf{e}_{\xi j'})_i$$

$$= [\Pi_{k\neq j}(\xi_k^2 - a_i^2)]\ \ (\xi_j/x_i) \,/\, [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]\ * [\Pi_{k\neq j'}(\xi_k^2 - a_i^2)]\ \ (\xi_{j'}/x_i) \,/\, [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]$$

$$= (\xi_j/x_i)\,(\xi_{j'}/x_i)\,[\Pi_{k\neq j}(\xi_k^2 - a_i^2)]\,[\Pi_{k\neq j'}(\xi_k^2 - a_i^2)] \,/\, [\,\Pi_{s\neq i}(a_i^2 - a_s^2)]^2$$

$$= (\xi_j \xi_{j'}/x_i^2) [\Pi_{k\neq j}(\xi_k^2-a_i^2)] [\Pi_{k\neq j'}(\xi_k^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)]^2$$

We now have to get rid of x_i^2 in favor of the ellipsoidal coordinates, so

$$= (\xi_j \xi_{j'} [\Pi_{s\neq i}(a_i^2-a_s^2)] / [\Pi_{k=1}^3(\xi_k^2-a_i^2)]) [\Pi_{k\neq j}(\xi_k^2-a_i^2)] [\Pi_{k\neq j'}(\xi_k^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)]^2$$

$$= \Sigma_i (\xi_j \xi_{j'} / [\Pi_k(\xi_k^2-a_i^2)]) [\Pi_{k\neq j}(\xi_k^2-a_i^2)] [\Pi_{k\neq j'}(\xi_k^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)]$$

$$= \Sigma_i (\xi_j \xi_{j'} / (\xi_j^2-a_i^2)) * [\Pi_{k\neq j'}(\xi_k^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)]$$

$$= \xi_j \xi_{j'} \Sigma_i (1 / (\xi_j^2-a_i^2)) * [\Pi_{k\neq j'}(\xi_k^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)]$$

How can this possibly be zero? It must be true that, for $j \neq j'$, we have

$$\Sigma_{i=1}^3 (1 / (\xi_j^2-a_i^2)) * [\Pi_{k\neq j'}(\xi_k^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)]$$

Let's try an example. Let $j = 1$ and $j' = 2$:

$$\Sigma_{i=1}^3 (1 / (\xi_1^2-a_i^2)) * [\Pi_{k\neq 2}(\xi_k^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)] = 0 ?$$

$$\Sigma_{i=1}^3 (1 / (\xi_1^2-a_i^2)) * [(\xi_1^2-a_i^2) (\xi_3^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)] = 0 ?$$

$$\Sigma_{i=1}^3 (\xi_3^2-a_i^2)] / [\Pi_{s\neq i}(a_i^2-a_s^2)] = 0 ?$$

$$(\xi_3^2-a_1^2) / [(a_1^2-a_2^2) (a_1^2-a_3^2)] + (\xi_3^2-a_2^2) / [(a_2^2-a_1^2) (a_2^2-a_3^2)] + (\xi_3^2-a_3^2) / [(a_3^2-a_1^2) (a_3^2-a_2^2)]$$

Well let's assume this last line is some object F. Then we have

$$F (a_1^2-a_2^2) (a_1^2-a_3^2) (a_2^2-a_3^2)$$

$$= (\xi_3^2-a_1^2) (a_2^2-a_3^2) + (\xi_3^2-a_2^2) (a_3^2-a_1^2) + (\xi_3^2-a_3^2) (a_1^2-a_2^2)$$

and this is 0 for the same reason we got above, "just do it".

Well, all I have really done here is say that the normals are orthogonal because the metric tensor is diagonal, and to show that I have to calculate the elements $g_{ij} = \mathbf{e}_{\xi j} \bullet \mathbf{e}_{\xi j'}$ of the metric tensor, but this doesn't explain much that we don't already know.

Well this is doing nothing for me whatsoever. Just the metric tensor being diagonal method.