Expansion of the Position Vector in Curvilinear coordinates PhL 9.6.12

The error:

"Comment Appendix. In tensor doc, I do mention polars as an example on page 166, but I never make a connection between v_θ and θ. I was there just showing the various notations of a vector, use of italics, and all that stuff.

So let's examine the connection right here. We are using primes-removed notation.

$$\mathbf{r} = r^n \mathbf{e}_n = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z = r\hat{\mathbf{r}} + \theta\, r\, \hat{\boldsymbol{\theta}} + z\, \hat{\mathbf{z}} \text{ "}$$

Correction of this error:

The above is wrong because in cylindrical coordinates we really have

$$\mathbf{r} = r\, \hat{\mathbf{r}} + z\, \hat{\mathbf{z}}$$

so what was I thinking? I was just making a parallel between vector **r** and vector **v**. So here is the **v** stuff

$$\mathbf{v} = v^n \mathbf{e}_n = v^\theta \mathbf{e}_\theta + v^r \mathbf{e}_r$$
$$\mathbf{v} = v^n \hat{\mathbf{e}}_n = v_\theta \hat{\mathbf{e}}_\theta + v_r \hat{\mathbf{e}}_r = v_\theta \hat{\boldsymbol{\theta}} + v_r \hat{\mathbf{r}}$$
$$\mathbf{v} = v_n \hat{\mathbf{n}} = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$$

The corresponding **r** stuff would be

$$\mathbf{r} = r^n \mathbf{e}_n = r^\theta \mathbf{e}_\theta + r^r \mathbf{e}_r \qquad = r^r \mathbf{e}_r$$
$$\mathbf{r} = r^n \hat{\mathbf{e}}_n = r_\theta \hat{\mathbf{e}}_\theta + r_r \hat{\mathbf{e}}_r = r_\theta \hat{\boldsymbol{\theta}} + r_r \hat{\mathbf{r}} \qquad = r_r \hat{\mathbf{r}}$$
$$\mathbf{r} = r_n \hat{\mathbf{n}} = r_x \hat{\mathbf{x}} + r_y \hat{\mathbf{y}} \qquad = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

So the main point is that $r^\theta = r_\theta = 0$ for polar coordinates.

$r^\theta = r_\theta = 0$ // correct

$r^\theta = r_\theta = \theta$ // totally wrong

Negative Theorem: The components of the position vector **r** in curvilinear coordinates are NOT simply the curvilinear coordinates. We write things like $\mathbf{r} = r,\theta,\varphi$ in spherical coordinates, but that does not mean that $\mathbf{r} = r\, \hat{\mathbf{r}} + \theta\, \hat{\boldsymbol{\theta}} + \varphi\, \hat{\boldsymbol{\varphi}}$. You can write $\mathbf{r} = r_r\, \hat{\mathbf{r}} + r_\theta\, \hat{\boldsymbol{\theta}} + r_\varphi\, \hat{\boldsymbol{\varphi}}$ if you want, but $r_\theta = 0$ and $r_\varphi = 0$. The confusion arises because for a general vector $\mathbf{v} = v_r, v_\theta, v_\varphi$ you *do* have $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}}$ and in general all three components are non-zero. The position vector is always a special case.

Where is this in tensor doc?

Maybe I have failed to bring out this fact in tensor doc and that is why I fell into this hole. My first mention that relates I guess would be in the expansion section 6 (f) or 7 (s). In my expansion sections, I show the expansion of a vector **V**, but in general **r** = **x** is not a vector as I state many times. That is, the position "vector" **x** does not transform as a vector under F as a general rule. Does this preclude writing an expansion for **x** ? I never address this topic! Well, it gets some mention on page 65 of tensor doc in the Comments. My implication is that V can be any N-tuple in the expansion and the expansion is valid! I do say that directly. I later show that this is valid for expansions on $\mathbf{e_n}$, not just on $\mathbf{u_n}$.

So the point is that you have to actually compute the dot products in the expansion of the position "vector" in any given curvilinear system.

$$\mathbf{V} = V'_1\ \mathbf{e}_1 + V'_2\ \mathbf{e}_2 + \ldots \quad = \Sigma_n\, V'_n\, \mathbf{e}_n \qquad \text{where} \quad \mathbf{E}_n \bullet \mathbf{V} = V'_n \qquad \mathbf{E}_n = g'_{ni}\, \mathbf{e}_i$$

A few lines later I do the unit vector stuff

$$\mathbf{V} = \mathcal{V}'_1 \hat{\mathbf{e}}_1 + \mathcal{V}'_2 \hat{\mathbf{e}}_2 + \ldots \quad = \Sigma_n\, \mathcal{V}'_n\, \hat{\mathbf{e}}_n \qquad \text{where} \qquad \sqrt{g'_{nn}}\ \mathbf{E}_n \bullet \mathbf{V} = \mathcal{V}'_n = \sqrt{g'_{nn}}\, V'_n$$

Later in Section 7 I state that

$$\mathbf{V} = V'^1\mathbf{e}_1 + V'^2\mathbf{e}_2 + \ldots \quad = \Sigma_n\, V'^n\, \mathbf{e}_n \qquad \text{where} \quad \mathbf{e}^n \bullet \mathbf{V} = V'^n \qquad \mathbf{e}^n = g'^{ni}\, \mathbf{e}_i$$

so this seems to say that

$$V'^n = \mathbf{e}^n \bullet \mathbf{V} = g'^{ni}\, \mathbf{e}_i \bullet \mathbf{V}$$

Eventually you have to go compute $\mathbf{e}_i \bullet \mathbf{V}$ for your vector **V** and if **V** = **x**, for vector **x**. I am looking for a general formula here valid for any curvilinear system. How about this

$$V'^n = R^n{}_m V^m$$

and even though x is not a real vector, according to my notes we still define

$$x'^n = R^n{}_m x^m \qquad \mathcal{x}'^n = h_n x'^n$$

$$\mathcal{x}'^n = h_n\, R^n{}_m x^m \qquad \text{// these are the unit vector components of the position vector in any system!}$$

In polar coordinates we have 1,2 = θ,r

$$R = S^{-1} = \begin{pmatrix} -\sin\theta/r & \cos\theta/r \\ \cos(\theta) & \sin\theta \end{pmatrix} \qquad h_\theta = r \quad h_r = 1$$

$$\begin{pmatrix} \mathcal{x}'^1 \\ \mathcal{x}'^2 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\theta/r & \cos\theta/r \\ \cos(\theta) & \sin\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos(\theta) & \sin\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\sin\theta\, x + \cos\theta\, y \\ \cos\theta\, x + \sin\theta\, y \end{pmatrix}$$

But,

$$-\sin\theta\, x + \cos\theta\, y = -\sin\theta\, r\cos\theta + \cos\theta\, r\sin\theta = 0$$

$$\cos\theta\, x + \sin\theta\, y = \cos\theta r\cos\theta + \sin\theta\, r\sin\theta = r$$

so we find that

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} x_\theta \\ x_r \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} = \begin{pmatrix} r_\theta \\ r_r \end{pmatrix}$$

so this is now we discover that values for r_r and r_θ. So the general formula is this:

$$x'^n = h_n R^n{}_m x^m$$

but more likely you would know all this from the way the curvilinear coordinates are defined! That is to say, for sphericals, you would know that $\mathbf{r} = r\hat{\mathbf{r}}$ without doing all this work.