History of curvilinear coordinates PhL 11.17.10

This book has some comments on Lamé and Heine and Mathieu and Weber all of whom did elliptic things. They were amazed at what you could do with "curvilinear coordinates".

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Mathematical thought from ancient to modern times, Volume 2

By Morris Kline // some excerpts on history of curvilinears.

5. *Curvilinear Coordinates*

Green introduced a number of major ideas whose significance extended far beyond the potential equation. Gabriel Lamé (1795–1870), a mathematician and engineer concerned primarily with the heat equation, introduced another major technique, the use of curvilinear coordinate systems, which could also be used for many types of equations. Lamé pointed out in 1833[14] that the heat equation had been solved only for conducting bodies whose surfaces are normal to the coordinate planes $x = \text{const.}$, $y = \text{const.}$, and $z = \text{const.}$ Lamé's idea was to introduce new systems of coordinates and the corresponding coordinate surfaces. To a very limited extent this had been done by Euler and Laplace, both of whom used spherical coordinates ρ, θ, and ϕ, in which case the coordinate surfaces $\rho = \text{const.}$, $\theta = \text{const.}$, and $\phi = \text{const.}$ are spheres, planes, and cones respectively. Knowing the equations that transform from rectangular to spherical coordinates, one can, as Euler and Laplace did, transform the potential equation from rectangular to spherical coordinates.

The value of the new coordinate systems and surfaces is twofold. First, a partial differential equation in rectangular coordinates might not be separable into ordinary differential equations in this system but might be separable in some other system. Secondly, the physical problem might call for a boundary condition on, say, an ellipsoid. Such a boundary is represented simply in a coordinate system wherein one family of surfaces consists of ellipsoids, whereas in the rectangular system a relatively complicated equation must be used. Moreover, after separation of variables in the proper coordinate system is employed, this boundary condition becomes applicable to just one of the resulting ordinary differential equations.

14. *Jour. de l'Ecole Poly.*, 14, 1833, 194–251.

Lamé introduced several new coordinate systems for the express purpose of solving the heat equation in these systems.[15] His chief system was the three families of surfaces given by the equations

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - c^2} - 1 = 0$$

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - c^2} - 1 = 0$$

$$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - b^2} + \frac{z^2}{\nu^2 - c^2} - 1 = 0,$$

where $\lambda^2 > c^2 > \mu^2 > b^2 > \nu^2$. These three families are ellipsoids, hyperboloids of one sheet, and hyperboloids of two sheets, all of which possess the same foci. Any surface of one family cuts all the surfaces of the other two orthogonally, and in fact cuts them in lines of curvature (Chap. 23, sec. 7). Any point in space accordingly has coordinates (λ, μ, ν), namely the λ, μ, and ν of the surfaces, one from each family, which go through that point. This new coordinate system is called ellipsoidal, though Lamé called it elliptical, a term now used for another system.

Lamé transformed the heat equation for the steady-state case (temperature independent of time), that is, the potential equation, to these coordinates, and showed that he could use separation of variables to reduce the partial differential equation to three ordinary differential equations. Of course these equations must be solved subject to appropriate boundary conditions. In a paper of 1839[16] Lamé studied further the steady-state temperature distribution in a three-axis ellipsoid and gave a complete solution of the problem treated in his 1833 paper. In this 1839 paper he also introduced another curvilinear coordinate system, now called the spheroconal system, wherein the coordinate surfaces are a family of spheres and two families of cones. This system too Lamé used to solve heat conduction problems. Lamé wrote many other papers on heat conduction using ellipsoidal coordinates, including a second one of 1839 in the same volume of the *Journal de Mathématiques*, in which he treats special cases of the ellipsoid.[17]

The subject of mutually orthogonal families of surfaces had such obvious importance in the solution of partial differential equations that it became a subject of investigation in and for itself. In a paper of 1834[18] Lamé considered the general properties of any three families of mutually orthogonal surfaces and gave a procedure for expressing a partial differen-

15. *Annales de Chimie et Physique*, (2), 53, 1833, 190–204.
16. *Jour. de Math.*, 4, 1839, 126–63.
17. *Jour. de Math.*, 4, 1839, 351–85.
18. *Jour de l'Ecole Poly.*, 14, 1834, 191–288.

tial equation in any orthogonal coordinate system, a technique used continually ever since.

(Heinrich) Eduard Heine (1821–81) followed in Lamé's tracks. Heine in his doctoral dissertation of 1842[19] determined the potential (steady-state temperature) not merely for the interior of an ellipsoid of revolution when the value of the potential is given at the surface, but also for the exterior of such an ellipsoid and for the shell between confocal ellipsoids of revolution.

Lamé was so much impressed with what he and others accomplished by the use of triply orthogonal coordinate systems that he thought all partial differential equations could be solved by finding a suitable system. Later he realized that this was a mistake. In 1859 he published a book on the whole subject, *Leçons sur les coordonnées curvilignes*.

Though the use of mutually orthogonal families of surfaces as the coordinate surfaces did not solve all partial differential equations, it did open up a new technique that could be exploited to advantage in many problems. The use of curvilinear coordinates was carried over to other partial differential equations. Thus Emile-Léonard Mathieu (1835–1900), in a paper of 1868,[20] treated the vibrations of an elliptic membrane, which involves the wave equation, and here introduced elliptic cylinder coordinates and functions appropriate to these coordinates, now called Mathieu functions (Chap. 29, sec. 2). In the same year Heinrich Weber (1842–1913), working with the equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + k^2 u = 0$, solved it[21] for a domain bounded by a complete ellipse and also for the region bounded by two arcs of confocal ellipses and two arcs of hyperbolas confocal with the ellipses. The special case in which the ellipses and hyperbolas become confocal parabolas was also considered, and here Weber introduced functions appropriate to expansions in this coordinate system, now called Weber functions or parabolic cylinder functions. In his *Cours de physique mathématique* (1873), Mathieu took up new problems involving the ellipsoid and introduced still other new functions.

Our discussion of the idea initiated by Lamé, the use of curvilinear coordinates, describes just the beginning of this work. Many other coordinate systems have been introduced; corresponding special functions that result from solving the ordinary differential equations, which arise from separation of variables, have also been studied.[22] Most of this theory of special functions was created by physicists as they needed the functions and their properties in concrete problems (see also Chap. 29).

19. *Jour. für Math.*, 26, 1843, 185–216.
20. *Jour. de Math.*, (2), 13, 1868, 137–203.
21. *Math. Ann.*, 1, 1869, 1–36.
22. See William E. Byerly, *An Elementary Treatise on Fourier Series*, Dover (reprint), 1959, and E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea (reprint), 1955.

From Byerly:

Of these the first is a paper by Mathieu[1] on the vibration of an elliptic membrane (1868), in which the functions of the elliptic cylinder occur for the first time.

This was followed in the same year by a paper on closely allied subjects by H. Weber,[2] in which not merely the case of the complete ellipse is briefly considered, but also that in which the boundary consists of two arcs of confocal ellipses and two arcs of hyperbolas confocal with them. The special case in which the ellipses and hyperbolas become confocal parabolas is also considered, whereby the functions of the parabolic cylinder are for the first time introduced.

degree. Wangerin[1] (1875–76) considered in connection with the theory of the potential, more general systems of curvilinear coördinates than had previously been used in physical questions, namely, *cyclidic* coördinates.[2] He showed, however, merely how to break up Laplace's equation into three ordinary differential equations.[3]