More Levi-Cevita stuff PhL 3.26.05

In my previous document on this subject, I state and prove some theorems involving the ε symbol in 3 dimensions. Here I shall attempt to generalize to N dimensions. It won't be an easy ride. My N=3 proofs are not too bad, but the generalization requires a bit more work.

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1. Permutations

Definition: $\mathcal{N}$ = {1,2,3...N } = the set of the first N positive integers. In what follows, it is assumed that each variable a,b,c...x is an element of $\mathcal{N}$, and when (abc..x) is a permutation of (123..N), then each variable is a *different* element of $\mathcal{N}$.

If (abc..x) is a permutation of 12..N we write

$$(abc..x) = P(12..N) \quad \text{and} \quad (12..N) = P^{-1}(abc..x)$$

It is possible to generate N! distinct permutations of 12...N in this manner: a can be one of N values, then b can be one of N-1 values, and so on.

It turns out that you can get to (abc..x) from (12..N) by doing a sequence of pairwise swaps. The sequence is not unique, but two things can be said:

(1.1) there is some minimum number of swaps needed to get from (abc..x) to (12..N) and by reversing the sequence it is clear that the same number of swaps is needed to get from (12..N) to (abc..x). So we can associate some minimum number of swaps with operator P and that number is the same for P^{-1}.

Let's start with abc...x. We look for the "1" wherever it is. If it is not in the first position, we do a swap a↔1 to get it there. We then look for the "2" wherever it is. If it is not in the second position, we do swap b↔2 to get it there. We keep doing this until we get 12...N. In the worst case, we might have to do a swap for every position except the last which will already be right. Thus, there is some well-defined "swap count" $S_{min} \leq$ N-1 it takes to unwind a permutation.

(1.2) It turns out that there might be many sequences of pairwise swaps which do the same job. These might have swap counts S,S',S" and so on. If we list off all such sequences, then S_{min} = min(S,S',S"...).

(1.3) It turns out that all these S's will be even, or all will be odd. This fact is not obvious, but it is true and can be verified in simple examples such as the following, in which $S_{min} = 1$:

321→123	// we did one swap 1↔3 and the job is done	S = 1
321→231	// we did swap 2↔3 and the job is not yet done	
231→213	// we did swap 1↔3 and the job is not yet done	
213→123	// we did swap 1↔2 and the job is done	S = 3

I downloaded "permutations.pdf" in which it is proved that all the S's are even or all are odd, but the proof requires a lot of background, such as "cycle notation" and such things. So let's accept it for now.

(1.4) We can define the **parity** of a permutation as $p \equiv (-1)^S$ where S is any of the swap counts S,S',S".. noted above. Of course it is clear that $(-1)^S = (-1)^{S'} = (-1)^{S_{min}}$ and so on. The parity is either +1 or -1. Let's make this very clear:

Every permutation P has an intrinsic parity $p = (-1)^{S_{min}}$ which is +1 or -1.

Thus, if we write out all the permutations of 12345, they could be partitioned them into two groups: those that have parity +1, and those that have parity -1. We might refer to these parities loosely as "even" or "odd", where it is understood that it is really S_{min} that is an even or odd number.

Fact: Since it takes the same number of swaps to go either way in a permutation, we know that the parity associated with P is the same as that associated with P^{-1}.

(1.5) It seems clear from the definition that if we concatenate two permutations $T = P_2P_1$, say, then if P_1 is associated with a swapcount S_1 and P_2 with a swapcount S_2, then one swap sequence for T would be to do the S_1 swaps of P_1 and then do the S_2 swaps of P_2, for a total of $S_1 + S_2$ swaps. Then the parity of T will be the product of the parities of P_1 and P_2 because $parity(T) = (-1)^{S1+S2} = (-1)^{S1}(-1)^{S2}$.

Rule (1.5): when you concatenate permutations, the parities multiply.

(1.6) We can talk about the action of P on a *function of N variables* this way

$P [f(\xi_1, \xi_2, \xi_3, ...\xi_N)] = f(\xi_a, \xi_b, \xi_c.....\xi_x)$ where P(12..N) = (abc..x)

Nothing new arises when we do this in terms of the permutation theory part of things. Here the permutation operator P acts on the subscripts of the variables treated as a set (12....N).

(1.7) We can also talk about the action of P directly on the indices of a tensor T

$P [T_{123..N}] = T_{abc..x}$ where P(12..N) = (abc..x)

where $T_{123..N}$ is just a number. An example we shall see very soon is

$$P [\varepsilon_{123..N}] = \varepsilon_{abc..x} \qquad \text{where} \quad P(12..N) = (abc..x)$$

where ε is the totally antisymmetric tensor of rank N.

(1.8) There is something else we could do with tensor indices, and it is a little more subtle.

$$P [T_{i_1 i_2 i_3 \ldots . i_n}] = T_{i_a i_b i_c \ldots . i_x} \qquad \text{where} \quad P(12..N) = (abc..x)$$

In this case, P is not acting on the tensor indices; it is acting on the indices *of* the indices!

(1.9) The Dummy Index Contraction Theorem

Suppose we have something like

$$\Sigma_{abc..x} \, U_{abc..x} \, V_{abc..x}$$

Surely we can write this as

$$\Sigma_{a'b'c'..x'} \, U_{a'b'c'..x'} \, V_{a'b'c'..x'} \qquad \text{where } (a'b'c'..x') = P(abc..x)$$

Notice that abc..x are not numbers, they are variable names. So here we have done some shuffle on the dummy variable names. Since $\Sigma_{a'b'c'..x'}$ is really a sum on each of the N indices, it is the same whether or not we shuffle the indices, so $\Sigma_{a'b'c'..x'} = \Sigma_{abc..x}$. Then we have

$$\Sigma_{abc..x} \, U_{abc..x} \, V_{abc..x} = \Sigma_{abc..x} \, P[U_{abc..x}] \, P[V_{abc..x}]$$

and if we use implied summation notation, this says

$$U_{abc..x} \, V_{abc..x} = P[U_{abc..x}] \, P[V_{abc..x}]$$

This just says we can throw in the same P on both factors and change nothing.

We could also apply this if $V_{abc..x}$ had some other non-contracted indices. Here is an example

$$\varepsilon_{abc..x} \, (M_{1a} M_{2b} M_{3c} .... M_{Nx}) = P[\varepsilon_{abc..x}] P_2 \, [\, (M_{1a} M_{2b} M_{3c} .... M_{Nx})]$$

Here the subscript 2 on P_2 means only that P is acting on the second indices of the object in [...]. It does not mean P_2 is some permutation different from P.

2. Definition of the ε symbol in N dimensions.

(2.1) Define $\mathcal{N} = \{1,2,3....N\}$, the set of integers. Then we can thing of ε as a *function* this way:

$\varepsilon: \mathcal{N}^{\mathbf{N}} \rightarrow \{1,0,-1\}$

In fact ε is a tensor with N indices, and we can write it in two ways if we want

$\varepsilon(abc..x) = \varepsilon_{\mathbf{abc..x}}$

(2.2) Definition of ε:

ε(abc..x) = + 1 if abc..x is an even permutation of 123...N (the identity permutation is even)
ε(abc..x) = - 1 if abc..x is an odd permutation of 123...N
ε(abc..x) = 0 otherwise

The third line says that if (abc..x) is not a permutation of (12...N), then ε(abc..x) = 0. The way for (abc..x) to not be a permutation is to have duplication in the set, for example, to have two indices the same, an example being ε(1123)=0 for N=4.

Assuming (abc..x) *is* a permutation of (12...N), we can interpret the first two lines as saying this:

$\varepsilon(abc..x) = P[\ \varepsilon(123..N)\ ] = p$ where P has parity p

In words: if the subscripts on $\varepsilon_{\mathbf{abc..x}}$ are a permutation of 012..N, then the value of $\varepsilon_{\mathbf{abc..x}}$ ***IS*** the parity of the permutation P(12..N) = (abc..x) which creates the subscripts from 012..N. So you might refer to ε as the "permutation parity tensor", at least when the subscripts are a permutation of 012..N.

Implications of the above:

(2.3) If two indices of $\varepsilon_{\mathbf{abc..x}}$ are the same, then $\varepsilon_{\mathbf{abc..x}} = 0$.
In this case, (abc..x) cannot possibly be a permutation of (12..N) and this ε = 0.

(2.4) Swapping any pair of indices changes the sign of ε.
This action adds 1 to the swapcount from 01..N and thus $(-1)^{\mathbf{s+1}} = -(-1)^{\mathbf{s}}$ so the parity negates.

(2.5) ε is a "totally antisymmetric tensor" in N dimensions.
This is so because swapping any pair of indices negates the tensor. True if tensor element is 0 as well.

Totally Antisymmetric Tensors. Consider a tensor $T_{\mathbf{abc\cdots x}}$ which has N indices, and where each index takes values in $\mathcal{N}$. If this tensor is "totally antisymmetric", then every element can be obtained from one of the elements. For example, if $T_{\mathbf{123}} = 5$, then $T_{\mathbf{213}} = -5$. Therefore, $T_{\mathbf{abc..x}}$ = constant * $\varepsilon_{\mathbf{abc..x}}$. Apart from an overall constant, there is only one "totally antisymmetric tensor" which has N indices in $\mathcal{N}$, and it is ε. There are of course many totally antisymmetric tensors with *fewer* than N indices For example, if N=3 and a tensor has 2 indices, we might write this on the left, which of course has the wrong number of indices to be an $\varepsilon_{\mathbf{ijk}}$ object.

$$T_{ij} = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}, \quad N=3 \qquad\qquad M_{ij} = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} = x\,\varepsilon_{ij}, \quad N=2$$

On the other hand, if N=2, an antisymmetric tensor with 2 indices is much more constrained, as shown above on the right.

(2.6) **Theorem (2.6).** Suppose (a'b'c'..x') = T(abc..x) with parity t. Then ε(a'b'c'..x') = t ε(abc..x).

Proof: Assume it takes some number τ of pair-wise swaps to get from (abc..x) to (a'b'c'..x'). We know that each such swap changes the sign of ε, so we end up with $\varepsilon(a'b'c'..x') = (-1)^{\tau}\,\varepsilon(abc..x)$. But of course this thing $(-1)^{\tau}$ is by definition the parity t of T, QED.

There are a few ways to write the conclusion of this theorem:

$$T[\varepsilon(abc..x)] = t\,\varepsilon(abc..x)$$

Since this is true for any T, it is also true for T^{-1}. But T^{-1} has the same parity as T, so

$$T^{-1}[\varepsilon(abc..x)] = t\,\varepsilon(abc..x)$$

3. Rules involving ε and the determinant of a matrix

(3.1) The basic determinant evaluation and its permutation sum form.

We start with these two ε-forms for determinant evaluation:

$$\det(M) = \varepsilon_{abc..x} M_{1a} M_{2b} M_{3c} \ldots M_{Nx} = \varepsilon_{abc..x} M_{a1} M_{b2} M_{c3} \ldots M_{xN} \qquad (1)$$

I have proven these facts in Matrix World and won't repeat those proofs here. Notice that there are N! non-zero terms in this expansion since a can take N values, but for a given a value, b can only take N-1 values to maintain ε ≠ 0, and for a given a and b, c can only take N-2 values, and so on. Each term is associated with one of the N! permutations of 123..N. There is a one-to-one correspondence between the non-vanishing terms in the above sum and the permutations of 123...N. Moreover, we know from the previous section that $\varepsilon_{abc..x} = p$, where p is the parity of P(12..N) = (abc..x). Also, we can write

$$M_{1a} M_{2b} M_{3c} \ldots M_{Nx} = P_2^{(abc..x)} (M_{11} M_{22} M_{33} \ldots M_{NN})$$

where P_2 acts on the second indices of the M_{ij} factors, and we temporarily label the permutation. Thus we can say

$$\det(M) = \Sigma_{abc..x}\ \varepsilon_{abc..x} M_{1a} M_{2b} M_{3c} \ldots M_{Nx}$$

$$= \Sigma_{abc..x}\ \varepsilon_{abc..x}\, P_2^{(abc..x)} [\, (M_{11} M_{22} M_{33} \ldots M_{NN})]$$

$$= \Sigma_{abc..x} \; p^{(abc..x)} \; P_2{}^{(abc..x)} [\, (M_{11}M_{22}M_{33}....M_{NN})]$$

$$= \Sigma_P \;\; p \;\; P_2 \, [\, (M_{11}M_{22}M_{33}....M_{NN})] \qquad \text{where (abc..x) = P(123..N) with parity p}$$

We show temporary labels on P_2 and p so one can see the both elements of each pair of summation indices in the third line. In the fourth line, we remove them, and let P itself represent a permutation. In this fourth line we are of course summing over all N! possible permutations P, and P_2 acts on the M_{ij} second indices. Had we started with our other expansion in (1), we could obtain the same result with P_1 instead of P_2. So we have now proven that the ε summation notation is equivalent to a permutation summation and

$$\det(M) = \Sigma_P \;\; p \;\; P_2 \, [\, (M_{11}M_{22}M_{33}....M_{NN})] \;\; = \;\; \Sigma_P \;\; p \;\; P_1 \, [\, (M_{11}M_{22}M_{33}....M_{NN})]$$

where P is defined by (abc..x) = P(123..N) and such P has parity p. For example, a few terms are

$$\begin{aligned}
\det(M) \;\; &= +1 \; M_{11}M_{22}M_{33}....M_{NN} && \text{// product of diagonal elements} \\
&\quad -1 \;\; M_{12}M_{21}M_{33}....M_{NN} && \text{// swapped 1↔2 from first term, parity = -1} \\
&\quad -1 \; M_{11}M_{23}M_{32}....M_{NN} && \text{// swapped 2↔3 from first term, parity = -1} \\
&\quad + M_{13}M_{21}M_{32}....M_{NN} && \text{// swapped 2↔3 then 1↔3 , parity = +1}
\end{aligned}$$

There are N! terms and the first term is the product of the diagonal elements of matrix M. This is just a compact way to write out the terms of a determinant. It is just notation. Sometimes this P summation form is easier to write than the ε form or than by drawing the entire determinant. We shall see examples below.

(3.2) **Theorem 3.2:**

$$\varepsilon_{abc..x} M_{Aa} M_{Bb} M_{Cc}....M_{Xx} \;\; = \;\; t \;\; \varepsilon_{abc..x} \, M_{1a} M_{2b} M_{3c}....M_{Nx}$$

where (ABC..X) = T(123..N) and the parity of T is t.

Proof: We know that (T_1 means T acts on the first indices)

$$M_{Aa}M_{Bb}M_{Cc}....M_{Xx} \;\; = \;\; T_1(\, M_{1a}M_{2b}M_{3c}....M_{Nx} \,)$$

Thus

$$\text{LHS} = \; \varepsilon_{abc..x} M_{Aa} M_{Bb} M_{Cc}....M_{Xx} \;\; = \varepsilon_{abc..x} \; T_1(\, M_{1a}M_{2b}M_{3c}....M_{Nx} \,)$$

Now use the Rule of (1.9) to throw in a pair of permutations on each tensor, and let's choose T^{-1}. We then get

$$= T^{-1}[\varepsilon_{abc..x}] \; M_{1a}M_{2b}M_{3c}....M_{Nx}$$

But according to Theorem (2.6) we then get

$= t[\varepsilon_{abc..x}]\ M_{1a}M_{2b}M_{3c}....M_{Nx} = RHS$ QED,

Another proof is to say that we have permuted the columns of the matrix in our determinant, and each swap makes a -1, so we get $(-1)^{swaps}$ but of course this is just t.

4. ε as the determinant of a matrix

Consider this N x N matrix M, where a,b,c...x is a permutation of 1,2,3 ...N : (**R** stands for Row)

$\delta_{a,1}$	$\delta_{a,2}$	$\delta_{a,3}$	...	$\delta_{a,N}$	$\mathbf{R_a}$	$(\mathbf{R_a})_i = \delta_{a,i} = M_{1i}$, i = 1..N
$\delta_{b,1}$	$\delta_{b,2}$	$\delta_{b,3}$	...	$\delta_{b,N}$	$\mathbf{R_b}$	$(\mathbf{R_b})_i = \delta_{b,i} = M_{2i}$, i = 1..N
$\delta_{c,1}$	$\delta_{c,2}$	$\delta_{c,3}$	...	$\delta_{c,N}$	$\mathbf{R_c}$	$(\mathbf{R_c})_i = \delta_{c,i} = M_{3i}$, i = 1..N
....						
$\delta_{x,1}$	$\delta_{x,2}$	$\delta_{x,3}$	...	$\delta_{x,N}$	$\mathbf{R_x}$	$(\mathbf{R_x})_i = \delta_{x,i} = M_{xi}$, i = 1..N

Some comments are needed at this point.

(4.1) We are using a,b,c...x to represent arbitrary integers in $\mathcal{N}$. Since we don't know what N is, we don't know what specific letter x *is* of the alphabet so we just call it x. A more systematic notation would be to replace a,b,c...x by a_1, a_2.....a_N. But then we have "subscripts on subscripts" and it just makes the notation ugly, even if more systematic. So whenever we see a list of variables like a,b,c....x we can *think of it* as a_1, a_2.....a_N if we like, but we don't write it that way. Notice that we are not claiming that all the a,b,c...x are different integers or anything like that. They are just each taken a value from $\mathcal{N}$.

(4.2) On the right we think of each row of matrix M as a row-vector like $\mathbf{R_a}$, and we show the row vector elements. So we can think of our matrix M as a collection of these N row vectors. Each of these row vectors has a 1 in only one of its columns and so is a unit vector. The 1 for row-vector $\mathbf{R_a}$ is in column a, for example.

Now, as for any matrix, we have a rule for writing the determinant, and we can apply that rule here

$$\det(M) = \varepsilon_{ABC..X}\, M_{1A}M_{2B}....M_{NX}$$

where M_{1A} is the first row, and so on, as shown on the far right above. But as shown above, we know that $M_{1A} = \delta_{a,A}$ and similarly for the other rows, so we have only a single term surviving in the sum:

$$\det(M) = \varepsilon_{ABC..X}\, \delta_{a,A}\, \delta_{b,B}....\ \delta_{x,X} \;=\; \varepsilon_{abc...x}$$

Thus we have shown that we can represent the function ε(abc...x) having N arguments as the determinant of a certain NxN matrix:

$$\varepsilon_{abc\ldots x} = \begin{vmatrix} \delta_{a,1} & \delta_{a,2} & \delta_{a,3} & \cdots & \delta_{a,N} \\ \delta_{b,1} & \delta_{b,2} & \delta_{b,3} & \cdots & \delta_{b,N} \\ \delta_{c,1} & \delta_{c,2} & \delta_{c,3} & \cdots & \delta_{c,N} \\ \cdots & & & & \\ \delta_{x,1} & \delta_{x,2} & \delta_{x,3} & \cdots & \delta_{x,N} \end{vmatrix}$$

We can write this in the permutation sum notation of (3.1) in this manner

$$\varepsilon_{abc\ldots x} = \Sigma_P \, p \;\; P_2 \, [\delta_{a,1}\, \delta_{b,2}\, \delta_{c,3} \ldots \delta_{x,N}]$$

where recall that what goes in [...] is the product of the diagonal elements of the matrix. We are summing over all permutations of the second indices on the δ's, and p is the parity of P.

5. The product of two ε factors as the determinant of a matrix.

As noted above, consider these two ε objects,

$$\varepsilon_{abc\ldots x} = \det \begin{pmatrix} \mathbf{R_a} \\ \mathbf{R_b} \\ \ldots \\ \mathbf{R_x} \end{pmatrix} \qquad \varepsilon_{a'b'c'\ldots x'} = \det \begin{pmatrix} \mathbf{R_{a'}} \\ \mathbf{R_{b'}} \\ \ldots \\ \mathbf{R_{x'}} \end{pmatrix}$$

We can then write

$$\varepsilon_{abc\ldots x}\, \varepsilon_{a'b'c'\ldots x'} = \det \begin{pmatrix} \mathbf{R_a} \\ \mathbf{R_b} \\ \ldots \\ \mathbf{R_x} \end{pmatrix} \det \begin{pmatrix} \mathbf{R_{a'}} \\ \mathbf{R_{b'}} \\ \ldots \\ \mathbf{R_{x'}} \end{pmatrix} = \det \begin{pmatrix} \mathbf{R_a} \\ \mathbf{R_b} \\ \ldots \\ \mathbf{R_x} \end{pmatrix} \det (\mathbf{R_{a'}}\, \mathbf{R_{b'}}\, \ldots . \mathbf{R_{x'}})$$

where we have used the fact that det(M) = det(M^T). Now we use detA detB = det(AB) to get

$$= \det \left\{ \begin{pmatrix} \mathbf{R_a} \\ \mathbf{R_b} \\ \ldots \\ \mathbf{R_x} \end{pmatrix} (\mathbf{R_{a'}}\, \mathbf{R_{b'}}\, \ldots . \mathbf{R_{x'}}) \right\}$$

The matrix here can be written as

$$\begin{matrix} \mathbf{R_a \bullet R_{a'}} & \mathbf{R_a \bullet R_{b'}} & \mathbf{R_a \bullet R_{c'}} & \ldots\ \mathbf{R_a \bullet R_{x'}} \\ \mathbf{R_b \bullet R_{a'}} & \mathbf{R_b \bullet R_{b'}} & \mathbf{R_b \bullet R_{c'}} & \ldots\ \mathbf{R_b \bullet R_{x'}} \\ \mathbf{R_c \bullet R_{a'}} & \mathbf{R_c \bullet R_{b'}} & \mathbf{R_c \bullet R_{c'}} & \ldots\ \mathbf{R_c \bullet R_{x'}} \\ \ldots & & & \end{matrix}$$

$\mathbf{R_x \bullet R_{a'}}$ $\mathbf{R_x \bullet R_{b'}}$ $\mathbf{R_x \bullet R_{c'}}$ ... $\mathbf{R_x \bullet R_{x'}}$

But we know that, for example,

$$\mathbf{R_c \bullet R_{b'}} = (\mathbf{R_c})_i(\mathbf{R_{b'}})_i = \delta_{c,i}\,\delta_{b',i} = \delta_{c,b'}$$

Thus we can write this same NxN matrix as

$$\begin{array}{llll} \delta_{a,a'} & \delta_{a,b'} & \delta_{a,c'} & .... \delta_{a,x'} \\ \delta_{b,a'} & \delta_{b,b'} & \delta_{b,c'} & .... \delta_{b,x'} \\ \delta_{c,a'} & \delta_{c,b'} & \delta_{c,c'} & .... \delta_{c,x'} \\ .... & & & \\ \delta_{x,a'} & \delta_{x,b'} & \delta_{x,c'} & .... \delta_{x,x'} \end{array}$$

and so we have now shown that

$$\varepsilon_{abc...x}\,\varepsilon_{a'b'c'...x'} = \begin{vmatrix} \delta_{a,a'} & \delta_{a,b'} & \delta_{a,c'} & .... \delta_{a,x'} \\ \delta_{b,a'} & \delta_{b,b'} & \delta_{b,c'} & .... \delta_{b,x'} \\ \delta_{c,a'} & \delta_{c,b'} & \delta_{c,c'} & .... \delta_{c,x'} \\ .... & & & \\ \delta_{x,a'} & \delta_{x,b'} & \delta_{x,c'} & .... \delta_{x,x'} \end{vmatrix}$$

which one can compare to our previous matrix

$$\varepsilon_{abc...x} = \begin{vmatrix} \delta_{a,1} & \delta_{a,2} & \delta_{a,3} & ... & \delta_{a,N} \\ \delta_{b,1} & \delta_{b,2} & \delta_{b,3} & ... & \delta_{b,N} \\ \delta_{c,1} & \delta_{c,2} & \delta_{c,3} & ... & \delta_{c,N} \\ .... & & & & \\ \delta_{x,1} & \delta_{x,2} & \delta_{x,3} & ... & \delta_{x,N} \end{vmatrix}$$

6. Contracting 1 index in the εε product.

If in the last result above we set a' = a and then sum on a, we find that (implied sum on a)

$$\varepsilon_{abc...x}\,\varepsilon_{ab'c'...x'}$$

is the determinant of this matrix (summed on a)

$$\begin{array}{llll} \delta_{a,a}=1 & \delta_{a,b'} & \delta_{a,c'} & .... \delta_{a,x'} \\ \delta_{b,a} & \delta_{b,b'} & \delta_{b,c'} & .... \delta_{b,x'} \\ \delta_{c,a} & \delta_{c,b'} & \delta_{c,c'} & .... \delta_{c,x'} \end{array}$$

....

$\delta_{x,a} \quad \delta_{x,b'} \quad \delta_{x,c'} \quad .... \delta_{x,x'}$

where we have just set a' = a in the first column. We can evaluate the determinant by going down the first column and looking at the cofactors in the usual way. The *claim* is that the resulting determinant comes entirely from the first cofactor piece, and that the remaining cofactor pieces add to 0 ! We are unable to show this in any reasonable way using matrix notation, but let 's assume for the moment that it is true and we will prove it some other way below. Then we will have shown that

$$\varepsilon_{abc...x}\,\varepsilon_{ab'c'...x'} = \begin{vmatrix} \delta_{b,b'} & \delta_{b,c'} & .... & \delta_{b,x'} \\ \delta_{c,b'} & \delta_{c,c'} & .... & \delta_{c,x'} \\ & & & \\ \delta_{x,b'} & \delta_{x,c'} & .... & \delta_{x,x'} \end{vmatrix} \qquad (1)$$

where this matrix is the lower right N-1 x N-1 submatrix of the starting matrix.

Before continuing, recall from (3.3) that we could write a determinant of an NxN matrix M in this manner

$\det(M) = \Sigma_P \; p \; P_2(M_{11}M_{22}M_{33}....M_{NN})$ p is the parity of P

Suppose M is an N-1 x N-1 matrix as the one shown in (1) above. Then the form of the determinant expansion is pretty much the same except for the indices on the last M:

$\det(M) = \Sigma_P \; p \; P_2(M_{11}M_{22}M_{33}....M_{N-1N-1})$ p is the parity of P

We are summing over all permutations of the second indices on the M_{ij}. The group of elements being permuted of course now only contains N-1 items, even though each item can be in 1..N still. If we *install* our diagonal matrix elements of the M shown in (1), we get

$\det(M) = \Sigma_P \; p \; P_2(\delta_{b,b'}\,\delta_{c,c'}\,\delta_{d,d'}...\,\delta_{x,x'})$ // there are N-1 δ factors

Now what does it mean to "sum over permutations of the second indices"? Let's look at some terms. The first term has P = 1 and p = + and is then

$\det(M)_{\text{first term}} = \delta_{b,b'}\,\delta_{c,c'}\,\delta_{d,d'}...\,\delta_{x,x'}$

Then if we look at some other term in this sum

$P_2(\delta_{b,b'}\,\delta_{c,c'}\,\delta_{d,d'}...\,\delta_{x,x'}) = \delta_{b,b''}\,\delta_{c,c''}\,\delta_{d,d''}...\,\delta_{x,x''}$

it will have parity p which goes with P(a'bc'..x') = (a"b"c"..x"). So we just add up all such terms to the first term and we have det(M). Then we are saying

$$\varepsilon_{abc\ldots x}\,\varepsilon_{ab'c'\ldots x'} = \Sigma_P\ p\ P_2(\delta_{b,b'}\,\delta_{c,c'}\,\delta_{d,d'}\ldots\delta_{x,x'}) \qquad p = \text{parity of P} \qquad (2)$$

Equation (2) is simply another way to write equation (1) -- they are equivalent .

Proof of equation (1)

Suppose two indices of the set abc...x are the same, for example, b = c. In this case of course the LHS = 0 since the first ε is 0. But the RHS is also 0 because the first two rows are then the same! (We could add a multiple of one row to the other and get a row of all 0's, for example). So we have proven that equation (1) is true in this case. Similarly, if two indices of the set a'b'...x' are the same, for example b'=c', then two *columns* are the same and again we get both sides = 0. So we have already proven a large part of (1) !

We now have only one "case" left to prove. In this case we have

bc...x are all different elements of $\mathcal{N}$ // there are N-1 items in the list
Let α be the element of $\mathcal{N}$ that is not included.

b'c'...x' are all different elements of $\mathcal{N}$ // there are N-1 items in the list
Let α' be the element of $\mathcal{N}$ that is not included.

Consider now this expression (recall there are N-1 factors)

$$P\,(\delta_{b,b'}\,\delta_{c,c'}\ldots\delta_{x,x'}) = \delta_{b,b''}\,\delta_{c,c''}\ldots\delta_{x,x''}$$

where P is some arbitrary permutation of the primed indices which we indicate by double primes. The only way this product can be non-zero is if we have b = b", c = c" ..... and finally x = x". Define

$\mathcal{N}$-α = the set of N-1 elements from which ab...x come

$\mathcal{N}$-α' = the set of N-1 elements from which a'b'...x' and thus a"b"...x" come

If these two sets are not the same, then how are we going to ever have b = b", c = c" etc ? For example, if the first set is {1,2,4} and the second set is {1,3,4}, we could get 1=1 and 4=4, but we fail on 2=3. In order to match ALL elements of two sets, the two sets have to be the same!

Therefore we have shown that if $\alpha \neq \alpha'$, $\delta_{b,b''}\,\delta_{c,c''}\ldots\delta_{x,x''} = 0$ because at least one of the δ's will be 0. Therefore $P\,(\delta_{b,b'}\,\delta_{c,c'}\ldots\delta_{x,x'})$ for *all* permutations P, and therefore the entire RHS of equation (2) is 0, and RHS equation (2) is the same as RHS equation (1).

But in this case we *also* know that the LHS of (1) is zero. The reason is that in the implied sum on a, we cannot ever have (at the same time) a = α and a = α' if $\alpha \neq \alpha'$. So no value of a in the sum will give a non-zero term.

So only one subcase remains to be shown. When α = α', why is (1) true? For this proof, we turn to the form (2) of equation 1, which is now

$$\varepsilon_{\alpha bc\ldots x}\,\varepsilon_{\alpha b'c'\ldots x'} = \Sigma_P\ p\ P_2\,(\delta_{b,b'}\,\delta_{c,c'}\ldots\delta_{x,x'})$$

where the implied sum over a is gone, and we have the first index of both ε's equal to α which was defined above. We want to show this is true.

Now, since bc...x and b'c'..x' are permutations of the same set of numbers (by our assumption for the special last case we are now considering), there will be *only one term* on the RHS that does not vanish. This is the one term that matches the elements of these two sets in every δ factor. In every *other* permutation of the δ's, at least one δ will miss and we get 0. The parity of this one term will be the same as the parity of the permutation which does (b'c'..x') = T(bc..x) which we might call t. Then we have

$$\text{RHS} = t \;\; 1*1*1...*1 = t$$

But now look at the LHS. We can write

$$\varepsilon_{\alpha bc\ldots x}\,\varepsilon_{\alpha b'c'\ldots x'} = \varepsilon_{\alpha bc\ldots x}\,T(\varepsilon_{\alpha bc\ldots x}) = \varepsilon_{\alpha bc\ldots x}\,\{\,t\,\varepsilon_{\alpha bc\ldots x}\,\}$$

using Theorem 2.6. Continuing, we have

$$= t(\varepsilon_{\alpha bc\ldots x})^2 = t$$

So we first showed that RHS = t, and now we have shown that LHS = t. Therefore we have shown that

$$\varepsilon_{\alpha bc\ldots x}\,\varepsilon_{\alpha b'c'\ldots x'} = \Sigma_P\, p\, P_2\,(\delta_{b,b'}\,\delta_{c,c'}\,\ldots.\,\delta_{x,x'})$$

and our last case is finished! This was a painful proof but I think a valid one.

Now let's go back to the result of Section 5 which was

$$\varepsilon_{abc\ldots x}\,\varepsilon_{a'b'c'\ldots x'} = \begin{vmatrix} \delta_{a,a'} & \delta_{a,b'} & \delta_{a,c'} & \ldots. & \delta_{a,x'} \\ \delta_{b,a'} & \delta_{b,b'} & \delta_{b,c'} & \ldots. & \delta_{b,x'} \\ \delta_{c,a'} & \delta_{c,b'} & \delta_{c,c'} & \ldots. & \delta_{c,x'} \\ \ldots. & & & & \\ \delta_{x,a'} & \delta_{x,b'} & \delta_{x,c'} & \ldots. & \delta_{x,x'} \end{vmatrix}$$

Suppose we give the matrix on the right a name: M^0. For example, $M^0_{11} = \delta_{a,a'}$. Then we can write the Section 5 result this way

$$\varepsilon_{abc\ldots x}\,\varepsilon_{ab'c'\ldots x'} = \det(M^0)$$

Now look at the result of our current section

$$\varepsilon_{abc\ldots x}\,\varepsilon_{ab'c'\ldots x'} = \begin{vmatrix} \delta_{b,b'} & \delta_{b,c'} & \ldots. & \delta_{b,x'} \\ \delta_{c,b'} & \delta_{c,c'} & \ldots. & \delta_{c,x'} \\ & & & \\ \delta_{x,b'} & \delta_{x,c'} & \ldots. & \delta_{x,x'} \end{vmatrix}$$

Suppose we give this matrix the name M^1 , $M^1{}_{11} = \delta_{b,b'}$. Since this matrix is the lower right part of the previous matrix, we see that

$$\det(M^1) = \min(M^0{}_{11}) = \text{cof}(M^0{}_{11})$$

That is, the number $\det(M^1)$ is just the minor of the top left element of M^0. This is just a compact notation to express what we have already learned. It is possible to write this cofactor as follows

$$\text{cof}(M^0{}_{11}) = \varepsilon_{1bc\ldots x} M^0{}_{2b} M^0{}_{3c} \ldots M^0{}_{Nx}$$

Of course in this last multiple sum, if any summation index takes the value 1, $\varepsilon = 0$, so in effect all the summations are from 2 to N.

Note Added: I might just mention this possibly useful fact from my M&M notes. Note that our result above is the cofactor of the upper left element of the full NxN matrix. Here is a quote from my M&M notes:

$$\text{cof}(a_{pq}) = \varepsilon_{ij\ldots.q\ldots.\ell}\, a_{1i}\, a_{2j} \ldots\ldots a_{p-1x}\, \cancel{(a_{py})}\, a_{p+1z} \ldots\ldots\ldots\ldots \quad \text{// in general}$$

where we have now tried to write a very general result. It has the number q in the pth ε subscript position, and the a_{py} factor is *missing* from the product. (we show it "struck out").

6a. An Application of Section 6.

It is possible to generalize the usual cross product used when N = 3 in the following manner

$$\mathbf{Q} = \mathbf{B} \times \mathbf{C} \times \ldots. \times \mathbf{X} \qquad \text{// N-1 vectors are crossed, there are N-2 x's}$$

$$Q_a = \varepsilon_{abc\ldots x} B_b C_c \ldots. X_x$$

where the meaning of the first line is defined by the second line, and the vectors are N dimensional vectors. The x notation is suggestive of a fact which is in fact true, namely, that **Q** is perpendicular to all the vectors from which it is made. For example

$$\mathbf{Q} \bullet \mathbf{C} = Q_a C_a = \varepsilon_{abc\ldots x} B_b C_c \ldots. X_x C_a = 0$$

because we contract the symmetric $C_c C_a$ against the antisymmetric ε.

Suppose we also have

Q' = **B'** x **C'** .... x **X'** // N-1 vectors are crossed, there are N-2 x's

$Q'_a = \varepsilon_{abc...x} B'_b C'_c ... X'_x$

Then

Q•Q' = (**B** x .... x **X**) • (**B'** x .... x **X'**)

$= (\varepsilon_{abc...x} B_b C_c .... X_x)(\varepsilon_{ab'c'...x'} B'_b C'_c .... X'_{x'})$

$= \varepsilon_{abc...x} \varepsilon_{ab'c'...x'} B_b C_c .... X_x \;\; B'_b C'_c .... X'_x$

$= \Sigma_P \, p \, P_2 \, (\delta_{b,b'} \delta_{c,c'} .... \delta_{x,x'}) B_b C_c .... X_x \;\; B'_b C'_c .... X'_x$

$= \Sigma_P \, p \, P_2$ **B•B'** .....**X•X'**

where P now permutes the vectors (**B'** ... **X'**) in all possible (N-1)! ways. Suppose we now define an NXN matrix M this way

$$M \equiv \begin{matrix} \mathbf{A•A'} & \mathbf{A•B'} & \mathbf{A•C'} & ..... & \mathbf{A•X'} \\ \mathbf{B•A'} & \mathbf{B•B'} & \mathbf{B•C'} & ..... & \mathbf{B•X'} \\ \mathbf{C•A'} & \mathbf{C•B'} & \mathbf{C•C'} & ..... & \mathbf{C•X'} \\ ..... & & & & \\ \mathbf{X•A'} & \mathbf{X•B'} & \mathbf{X•C'} & ..... & \mathbf{X•X'} \end{matrix}$$

We see that $\Sigma_P \, p \, P_2$ **B•B'** .....**X•X'** is the determinant of the submatrix formed by crossing out the first row and the first column. But this has a name: it is the minor (or cofactor) of M_{11}. Thus we have shown that

Q•Q' = min(M_{11}) = cof(M_{11})

Metric Tensor Special Case

As a special case of this rule, suppose we have

A = $\mathbf{e}_1$ **B** = $\mathbf{e}_2$ ..... **X** = $\mathbf{e}_N$
A' = $\mathbf{e}_1$ **B'** = $\mathbf{e}_2$ ..... **X'** = $\mathbf{e}_N$

where the $\mathbf{e}_i$ are the tangent base vectors of some curvilinear coordinate system. In this case, we have that $\mathbf{e}_a \bullet \mathbf{e}_b = g'_{ab}$ where g' is the metric tensor, so we have M = g'.

$\mathbf{Q}^{(1)} \equiv \mathbf{e}_2 \times \mathbf{e}_3 \times .... \times \mathbf{e}_N$

We find that

$\mathbf{Q}^{(1)} \bullet \mathbf{Q}^{(1)} = \text{cof}(g'_{11}) = \min(g'_{11})$

Now consider instead the case where $\mathbf{e}_2$ is missing from the cross product

$\mathbf{Q}^{(2)} \equiv \mathbf{e}_1 \times \mathbf{e}_3 \times .... \times \mathbf{e}_N$

We take the same matrix M as shown above with the same $\mathbf{e}_k$ identifications.

More generally we can show that

$\mathbf{Q}^{(k)} \bullet \mathbf{Q}^{(k)} = \text{cof}(g'_{kk})$

where

$\mathbf{Q}^{(k)} \equiv \mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_3 \times .... \times \mathbf{e}_N$ // where $\mathbf{e}_k$ is missing

7. Contracting 2 indices in the εε product.

The conclusion of Section 6 was this:

$$\varepsilon_{abc...x}\,\varepsilon_{ab'c'...x'} = \begin{vmatrix} \delta_{b,b'} & \delta_{b,c'} & .... & \delta_{b,x'} \\ \delta_{c,b'} & \delta_{c,c'} & .... & \delta_{c,x'} \\ & & & \\ \delta_{x,b'} & \delta_{x,c'} & .... & \delta_{x,x'} \end{vmatrix}$$

where we have contracted on the a index only. What happens if we set b' = b and then sum on the b index as well? This is a "recursion" of what we just proved in Section 6 at the N-1/N-2 level instead of the N/N-1 level. The result will be that you get the determinant of the matrix which is the minor of the top-left element of the above matrix. That is to say, we are interested now in $\varepsilon_{abcd...x}\,\varepsilon_{abc'd'...x'}$ and the result is going to be (2! to be explained below)

$$\varepsilon_{abcd\ldots x}\,\varepsilon_{abc'd'\ldots x'} = 2!\begin{vmatrix} \delta_{c,c'} & \ldots. \delta_{c,x'} \\ \delta_{x,c'} & \ldots. \delta_{x,x'} \end{vmatrix}$$

The proof would go much as above. Both sides will be 0 unless c,d...x are all different elements of the set $\mathcal{N}$. So we might then define α,β as the *two* "other elements" for cd... and α',β' for the c'd'... set. Consider the double sum Σ_{ab} implied above

$$\Sigma_{ab}\,\varepsilon_{abcd..x}\,\varepsilon_{abc'd'..x'}$$

Suppose {α,β} = {1,2} and {α',β'} = {3,4}. Our only options for non-zero terms are then

$$\varepsilon_{12cd..x}\,\varepsilon_{34c'd'..x'}$$
$$\varepsilon_{12cd..x}\,\varepsilon_{43c'd'..x'}$$
$$\varepsilon_{21cd..x}\,\varepsilon_{34c'd'..x'}$$
$$\varepsilon_{21cd..x}\,\varepsilon_{43c'd'..x'}$$

But none of these four terms fits into the template shown above in which the first two indices on each ε have to be the same, so none of these terms contributes.

Suppose {α,β} = {1,2} and {α',β'} = {1,4}. Our only options for non-zero terms are then

$$\varepsilon_{12cd..x}\,\varepsilon_{14c'd'..x'}$$
$$\varepsilon_{12cd..x}\,\varepsilon_{41c'd'..x'}$$
$$\varepsilon_{21cd..x}\,\varepsilon_{14c'd'..x'}$$
$$\varepsilon_{21cd..x}\,\varepsilon_{41c'd'..x'}$$

Although the first form has the first index the same (the only one of the four lines), it does not have the second case the same, so again there is no contribution.

Suppose {α,β} = {1,2} and {α',β'} = {1,2}. Our only options for non-zero terms are then

$$\varepsilon_{12cd..x}\,\varepsilon_{12c'd'..x'}$$
$$\varepsilon_{12cd..x}\,\varepsilon_{21c'd'..x'}$$
$$\varepsilon_{21cd..x}\,\varepsilon_{12c'd'..x'}$$
$$\varepsilon_{21cd..x}\,\varepsilon_{21c'd'..x'}$$

Now the **first and last** items fit our mold! So our double sum then gets 2 "hits". By swapping the indices in the last hit, the ε product is the same as the first in the list. So basically this causes an unexpected factor of 2 in our equation!

8. Contracting 3 indices in the εε product.

This will give a result similar to the above, but with a 3! out front. And so on. The matrix keeps shrinking by one dimension each time as we move toward the lower right corner.

9. Contracting N-1 indices in the εε product.

If we follow the pattern, we should find that

$$\varepsilon_{abc\ldots x}\,\varepsilon_{abc..x'} = (N-1)!\,\delta_{x,x'}$$

This is of course correct! In the multiple sum, we get (N-1)! hits, all giving the same contribution. If we look at our earlier Levi-Civita paper which deals with the case N = 3 only, we got there

$$\varepsilon_{ijk}\,\varepsilon_{ijk'} = \begin{vmatrix} \delta_{jj} & \delta_{jk'} \\ \delta_{kj} & \delta_{kk'} \end{vmatrix} = \delta_{jj}\,\delta_{kk'} - \delta_{jk'}\,\delta_{kj} = 3\delta_{kk'} - \delta_{kk'} = 2\,\delta_{kk'}$$

and that 2 is our 2!

10. Contracting N indices in the εε product.

If we sum on all indices, we get

$$\varepsilon_{abc\ldots x}\,\varepsilon_{abc..x} = N!$$

which is again obviously correct: there are N! permutations of (abc..x) and each contributes $\varepsilon^2 = 1$ to the sum for a total of N!.

11. An application of