A non-Stackel method of doing separation PhL 8.30.11

This started out being Section 15 of my Stackel document, but it just did not fit. First, here is that Section 15 moved here:

Review of "Section 15" below

The stuff below does in fact depend on earlier equations of my Stackel document, but it just fails to go anywhere useful. For simple R=1 separation I obtained a result I *thought* was going to be very useful, namely that I could write $L = \Sigma_n\ (1/h_n^2)\ \mathcal{L}_n$ where $\mathcal{L}_n$ and the L_n with separation constants set to 0. That appears at the end of Section A below.

I then have in Section B what I think is a very good review of Sturm-Liouville theory with a reasonable amount of detail. I did this because I wanted to apply that theory to the $\mathcal{L}_n$.

Then in Section C I imagine having a SL EV problem at first in ALL of the coordinates ξ_n. I suppose that might be possible in a problem where the boundaries all align with level surfaces. And of course I am already assuming we get simple separation. I need to return to this question later, I don't know what it really means.

Next I allow as maybe only ξ_1 is a SL problem and I assume ψ where X_1 is an EV. When I use this form $L = \Sigma_n\ (1/h_n^2)\ \mathcal{L}_n$, this DOES let me remove $\mathcal{L}_1$ from the problem, so L is simplified in that sense. But I just don't think this buys us much. I go right to the simplest case of N=2 to see what it can do for me. I find that, under certain extra conditions, I can form an ODE involving $\mathcal{L}_2$. But what is the point of doing all that work, if I already have separated operators L_1 and L_2 ! It is all a pointless exercise. I apply it to polars and yes it works, but so what!

It is true that Stakgold several times uses this method to solve a PDE in polar coordinates. He always does this by the "partial eigenfunction expansion" method. This DOES mean that you find some eigenfunctions in one coordinate, and then lincomb them with coefficients like $a_n(r)$ and then the PDE becomes an ODE in r. But with Stackel, I *know all about* doing the separation. I was thinking that this was going to produce something new. With Stackel, separation tell us this instantly

$$r^2R'' + rR' + [\ r^2 a^2 - b^2\ ]R = 0 \qquad \text{or} \qquad -(rR')' + b^2\,(R/r) - a^2 rR = 0$$
$$\Theta'' + b^2\Theta = 0$$

where $a^2 = K_1^2$. There is only one separation variable b^2, and you can identify it with your eigenvalue for either equation as your starting point. We know the "atoms" right off the bat and I wrote them down in Section 14.

Is a Green's Function Equation Really separable?

This subject is really treated in the Schrodinger section where we talk about adding a function in there with K_1^2. The conclusion was that things are NOT separable unless that function unless the driving term has the form $-f(123) = \Sigma_n\ s_n(n)/h_n^2$, and the Green's Function driver does NOT have this form.

On the other hand, at all points in space away from the Green's Function, we have the homogeneous equation and we ARE separable.

15. Title not known yet!

A. Expressing the Helmholtz operator L in terms of operators $\mathcal{L}_n$

We define $\mathcal{L}_n$ to be L_n with all separation constants set to 0. Thus, from (4.8) we find

$$\mathcal{L}_n \equiv (1/f_n)\partial_n[f_n(\partial_n)] + \kappa_1^2\Phi_{n1}(n) \qquad (15.1)$$

then (3.10) can be written

$$(L\psi)/\psi = \Sigma_n(1/[h_n^2X_n])\,(1/f_n)\partial_n[f_n(\partial_nX_n)] + k_1^2/Q + K_1^2 = 0, \qquad (3.10)$$

$$(L\psi)/\psi = \Sigma_n(1/[h_n^2X_n])\,[\mathcal{L}_n - \kappa_1^2\Phi_{n1}(n)]\,X_n + k_1^2/Q + K_1^2 = 0$$

$$(L\psi)/\psi = \{\,(1/[h_1^2X_1])\,[\mathcal{L}_1 - \kappa_1^2\Phi_{11}(1)]\,X_1 + \text{cyclic}\,\} + k_1^2/Q + K_1^2 = 0$$

We now multiply through by $\psi = X_1X_2X_3...X_n/R$ to get

$$L\psi = \{\,[X_1X_2X_3...X_n/R]\,(1/[h_1^2X_1])\,[\mathcal{L}_1 - \kappa_1^2\Phi_{11}(1)]\,X_1 + \text{cyclic}\,\} + [k_1^2/Q + K_1^2]\psi = 0$$

$$L\psi = \{\,(1/[Rh_1^2])\,[\mathcal{L}_1 - \kappa_1^2\Phi_{11}(1)]\,X_1X_2X_3..X_n + \text{cyclic}\,\} + [k_1^2/Q + K_1^2]\psi = 0$$

$$L\psi = \{\,(1/[Rh_1^2])\,[\mathcal{L}_1 - \kappa_1^2\Phi_{11}(1)]\,(R\psi) + \text{cyclic}\,\} + [k_1^2/Q + K_1^2]\psi = 0$$

$$L\psi = \{\,(1/[Rh_1^2])\,[\mathcal{L}_1 - \kappa_1^2\Phi_{11}(1)] + \text{cyclic}\,\}\,(R\psi) + [k_1^2/Q + K_1^2]\psi = 0$$

$$L\psi = \Sigma_n\,(1/[Rh_n^2])\,[\mathcal{L}_n - \kappa_1^2\Phi_{21}(2)]\,(R\psi) + [k_1^2/Q + K_1^2]\psi = 0$$

We would *like* to be able to express the L operator as some linear combination of the $\mathcal{L}_n$, but in the general case of R-separation this is not possible due to the R sitting in the (Rψ) factor. Therefore, we shall now restrict out interest to simple separation and continue along:

Problem A (simple-separation of Helmholtz) : R = 1, Q = 1, $k_1^2 = 0$, $\kappa_1^2 = K_1^2$

$$L\psi = \Sigma_n\,(1/h_2^2)\,[\mathcal{L}_n - K_1^2\Phi_{21}(2)]\,\psi + K_1^2\psi = 0$$

$$L = \Sigma_n\,(1/h_n^2)\,\mathcal{L}_n + K_1^2\{\,1 - \Sigma_n\,(1/(h_n^2)\,\Phi_{n1}(n)\,\}$$

From (5.10) we recognize the second sum $\Sigma_n\,(1/[h_n^2])\,\Phi_{n1}(n)$ as Q, and Q = 1, so we have

$$L = \Sigma_n\,(1(h_n^2)\,\mathcal{L}_n + K_1^2\{\,1 - 1\,\}$$

$$L = \Sigma_n\,(1/h_n^2)\,\mathcal{L}_n \qquad // \; L = \nabla^2 + K_1^2 \qquad (15.2)$$

and now we have succeeded in expressing L as a linear combination of the $\mathcal{L}_n$ for the simple-separation Helmholtz problem. Our path to (15.2) was by design a bit indirect because we wanted to show that such a result does not exist for R-separation. We can verify (15.2) more directly as follows

$$L = \Sigma_n \ (1/(h_n^2)\, \mathcal{L}_n \ = \Sigma_n \ (1/(h_n^2)\ \{ (1/f_n)\partial_n[f_n(\partial_n)] \ + K_1^2\Phi_{n1}(n)\}$$

$$= \Sigma_n \ (1/(h_n^2)\ \{ (1/f_n)\partial_n[f_n(\partial_n)] \ + K_1^2\ \Sigma_n \ (1/(h_n^2)\ \Phi_{n1}(n)$$

$$= \qquad\qquad \nabla^2 \qquad\qquad + \qquad K_1^2$$

where in the last step we used (3.7a) and (5.10) with Q = 1.

Our interest in operators $\mathcal{L}_n$ is that, given some set of boundary conditions, we might be able to form eigenvalue problems which are Sturm Liouville problems which can serve to simplify the solution of our Helmholtz equation.

B. Review of 1D Sturm-Liouville Theory

The requirements of a 1D Sturm-Liouville problem are these:

(1) $\mathcal{L}$ is a self-adjoint differential operator of the form $\mathcal{L}u = (pu')'+q$.

(2) $\mathcal{L}$ acts on L^2-normalizable (possibly complex) functions on some interval (a,b). L^2 is a Hilbert Space of such normalizable functions with scalar product (physics convention) $\langle f,g\rangle = \int_a^b dx\, f^*(x)g(x)$.

(3) The eigenvalue problem is $\mathcal{L}\varphi_\lambda = s(x)\lambda\varphi_\lambda$ where s(x) is some weight function. The functions p,q,s are all real and "reasonable", and p and s must be non-negative on (a,b). At each end of the interval we have an unmixed boundary condition such as $A\varphi_\lambda(a) + B\ \varphi_\lambda'(a) = 0$, where A and B are real. If either of the endpoints is "singular", such as $b=\infty$ or $p(b) = 0$, the boundary condition for that endpoint is replaced by a requirement that φ_λ be finite at that endpoint. This can happen in two ways called "limit circle" and "limit point", but the distinction does not concern us at the moment.

The eigenfunctions of such a problem form a complete set: the φ_λ span the infinite dimensional Hilbert Space of L^2 of functions on (a,b) which meet the boundary conditions. The spectrum of eigenvalues of λ can in general be "mixed", consisting of a point spectrum and a continuous spectrum, both on the positive real axis in the λ plane, though in practice one usually has *either* discrete or continuous. We will assume the general mixed case, so we shall write the eigenvalue problem on two lines

$$\mathcal{L}\varphi_\lambda\,(x) \ = \lambda\ s(x)\ \varphi_\lambda\,(x) \qquad \text{// spectrum } \lambda_n = \text{continuous range of real values} \qquad (15.3)$$
$$\mathcal{L}\varphi_{\lambda i}\,(x) = \lambda_i\ s(x)\ \varphi_{\lambda i}(x) \qquad \text{// spectrum } \lambda_{ni} = \text{discrete set of real values } i = 1,2,3....$$

The statements of completeness and orthonormality of the eigenfunctions are these

$$\Sigma_i\ \varphi_{\lambda i}{}^*(x)\varphi_{\lambda i}(\xi) \ + \int d\lambda\ \varphi_\lambda(x)^*\ \varphi_\lambda(\xi) \ = \delta(x-\xi)/s(x) \qquad \text{// completeness} \qquad (15.4)$$

$$\langle \varphi_{\lambda i}, s\varphi_{\lambda j} \rangle = \delta_{i,j}$$
$$\langle \varphi_{\lambda}, s\varphi_{\lambda'} \rangle = \delta(\lambda-\lambda')$$
$$\langle \varphi_{\lambda}, s\varphi_{\lambda n} \rangle = 0 \qquad \text{// orthogonality} \qquad (15.5)$$

where one should take note of the weight function s(x) in both equations.

Such relations immediately imply the existence of a "transform" which we write as

$$F_{\lambda i} \equiv \int_a^b dx\, s(x)\, \varphi_{\lambda i}^*(x)\, f(x)$$
$$F_{\lambda} \equiv \int_a^b dx\, s(x)\, \varphi_{\lambda}^*(x)\, f(x) \qquad \text{// projections} \qquad (15.6)$$
$$f(x) = \Sigma_i\, F_{\lambda i}\, \varphi_{\lambda i}(x) + \int d\lambda\, F_{\lambda}\, \varphi_{\lambda}(x) \qquad \text{// expansion} \qquad (15.7)$$

where we ignore issues of convergence of sums and integrals and assume f(x) is suitable such that its projections exist, and that the expansion converges. The above transform can easily be "verified" by inserting either line into the other and using the completeness and orthogonality given above.

The reader is reminded that every Sturm-Liouville problem is associated with its own private transform, though frequently occurring transforms have people's names attached to them such as Fourier Sine Series Transform, Fourier Integral Transform, Mellin Transform, Hankel Transform, and so on.

Having been explicit with the mixed spectrum, we now adopt a more compact notation where the reader understands that the spectrum can be discrete, continuous, or mixed:

$$F_{\lambda i} \equiv \int_a^b dx\, s(x)\, \varphi_{\lambda i}^*(x)\, f(x) \qquad \text{// projection}$$
$$f(x) = \Sigma_{\lambda i}\, F_{\lambda i}\, \varphi_{\lambda i}(x) \qquad \text{// expansion}$$

Here the λ_i in the projection includes discrete and continuous, and the notation $\Sigma_{\lambda i}$ implies a sum on the discrete part plus an integral on the continuous part. Even more compactly we can write

$$F_{\lambda i} \equiv \int_a^b dx\, s\, \varphi_{\lambda i}^*\, f \qquad \text{// projection}$$
$$f = \Sigma_{\lambda i}\, F_{\lambda i}\, \varphi_{\lambda i} \qquad \text{// expansion}$$

where one keeps in mind which symbols are functions of x. If we *knew* there was only a continuous spectrum, we might write this as

$$F(\lambda) \equiv \int_a^b dx\, s\, \varphi_{\lambda}^*\, f \qquad \text{// projection}$$
$$f = \int d\lambda\, F(\lambda)\, \varphi_{\lambda} \qquad \text{// expansion}$$

but we shall retain the $F_{\lambda i}$ notation. The $F_{\lambda i}$ are sometimes called "the coefficients", or "the transform". Basically the variable x of f(x) is traded out for the variable λ_i of $F_{\lambda i}$, just as in a spatial Fourier

transform the variable x is traded out for $\lambda = k^2$. One must be a little careful with the distinction between λ and a convenient variable used to label λ, in this case k.

In the same vein, we can compact down our completeness and orthogonality this way

$$\Sigma_{\lambda i}\, \varphi_{\lambda i}^*(x)\varphi_{\lambda i}(\xi) = \delta(x-\xi)/s(x) \qquad \text{// completeness}$$
$$\int_a^b dx\, s(x)\, \varphi_{\lambda i}^*(x)\, \varphi_{\lambda i}'(x) = \delta_{\lambda i,\lambda i'} \qquad \text{// orthogonality}$$

The notations $\Sigma_{\lambda i}$ and $\delta_{\lambda i,\lambda i'}$ are just shorthands for the fuller equations. We can gather up the above results:

$$\mathcal{L}\varphi_{\lambda i}(x) = \lambda_i\, s(x)\, \varphi_{\lambda i}(x) \qquad \text{// eigenvalue problem (with BC's) on (a,b)}$$

$$\Sigma_{\lambda i}\, \varphi_{\lambda i}^*(x)\varphi_{\lambda i}(\xi) = \delta(x-\xi)/s(x) \qquad \text{// completeness}$$
$$\int_a^b dx\, s(x)\, \varphi_{\lambda i}^*(x)\, \varphi_{\lambda i}'(x) = \delta_{\lambda i,\lambda i'} \qquad \text{// orthogonality}$$
$$F_{\lambda i} \equiv \int_a^b dx\, s(x)\, \varphi_{\lambda i}^*(x)\; f(x) \qquad \text{// projection}$$
$$f(x) = \Sigma_{\lambda i}\, F_{\lambda i}\, \varphi_{\lambda i}(x) \qquad \text{// expansion} \qquad (15.8)$$

We should mention that there is a standard technique used to find the eigenfunctions φ_λ. One first solves this Green's Function problem

$$[\mathcal{L}_x - \lambda s(x)]\, g(x|\xi;\lambda) = \delta(x-\xi) \qquad (15.9)$$

using the usual method of finding a boundary-condition-matching homogeneous solution to the left and to the right of $x = \xi$, and then matching the jump condition in the first derivative at $x = \xi$.

$$g(x|\xi;\lambda) = A(\lambda)\, u_{left}(x_<)\, u_{right}(x_>) \qquad \Delta g'|^{x=\xi} = -1/p$$

Once this $g(x|\xi;\lambda)$ is found, one can deduce the normalized eigenfunctions by matching the two sides of this equation, where C is a counterclockwise great circle contour in the λ plane,

$$-(1/2\pi i) \int_C d\lambda\, g(x|\xi;\lambda) = \Sigma_{\lambda n}\, \varphi_{\lambda n}^*(x)\varphi_{\lambda n}(\xi) + \int d\lambda\, \varphi_\lambda(x)^*\, \varphi_\lambda(\xi) \qquad (15.10)$$

If $g(x|\xi;\lambda)$ has a branch cut along the real axis, the contour wraps this cut and one obtains a real integral of the discontinuity of g across the cut, and that becomes the second term on the RHS. Again, care is needed to distinguish the use of λ on the LHS as a complex contour integral variable, and on the RHS as the variable of a real axis integration. Poles in $g(x|\xi;\lambda)$ give rise to the first term on the RHS. We would be remiss to omit the following standard expansion for $g(x|\xi;\lambda)$,

$$g(x|\xi;\lambda) = \Sigma_i\, \varphi_{\lambda i}(x)^*\, \varphi_{\lambda i}(\xi)/(\lambda_i - \lambda) + \int d\lambda'\, \varphi_\lambda(x)^*\, \varphi_{\lambda'}(\xi)/(\lambda' - \lambda)] \qquad (15.11)$$

In the second term it is the "continuum of poles" that creates the branch cut whose discontinuity is then picked up as the second term in (15.10).

Application to the Separation Problem

In (15.1) we defined some operators $\mathcal{L}_n$. The motivation was the hope that in one or more of our curvilinear coordinates ξ_n the boundary conditions of our Helmholtz equation problem might define a Sturm Liouville problem in that variable. We need to power up our notation a bit to allow for the extra label n associated with $\mathcal{L}_n$. We maintain the possible implication of a mixed spectrum:

$$\mathcal{L}_n \varphi_{\lambda i,n}(\xi_n) = \lambda_{i,n}\, s_n(\xi_n)\, \varphi_{\lambda i,n}(\xi_n) \qquad \text{// eigenvalue problem (with BC's) on } (a_n,b_n)$$

$$\Sigma_{\lambda i}\, \varphi_{\lambda i,n}{}^*(\xi_n)\varphi_{\lambda i,n}(\xi_n') = \delta(\xi_n - \xi_n')/s_n(\xi_n) \qquad \text{// completeness}$$

$$\int_{an}^{bn} d\xi_n\, s_n(\xi_n)\, \varphi_{\lambda i,n}{}^*(\xi_n)\, \varphi_{\lambda i',n}(\xi_n) = \delta_{\lambda i,\lambda i'} \qquad \text{// orthogonality}$$

$$F_{\lambda i,n} \equiv \int_{an}^{bn} d\xi_n\, s_n(\xi_n)\, \varphi_{\lambda i,n}{}^*(\xi_n)\; f(\xi_n) \qquad \text{// projection}$$

$$f(\xi_n) = \Sigma_{\lambda i}\, F_{\lambda i,n}\, \varphi_{\lambda i,n}(\xi_n) \qquad \text{// expansion} \qquad (15.8)$$

Naturally, our specific interest is to expand not just any function $f(\xi_n)$, but the particular function $X_n(\xi_n)$. So the transform can then be written

$$X_{\lambda i,n} \equiv \int_{an}^{bn} d\xi_n\, s_n(\xi_n)\, \varphi_{\lambda i,n}{}^*(\xi_n)\; X_n(\xi_n) \qquad \text{// projection}$$

$$X_n(\xi_n) = \Sigma_{\lambda i}\, X_{\lambda i,n}\, \varphi_{\lambda i,n}(\xi_n) \qquad \text{// expansion} \qquad (15.8)$$

Now in our simple-separation (R = 1) N-variable Stackel theory, let's assume for the sake of argument that variable ξ_1 supports a Sturm Liouville problem, so we have

$$\mathcal{L}_1 \varphi_{\lambda i,1}(1) = \lambda_{i,1}\, s_1(1)\, \varphi_{\lambda i,1}(1) \qquad \text{// eigenvalue problem}$$

$$X_{\lambda i,1} \equiv \int_{a1}^{b1} d\xi_1\, s_1(1)\, \varphi_{\lambda i,1}{}^*(1)\; X_1(1) \qquad \text{// projection}$$

$$X_1(1) = \Sigma_{\lambda i}\, X_{\lambda i,1}\, \varphi_{\lambda i,1}(1) \qquad \text{// expansion} \qquad (15.8)$$

where we have reverted to our compact coordinate notation as in earlier Sections. Let's now try to solve for this function, where for X_1 we pick one of the ξ_1 eigenfunctions,

$$\psi_{\lambda i,1}(12...N) \equiv \varphi_{\lambda i,1}(1)\, X_2(2) X_3(3)...X_N(N)$$

We are trying to solve our simple-separation Helmholtz equation using (15.2) applied to this special ψ

$$L\, \psi_{\lambda i,1} = \Sigma_n\, (1/h_n{}^2)\, \mathcal{L}_n\; \psi_{\lambda i,1} \qquad = 0$$

$$L\, \psi_{\lambda i,1} = \Sigma_{n=1}{}^N\, (1/h_n{}^2)\, \mathcal{L}_n\, \psi_{\lambda i,1} = (1/h_1{}^2)\, \mathcal{L}_1\, \psi_{\lambda i,1} + \Sigma_{n=2}{}^N\, (1/h_n{}^2)\, \mathcal{L}_n\, \psi_{\lambda i,1}$$

$$= (1/h_1^2)\, \mathcal{L}_1(\varphi_{\lambda i,1}(1)...X_n) + \Sigma_{n=2}^{N}\ (1/h_n^2)\, \mathcal{L}_n(\varphi_{\lambda i,1}(1)...X_n)$$

$$= [(X_2X_3...X_N)/h_1^2]\ \mathcal{L}_1\, \varphi_{\lambda i,1}(1) + \Sigma_{n=2}^{N}\ (1/h_n^2)\, \mathcal{L}_n(\varphi_{\lambda i,1}(1)...X_n)$$

$$= [(X_2X_3...X_N)/h_1^2]\ \lambda_{i,1}\, s_1(1)\, \varphi_{\lambda i,1}(1) + \Sigma_{n=2}^{N}\ (1/h_n^2)\, \mathcal{L}_n(\varphi_{\lambda i,1}(1)...X_n)$$

Let's now go immediately to the case N = 2 and then write

$$L\, \psi_{\lambda i,1} = [(X_2)/h_1^2]\ \lambda_{i,1}\, s_1(1)\, \varphi_{\lambda i,1}(1) + \Sigma_{n=2}^{2}\ (1/h_n^2)\, \mathcal{L}_n(\varphi_{\lambda i,1}(1)...X_n) = 0$$

$$L\, \psi_{\lambda i,1} = [(X_2)/h_1^2]\ \lambda_{i,1}\, s_1(1)\, \varphi_{\lambda i,1}(1) + (1/h_2^2)\, \mathcal{L}_2[\, \varphi_{\lambda i,1}(1)X_2] = 0$$

$$L\, \psi_{\lambda i,1} = [(X_2)/h_1^2]\ \lambda_{i,1}\, s_1(1)\, \varphi_{\lambda i,1}(1) + (1/h_2^2)\, \varphi_{\lambda i,1}(1)\, \mathcal{L}_2X_2 = 0$$

Since the $\{ \varphi_{\lambda i,1}(1)\}$ form a complete set, the above implies that

$$[L\, \psi_{\lambda i,1}]/\, \varphi_{\lambda i,1}(1) = [(X_2)/h_1^2]\ \lambda_{i,1}\, s_1(1) + (1/h_2^2)\, \mathcal{L}_2X_2 = 0$$

which says

$$\mathcal{L}_2X_2 - \lambda_{i,1}\, s_1(1)\, [\, h_2^2/h_1^2]\, X_2 = 0$$

If it happens that we can write

$$h_2^2/h_1^2 = A(1)B(2)$$

Then we can select $s_1(1) = 1/A(1)$ and we end up then with

$$\mathcal{L}_2X_2 - \lambda_{i,1}\, B(2)\, X_2 = 0$$

and this is an ODE in variable ξ_2 which we can probably solve quite easily for X_2 .

If we swap 1↔2 in the above analysis, we obtain these equations:

$$\psi_{\lambda i,2}(12) \equiv X_1(1)\, \varphi_{\lambda i,2}(2)$$

$$\mathcal{L}_2\varphi_{\lambda i,2}(2) = \lambda_{i,1}\, s_2(2)\, \varphi_{\lambda i,2}(2) \qquad \text{// eigenvalue problem}$$

$$X_{\lambda i,2} \equiv \int_{a2}^{b2} d\xi_2\, s_2(2)\, \varphi_{\lambda i,2}^*(2)\ X_2(2) \qquad \text{// projection}$$

$$X_2(2) = \Sigma_{\lambda i}\, X_{\lambda i,2}\, \varphi_{\lambda i,1}(2) \qquad \text{// expansion} \qquad (15.8)$$

$$\mathcal{L}_1X_1 - \lambda_{i,2}\, s_2(2)\, [\, h_1^2/h_2^2]\, X_1 = 0$$

If it happens that we can write

$h_1^2/h_2^2 = A(2)B(1)$

Then we can select $s_2(2) = 1/A(2)$ and we end up then with

$\mathcal{L}_1 X_1 - \lambda_{i,2} B(1) X_1 = 0$

and this is an ODE in variable ξ_1 which we can probably solve quite easily for X_1 .

Example: Polar Coordinates. Let $(\xi_1,\xi_2) = (r,\theta)$, in which case we have this data

$h_1 = 1 \quad h_2 = \xi_1 \qquad \Phi = \begin{pmatrix} 1 & -\xi_1^{-2} \\ 0 & 1 \end{pmatrix} \qquad f_1 = \xi_1 \qquad f_2 = 1$

$h_1^2/h_2^2 = \xi_1^{-2} = A(2)B(1) \qquad \Rightarrow \qquad A(2) = 1, \quad s_2(2) = 1, \quad B(1) = \xi_1^{-2}$

If we assume $(0,2\pi)$ full range for $\xi_2 = \theta$, we have (changing EF label from i to m)

$\mathcal{L}_2 \varphi_{\lambda m,2}(2) = \lambda_{m,2}\, \varphi_{\lambda m,2}(2)$

$\mathcal{L}_2 \varphi_{\lambda m,2}(2) = \lambda_{m,2}\, \varphi_{\lambda m,2}(2)$

$\mathcal{L}_n \equiv (1/f_n)\partial_n[f_n(\partial_n)] + \kappa_1^2 \Phi_{n1}(n)$

$\mathcal{L}_2 \equiv (1/f_2)\partial_2[f_2(\partial_2)] + K_1^2 \Phi_{21}(2)$
$= (1/1)\partial_2[1(\partial_2)] + K_1^2\, 0$
$= \partial_2^2$

$\Rightarrow \partial_2^2\, \varphi_{\lambda m,2}(2) - \lambda_{m,2}\, \varphi_{\lambda m,2}(2) = 0$

Solutions are of the type $\exp(\sqrt{\lambda_{m2}}\, \xi_2)$ and the BC requires $\exp(\sqrt{\lambda_{m2}}\, \xi_2) = \exp(\sqrt{\lambda_{m2}}\, [\xi_2+2\pi])$ which in turn requires $\exp(\sqrt{\lambda_{m2}}\, 2\pi) = 1$ which in turn requires $\sqrt{\lambda_{m2}} = m$, m = 0,1,2.... Therefore

$\varphi_{\lambda m,2}(2) = [\sin(m\xi_2), \cos(m\xi_2)] \qquad \lambda_{m,2} = m^2$, m = 0,1,2...

Now we turn to the ξ_1 problem

$\mathcal{L}_1 X_1 - \lambda_{m,2} B(1) X_1 = 0$

$\mathcal{L}_1 \equiv (1/f_1)\partial_1[f_1(\partial_1)] + K_1^2 \Phi_{11}(1)$

$= (1/\xi_1)\partial_1[\xi_1(\partial_1)] + K_1^2$

Putting in the pieces we then have

$$\{ (1/\xi_1)\partial_1[\xi_1(\partial_1)] + K_1^2\}X_1 - m^2 \xi_1^{-2}X_1 = 0$$

$$(1/\xi_1)\partial_1[\xi_1(\partial_1 X_1)] + K_1^2 X_1 - m^2 \xi_1^{-2}X_1 = 0$$

$$\partial_1[\xi_1(\partial_1 X_1)] + K_1^2\xi_1 X_1 - m^2 \xi_1^{-1}X_1 = 0$$

$$\xi_1\partial_1^2 X_1 + \partial_1 X_1 + K_1^2\xi_1 X_1 - m^2 \xi_1^{-1}X_1 = 0$$

$$\xi_1^2\partial_1^2 X_1 + \xi_1\partial_1 X_1 + K_1^2\xi_1^2 X_1 - m^2 X_1 = 0$$

$$\xi_1^2\partial_1^2 X_1 + \xi_1\partial_1 X_1 + (K_1^2\xi_1^2 X_1 - m^2) X_1 = 0$$

This is a scaled Bessel equation with solution of the type $J_m(K_1\xi_1)$. Therefore by this method we have found separated solutions of the form

$$\psi_{\lambda i,2}(12) \equiv X_1(1)\, \varphi_{\lambda i,2}(2) = [J_m(K_1\xi_1), Y_m(K_1\xi_1) [\sin(m\xi_2),\cos(m\xi_2)]$$