Lemma: How do you enumerate a set of permutations in a systematic way by using swaps?

Here is what can do with just a single swap

12345	00000
21345	11000
13245	01100
12435	00110
12354	00011
32145	10100
14325	01010
12543	00101
42315	10010
15342	01001
52341	10001

Suppose we just alternate swapping in the last three positions this way:

12345		ε_{12345}
12354	1	
12534	1	
12543		
12453		
12435		

We get 5 new terms. If we try to continue, we replicate the first term, so this cycle is exhausted.

$P_{12}f(123) = f(213)$
$P_{23}f(123) = f(132)$
$P_{13}f(123) = f(321)$
$P_{12}P_{23}f(123) = P_{12}\, f(132) = f(231)$
$P_{13}P_{12}f(123) = P_{13}\, f(213) = f(231)$

Lemma: Consider some function $f(m_1, m_2 \ldots m_N)$. Let P represent some permutation of the arguments. Let p indicate the number of pair swaps needed to get from the original order to that permutation, so p is determined by P. There are N! permutations of f(123...N) including the original term. We could write a simple sum of all N! permutations using this notation

$$f(123...N) + f(21...N) + \ldots\ldots = \Sigma_P\, P\, f(123...N)$$

It is totally unambigous what this sum is. We could create another sum in which all the terms with an odd number of pair swaps have minus signs.

$$f(123...N) - f(21...N) + ... = \Sigma_P (-1)^P P\, f(123...N)$$

There are still N! terms. Here is the series for N = 2 and N=3,

$$\Sigma_P (-1)^P P\, f(123) = f(123) - f(213) + f(231) - f(312) + f(321) - f(231)$$

Can we represent this using the ε symbol somehow?

$$\varepsilon_{ijk} f(ijk) = \varepsilon_{123} f(123) + \varepsilon_{213} f(213) + ...$$

This seems to work, so I think we can say

$$f(123...N) - f(21...N) + ... = \Sigma_P (-1)^P P\, f(123...N) = \Sigma_{ijk}\, \varepsilon_{ijk}\, f(ijk) = \varepsilon_{ijk}\, f(ijk)$$

where each sum is over 1...N.

Now suppose you have a different kind of function with only N-1 arguments:

$$g(i_1 i_2 .....i_{N-1})$$

where each i_n takes a value in 1,2....N. How now, brown cow, do we list the permutations of these N-1 indices? There are only N-1 positions subject to swapping. We could still write

$$g(i_1 i_2 .....i_{N-1}) - g(i_2 i_1 .....i_{N-1}) + ... = \Sigma_P (-1)^P P\, g(i_1 i_2 .....i_{N-1})$$

where P permutes the N-1 objects. The question now is how you use the ε symbol to indicate the sign! The first thing you might try is this

$$(-1)^P = \varepsilon_{i_1 i_2 i_3 \ldots . i_{N-1}}$$

But the problem here is that this ε only has N-1 indices, but we want to have N integers involved, so there is a mismatch.

Let's just stick to N = 3. We have some $g(i_1 i_2)$ object where each i is 1,2 or 3. Our answer is just this

$$\Sigma_P (-1)^P P\, g(i_1 i_2) = g(i_1 i_2) - g(i_2 i_1) = \varepsilon_{12}\, g(i_1 i_2) + \varepsilon_{21}\, g(i_2 i_1) \quad ?$$

Here the ε indices are the subscripts on the i_n indices. The ε indices are not the values of the i_n . Since there are only two arguments for g, we use the ε_{ij} tensor with each index doing 1 or 2. Hmmm.

$$\Sigma_P (-1)^P P\, g(i_1 i_2 i_3) = \varepsilon_{\ell mn}\, g(i_\ell\, i_m\, i_n) = \varepsilon_{abc}\, g(i_a i_b i_c)$$

Now think of

$$\delta_{i_1, m_1}\, \delta_{i_2, m_2} ..... \delta_{i_{N-1}, m_{N-1}} = f(m_1\, m_2 .... m_{N-1})$$

Suppose we want to add to this term all the permutations of the m's. We could say

series = $f(m_{i1}\, m_{i2}.... mi_{N-1})\varepsilon_{i1i2...}$

It is sort of "indirect addressing" or pointer stuff. Maybe write as

series = $f(m_a m_b.....m_x)\ \varepsilon_{abc...x}$

Then we get

$$\delta_{i_1,m_a}\, \delta_{i_2,m_b}..... \delta_{i_{N-1},m_x}\, \varepsilon_{abc...x}$$

So here is a new statement of the Theorem below

$$\Sigma_j\, \varepsilon_{ji_1i_2i_3....i_{N-1}}\, \varepsilon_{jm_1m_2m_3....m_{N-1}}$$

$$= \delta_{i_1,m_1}\, \delta_{i_2,m_2}..... \delta_{i_{N-1},m_{N-1}} \ + \text{ other terms} \qquad \text{// line 1}$$

$$= \delta_{i_1,m_a}\, \delta_{i_2,m_b}..... \delta_{i_{N-1},m_x}\, \varepsilon_{abc...x} \qquad \text{// line 2}$$

where $\varepsilon_{abc...x}$ has N-1 indices each of which takes a value in (1...N). Example

$$\Sigma_j\, \varepsilon_{ji_1i_2}\, \varepsilon_{jm_1m_2} \quad = \quad \delta_{i_1,m_a}\, \delta_{i_2,m_b}\varepsilon_{ab}$$

$$= \delta_{i_1,m_1}\, \delta_{i_2,m_2}\, \varepsilon_{12} + \delta_{i_1,m_2}\, \delta_{i_2,m1_2}\, \varepsilon_{21}$$

so I think that does it. I think trying to replace the abc with a_1a_2 is too complicated.

Theorem 1: In this theorem we assume all indices are present. The theorem claims that

$$\Sigma_j\, \varepsilon_{ji_1i_2i_3....i_{N-1}}\, \varepsilon_{jm_1m_2m_3....m_{N-1}}$$

$$= \delta_{i_1,m_1}\, \delta_{i_2,m_2}..... \delta_{i_{N-1},m_{N-1}} \ + \text{ other terms} \qquad \text{// line 1}$$

$$= \delta_{i_1,s_1}\, \delta_{i_2,s_2}..... \delta_{i_{N-1},s_{N-1}}\, \varepsilon_{as_1s_2s_3....s_{N-1}}\ \varepsilon_{bm_1m_2m_3....m_{N-1}} \qquad \text{// line 2}$$

where a is the number in {1,2...N} not present among the i_n , and b the number not in the m_n .

(1) Each ε on the LHS has N indices, and each index has must take a value in the set {1,2,3...N}.

(2a) In order for the first ε to be non-zero, all N-1 i_n indices must be different from each other. If this is the case, then only one number in the set {1,2,3...N} does not appear among the i_n. Call this number a.

(2b) In order for the second ε to be non-zero, all N-1 m_n indices must be different from each other. If this is the case, then only one number in the set {1,2,3...N} does not appear among the i_n. Call this number b.

(3) In order for the *product* of the two ε's to be non-zero, the value set taken by the i_n must omit the same integer that the value set taken by the m_n omits. Only then can we set j equal to the omitted value so that *both* ε's can be non-zero.

(4) Assuming this is the case, one possibility is that we have $i_1 = m_1$, $i_2 = m_2$ and so on to $i_{N-1} = m_{N-1}$. Both sets are omitting the same value, though we don't know what that value is. In the j sum, j will take on that omitted value to generate a non-zero εε product. In this case, for the one contributing term in the j sum, the indices on each ε are *identical*, so whether than ε is +1 or -1, we know that $\varepsilon^2 = +1$. So this is where the first term on line 1 comes from.

(5) Another possibility would be $i_1 = m_2$, $i_2 = m_1$ and all other indices as they were in (4). In this case the indices on the two ε's don't match. But if we swap indices m_1 and m_2 on the second ε, they would match. But this creates a minus sign! Thus, another term on line 1 would be $-\delta_{i_1,m_2}\,\delta_{i_2,m_1}\ldots..\,\delta_{i_{N-1},m_{N-1}}$ and it is contained on the "other terms" of this line.

(6) In general, the sign of a term will be minus(plus) if an odd(even) number of m_n index swaps is needed to make it look like the original first term.

(7) Now look at the RHS of the theorem. One term in the implied s sums has $s_1 = m_1$, $s_2 = m_2$ and so on. The ε for that term will be then $\varepsilon_{m_1 m_2 m_3 \ldots . m_{N=1}}$. Since we have added a factor which is this exact same ε, we will have ε^2 and then we obtain the term shown be.

(8) If we look at other terms in the multiple s sums, the $\varepsilon_{s_1 s_2 s_3 \ldots . s_{N-1}}$ factor generates exactly the sign we need as stated in (6) above.

(9) So we have now shown that the theorem is true *in the case* that the i_n and m_n sets meet our conditions (2), (2') and (3) above: the i_n are all different, the m_n are all different, and these two sets exclude the same element. We still need to show that our theorem is true if these rules are not met, in which case we know that the LHS = 0. So we have to show that the RHS is also 0 in these cases.

(10a) Suppose two m_n indices are the same. The RHS is zero due to $\varepsilon_{m_1 m_2 m_3 \ldots . m_{N=1}}$.

(10b) Suppose two i_n indices are the same. For example, suppose $i_1 = i_2$. We then have symmetric $\delta_{i_1,s_1}\,\delta_{i_1,s_2}$ contracted against antisymmetric ε with the s's, so we get 0.

(10c) Suppose the two sets don't exclude the same value? Consider $\delta_{i_1,s_1}\,\delta_{i_2,s_2}\ldots..\,\delta_{i_{N-1},s_{N-1}}$. In order for this product of N-1 δ's to be non-zero, every single δ must be 1. We must have therefore

$$i_1 = s_1\,,\quad i_2 = s_2.\ \ldots\quad i_{N-1} = s_{N-1}$$

Suppose the i_n set omits value 3, and the m_n set omits value 4. Then the δ product $\delta_{i_1,s_1}\,\delta_{i_2,s_2}\ldots..\,\delta_{i_{N-1},s_{N-1}}$ can have no factor $\delta_{3,3}$.

QED. Let's now write this theorem for N = 3 where it will take on a perhaps familiar form.:

$$\Sigma_j\,\varepsilon_{ji_1i_2}\,\varepsilon_{jm_1m_2}$$

$$= \delta_{i_1,m_1}\,\delta_{i_2,m_2} + \text{other terms} \qquad \text{// line 1}$$

$$= \delta_{i_1,s_1}\,\delta_{i_2,s_2}\ \varepsilon_{s_1s_2}\,\varepsilon_{m_1m_2} \qquad \text{// line 2}$$

$$= \delta_{i_1,m_1}\,\delta_{i_2,m_2} - \delta_{i_1,m_2}\,\delta_{i_2,m_1}$$

Suppose 2 is missing from the i_n set and 3 is missing from the m_n set. What makes the RHS be zero? Specifically, suppose we have this

$$\Sigma_j\,\varepsilon_{j13}\,\varepsilon_{j12} = \delta_{1,s_1}\,\delta_{3,s_2}\ \varepsilon_{s_1s_2} * \varepsilon_{12} = \delta_{1,s_1}\,\delta_{3,s_2}\ \varepsilon_{s_1s_2}$$

$$= \delta_{1,1}\,\delta_{3,3}\ \varepsilon_{13}\ ???$$

Consider our tensor application now

$$= \varepsilon_{ji_1i_2i_3\ldots.i_N}\,\varepsilon_{jm_1m_2m_3\ldots.m_N}\ [(\mathbf{e}_1)_{i_1}(\mathbf{e}_1)_{m_1}][(\mathbf{e}_2)_{i_2}(\mathbf{e}_2)_{m_2}]\ \ldots..\ [(\mathbf{e}_N)_{i_N}(\mathbf{e}_N)_{m_N}]$$

$$\varepsilon_{ji_1i_2i_3\ldots.i_N}\,\varepsilon_{jm_1m_2m_3\ldots.m_N} = \delta_{i_1,m_a}\,\delta_{i_2,m_b}\ldots..\,\delta_{i_{N-1},m_x}\,\varepsilon_{abc\ldots x}$$

So we then have

$$\delta_{i_1,m_a}\,\delta_{i_2,m_b}\ldots..\,\delta_{i_{N-1},m_x}\,\varepsilon_{abc\ldots x}\,[(\mathbf{e}_1)_{i_1}(\mathbf{e}_1)_{m_1}][(\mathbf{e}_2)_{i_2}(\mathbf{e}_2)_{m_2}]\ \ldots..\ [(\mathbf{e}_N)_{i_N}(\mathbf{e}_N)_{m_N}]$$

Maybe it is better to just use the P notation. Maybe use this

$$[\mathbf{e}_1\bullet\,\mathbf{e}_1]\,[\mathbf{e}_2\bullet\,\mathbf{e}_2]\ldots..\,[\mathbf{e}_N\bullet\,\mathbf{e}_N] + \text{other terms}$$

$= [\mathbf{e}_1 \bullet \mathbf{e}_a]\,[\mathbf{e}_2 \bullet \mathbf{e}_b] \ldots [\mathbf{e}_N \bullet \mathbf{e}_x]\ \varepsilon_{abc\ldots x}$

And in our application, k is missing, so there are only N-1 factors, and ε has N-1 indices. Then

$= g'_{1a} g'_{2b} \ldots g'_{Nx}\ \varepsilon_{abc\ldots x}$ This is (N-1)! terms.

Example: suppose N = 3 and k=1 is missing. Then we have

$g'_{2b} g'_{3c}\ \varepsilon_{bc} = g'_{22} g'_{33} \varepsilon_{23} + g'_{23} g'_{32} \varepsilon_{32} = g'_{22} g'_{33} - g'_{23} g'_{32}$ = correct answer

But here the indices on ε_{bc} only take values 2 and 3 since only those are in the set.

$g'_{1a}\ g'_{2b}\ g'_{2c} \ldots g'_{Nx}\ \varepsilon_{abc\ldots x}$

But we know that k is missing, so why not this

$g'_{1a}\ g'_{2b}\ g'_{2c} \ldots g'_{Nx}\ \varepsilon_{kabc\ldots x}$

where now ε has the full N indices. Maybe we put it in its usual place

$[\ (g'_{1a}\ g'_{2b}\ g'_{2c} \ldots g'_{Nx}) / g'_{2k}\ ]\ \varepsilon_{abc..k..x}$

So try this with N=3 and k=1 missing

$[\ (g'_{1a}\ g'_{2b}\ g'_{3c}) / g'_{1a}\ ]\ \varepsilon_{1bc} = g'_{2b}\ g'_{3c}\ \varepsilon_{1bc} = g'_{22}\ g'_{33}\ \varepsilon_{123}$ etc and it works!

So maybe I have

$(\Pi_{x;\, i\neq k}\ \mathbf{e}_i) \bullet (\Pi_{x;\, i\neq k}\ \mathbf{e}_i) = ([\mathbf{e}_1 \bullet \mathbf{e}_1]\,[\mathbf{e}_2 \bullet \mathbf{e}_2] \ldots [\mathbf{e}_N \bullet \mathbf{e}_N]) / [\mathbf{e}_k \bullet \mathbf{e}_k]$ + other terms

$(g'_{1a}\ g'_{2b}\ g'_{2c} \ldots g'_{Nx})\ \varepsilon_{abc..k..x}$ // g'_{ky} is missing from the product.

$[\ (g'_{1a_1}\ g'_{2a_2} \ldots g'_{Na_N}) / g'_{ka_k}\ ]\ \varepsilon_{a_1 a_2 \cdots k..a_N}$

Assuming k is out in the middle somewhere, the first term here is

$(g'_{11}\ g'_{22} \ldots g'_{NN}) / g'_{ka_k}\ \varepsilon_{12 \cdots k..N}$

Perhaps write instead as

$[\ \Pi_{i\neq k}\ (g'_{ia_i})\ ]\ \varepsilon_{a_1 a_2 \cdots k..a_N}$

Example N=3 and k = 1:

$[\ \Pi_{i\neq 1}\ (g'_{ia_i})\ ]\ \varepsilon_{1a_2a_3}$

$= (g'_{2a_2})(g'_{3a_3})\ \varepsilon_{1a_2a_3}$

$= (g'_{22})(g'_{33})\ \varepsilon_{123}\ +\ (g'_{23})(g'_{32})\ \varepsilon_{132}$

$= (g'_{22})(g'_{33}) - (g'_{23})(g'_{32})$ agrees.

Example N = 4 and k = 2

$[\ \Pi_{i=1,N;\ i\neq k}\ (g'_{ia_i})\ ]\ \varepsilon_{a_1a_2\cdots k..a_N}$

$= [\ \Pi_{i=1,4;\ i\neq 2}\ (g'_{ia_i})\ ]\ \varepsilon_{a_1 2 a_3 a_4}$

$= [(g'_{1a_1})(g'_{3a_3})(g'_{4a_4})\ ]\ \varepsilon_{a_1 2 a_3 a_4}$

$= [(g'_{11})(g'_{33})(g'_{44})\ ]\ \varepsilon_{1234} + [(g'_{11})(g'_{34})(g'_{43})\ ]\ \varepsilon_{1243}$

$+ [(g'_{13})(g'_{31})(g'_{44})\ ]\ \varepsilon_{3214} + [(g'_{13}(g'_{34}(g'_{41})\ ]\ \varepsilon_{3241}$

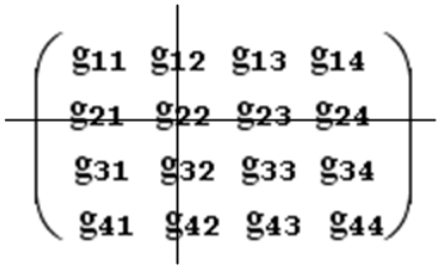
$+ [(g'_{14}(g'_{31})(g'_{43}\ ]\ \varepsilon_{4213} + [(g'_{14})(g'_{33})(g'_{41})\ ]\ \varepsilon_{4231}$

Stop. This looks like a minor of a determinant.

I have looked at my matrix notes, so go back to

$\varepsilon_{a_1 2 a_3 a_4}[(g'_{1a_1})(g'_{3a_3})(g'_{4a_4})\ ] = \text{cof}(g'_{22})$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$



It works. So here is a much simpler notation

$$[\ \Pi_{i=1,N;\ i\neq k}\ (g'_{i a_i})\ ]\ \varepsilon_{a_1 a_2 \cdots k \ldots a_N} = \mathrm{cof}(g'_{kk})$$

Theorem:

$$\varepsilon_{j i_1 i_2 i_3 \ldots . i_{N-1}}\ \varepsilon_{j m_1 m_2 m_3 \ldots . m_{N-1}} = \delta_{i_1, m_a}\ \delta_{i_2, m_b} \ldots\ldots\ \delta_{i_{N-1}, m_x}\ \varepsilon_{abc \ldots x}$$

$$= \delta_{i_1, m_1}\ \delta_{i_2, m_2} \ldots\ldots\ \delta_{i_{N-1}, m_{N-1}} + \text{other terms}$$

If nothing were "missing", I think this is the determinant of this matrix

$$\begin{vmatrix} \delta_{i_1,m_1} & \delta_{i_1,m_2} & \delta_{i_1,m_3} & \ldots\ldots & \delta_{i_1,m_{N-1}} \\ \delta_{i_2,m_1} & \delta_{i_2,m_2} & \delta_{i_2,m_3} & \ldots\ldots & \delta_{i_2,m_{N-1}} \\ \delta_{i_3,m_1} & \delta_{i_3,m_2} & \delta_{i_3,m_3} & \ldots\ldots & \delta_{i_3,m_{N-1}} \\ \bullet\bullet\bullet & & & & \\ \delta_{i_{N-1},m_1} & \delta_{i_{N-1},m_2} & \delta_{i_{N-1},m_3} & \ldots\ldots & \delta_{i_{N-1},m_{N-1}} \end{vmatrix}$$

This matrix is $(N-1)^2$.

The result quoted in my Levi section would have N = 2.

Suppose we have our εε object summed against a product of matrix elements this way

$$\varepsilon_{j i_1 i_2 i_3 \ldots . i_N}\ \varepsilon_{j m_1 m_2 m_3 \ldots . m_N}\ g_{i_1, m_1}\ g_{i_2, m_2} \ldots\ldots\ g_{i_N, m_N}$$

Then this is the determinant of a matrix you could construct just like the δ matrix above. We have

$$\Sigma_i\ \Sigma_m\ \varepsilon_{j i_1 i_2 i_3 \ldots . i_N}\ \varepsilon_{j m_1 m_2 m_3 \ldots . m_N}\ g_{i_1 m_1}\ g_{i_2 m_2} \ldots\ldots\ g_{i_N m_N}$$

$$= \Sigma_i\ \Sigma_m\ \delta_{i_1, m_a}\ \delta_{i_2, m_b} \ldots\ldots\ \delta_{i_{N-1}, m_x}\ \varepsilon_{abc \ldots x}\ g_{i_1 m_1}\ g_{i_2 m_2} \ldots\ldots\ g_{i_N m_N}$$

$$= \Sigma_m\ \varepsilon_{abc \ldots x}\ g_{m_a m_1}\ g_{m_b m_2} \ldots\ldots\ g_{m_x m_N}$$

$$= \Sigma_m\ [\ g_{m_1 m_1}\ g_{m_2 m_2} \ldots\ldots\ g_{m_N m_N} + \text{other terms}\ ]$$

Is this perhaps N! times det g ? Each [..] object is the determinant of the g matrix with the rows and columns labeled by m_1, m_2 .... instead of by 1,2,3... I am sure that det of a matrix does not change if you permute the rows AND the columns by some arbitrary permutation, same for both.