More on Stakgold's half-plane problem. PhL 3.26.05

Right now in my Stackel paper I have some comments on this that don't quite seem right still. I have dont things Stak's way my chapter 7 raw2 notes and all equations are correct, but the method is hazy. Stak uses the "Dirichlet method" of solving the problem because the added function v does solve the homo equation even at the Green's point and is therefore "fully separable."

Let's start over on this problem both ways focusing on the notion of separation. We have these separated equations, where $a^2 = K_1^2$ and b^2 is the separation constant.

$$r^2R'' + rR' + [\, r^2 a^2 - b^2 \,]R = 0 \qquad \text{or} \qquad -(rR')' + b^2 (R/r) - a^2 rR = 0 \qquad (14.2)$$
$$\Theta'' + b^2\Theta = 0$$

We are trying to solve $(\nabla^2 + a^2)g = \delta^{(2)}(\mathbf{r}-\mathbf{r_o})$ and we have our little wedge region. We know the atoms are these

$$\psi \sim X_1X_2 \sim [J_b(ar),\, Y_b(ar)]\ [\sin(b\theta),\cos(b\theta)] \qquad a = \text{real} \quad b = \text{real} \qquad (14.3a)$$
$$\psi \sim X_1X_2 \sim [I_b(\alpha r),\, K_b(\alpha r)]\ [\sin(b\theta),\cos(b\theta)] \qquad a = \text{imaginary} \quad b = \text{real} \qquad (14.3b)$$
$$\psi \sim X_1X_2 \sim [J_{i\beta}(ar),\, Y_{i\beta}(ar)]\ [\sinh(\beta\theta),\cosh(\beta\theta)] \qquad a = \text{real} \quad b = \text{imaginary} \qquad (14.3c)$$
$$\psi \sim X_1X_2 \sim [I_{i\beta}(\alpha r),\, K_{i\beta}(\alpha r)]\ [\sinh(\beta\theta),\cosh(\beta\theta)] \qquad a = \text{imaginary} \quad b = \text{imaginary} \qquad (14.3d)$$

I like my Smythian form written this way

$$g(r,\theta|r',\theta') = \Sigma_{n=1}^{\infty} C_n(r',\theta') \sin(b_n\theta)\, I_b(\alpha r_<)\, K_b(\alpha r_>) \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

because it puts our little dotted math boundary as a circular surface at r = r' on which the Green's Function lies.

Review of Pillbox Theory in 3D

Now what about my little method of putting a box around the point source as used in the oblate work. This was the "pillbox" idea. Maybe I should rederive that pillbox concept in a more general sense. It is done pretty well in the "on axis greens for oblate spheroid" doc, and here are the key results:

$$\rho = q\,\delta(\boldsymbol{\xi}-\boldsymbol{\xi}') = (q/H) * \delta(\xi_1-\xi_1')\,\delta(\xi_2-\xi_2')\,\delta(\xi_3-\xi_3')$$

$$\sigma_1 = (q/h_2h_3) * \delta(\xi_2-\xi_2')\,\delta(\xi_3-\xi_3')$$

The first line gives the 3D source density of a point source of size q. Now imagine we draw a dotted line through our point source such that the dotted surface at the point source is normal to direction ξ_1. Then σ_1 is the effective surface source density on that surface, where we will only use it right near the point source.

Now the next step is supposed to be "Gauss's Law", but now we are doing Helmholtz and not Laplace, so how will that change? The idea is that we have this (going back to 3D)

$$(\nabla^2 + a^2)g(\mathbf{r}|\mathbf{r}_0) = q\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_0)$$

We integrate both sides over a tiny sphere centered at $\mathbf{r}_0$, there the point source is located. The RHS obviously gives q. The a^2 term gives this

$$\int dV\ [a^2\, g(\mathbf{r}|\mathbf{r}_0)\ ] = a^2 \int dV\ g(\mathbf{r}|\mathbf{r}_0)$$

This term is not so obvious. In 1D we would say that we know g is continuous, and then the integral would vanish. But think of potential theory with a point charge and let $r_0 = 0$ and then $g(\mathbf{r}|\mathbf{0}) = 1/r$. We put a sphere around this thing and shrink it. We get

$$\int dV\ g(\mathbf{r}|\mathbf{0}) \;=\; \int r^2 dr\, d\Omega\ (1/r) \;\rightarrow 0$$

so the volume element overwhelms the divergence of g in this case. For Helmholtz the corresponding Green's function is not 1/r but I think e^{-ar}/r (5.121) maybe with an i in the exponent, and this will then give the same result!. So I think we can therefore ignore the a^2 term.

We are then left with something we can send into the divergence equation

$$\int dV\ \nabla^2 g(\mathbf{r}|\mathbf{r}_0) \;=\; \int d\mathbf{S} \bullet \nabla g \;=\; \int dS\ \partial_n g$$

where n is the outgoing normal at the surface. So NOW we assume our pillbox surface instead of the spherical one. Our pillbox is very flat and the sides contribute nothing, so we only care about the two ends of the pillbox. On the "top" we get $dA\ \partial_n g^{top}$ and on the bottom $dA\ \partial_n g^{bot}$, so this term gives

$$\int_{pillbox} dV\ \nabla^2 g(\mathbf{r}|\mathbf{r}_0) \;=\; \int dA\ \partial_n g^{top} - \int dA\ \partial_n g^{bot} \;=\; \int dA\ [\partial_n g^{top} - \partial_n g^{bot}]$$

Although dA is small, I still want to show that we are integrating over this tiny area. The reason is that even though the area dA is tiny, there is still "violent action" going on as we move around on this little area, because at the center of the area we get right next to our point source.

Now what happens on the RHS of our Green's equation with a pillbox? It captures the surface source in the box. We get

$$\int_{pillbox} dV\ q\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_0) \;=\; ?$$

Why don't we just get q ? If we just do the above in the obvious way, we do get q. But if we think of the equivalent σ, then as should get σdA where $\sigma_1 = (q/h_2h_3) * \delta(\xi_2-\xi_2')\,\delta(\xi_3-\xi_3')$.

Well this is a little paradox and we resolve it as follows. Write

$$\int_{pillbox} dV\ q\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_0) \;=\; \int dA\ \sigma \;=\; \int dA\ \ (q/h_2h_3) * \delta(\xi_2-\xi_2')\,\delta(\xi_3-\xi_3')$$

It is true that if we carry out the dA integral, we will get just q. But the point is that we DON'T do this integral we just leave it as an integral over our area. Then we have

$$\int dA\, [\partial_n g^{top} - \partial_n g^{bot}] \quad = \quad \int dA\ (q/h_2 h_3) * \delta(\xi_2 - \xi_2') \, \delta(\xi_3 - \xi_3')$$

Then we argue that this must be valid for dA no matter how small we make it, so the integrands must then match, and THEN we make the claim

$$\partial_1 g^{top} - \partial_1 g^{bot} = (q/h_2 h_3) * \delta(\xi_2 - \xi_2') \, \delta(\xi_3 - \xi_3')$$

Review of Pillbox Theory in 2D

I don't think I have ever done this before.