(b1) Area. (general N)

In N dimensions for N≠3, the notation **q** = **A** x **B** has no meaning so we must replace it with something that does have meaning. Consider the N=3 case again where the cross product is written this way (repeated indices are summed unless stated otherwise)

$$\mathbf{q} = \mathbf{A} \times \mathbf{B}$$

$$q_k = \varepsilon_{kab} A_a B_b$$

The generalization to N dimensions is this

$$\mathbf{q} = \mathbf{A} \times \mathbf{B} \times \mathbf{C} \times \ldots \times \mathbf{X}$$

$$q_k = \varepsilon_{kabc..x} A_a B_b C_c \ldots X_x$$

There must be N-1 vectors crossed together, so there will then be N-2 cross's. If N were 5, we would use **X** = **D**, the 4th letter of the alphabet , but since we don't know N, we just use **X**.

A more proper notation would be to write

$$q_k = \varepsilon_{ki_1i_2i_3\ldots.i_{N-1}} A^{(1)}{}_{i_1} A^{(2)}{}_{i_2} \ldots. A^{(N-1)}{}_{i_{N-1}}$$

but this requires a lot more symbols which cause much clutter as we continue, so we will try to stick with the simpler though less precise notation.

ε is the totally antisymmetric tensor in N dimensions (it has N indices), meaning that if any two indices are swapped, it changes sign.

The cross product notation suggests that **q** might be perpendicular to all the vectors from which it is composed, and this is in fact the case. For example, consider

$$\mathbf{q} \bullet \mathbf{B} = q_k B_k = \varepsilon_{kabc..x} A_a B_b C_c \ldots X_x B_k = \Sigma_{cd..}[\Sigma_{ab} B_b B_k \varepsilon_{kabc..x}] A_a C_c \ldots X_x$$

But $[\Sigma_{ab} B_b B_k \varepsilon_{kabc..x}] = 0$ since we have a symmetric object $B_b B_k$ contracted on both indices against an antisymmetric object. Therefore, $\mathbf{q} \bullet \mathbf{B} = 0$ and similarly q is perpendicular to the other vectors from which **q** is built.

We are then led to this obvious generalization of the formula for differential area in N dimensions,

$$d\mathbf{S}_1 = d\mathbf{s}_2 \times d\mathbf{s}_3 \times d\mathbf{s}_4 \ldots\ldots\ldots \times d\mathbf{s}_N$$
$$d\mathbf{S}_2 = d\mathbf{s}_1 \times d\mathbf{s}_3 \times d\mathbf{s}_4 \ldots\ldots\ldots \times d\mathbf{s}_N$$
$$\ldots.$$
$$dS_N = d\mathbf{s}_1 \times d\mathbf{s}_2 \times d\mathbf{s}_3 \times d\mathbf{s}_4 \ldots d\mathbf{s}_{N-1}$$

We can summarize these N equations by one equation

$$d\mathbf{S}_k = \Pi_{x;i\neq k}\, ds_i \qquad (1)$$

where the first subscript x of Π indicates "cross products", and the second tells us to omit $d\mathbf{s}_k$. [Please don't confuse this x with the x used above as an ε subscript.] If we omit the first subscript on a Π symbol, we imply regular multiplication of the factors.

The N vectors $d\mathbf{s}_i$ scaffold an N dimensional parallelogram in N-space, and each of these vectors can be written in terms of its tangent base vector $\mathbf{e}_i$

$$d\mathbf{s}_i = \mathbf{e}_i\ dq^i \qquad i = 1...N \qquad (2)$$

so we can write the area element this way

$$d\mathbf{S}_k = \Pi_{x;i\neq k}(x)\ d\mathbf{s}_i = (\Pi_{x;i\neq k}\ \mathbf{e}_i)\ (\Pi_{i\neq k}\ dq^i) \qquad (3)$$

For example, in N=5 dimensions we would have

$$d\mathbf{S}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_4 \times \mathbf{e}_5\ dq^1 dq^2 dq^4 dq^5$$

We are interested in the magnitude dS_k so we have to compute the magnitude^2 of $(\Pi_{x;i\neq k}\ \mathbf{e}_i)$:

$$(\Pi_{x;i\neq k}\ \mathbf{e}_i) \bullet (\Pi_{x;i\neq k}\ \mathbf{e}_i) = (\Pi_{x;i\neq k}\ \mathbf{e}_i)_j\ (\Pi_{x;i\neq k}\ \mathbf{e}_i)_j$$

Now using our cross product ε formula from above we can write

$$(\Pi_{x;i\neq k}\ \mathbf{e}_i)_j = \varepsilon_{j i_1 i_2 i_3 \ldots i_N}\ (\mathbf{e}_1)_{i_1}\ (\mathbf{e}_2)_{i_2}\ \ldots..\ (\mathbf{e}_N)_{i_N} \qquad //\ i_k \text{ and } (\mathbf{e}_k)_{i_k} \text{ missing}$$

where ε is missing index i_k and the product is missing $(\mathbf{e}_k)_{i_k}$ Then we have

$$(\Pi_{x;i\neq k}\ \mathbf{e}_i)_j\ (\Pi_{x;i\neq k}\ \mathbf{e}_i)_j =$$

$$(\varepsilon_{j i_1 i_2 i_3 \ldots . i_N}\ (\mathbf{e}_1)_{i_1}\ (\mathbf{e}_2)_{i_2}\ \ldots..\ (\mathbf{e}_N)_{i_N})(\varepsilon_{j m_1 m_2 m_3 \ldots . m_N}\ (\mathbf{e}_1)_{m_1}\ (\mathbf{e}_2)_{m_2}\ \ldots..\ (\mathbf{e}_N)_{m_N})$$

$$= \varepsilon_{j i_1 i_2 i_3 \ldots . i_N}\ \varepsilon_{j m_1 m_2 m_3 \ldots . m_N}\ [(\mathbf{e}_1)_{i_1}(\mathbf{e}_1)_{m_1}][(\mathbf{e}_2)_{i_2}(\mathbf{e}_2)_{m_2}]\ \ldots..\ [(\mathbf{e}_N)_{i_N}(\mathbf{e}_N)_{m_N}] \qquad (4)$$

$$//\ i_k \text{ and } (\mathbf{e}_k)_{i_k} \text{ missing AND } m_k \text{ and } (\mathbf{e}_m)_{i_m} \text{ missing}$$

Now we have to deal with this εε product.

For the case where nothing is "missing", we show elsewhere that if we contract the product of two ε symbols only on the first index, we get

$$\varepsilon_{ji_1i_2i_3....i_N}\,\varepsilon_{jm_1m_2m_3....m_N}$$

$$= \Sigma_P\ p\ P_2(\delta_{i_1,m_1}\delta_{i_2,m_2}\,\delta_{i_3,m_3}...\,\delta_{i_N,m_N}) \qquad p = \text{parity of P} \qquad (5)$$

By way of explanation, P_2 represents an arbitrary permutation P acting on the 2nd indices of the δ's. We might write such a permutation this way

$$P_2(\delta_{i_1,m_1}\delta_{i_2,m_2}\,\delta_{i_3,m_3}...\,\delta_{i_N,m_N}) = (\delta_{i_1,m_1}"\delta_{i_2,m_2}"\,\delta_{i_3,m_3}"...\,\delta_{i_N,m_N}")$$

where $P(m_1m_2m_3....m_N) = m_1"m_2"m_3"....m_N"$ with parity p

The parity of a permutation is $p = (-1)^S$ where S is the number of pairwise swaps it takes to get from the original order to the permuted order. Although there might be many pairwise swap sequences to do this, they will all have S even or all have S odd so the parity is well-defined for a given permutation P. Notice that the first term in the sum is obtained with P = 1 and p = 1 so we can write

$$\varepsilon_{ji_1i_2i_3....i_N}\,\varepsilon_{jm_1m_2m_3....m_N} = \delta_{i_1,m_1}\delta_{i_2,m_2}\,\delta_{i_3,m_3}...\,\delta_{i_N,m_N} + \text{other signed terms}$$

Hopefully the reader is convinced of the validity of (5) . Since we know that

$$\Sigma_P\ p\ P_2(M_{11}M_{22}M_{33}...M_{NN}) = \det(M)$$

we can "draw" $\varepsilon_{ji_1i_2i_3....i_N}\,\varepsilon_{jm_1m_2m_3....m_N}$ as the determinant of a certain matrix, but it takes a lot of space to draw such things, and the permutation notation seems more compact. Here is an example of such a determinant with simpler subscript notation

$$\varepsilon_{abc...x}\,\varepsilon_{ab'c'...x'} = \begin{vmatrix} \delta_{b,b'} & \delta_{b,c'} & .... & \delta_{b,x'} \\ \delta_{c,b'} & \delta_{c,c'} & .... & \delta_{c,x'} \\ & & & \\ \delta_{x,b'} & \delta_{x,c'} & .... & \delta_{x,x'} \end{vmatrix}$$

Now in our case, we have certain items "missing" , but the permutation sum formula "still works" and we can write

$$\varepsilon_{ji_1i_2i_3....i_N}\,\varepsilon_{jm_1m_2m_3....m_N} \qquad // \ i_k \text{ and } (\mathbf{e}_k)_{i_k} \text{ missing AND } m_k \text{ and } (\mathbf{e}_m)_{i_m} \text{ missing}$$

$$= \Sigma_P\ p\ P_2(\delta_{i_1,m_1}\delta_{i_2,m_2}\,\delta_{i_3,m_3}...\,\delta_{i_N,m_N}) \qquad p = \text{parity of P} \qquad (6)$$

where now there are only N-1 δ's in the product and δ_{i_k,m_k} is missing. The P operator is now acting on the second indices of the δ's, but there are only N-1 such indices, so P permutes N-1 variables, and as before the parity p is determined by P.

We now apply (6) to (4) to get

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i)_j\,(\Pi_{x;i\neq k}\, \mathbf{e}_i)_j \;=$$

$$(\varepsilon_{ji_1i_2i_3\ldots i_N}\;(\mathbf{e}_1)_{i_1}(\mathbf{e}_2)_{i_2}\;\ldots.\,(\mathbf{e}_N)_{i_N})(\varepsilon_{jm_1m_2m_3\ldots m_N}\;(\mathbf{e}_1)_{m_1}(\mathbf{e}_2)_{m_2}\;\ldots.\,(\mathbf{e}_N)_{m_N})$$

$$= \varepsilon_{ji_1i_2i_3\ldots i_N}\,\varepsilon_{jm_1m_2m_3\ldots m_N}\;[(\mathbf{e}_1)_{i_1}(\mathbf{e}_1)_{m_1}][(\mathbf{e}_2)_{i_2}(\mathbf{e}_2)_{m_2}]\;\ldots..\,[(\mathbf{e}_N)_{i_N}(\mathbf{e}_N)_{m_N}] \qquad (4)$$

// i_k and $(\mathbf{e}_k)_{i_k}$ missing AND m_k and $(\mathbf{e}_m)_{i_m}$ missing

$$= \Sigma_P\;p\;P_2(\delta_{i_1,m_1}\delta_{i_2,m_2}\,\delta_{i_3,m_3}\ldots\,\delta_{i_N,m_N})\;\;[(\mathbf{e}_1)_{i_1}(\mathbf{e}_1)_{m_1}][(\mathbf{e}_2)_{i_2}(\mathbf{e}_2)_{m_2}]\;\ldots..\,[(\mathbf{e}_N)_{i_N}(\mathbf{e}_N)_{m_N}]$$

$$= \Sigma_P\;p\;P_2\,[\,(\mathbf{e}_1\bullet\mathbf{e}_1)\,(\mathbf{e}_2\bullet\mathbf{e}_2)\,(\mathbf{e}_3\bullet\mathbf{e}_3)\ldots.\,(\mathbf{e}_N\bullet\mathbf{e}_N)]$$

$$= \Sigma_P\;p\;P_2\,[\,g'_{11}g'_{22}\,g'_{22}\ldots.g'_{nn}] \qquad \text{// } g'_{kk} \text{ is missing}$$

This is the determinant of the metric tensor submatrix obtained by crossing out the kth row and the kth column! The leading term is $g'_{11}g'_{22}\,g'_{22}\ldots.g'_{nn}$ where g'_{kk} is missing. Therefore in this very long-winded discussion we have shown that

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i)_j\,(\Pi_{x;i\neq k}\, \mathbf{e}_i)_j \;=\; \min(g'_{kk})$$

Each of the "other terms" has the same form as the first term, but the set of m_i labels has been permuted (and the i_i labels stay fixed). Such a term has a sign ± depending on whether this permutation is obtained by an even or odd number of index swaps from the original order. Including the first term, the total number of permutations is (N-1)!, so the portion "other terms" contains (N-1)! - 1 terms. For N=3, there is only one "other term".

Notice that no m_i in any term (including the first term) ever takes the value k

needs repair

Therefore, we have shown that

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i)\bullet(\Pi_{x;i\neq k}\, \mathbf{e}_i) \;=\; [\mathbf{e}_1\bullet\mathbf{e}_1]\,[\mathbf{e}_2\bullet\mathbf{e}_2]\ldots..\,[\mathbf{e}_N\bullet\mathbf{e}_N] \;+\text{ other terms}$$

where again $[\mathbf{e}_k \bullet \mathbf{e}_k]$ is missing. In those other terms, the second set of **e**'s is permuted. We recognize that dot products as metric tensor elements, so we have

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i) \bullet (\Pi_{x;i\neq k}\, \mathbf{e}_i) = g'_{11}\ g'_{22}\ldots..\ g'_{NN} + \text{other terms}$$

Now that we have a relatively simple product, we can account for the signs due to the swaps mentioned above with and ε symbol having N-1 indices. Thus

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i) \bullet (\Pi_{x;i\neq k}\, \mathbf{e}_i) = g'_{1m_1}\ g'_{2m_2}\ldots..\ g'_{Nm_N}\ \varepsilon_{m_1 m_2 m_3 \ldots . m_N} \qquad // \ m_k \text{ missing}$$

For example, if N=4 and k=2 then each index m_i in the implied sums only take the values m_i = 1,3,4 because we can never have

In the implied sums here over the m_i, we are listing off the permutations of $\{m_i...\}$ = { 1,2....N} where k is missing from the list. Therefore the sums are all of the form $\Sigma_{m_i \neq m_k}$. We can add this reminder to the notation by writing

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i) \bullet (\Pi_{x;i\neq k}\, \mathbf{e}_i) = \Sigma_{m_i \neq m_k}\ g'_{1m_1}\ g'_{2m_2}\ldots..\ g'_{Nm_N}\ \varepsilon_{m_1 m_2 m_3 \ldots . m_N} \qquad // \ m_k \text{ missing}$$

So there are two issues here.

(1) ε is missing index m_k and g'_{km_k} is missing from the product of g' objects.

(2) each sum excludes m_k

We have therefore shown that

$$d\mathbf{S}_k = \Pi_{x;i\neq k}(x)\ d\mathbf{s}_i = (\Pi_{x;i\neq k}\, \mathbf{e}_i)$$

$$dS_k = [\Sigma_{m_i \neq m_k}\ g'_{1m_1}\ g'_{2m_2}\ldots..\ g'_{Nm_N}\ \varepsilon_{jm_1 m_2 m_3 \ldots . m_N}]^{1/2} (\Pi_{i\neq k}\, dq^i) \qquad // \ m_k \text{ missing}$$

As a check on this result, for N=2 it says

$$dS_1 = [\Sigma_{m_i \neq 1} g'_{2m_2}\ g'_{3m_3}\ \varepsilon_{m_2 m_3}]^{1/2}\ dq^2\ dq^3$$

$$= [g'_{2m_2}\ g'_{3m_3}\ \varepsilon_{m_2 m_3}]^{1/2}\ dq^2\ dq^3$$

$$g'_{2m_2}\ g'_{3m_3}\ \varepsilon_{m_2 m_3}$$