(b1) Area. (general N)

We saw at the end of Section 11 the notion of a cross product in N dimensions. For example, in (6') we were able to express the reciprocal base vectors in terms of the tangent base vectors this way

$$\mathbf{e}^{\mathbf{i}} = [1/(N-1)!]\, \varepsilon^{\mathbf{iab}\cdots}\, \mathbf{e}_{\mathbf{a}} \times \mathbf{e}_{\mathbf{b}} \times \mathbf{e}_{\mathbf{c}} \ldots \ /v \qquad (6')$$

for example

$$e^{1} = \mathbf{e}_2 \times \mathbf{e}_3 \times \mathbf{e}_4 \ \ldots \times e_N / v$$

This suggests the following form for our differential area element #1 :

$$d\mathbf{S}_1 = d\mathbf{s}_2 \times d\mathbf{s}_3 \times d\mathbf{s}_4 \ldots \times d\mathbf{s}_N \qquad \text{where} \quad d\mathbf{s}_k = \mathbf{e}_k\, dq^k$$

where then $d\mathbf{S}_1$ is orthogonal to all the differential distance vectors from which it is formed, as shown at the end of Section 11. We could then write this as

$$d\mathbf{S}_1 = (dq^2\, \mathbf{e}_2) \times (dq^3 \mathbf{e}_3) \times (dq^4 \mathbf{e}_4) \ldots \times (dq^N\, e_N)$$

$$= (dq^2 dq^3 dq^4 \ldots dq^N)\, \mathbf{e}_2 \times \mathbf{e}_3 \times \mathbf{e}_4 \ \ldots \times e_N$$

$$= (dq^2 dq^3 dq^4 \ldots dq^N)\, [\, v\, \mathbf{e}^1]$$

The magnitude dS_1 can be determined from

$$(dS_1)^2 = d\mathbf{S}_1 \bullet d\mathbf{S}^1 = (dq^2 dq^3 dq^4 \ldots dq^N)^2\, v^2\, e^1 \bullet e_1 = (dq^2 dq^3 dq^4 \ldots dq^N)^2\, v^2$$

so that

$$dS_1 = (dq^2 dq^3 dq^4 \ldots dq^N)v = (dq^2 dq^3 dq^4 \ldots dq^N)\sqrt{\det(g')} = (\Pi_{i\neq 1}\, dq^i)\, \sqrt{\det(g')}$$

In general, then, we have

$$d\mathbf{S}_a = [1/(N-1)!]\, \varepsilon^{\mathbf{abc}\cdots}\, d\mathbf{s}_b \times d\mathbf{s}_c \ \ldots\ldots$$

$$dS_a = |\, d\mathbf{S}_a| = (\Pi_{i\neq k}\, dq^i)\, \sqrt{\det(g')}$$

In N dimensions for N≠3, the notation $\mathbf{q} = \mathbf{a} \times \mathbf{b}$ has no meaning so we must replace it with something that does have meaning. Consider the N=3 case again for two vectors called $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$. The cross product is written this way (repeated indices are summed unless stated otherwise)

$\mathbf{q} = \mathbf{a}^{(1)} \times \mathbf{a}^{(2)}$ $\qquad\qquad$ $\mathbf{q} = \mathbf{A} \times \mathbf{B}$

$q_k = \varepsilon_{ki_1i_2}\ a^{(1)}{}_{i_1}\, a^{(2)}{}_{i_2}$ $\qquad\qquad$ $q_k = \varepsilon_{kab}\, A_a\, B_b$

The generalization to N dimensions is this (there must be N-1 vectors crossed together, so there will then be N-2 cross's)

$\mathbf{q} = \mathbf{a}^{(1)} \times \mathbf{a}^{(2)} \times \mathbf{a}^{(3)} \times .... \times \mathbf{a}^{(N-1)}$ $\qquad\qquad$ $\mathbf{q} = \mathbf{A} \times \mathbf{B} \times \mathbf{C} \times ... \times \mathbf{X}$

$q_k = \varepsilon_{ki_1i_2i_3....i_{N-1}}\ a^{(1)}{}_{i_1}\, a^{(2)}{}_{i_2}\ ....\ a^{(N-1)}{}_{i_{N-1}}$ $\qquad\qquad$ $q_k = \varepsilon_{kabc..x}\, A_a\, B_b C_c \ldots X_x$

where this ε is the totally antisymmetric tensor in N dimensions (it has N indices), meaning that if any two indices are swapped, it changes sign. It is easy to show that this N-vector **q** is perpendicular to all the $\mathbf{a}^{(i)}$, just as we are used to having when N = 3:

$$\mathbf{a}^{(2)} \bullet q = a^{(2)}{}_k\, q_k = a^{(2)}{}_k\, \varepsilon_{ki_1i_2i_3....i_{N-1}}\ a^{(1)}{}_{i_1}\, a^{(2)}{}_{i_2}\ ....\ a^{(N-1)}{}_{i_N}$$

$$= \varepsilon_{ki_1i_2i_3....i_{N-1}}\ a^{(1)}{}_{i_1}\, [\, a^{(2)}{}_k\, a^{(2)}{}_{i_2}\,]\ ....\ a^{(N-1)}{}_{i_N}$$

In terms just of indices k and i_2 and their sums, we can thing of ε as being A_{ki_2} , an Antisymmetric tensor in these indices. Then we have

$$A_{ki_2}[\, a^{(2)}{}_k\, a^{(2)}{}_{i_2}\,] = -A_{i_2k}[\, a^{(2)}{}_k\, a^{(2)}{}_{i_2}\,] = -A_{ki_2}[\, a^{(2)}{}_k\, a^{(2)}{}_{i_2}\,]$$

and something that is minus itself must be 0. In the last step we just swapped the names of the dummy summation indices. In general, $A_{ij}S_{ij} = 0$ if A is antisymmetric and S is symmetric. Therefore, we have shown that $\mathbf{a}^{(2)} \bullet \mathbf{q} = 0$, and in general **q** is perpendicular to all the $\mathbf{a}^{(m)}$ which appear in the cross product.

We are then led to this obvious generalization of the formula for differential area in N dimensions

$$d\mathbf{S}_1 = d\mathbf{s}_2 \times d\mathbf{s}_3 \times d\mathbf{s}_4 \ldots. \times d\mathbf{s}_N$$

where as desired this area element is perpendicular to all the vector distances $d\mathbf{s}_k$ from which it is constructed. The general case can be written

$$d\mathbf{S}_k = \Pi_{x;\, i \neq k}(x)\ d\mathbf{s}_i$$

where the x indicates the mulitplicative operation indended by Π. As for N=3, the vectors $d\mathbf{s}_i$ skaffold an N dimensional parallelogram in N-space, and each of these vectors can be written in terms of its tangent base vector $\mathbf{e}_i$

$d\mathbf{s}_i = \mathbf{e}_i\ dq^i$ $\qquad\qquad$ i = 1...N

so we can write the area element this way

$$d\mathbf{S}_k = \Pi_{x;i\neq k}(x)\, ds_i = (\Pi_{x;i\neq k}\, \mathbf{e}_i)\,(\Pi_{i\neq k}\, dq^i)$$

For example, in N=5 dimensions we would have

$$d\mathbf{S}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_4 \times \mathbf{e}_5\; dq^1 dq^2 dq^4 dq^5$$

We are interested in the magnitude dS_k so we have to compute the magnitude^2 of $(\Pi_{x;i\neq k}\, \mathbf{e}_i)$:

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i) \bullet (\Pi_{x;i\neq k}\, \mathbf{e}_i) = (\Pi_{x;i\neq k}\, \mathbf{e}_i)_j\, (\Pi_{x;i\neq k}\, \mathbf{e}_i)_j$$

Now using our cross product ε formula from above we can write

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i)_j = \varepsilon_{j i_1 i_2 i_3 \ldots . i_N}\; (\mathbf{e}_1)_{i_1}\, (\mathbf{e}_2)_{i_2}\; \ldots . (\mathbf{e}_N)_{i_N}$$

where ε is missing index i_k and the product is missing $(\mathbf{e}_k)_{i_k}$; the reader will just have to keep this in mind so we don't have to show it in the notation. Then we have

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i)_j\, (\Pi_{x;i\neq k}\, \mathbf{e}_i)_j =$$

$$(\varepsilon_{j i_1 i_2 i_3 \ldots . i_N}\; (\mathbf{e}_1)_{i_1}\, (\mathbf{e}_2)_{i_2}\; \ldots . (\mathbf{e}_N)_{i_N})(\varepsilon_{j m_1 m_2 m_3 \ldots . m_N}\; (\mathbf{e}_1)_{m_1}\, (\mathbf{e}_2)_{m_2}\; \ldots . (\mathbf{e}_N)_{m_N})$$

$$= \varepsilon_{j i_1 i_2 i_3 \ldots . i_N}\, \varepsilon_{j m_1 m_2 m_3 \ldots . m_N}\; [(\mathbf{e}_1)_{i_1}(\mathbf{e}_1)_{m_1}][(\mathbf{e}_2)_{i_2}(\mathbf{e}_2)_{m_2}]\; \ldots . [(\mathbf{e}_N)_{i_N}(\mathbf{e}_N)_{m_N}]$$

Now we have to deal with this εε product. So we pause to consider:

Each of the "other terms" has the same form as the first term, but the set of m_i labels has been permuted (and the i_i labels stay fixed). Such a term has a sign ± depending on whether this permutation is obtained by an even or odd number of index swaps from the original order. Including the first term, the total number of permutations is (N-1)!, so the portion "other terms" contains (N-1)! - 1 terms. For N=3, there is only one "other term".

Notice that no m_i in any term (including the first term) ever takes the value k

needs repair

Therefore, we have shown that

$$(\Pi_{x;i\neq k}\, \mathbf{e}_i) \bullet (\Pi_{x;i\neq k}\, \mathbf{e}_i) = [\mathbf{e}_1 \bullet \mathbf{e}_1]\, [\mathbf{e}_2 \bullet \mathbf{e}_2] \ldots . [\mathbf{e}_N \bullet \mathbf{e}_N] \; + \text{other terms}$$

where again $[\mathbf{e}_k \bullet \mathbf{e}_k]$ is missing. In those other terms, the second set of **e**'s is permuted. We recognize that dot products as metric tensor elements, so we have

$(\Pi_{x;i\neq k}\, e_i) \bullet (\Pi_{x;i\neq k}\, e_i) = g'_{11}\ g'_{22}..... g'_{NN}$ + other terms

Now that we have a relatively simple product, we can account for the signs due to the swaps mentioned above with and ε symbol having N-1 indices. Thus

$(\Pi_{x;i\neq k}\, e_i) \bullet (\Pi_{x;i\neq k}\, e_i) = g'_{1m_1}\ g'_{2m_2}..... g'_{Nm_N}\ \varepsilon_{m_1 m_2 m_3 .... m_N}$ // m_k missing

For example, if N=4 and k=2 then each index m_i in the implied sums only take the values m_i = 1,3,4 because we can never have

In the implied sums here over the m_i, we are listing off the permutations of $\{m_i...\}$ = { 1,2....N} where k is missing from the list. Therefore the sums are all of the form $\Sigma_{m_i\neq m_k}$. We can add this reminder to the notation by writing

$(\Pi_{x;i\neq k}\, e_i) \bullet (\Pi_{x;i\neq k}\, e_i) = \Sigma_{m_i\neq m_k}\ g'_{1m_1}\ g'_{2m_2}..... g'_{Nm_N}\ \varepsilon_{m_1 m_2 m_3 .... m_N}$ // m_k missing

So there are two issues here.

(1) ε is missing index m_k and g'_{km_k} is missing from the product of g' objects.

(2) each sum excludes m_k

We have therefore shown that

$dS_k = \Pi_{x;i\neq k}(x)\, ds_i = (\Pi_{x;i\neq k}\, e_i)$

$dS_k = [\Sigma_{m_i\neq m_k}\ g'_{1m_1}\ g'_{2m_2}..... g'_{Nm_N}\ \varepsilon_{jm_1 m_2 m_3 .... m_N}]^{1/2} (\Pi_{i\neq k}\, dq^i)$ // m_k missing

As a check on this result, for N=2 it says

$dS_1 = [\Sigma_{m_i\neq 1} g'_{2m_2}\ g'_{3m_3}\ \varepsilon_{m_2 m_3}]^{1/2}\, dq^2\, dq^3$

$= [g'_{2m_2}\ g'_{3m_3}\ \varepsilon_{m_2 m_3}]^{1/2}\, dq^2\, dq^3$

$g'_{2m_2}\ g'_{3m_3}\ \varepsilon_{m_2 m_3}$

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Contradiction:

(1) In the N=3 area section I write

$$dS_i = \tfrac{1}{2}\,\varepsilon_{ijk}\,|\,\mathbf{e}_j \times \mathbf{e}_k\,|\,dq^j\,dq^k$$

We go off then and grab our vector identity,

$$(\mathbf{A} \times \mathbf{B}) \bullet (\mathbf{A} \times \mathbf{B}) = A^2B^2 - (\mathbf{A}\bullet\mathbf{B})^2$$

so that

$$|\,\mathbf{e}_j \times \mathbf{e}_k\,|^2 = (\mathbf{e}_j \times \mathbf{e}_k) \bullet (\mathbf{e}_j \times \mathbf{e}_k) = (e_j\bullet e_j)\,(e_k\bullet e_k) - (e_j\bullet e_k)^2 = g'_{jj}g'_{kk} - g'^2_{jk}$$

giving the result

$$dS_i = \tfrac{1}{2}\,\varepsilon_{ijk}\,\sqrt{g'_{jj}g'_{kk} - g'^2_{jk}}\,dq^j\,dq^k$$

(2) An example of the above would be

$$dS_1 = |\,\mathbf{e}_2 \times \mathbf{e}_3\,|\,dq^2\,dq^3 = \sqrt{g'_{22}g'_{33} - g'^2_{23}}\;dq^2\,dq^3$$

(3) But I can do this another way and get a different answer:

$$\mathbf{e}^1 \equiv \mathbf{e}_2 \times \mathbf{e}_3/v$$

$$|\,\mathbf{e}^1|^2 = e^1\bullet e^1 = g'^{11} \qquad \Rightarrow \qquad |\,\mathbf{e}^1| = \sqrt{g'^{11}}$$

And therefore

$$dS_1 = |\,\mathbf{e}_2 \times \mathbf{e}_3\,|\,dq^2\,dq^3 = v\,|\,\mathbf{e}^1|\,dq^2\,dq^3 = \sqrt{\det(g')}\,\sqrt{g'^{11}}\;dq^2\,dq^3$$

And this disagrees with result (2) above. This is an N=3 disagreement!!!

This is going to be another 20 hour problem, I see it coming.

Contradiction B. Boil it down to something simpler.

(1) On the one hand, we have

$$|\,\mathbf{e}_2 \times \mathbf{e}_3\,|^2 = (\mathbf{e}_2 \times \mathbf{e}_3) \bullet (\mathbf{e}_2 \times \mathbf{e}_3) = (e_2\bullet e_2)\,(e_3\bullet e_3) - (e_2\bullet e_3)^2 = g'_{22}g'_{33} - g'^2_{23}$$

(2) On the other hand, we have

$$|\,\mathbf{e}_2 \times \mathbf{e}_3\,|^2 = |\,v\;\mathbf{e}^1\;|^2 = v^2\,e^1\bullet e^1 = v^2\,g'^{11} = \det(g')\,g'^{11}$$

where det(g') has lower index g'_{ij} in it, at least that is how I derived it. Is it possible that

$g'^{11} =$