M&M Chapter 4,5 Ponderings PhL 3.26.05

Here I wrote down questions that occurred to me, and then my answers to these questions.

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Question #1: What is *meant* by the following question: "What does ∇xA look like in curvilinear coordinates? "

Answer #1. This is the wrong answer! But I did the math, so will keep it here. See Question #1.6 below, and see conclusions at the end of this answer. We are wondering here about the form invariance of the equation **B** = **∇**x**A** under an arbitrary transformation. We know the answer, but here it is anyway.

I compute the thing in Cartesian coordinates. I then do an arbitrary transform to some x' coordinates, and I just write the thing out then in those x' coordinates and that is what it "looks like".

Let's treat this as if we had this starting equation

$$\mathbf{B} = \nabla \mathrm{x} \mathbf{A}$$

and we want to get everything primed to see this in a "transformed frame". We have seen how this comes through with a simple rotation (see question 1.5 below), but here we shall do a general transformation.

Let's carry this out now. In Cartesians we know we have this:

$$\nabla \mathrm{x}\mathbf{A} = \hat{\mathbf{u}}_{(\mathbf{x})}(\partial_{\mathbf{y}} A^{\mathbf{z}} - \partial_{\mathbf{z}} A^{\mathbf{z}}) + \text{cyclic} \quad = \hat{\mathbf{u}}_{\mathbf{i}}\, \varepsilon_{\mathbf{ijk}}\, \partial_{\mathbf{j}} A^{\mathbf{k}}(\mathrm{x})$$

$$\mathbf{B} = B^{i}\,\hat{\mathbf{u}}_{i}$$

Here I am using my knowledge that we really mean contravariant indices on vectors, so I show this now explicitly in the above. And of course ∂_{j} means $\partial/\partial x^{j}$.

Now let's apply our usual "inverse transform rules" to show all this in primed coordinates. Those rules say the following, and I try to vary the name of the dummy index anticipating insertion into the above. Notice that the forward transform is $x'^{a} = T^{a}{}_{b}\, x^{b}$ so T has all its usual meanings.

$$x^{a} = T_{b}{}^{a}\, x'^{b} \qquad x_{a} = T^{b}{}_{a}\, x'_{b} \qquad T^{a}{}_{b} \equiv \frac{\partial x'^{a}}{\partial x^{b}} \qquad T_{a}{}^{b} \equiv \frac{\partial x^{b}}{\partial x'^{a}}$$

$$(\hat{\mathbf{u}}_{i})^{a} = T_{b}{}^{a}\ (\mathbf{e}_{i})^{b} \qquad \partial_{j} = T^{c}{}_{j}\,\partial'_{c}$$

$$A^{k}(x) = T_{e}{}^{k}\, A'^{e}(x') \qquad \varepsilon_{ijk} = (1/\sqrt{g}\,)\,\varepsilon'_{ijk} \qquad \text{(see question \#2 below)}$$

Now, in the starting Cartesian frame we have the obvious basis unit vectors $\hat{\mathbf{u}}_{i}$, but we know that in our transformed frame we will have the $\mathbf{e}_{i}$ basis vectors (non-orthogonal and non-unit), where the reverse transformation is this

$$(\hat{\mathbf{u}}_{i})^{a} = T_{b}{}^{a}\ (\mathbf{e}_{i})^{b}$$

as shown in the above equation set.

Thus we start off with

$$B^{i}\,\hat{\mathbf{u}}_{i}{}^{a} = \varepsilon_{ijk}\,\partial_{j}(A^{k})\,\hat{\mathbf{u}}_{i}{}^{a}$$

$$B^{i} = \varepsilon_{ijk}\,\partial_{j}(A^{k}) \qquad \text{// starting point}$$

and we make all these replacements:

$$B^{i} = T_{b}{}^{i}\, B'^{b}$$
$$\varepsilon_{ijk} = (1/\sqrt{g}\,)\,\varepsilon'_{ijk}$$
$$\partial_{j} = T^{e}{}_{j}\,\partial'_{e}$$
$$A^{k} = T_{r}{}^{k}\, A'^{r}$$

So install all these things to get:

$$T_{b}{}^{i}\, B'^{b} = (1/\sqrt{g'}\,)\,\varepsilon'_{ijk}\, T^{e}{}_{j}\,\partial'_{e}(T_{r}{}^{k}\, A'^{r})$$

Now we want to clear the T from the left side, so apply $T^{s}{}_{i}$ and sum on i to get $\delta_{b}{}^{s}$ so we get

$$B'^{s} = (1/\sqrt{g'})\, T^{s}{}_{i}\ \varepsilon'_{ijk}\, T^{e}{}_{j}\, \partial'_{e}(T_{r}{}^{k}\, A'^{r}) \qquad \text{"question 1 form"}$$

I now know the **correct answer** to the question which is this:

$$B'^{s} = (1/\sqrt{g'})\, \varepsilon'^{sjk}\, \partial'_{j}(\, g'_{ka} A'^{a}) \qquad \text{"M\&M form"}$$

where I put primes on everything (x' = q). In this last "correct" result, vectors A' and B' are those expanded directly on the e_i (which are not unit vectors).

It is hard to imagine that these results agree. We would have to show the following:

$$T^{s}{}_{i}\ \varepsilon'_{ijk}\, T^{e}{}_{j}\, \partial'_{e}(T_{r}{}^{k}\, A'^{r}) = \varepsilon'_{sek}\, \partial'_{e}(\, g'_{kr} A'^{r})$$

Let's now remove all primes keeping in mind that we are really talking q-space q = x';

$$T^{s}{}_{i}\ \varepsilon_{ijk}\, T^{e}{}_{j}\, \partial_{e}(T_{r}{}^{k}\, A^{r}) = \varepsilon_{sek}\, \partial_{e}(\, g_{kr} A^{r})$$

I do know the following "covariant forwards" transform

$$Q'_{ab} = T_{a}{}^{c}\ T_{b}{}^{d}\, Q_{cd}$$

so apply this to the metric tensor to ge

$$g'_{ab} = T_{a}{}^{c}\ T_{b}{}^{d}\, \delta_{cd} = T_{a}{}^{c}\ T_{b}{}^{c}$$

$$\Rightarrow\ g'_{kr} = T_{k}{}^{t}\ T_{r}{}^{t}$$

which could be installed on the RHS to give

$$T^{s}{}_{i}\ \varepsilon_{ijk}\, T^{e}{}_{j}\, \partial_{e}(T_{r}{}^{k}\, A^{r}) = \varepsilon_{sek}\, \partial_{e}(T_{k}{}^{t}\, T_{r}{}^{t}\, A^{r})$$

This looks unlikely because we have three T on the left and only two T on the right. But let's continue until we know for sure we are dead meat:

Let's swap the t and k summation index names on the RHS

$$T^{s}{}_{i}\ \varepsilon_{ijk}\, T^{e}{}_{j}\, \partial_{e}(T_{r}{}^{k}\, A^{r}) = \varepsilon_{set}\, \partial_{e}(T_{t}{}^{k}\, T_{r}{}^{k}\, A^{r})$$

LHS = RHS

Now blow out all the derivatives on both sides

$$LHS = T^{s}{}_{i}\ \varepsilon_{ijk}\, T^{e}{}_{j}\, \partial_{e}(T_{r}{}^{k})\, A^{r} + T^{s}{}_{i}\ \varepsilon_{ijk}\, T^{e}{}_{j}\, T_{r}{}^{k}\, \partial_{e}(A^{r})$$

$$RHS = \varepsilon_{set}\, \partial_{e}(T_{t}{}^{k})\, T_{r}{}^{k}\, A^{r} + \varepsilon_{set}\, T_{t}{}^{k}\, \partial_{e}(T_{r}{}^{k})\, A^{r} + T^{s}{}_{i}\ \varepsilon_{ijk}\, T^{e}{}_{j}\, T_{r}{}^{k}\, \partial_{e}(A^{r})$$

Inserted comment: suppose T is constant in space. then have
$T^{s}{}_{i}\ \varepsilon_{ijk}\ T^{e}{}_{j}\ T_{r}{}^{k}\ \partial_{e}(A^{r}) = T^{s}{}_{i}\ \varepsilon_{ijk}\ T^{e}{}_{j}\ T_{r}{}^{k}\ \partial_{e}(A^{r})$ = true.

Now A is supposed to be an arbitrary vector field, so the A^{r} and $\partial_{e}(A^{r})$ must separately be equal,

$$T^{s}{}_{i}\ \varepsilon_{ijk}\ T^{e}{}_{j}\ \partial_{e}(T_{r}{}^{k})\ A^{r} = \varepsilon_{set}\ \partial_{e}(T_{t}{}^{k})\ T_{r}{}^{k}\ A^{r} + \varepsilon_{set}\ T_{t}{}^{k}\ \partial_{e}(T_{r}{}^{k})\ A^{r} \qquad (1)$$

$$T^{s}{}_{i}\ \varepsilon_{ijk}\ T^{e}{}_{j}\ T_{r}{}^{k}\ \partial_{e}(A^{r}) = \varepsilon_{set}\ T_{t}{}^{k}\ T_{r}{}^{k}\ \partial_{e}(A^{r}) \qquad (2)$$

Inserted comment: suppose T is constant in space. both are true.
since we revert to $T_{t}{}^{k}\ T_{r}{}^{k} = g_{tr} = \delta_{tr}$ so (2)R = $\varepsilon_{ser}\ \partial_{e}(A^{r})$,
and (2)L will make a detT thing.

But again, since they are arbitrary, we need this to be true

$$T^{s}{}_{i}\ \varepsilon_{ijk}\ T^{e}{}_{j}\ \partial_{e}(T_{r}{}^{k}) = \varepsilon_{set}\ \partial_{e}(T_{t}{}^{k})\ T_{r}{}^{k} + \varepsilon_{set}\ T_{t}{}^{k}\ \partial_{e}(T_{r}{}^{k}) \qquad (1)$$

$$T^{s}{}_{i}\ \varepsilon_{ijk}\ T^{e}{}_{j}\ T_{r}{}^{k} = \varepsilon_{set}\ T_{t}{}^{k}\ T_{r}{}^{k} \qquad (2)$$

Inserted comment: Again, will say ε = ε, so (2) is still true for rotation

The simplest of these is (2), and the number of T's is just **wrong**! We can absorb three T's into nothing sometimes as a det, and we might absorb two into a g, but I don't see how this can possibly balance 3T against 2T. Something is not right.

Comments: In question 1.6 below, I show that the very simple equation **B** = **C** x **A** is form-invariant under the transformation T *only* if T is a rotation, which means $g_{ij} = \delta_{ij}$. Therefore, we can think without much stretch that **B** = ∇x**A** *might* retain its form in the primed system *under* these same circumstances, or it might not.

If yes, we can surely show that our two results (question 1 and M&M) above agree:

question 1: $B'^{s} = (1/\sqrt{g'})\ T^{s}{}_{i}\ \varepsilon'_{ijk}\ T^{e}{}_{j}\ \partial'_{e}(T_{r}{}^{k}\ A'^{r})$

MM: $B'^{s} = (1/\sqrt{g'})\ \varepsilon'^{sjk}\ \partial'_{j}(\ g'_{ka} A'^{a})$

We have g' = 1 and $g'_{ka} = \delta_{ka}$ so write as

question 1: $B'^{s} = T^{s}{}_{i}\ \varepsilon_{ijt}\ T^{e}{}_{j}\ \partial_{e}(T_{k}{}^{t}\ A^{k})$

MM: $B'^{s} = \varepsilon^{sek}\ \partial_{e}(A^{k})$

where we shuffled index names and removed primes just to make it easier to fiddle with. The above equations are the same if T is the identity, at least we have that much!

We can write the question 1 version as two terms:

$$B'^{s} = T^{s}{}_{i}\ \varepsilon_{ijt}\, T^{e}{}_{j}\, \partial_{e}(T_{k}{}^{t})\, A^{k} + T^{s}{}_{i}\ \varepsilon_{ijt}\, T^{e}{}_{j}\, T_{k}{}^{t}\, \partial_{e}(A^{k})$$

$$= (\varepsilon_{ijt}\, T^{s}{}_{i}\ T^{e}{}_{j})\, \partial_{e}(T_{k}{}^{t})\, A^{k} + (\varepsilon_{ijt}\, T^{s}{}_{i}\, T^{e}{}_{j}\, T_{k}{}^{t})\, \partial_{e}(A^{k})$$

Now write

$$T_{k}{}^{t} = (T^{-1})^{t}{}_{k} = (T^{-1,T})^{k}{}_{t} = T^{k}{}_{t}$$

if T unitary. Then we have for our question 1 form

$$= (\varepsilon_{ijt}\, T^{s}{}_{i}\ T^{e}{}_{j})\, \partial_{e}(T^{k}{}_{t})\, A^{k} + (\varepsilon_{ijt}\, T^{s}{}_{i}\, T^{e}{}_{j}\, T^{k}{}_{t}\ )\, \partial_{e}(A^{k})$$

$$= \varepsilon^{sek}\, \partial_{e}(A^{k}) + (\varepsilon_{ijt}\, T^{s}{}_{i}\ T^{e}{}_{j})\, \partial_{e}(T^{k}{}_{t})\, A^{k}$$

since t unimodular. So the first term is what we want, but now we have picked up a second term due to the possible spatial dependence of our rotation ! Here is the extra term where I restore primes.

$$\text{extra} = (\varepsilon_{ijt}\, T^{s}{}_{i}\ T^{e}{}_{j})\, \partial'_{e}(T^{k}{}_{t})\, A'^{k} \qquad\qquad T^{a}{}_{b} \equiv \frac{\partial x'^{a}}{\partial x^{b}}$$

so our derivative does not have any special symmetry, and we cannot argue extra = 0.

Conclusion: We now know that $\mathbf{B} = \nabla \mathrm{x} \mathbf{A}$ is form invariant only under a rotation T which is the same at all points x' so there is no derivative. If this last were not the case, we would only have a rotation at one local point anyway, so they really go together.

So, here is what this says. Suppose we have a "vector identity" $\mathbf{B} = \nabla \mathrm{x} \mathbf{A}$. If we do a transformation from the Cartesian world to, say, spherical coordinates even (which are orthogonal), we do not end up with the equation $\mathbf{B'} = \nabla' \mathrm{x} \mathbf{A'}$. But of course we knew that already. You have to first *go* into the new coordinate system, and then *compute the curl in that system from the definition* of the curl! That definition is *operationally* the same in both Spherical and Cartesian coordinates. This is pretty much what M&M do in equation 5-22a on page 197, and of course what I did in "divergence...doc"

Question #1.4 Show that A = B x C is form invariant under rotations.

This was much harder than I thought it would be: Here we go. Start here

$$A_{i} = \varepsilon_{ijk} B_{j} C_{k}$$

Then install $A_{i} = R_{ia} A'_{a}$ in three places

$$R_{ia} A'_{a} = \varepsilon_{ijk}\, R_{jb} B'_{b}\, R_{kc} C'_{c}$$

Now apply R_{ei} and sum i on the left to get

$$A'_{e} = R_{ei}\, \varepsilon_{ijk}\, R_{jb} B'_{b}\, R_{kc} C'_{c} = \varepsilon_{ijk}\, R_{ei} R_{jb} R_{kc}\, B'_{b} C'_{c}$$

Now we make our familiar observation that

$\varepsilon_{ijk} R_{ei} R_{jb} R_{kc} = F_{ebc}$ = totally antisym = $\varepsilon_{ebc} F_{123}$ so we get

$A'_e = \varepsilon_{ebc} F_{123} B'_b C'_c = \varepsilon_{ebc} \varepsilon_{ijk} R_{1i} R_{2b} R_{3c} B'_b C'_c = \varepsilon_{ebc} \det(R) B'_b C'_c = \varepsilon_{ebc} B'_b C'_c$

Therefore, in the new frame we have **A'** = **B'** x **C'**, as expected.

So this is an example of an equation that is form-invariant under a rotation.

Question #1.5. Show that the equation A =∇xC is form invariant under rotations. There is a very slight change here, but we get the result we want. Let's look at how ∇ transforms

$dx_i = R_{ia} dx'_a$ // from above

$dx'_j = R^{-1}_{ji} dx_i$

$d/dx_i = dx'_j/dx_i * d/dx'_j = R^{-1}_{ji} \; d/dx'_j$

$\partial_i = R^{-1}_{ji} \partial'_j = R^{T}_{ji} \partial'_j = R_{ij} \; \partial'_j$ // R is unitary

So we can compare

$\partial_i = R_{ij} \; \partial'_j$
$B_i = R_{ij} B'_j$

Because R is unitary (real orthogonal), we find that ∇ transforms as a vector just like B, so our result is the same, and in the new frame we find

A' =∇'x C'

So this was an example of a very simple "vector identity" that is form invariant under rotations.

Question #1.6 Show that A = B x C is NOT form invariant under an arbitrary transformation!

Start with the inverse transforms

$A^i = T_a{}^i \, A'^a$
$B^j = T_b{}^j \, B'^b$
$C^k = T_c{}^k \, C'^c$

$A^i = \varepsilon^{ijk} B^j C^k$ =>

$$T_a^{\,i}\, A'^{a} = \varepsilon^{ijk}\, T_b^{\,j}\, B'^{b}\, T_c^{\,k}\, C'^{c}$$

Now apply $T^{s}_{\ i}$ to the left and sum on i to get $\delta_a^{\,s}$ so that we have

$$A'^{s} = T^{s}_{\ i}\, \varepsilon^{ijk}\, T_b^{\,j}\, B'^{b}\, T_c^{\,k}\, C'^{c}$$

which we can rewrite as

$$A'^{s} = \varepsilon^{ijk}\ T^{s}_{\ i}\, T_b^{\,j}\, T_c^{\,k}\ B'^{b}\, C'^{c}$$

The first T is not quite right for our det thing, so what can we do about that? We know that

$$T^{i}_{\ s} = (T^{-1})_s^{\,i} \Rightarrow$$
$$T^{s}_{\ i} = (T^{-1,T})_s^{\,i}$$

So we then have

$$A'^{s} = \varepsilon^{ijk}\, (T^{-1,T})_s^{\,i}\, T_b^{\,j}\, T_c^{\,k}\ B'^{b}\, C'^{c}$$

Now *if it were true* that $T^{-1} = T^{T}$ we could conclude that

$$A'^{s} = \varepsilon^{ijk}\, T_s^{\,i}\, T_b^{\,j}\, T_c^{\,k}\ B'^{b}\, C'^{c} = F_{sbc}\, B'^{b}\, C'^{c} = \varepsilon_{sbc} F_{123}\, B'^{b}\, C'^{c}$$

$$= \varepsilon_{sbc}\, \varepsilon^{ijk}\, T_1^{\,i}\, T_2^{\,j}\, T_3^{\,k}\ B'^{b}\, C'^{c} = \det(T_a^{\,b})\, \varepsilon_{sbc}\, B'^{b}\, C'^{c}$$

Conclusion: The equation **A** = **B** x **C** is form invariant under $A^{i} = T_a^{\,i}\, A'^{a}$ *only if* the following things are true about $T \equiv T_a^{\,b}$

(1) T is real orthogonal so $T^{-1} = T^{T}$
(2) T is unimodular so det(T) = 1

In other words, this equation **A** = **B** x **C** is form invariant only of T is a rotation. Under a general transformation, it is not form invariant.

Question #2: Is ε^{ijk} really a tensor? Transforms like one? What is ε'^{ijk} in the transformed frame?

To answer this question, let's "give it the test" :

$$\varepsilon'_{ijk} = T_i^{\,a}\ T_j^{\,b}\, T_k^{\,c}\, \varepsilon_{abc}$$

We can see from inspection that ε'_{ijk} is totally antisymmetric, so we can write it as (see Question #3)

$$\varepsilon'_{ijk} = \varepsilon_{ijk}\, \varepsilon'_{123} = \varepsilon_{ijk}\, T_1^{\,a}\ T_2^{\,b}\, T_3^{\,c}\, \varepsilon_{abc} = \varepsilon_{ijk}\, |T^{a}_{\ b}| = \sqrt{g}\ \varepsilon_{ijk}$$

So I think the answer is this: Yes, ε_{123} is a tensor, and in any frame it is totally antisymmetric, but it picks up a factor of $\sqrt{g}$ in scale as you move from the Cartesian frame to another frame.

Question #3: How does "the Jacobian" fit into things?

We can write the Jacobian J in this way

$$J = \det\left(\frac{\partial x^b}{\partial x'^a}\right) = \det(T_a{}^b) \qquad \text{and} \qquad dV = \text{jac}\, dV'$$

In the support document I have proven this fact:

$$J = \sqrt{g}$$

as expected. So we now have three different ways to think about the volume change

$$J = \sqrt{g} = v$$

where

$$J = |T_a{}^b| \qquad g = |g_{ij}| \qquad v = \mathbf{e}_1 \bullet (\mathbf{e}_2 \times \mathbf{e}_3)$$

So this was a good question to ask I think.,

Question #4: How does the volume element transform, and what does this even mean?

See previous question where all three forms are shown. You would use this in transforming a volume integral from one set of coordinates to another.

Question #5: What is det(T) for an arbitrary transformation?

I don't think we have any idea what this is. For a rotation it is 1 since $TT^T = 1$ which says $\det T = \pm 1$. I think we again must conclude that T must be real orthogonal in order to have det(T+ 0 1

Question #6: Are the vector identities true in all coordinates? What does this question mean ?

Ouch, this is the tough one. Here is one possible answer:

Answer A: All the vector identities are true in an arbitrary curvilinear system *as long as* you use the correct forms for these objects: grad, div, Laplacian, curl, $A \bullet \nabla$ *and as long as* you interpret the vectors as

being distributed onto the curvilinear basis vectors meaning $\mathbf{A} = A^i\mathbf{e}_i = A_i\mathbf{e}^i$. You never want to work with a Cartesian component of any vector if you are in curvilinear coordinates!

We interpret the identities as being about the *concepts* of these operators, so they are always true.

But I have my doubts. Let's try to prove some vector identities !!!

Test #1: $\nabla\bullet(\varphi\mathbf{F}) = \mathbf{F}\bullet\nabla\varphi + \varphi\nabla\bullet\mathbf{F}$

Verify in curvilinear coordinates this vector identity:

$$\nabla\bullet(\varphi\mathbf{F}) = \mathbf{F}\bullet\nabla\varphi + \varphi\nabla\bullet\mathbf{F}$$

Now let's bring in our general formulas for grad and div:

$$\nabla\bullet\,\mathbf{A} = (1/\sqrt{g})\,\partial_i(\sqrt{g}\,A^i) \qquad \text{where } \mathbf{A} = A^i\mathbf{e}_i$$

$$\nabla(\varphi) = (\partial_i\varphi)\,\mathbf{e}^i$$

So here we go

$$\text{LHS} = \nabla\bullet(\varphi\mathbf{F}) = (1/\sqrt{g})\,\partial_i(\sqrt{g}\,\varphi F^i)$$

$$\text{RHS} = \mathbf{F}\bullet\nabla\varphi + \varphi\nabla\bullet\mathbf{F}$$
$$= \mathbf{F}\bullet(\partial_i\varphi)\,\mathbf{e}^i + \varphi\,(1/\sqrt{g})\,\partial_i(\sqrt{g}\,F^i)$$

$$= F^i(\partial_i\varphi) + \varphi\,(1/\sqrt{g})\,\partial_i(\sqrt{g}\,F^i)$$

Now expand the LHS

$$\text{LHS} = (1/\sqrt{g})\,\partial_i(\varphi\sqrt{g}\,F^i) = (1/\sqrt{g})\,\partial_i(\varphi)\sqrt{g}\,F^i + (1/\sqrt{g})\,\varphi\partial_i(\sqrt{g}\,F^i)$$

$$= \partial_i(\varphi)\,F^i + \varphi\,(1/\sqrt{g})\,\partial_i(\sqrt{g}\,F^i)$$

And sure enough, we have the LHS = RHS and this vector identity is confirmed in a general curvilinear system.

Test #2: $\nabla\bullet(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\bullet(\nabla\times\mathbf{A}) - \mathbf{A}\bullet(\nabla\times\mathbf{B})$

Verify in curvilinear coordinates this vector identity:

$$\nabla\bullet(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\bullet(\nabla\times\mathbf{A}) - \mathbf{A}\bullet(\nabla\times\mathbf{B})$$

Here are our general formulas:

$$[\nabla x \mathbf{F}]^{i} = (1/\sqrt{g})\, \varepsilon^{ijk}\, \partial_j F_k$$

$$\nabla \bullet \mathbf{F} = (1/\sqrt{g})\, \partial_i(\sqrt{g}\, F^i) \qquad \text{where } \mathbf{F} = F^i \mathbf{e}_i$$

So away we go. Oops. First we need to ask: What is **A**x**B**?

$$\mathbf{A}x\mathbf{B} = A_i \mathbf{e}^i \times B_k \mathbf{e}^k = A_i B_k\ \mathbf{e}^i \times \mathbf{e}^k$$

but we know that

$$\mathbf{e}^i \times \mathbf{e}^j = (1/v)\, \varepsilon^{ijk}\, \mathbf{e}_k = (1/\sqrt{g})\, \varepsilon^{ijk}\, \mathbf{e}_k$$

so get

$$\mathbf{A}x\mathbf{B} = A_i B_j\ \mathbf{e}^i \times \mathbf{e}^j = (1/\sqrt{g})\, \varepsilon^{ijk}\, A_i B_j\, \mathbf{e}_k$$

So we get this important result we would use in any vector identity

$$\mathbf{A}x\mathbf{B} = (1/\sqrt{g})\, \varepsilon^{ijk}\, A_i B_j\, \mathbf{e}_k$$

Now we can say that

$$\mathbf{A}x\mathbf{B} = [\mathbf{A}x\mathbf{B}]^i\, \mathbf{e}_i =$$

Dot both sides with $\mathbf{e}^k$ to get

$$\mathbf{A}x\mathbf{B} \bullet \mathbf{e}^k = [\mathbf{A}x\mathbf{B}]^k$$

so that

$$[\mathbf{A}x\mathbf{B}]^a = \mathbf{A}x\mathbf{B} \bullet \mathbf{e}^a = (1/\sqrt{g})\, \varepsilon^{ijk}\, A_i B_j\, \mathbf{e}_k \bullet \mathbf{e}^a = (1/\sqrt{g})\, \varepsilon^{ija}\, A_i B_j$$

or

$$[\mathbf{A}x\mathbf{B}]^k = (1/\sqrt{g})\, \varepsilon^{ijk}\, A_i B_j$$

It sure took me a long time to get this simple result. But just compare it to the curl formula which can be written exactly this way:

$$[\nabla x \mathbf{B}]^k = (1/\sqrt{g})\, \varepsilon^{ijk}\, \partial_i B_j$$

and now it looks very very good, they have exactly the same form.

Now finally we can apply our divergence formula

$$\nabla \bullet \mathbf{F} = (1/\sqrt{g})\, \partial_k(\sqrt{g}\, F^k)$$

to get

$$\text{LHS} = \nabla \bullet (\mathbf{A} \times \mathbf{B}) = (1/\sqrt{g})\, \partial_k(\sqrt{g}\, [(1/\sqrt{g})\, \varepsilon^{ijk} A_i B_j])$$

$$= (1/\sqrt{g})\, \varepsilon^{ijk}\, \partial_k(A_i B_j)$$

Meanwhile, our curl formula is

$$[\nabla \times \mathbf{F}]^i = (1/\sqrt{g})\, \varepsilon^{ijk}\, \partial_j F_k$$

so we have

$$\text{RHS} = B_i [\nabla \times \mathbf{A}]^i - A_i [\nabla \times B]^i \qquad \text{// why one index up and one down for dot product ???}$$

$$= B_i\, (1/\sqrt{g})\, \varepsilon^{ijk}\, \partial_j A_k - A_i\, (1/\sqrt{g})\, \varepsilon^{ijk}\, \partial_j B_k$$

$$= (1/\sqrt{g})\varepsilon^{ijk}\, \{ B_i\, \partial_j A_k - A_i\, \partial_j B_k \} \qquad \text{// swap j and k in second term ....}$$

$$= (1/\sqrt{g})\varepsilon^{ijk}\, \{ B_i\, \partial_j A_k + A_i\, \partial_k B_j \} \qquad \text{// .... and away goes that minus sign}$$

Now the LHS is this:

$$\text{LHS} = (1/\sqrt{g})\, \varepsilon^{ijk}\, \partial_k(A_i B_j)$$

so our vector identity will be proven if we can show that

$$\varepsilon^{ijk}\, \partial_k(A_i B_j) = \varepsilon^{ijk}\, \{ B_i\, \partial_j A_k + A_i\, \partial_k B_j \}$$

Now in the first term on the RHS cycle all the dummy index names ahead one

$$\varepsilon^{ijk}\, \{ B_i\, \partial_j A_k \} = \varepsilon^{jki}\, \{ B_j\, \partial_k A_i \} = \varepsilon^{ijk}\, \{ B_j\, \partial_k A_i \}$$

so we now have to show that

$$\varepsilon^{ijk}\, \partial_k(A_i B_j) = \varepsilon^{ijk}\, \{ B_j\, \partial_k A_i + A_i\, \partial_k B_j \}$$

but this is obviously true and our vector theorem is finally verified!

Footnote to Test #2:

In trying to show Test #2, I was erroneously doing this kind of thing:

"But we will need to know the contravariant component of this, which is

$$[\mathbf{A}\mathrm{x}\mathbf{B}\,]^a = (1/\sqrt{g})\,\varepsilon^{ijk} A_i B_j\ (\mathbf{e}_k)^a$$

But what is this last factor?

$$(\mathbf{e}_k)^i\,(\mathbf{e}_j)^i = g_{kj} \qquad = \Sigma_i\, T_k{}^i\, T_j{}^i \qquad \text{// how do I know both have contravariant indices? Regular } \bullet \text{ ?}$$

which suggests that

$$(\mathbf{e}_k)^i = T_k{}^i$$

Do I believe this. We probably have this fact:

$$(\mathbf{e}_1)^i = T^i{}_k(\,\hat{\mathbf{u}}_1)^k = T^i{}_k\,\delta_1{}^k = T^i{}_1 \qquad \Rightarrow \qquad (\mathbf{e}_k)^i = T^i{}_k$$

We seem to disagree here on the tilt direction! Let's just pick one to use for now.

Then we have

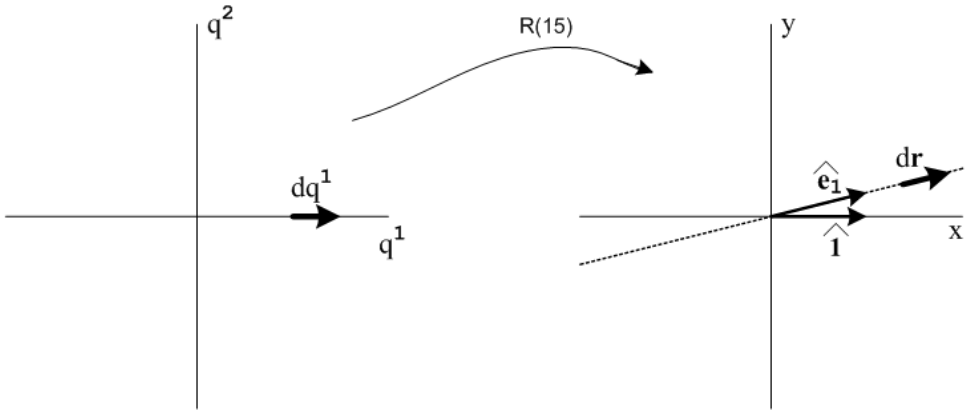
$$[\mathbf{A}\mathrm{x}\mathbf{B}\,]^a = (1/\sqrt{g})\,\varepsilon^{ijk} A_i B_j\ (\mathbf{e}^k)^a = (1/\sqrt{g})\,\varepsilon_{ijk} A_i B_j\, T^k{}_a$$

which seems rather unusual and painful. Now we have..."

This mistake here is that on the first line, I am specifying a *Cartesian* component $[\mathrm{AxB}]^a$, but what we really want of course is the *curvilinear system component* $[\mathrm{AxB}]^a$ and they are very different!

Question #7: What are the components of the various basis vectors?

Before attempting an answer this question, we have to clarify our various "spaces". For this purpose, let's consider a mapping which is a simple rotation. We draw this picture:



All the basis vectors $\mathbf{e}_i$ and $\mathbf{e}^i$ *and* the Cartesian unit vectors we are calling $\hat{\mathbf{i}}$ exist only in the Cartesian space on the right. A position change of $d\mathbf{q}^1$ in the q-space results in a position change $d\mathbf{r}$ in the x-space as shown. We see that a vector in q-space like $dq^1\hat{\mathbf{q}}^1$ is getting rotated $+15^0$ (RHR) where it maps into $d\mathbf{r}$. We write this rotation as R(15). This is the "active" rotation matrix if you wanted to write it out.

The above picture is based on this rule for mapping vectors:

$$x^i = R(15)^i{}_j q^i = R(15)^i{}_j x'^i$$

where we do our usual identification of x' = q. *However*, our standard transform notation which defines our transformation matrix T is this

$$x'^a = T^a{}_b x^b$$

Therefore it follows that

$$T^a{}_b = R(-15)^a{}_b$$

from which we conclude that

$$T = R^{-1}(+15) \qquad \text{and} \qquad T^{-1} = R(15)$$

Sometimes this inverse relationship can be confusing, but there it is.

Cartesian components of the basis vectors

In this section, we use square brackets like $[\mathbf{e}_i]^n$ to denote a Cartesian component.

Now, the definition of $\mathbf{e}_1$ given in equation A on M&M page 193 says, for example,

$$\mathbf{e}_1 = \partial\mathbf{r}/\partial q^1 \;=> \qquad \mathbf{e}_1 \bullet \hat{\mathbf{n}} = [\mathbf{e}_1]^n = \partial x^n/\partial q^1 = \partial x^n/\partial x'^1 = T_1{}^n$$

so we have found that our Cartesian components are given by

$$[\mathbf{e}_i]^n = T_i{}^n$$

Notice, making use of $T_a{}^b = (T^{-1})^b{}_a$ after the 4th equal sign, that:

$$[\mathbf{e}_1]^n = T_1{}^n = T_i{}^n \delta^i{}_1 = T_i{}^n (\hat{\mathbf{1}})^i = (T^{-1})^n{}_i (\hat{\mathbf{1}})^i = R(15)^n{}_i (\hat{\mathbf{1}})^i$$

so we recover the fact obvious from our picture that $\mathbf{e}_1 = R(15)\,\hat{\mathbf{1}}$. Thus we have

$$\mathbf{e}_k = R(15)\,\hat{\mathbf{k}} = T^{-1}\hat{\mathbf{k}}$$

So the covariant basis vectors $\mathbf{e_k}$ is just the Cartesian unit vector $\hat{\mathbf{k}}$ transformed by $T^{-1} = R(15)$ due to the way we have defined our transformation matrix T, as explained above.

Of course in Cartesian space where $g_{ij} = 1$, we have

$$[\mathbf{e_i}]^n = (T^{-1})^n{}_i = [\mathbf{e_i}]_n = T_i{}^n$$

So when we are talking about Cartesian components of our basis vectors e_i, it does not matter whether we put the index up or down.

What about the "dual" basis vectors $\mathbf{e^i}$? Well, consider

$$\delta^i{}_j = \mathbf{e^i} \bullet \mathbf{e_j} = [\mathbf{e^i}]_n [\mathbf{e_j}]_n = [\mathbf{e^i}]_n (T^{-1})^n{}_i$$

We can then apply from the right the factor $T^i{}_k$ and sum on i to get

$$T^j{}_k = [\mathbf{e^i}]_k = [\mathbf{e^i}]^k$$

We can summarize our Cartesian component results as follows:

$$[\mathbf{e_i}]_n = [\mathbf{e_i}]^n = T_i{}^n = T_{in}$$

$$[\mathbf{e^i}]_n = [\mathbf{e^i}]^n = T^i{}_n = T^{in}$$

See discussion in Question #8 below for why second index on T can be up or down with no difference. It exactly makes sense in this context! Remember that T is not a tensor so these Cartesian component objects are not tensors.

Notice that

$$\mathbf{e_i} \bullet \mathbf{e_j} = [\mathbf{e_i}]_n [\mathbf{e_j}]_n = T_i{}^n T_j{}^n = g_{ij} \qquad \text{// can put any n up or down for free}$$

so we have verified the idea that we usually only think about in terms of the basis vectors. A corresponding rule is this:

$$\mathbf{e^i} \bullet \mathbf{e^j} = [\mathbf{e^i}]_n [\mathbf{e^j}]_n = T^i{}_n T^j{}_n = g^{ij} \qquad \text{// can put any n up or down for free}$$

In fact, we can regard this as a tensor equation set

$$\mathbf{e_i} \bullet \mathbf{e_j} = T_i{}^n T_j{}^n = g_{ij}$$

Each of these three things is a way to write the same tensor g_{ij} and is of course a tensor.

Components of the basis vectors with respect to the basis vectors!

In this section, we use square brackets like $(\mathbf{e}_i)^n$ to denote basis vector component.

First, let's just collect facts obtained in divergence etc.doc

$\mathbf{e}_i \bullet \mathbf{e}_j = g_{ij}$ ` $\mathbf{e}^i \bullet \mathbf{e}^j = g^{ij} = G^{ij}/g$ G = cofactor(g) g = det(g)

$\mathbf{e}_i \bullet \mathbf{e}_i = g_{ii} = e_i^2 = Q_i^2$ $\mathbf{e}^i \bullet \mathbf{e}^i = g^{ii} = G^{ii}/g$ $v^2 = \det(g)$

$v \equiv \mathbf{e}_3 \bullet \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_1 \times \mathbf{e}_2 \bullet \mathbf{e}_3 = \mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3 = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3]$

$\mathbf{e}^1 \equiv \mathbf{e}_2 \times \mathbf{e}_3 / v$
$\mathbf{e}^2 \equiv \mathbf{e}_3 \times \mathbf{e}_1 / v$
$\mathbf{e}^3 \equiv \mathbf{e}_1 \times \mathbf{e}_2 / v$

$e^i \bullet e_j = \delta_{ij}$

$\mathbf{e}_1 = v\ \mathbf{e}^2 \times \mathbf{e}^3$
$\mathbf{e}_2 = v\ \mathbf{e}^3 \times \mathbf{e}^1$
$\mathbf{e}_3 = v\ \mathbf{e}^1 \times \mathbf{e}^2$

$1/v = \mathbf{e}^3 \bullet \mathbf{e}^1 \times \mathbf{e}^2 = \mathbf{e}^1 \times \mathbf{e}^2 \bullet \mathbf{e}^3 = \mathbf{e}^1 \bullet \mathbf{e}^2 \times \mathbf{e}^3 = [\mathbf{e}^1\ \mathbf{e}^2\ \mathbf{e}^3] = v'$

Now, expand these basis vectors onto the basis vectors as follows:

$\mathbf{e}_i = (\mathbf{e}_i)^k\ \mathbf{e}_k$ $\Rightarrow \mathbf{e}_i \bullet \mathbf{e}^n = (\mathbf{e}_i)^k\ \mathbf{e}_k \bullet \mathbf{e}^n \Rightarrow g_i^{\ n} = (\mathbf{e}_i)^n$

$\mathbf{e}_i = (\mathbf{e}_i)_k\ \mathbf{e}^k$ $\Rightarrow \mathbf{e}_i \bullet \mathbf{e}_n = (\mathbf{e}_i)_k\ \mathbf{e}^k \bullet \mathbf{e}_n \Rightarrow g_{in} = (\mathbf{e}_i)_n$

$\mathbf{e}^i = (\mathbf{e}^i)^k\ \mathbf{e}_k$ $\Rightarrow \mathbf{e}^i \bullet \mathbf{e}^n = (\mathbf{e}^i)^k\ \mathbf{e}_k \bullet \mathbf{e}^n \Rightarrow g^{in} = (\mathbf{e}_i)^n$

$\mathbf{e}^i = (\mathbf{e}^i)_k\ \mathbf{e}^k$ $\Rightarrow \mathbf{e}^i \bullet \mathbf{e}_n = (\mathbf{e}^i)_k\ \mathbf{e}^k \bullet \mathbf{e}_n \Rightarrow g^i_{\ n} = (\mathbf{e}^i)_n$

All this can be summarized in this single statement:

$(\mathbf{e}_i)_j = g_{ij}$ then fiddle the indices any way you want.

Thus, the component $(\mathbf{e}_i)_j$ is just a representation of the metric tensor g_{ij} and as such is itself a rank 2 tensor. This is in stark contrast to the Cartesian component $[\mathbf{e}_i]_j$ which is not a tensor as was shown above. So we end up with this interesting situation

$\mathbf{e}_i \bullet \mathbf{e}_j = (\mathbf{e}_i)_j = g_{ij}$

This is really much ado about nothing, but I was confused about "components" and here we have done them every which way.

Question #8. Is $T^a{}_b$ a Tensor? How do you raise and lower indices on $T^a{}_b$?

Consider the transformation of a tensor X^{ij}going from Cartesian space to some x' space, so

$$X'^{ab} = T^a{}_i \; T^b{}_j \; X^{ij}$$

In the Cartesian space our metric tensor is $g_{ij} = \delta_{ij}$. In the x' space, the metric tensor is something we shall call $g'_{ij} \neq \delta_{ij}$ In fact we know that

$$g'^{ab} = T^a{}_i \; T^b{}_j \; \delta^{ij}$$

and this transforms the metric tensor itself from the Cartesian space to the x' space.

Now here is the question: *which* metric tensor should we use to raise and lower indices on tensors? I would like to say this: for a tensor like X'^{ab} which is in the x' space, use g'_{ij} to raise and lower indices. And for a tensor like X^{ij} , use δ_{ij} to raise and lower indices. According to this idea, we would have

$$X_{ij} = X_i{}^j = X_{ij} = X^{ij}$$

and I think this is correct. We are used to saying for a vector that $V_i = V^i$ in a Cartesian space. So this is the equivalent thing for a rank-2 tensor.

Then here is the zinger: *which* metric tensor do you use to raise and lower indices on $T^a{}_b$??? Or is this thing itself not really a tensor?

In looking for an answer, consider this:

$$T_a{}^b \equiv \frac{\partial x^b}{\partial x'^a} \qquad\qquad T^a{}_b \equiv \frac{\partial x'^a}{\partial x^b}$$

I think the answer is going to be this:

(1) for $T_a{}^b$ you use g' on the first index, but δ on the second index, just looking at the definition as a partial derivative. Then we would say

$$T_a{}^b = T_{ab} \qquad \text{but} \qquad T^{ab} = g'^{ak} \, T_a{}^b$$

2) for $T^a{}_b$ the same is true, the second index can be up or down

$$T^a{}_b = T^{ab} \qquad \text{but} \qquad T_{ab} = g'_{ak} \; T^k{}_b$$

Conclusion: For either $T_{\mathtt{a}}{}^{\mathtt{b}}$ or $T^{\mathtt{a}}{}_{\mathtt{b}}$, the second index goes with $dx^{\mathtt{b}} = dx_{\mathtt{b}}$ and so it can be either up or down with no change. This is only true if T is a transformation from Cartesian space to x' space.

More generally, for T being a transformation from x-space to x'-space, the first index of $T_{\mathtt{a}}{}^{\mathtt{b}}$ or $T^{\mathtt{a}}{}_{\mathtt{b}}$ is raised and lowered by the g' of the x' space, and the second index is raised or lowered by the g of the x space. This is a special object that has one "leg" in each space, so it is NOT a rank-2 tensor.