M&M Chapters 4 & 5, support items PhL 9.1.08

1. Brute Force Proof that $V_{b,a}$ is a tensor 1
   Comments on Proof of Covariance of $V_{b,a}$ 1
2. Proof that jac = $\sqrt{g}$ 10

1. Brute Force Proof that $V_{b,a}$ is a tensor

I took these raw notes out of the main Chapter notes file. I finally got the proof done, it was not easy. I copied from the main file to here my summary of these raw notes, then come the raw notes:

Comments on Proof of Covariance of $V_{b,a}$

I did a brute force proof on my own to show that $V_{b,a}$ transforms like a tensor (see M&M Chapter 5 "support" doc). The proof was very messy, but I just wanted to try it without looking at the M&M solution which is highly tuned and delicate. I learned many useful things along the way. Here is an outline of my proof;

(1) By requiring the tensor transformation, I was quickly led to the need for the following to be true:

$$\{ab,c\}'V'_{c} = T_{a}^{d}(\partial_{d}\, T_{b}^{c})V_{c} - T_{a}^{d}\, T_{b}^{c}\{dc,e\}V_{e}$$

(2) I was disturbed by the fact that $T_{a}^{d}(\partial_{d}\, T_{b}^{c})V_{c}$ did not appear symmetric in a,b (while the other two terms are), so this led me off on a digression where I found that I could rewrite this term in a manifestly symmetric form:

$$T_{a}^{d}(\partial_{d}\, T_{b}^{c})V_{c} = -T_{a}^{s}\, T_{b}^{g}\, T_{f}^{c}\, S^{f}{}_{sg}\, V_{c} \qquad \text{where } S^{f}{}_{sg} = (\partial_{s} T^{f}{}_{g}) = \frac{\partial^{2} x'^{f}}{\partial x^{s} \partial x^{g}} = \text{sym(s,g)}$$

Showing this required me to derive and then use a little "fact #2" based on $\partial_{s}(T^{a}{}_{b}\, T_{c}^{b}) = 0$, namely,

$$(\partial_{s}\, T_{c}^{e}) = -T_{a}^{e}\, T_{c}^{b}\, (\partial_{s} T^{a}{}_{b}) = -T_{a}^{e}\, T_{c}^{b}\, S^{a}{}_{sb} \qquad \text{"Fact 2"}$$

This "fact" showed how to relate $\partial_{s}\, T_{\star}^{\star}$ to that of $\partial_{s}\, T^{\star}{}_{\star}$. So the problem then became to show this to be true:

$$\{ab,c\}'V'_{c} = -T_{a}^{s}\, T_{b}^{g}\, T_{f}^{c}\, S^{f}{}_{sg}\, V_{c} + T_{a}^{d}\, T_{b}^{c}\{dc,e\}V_{e}$$
$$\text{LHS} = \text{RHS}_{2} + \text{RHS}_{1}$$

(3) I then expanded the LHS and ended up with two big terms $LHS_1 + LHS_2$. LHS_1 contained the three terms having factors of the form $(\partial_{\star} g_{\star\star})$. I then compared this to RHS_1,

$$LHS_1 = \tfrac{1}{2} g^{eh} * V_{e} \{ T_{a}^{a'} T_{b}^{b'} (\partial_{a'} g_{b'h}) + T_{b}^{a'} T_{a}^{b'} (\partial_{a'} g_{b'h}) - T_{a}^{b'} T_{b}^{c'} (\partial_{h} g_{b'c'}) \}$$

$$RHS_1 = \tfrac{1}{2}\, T_a{}^d\, T_b{}^c\; g^{es}\,(\;\partial_d g_{cs} + \partial_c g_{ds} - \partial_s g_{dc}\;)V_e$$

and, by renaming indices, it was easy to show these terms were the same. So what remained to show was that $LHS_2 = RHS_2$. At this point then I had:

$$RHS_2 = -T_a{}^s\, T_b{}^g\, T_f{}^c\, S^f{}_{sg}\, V_c$$

$$\begin{aligned} LHS_2 \;=\; & +\tfrac{1}{2} T^d{}_h\; g^{eh} * V_e \,\{\, T_a{}^{a'} T_b{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} + T_b{}^{a'} T_a{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} \,\} \\ & + \tfrac{1}{2} g^{eh} * V_e \; * \\ & \{\; T_a{}^{a'} (\partial_{a'} T_b{}^{b'})\, g_{b'h} + T_b{}^{a'} (\partial_{a'} T_a{}^{b'})\, g_{b'h} \\ & -(\partial_h T_a{}^{b'}) T_b{}^{c'} g_{b'c'} - T_a{}^{b'} (\partial_h T_b{}^{c'}) g_{b'c'} \} \end{aligned}$$

where it took quite a bit of routine but detailed work to get this form for LHS_2. There are 6 terms here, and each pair of terms shows the a,b symmetry. BUT, it was hard for me to see how this was going to equal RHS_2 because I could see no cancellations of terms.

(4) I then processed LHS_2 into the following form, exposing the $(\partial_s T_d{}^c)$ type factors to which I knew I could apply Fact #2. This gave:

$$\begin{aligned} LHS_2 = \;& +\tfrac{1}{2} T^d{}_h\, V^h \,\{\, T_a{}^s T_{bc} + T_b{}^s T_{ac} \,\} (\partial_s T_d{}^c) && (\partial_s T_d{}^c) = -T_f{}^c\, T_d{}^g\, S^f{}_{sg} \\ & +\tfrac{1}{2} [\, V_c\, T_a{}^s - V^s\, T_{ac} \,] (\partial_s T_b{}^c) && (\partial_s T_b{}^c) = -T_f{}^c\, T_b{}^g\, S^f{}_{sg} \\ & +\tfrac{1}{2} [\; V_c\, T_b{}^s - V^s\, T_{bc} \,] (\partial_s T_a{}^c) && (\partial_s T_a{}^c) = -T_f{}^c\, T_a{}^g\, S^f{}_{sg} \end{aligned}$$

(5) I then installed Fact #2 three times on the right, and was able the express LHS_2 as the sum of 6 terms:

$$\begin{aligned} 1 &= -\tfrac{1}{2} V^g\; T_a{}^s T_{bc} \quad T_f{}^c\, S^f{}_{sg} \\ 2 &= -\tfrac{1}{2} V^g\; T_b{}^s T_{ac} \quad T_f{}^c\, S^f{}_{sg} \\ 3 &= -\tfrac{1}{2} V_c\; T_a{}^s\, T_b{}^g \quad T_f{}^c\, S^f{}_{sg} \\ 4 &= +\tfrac{1}{2} V^s\, T_{ac}\, T_b{}^g \quad T_f{}^c\, S^f{}_{sg} \\ 5 &= -\tfrac{1}{2} V_c\, T_b{}^s\, T_a{}^g \quad T_f{}^c\, S^f{}_{sg} \\ 6 &= +\tfrac{1}{2} V^s\, T_{bc}\, T_a{}^g \quad T_f{}^c\, S^f{}_{sg} \end{aligned}$$

$$T_f{}^c\, S^f{}_{sg} = \frac{\partial x^c}{\partial x'^f}\,\frac{\partial^2 x'^f}{\partial x^s \partial x^g} = \text{not simple}$$

(6) At this point, I made use of the symmetry of $S^f{}_{sg}$ (main point!) to rewrite terms 4,5 and 6, then getting

$$\begin{aligned} 1 &= -\tfrac{1}{2} V^g\; T_a{}^s T_{bc} \quad T_f{}^c\, S^f{}_{sg} \\ 2 &= -\tfrac{1}{2} V^g\; T_b{}^s T_{ac} \quad T_f{}^c\, S^f{}_{sg} \\ 3 &= -\tfrac{1}{2} V_c\; T_a{}^s\, T_b{}^g \quad T_f{}^c\, S^f{}_{sg} \\ 4 &= +\tfrac{1}{2} V^g\, T_{ac}\, T_b{}^s \quad T_f{}^c\, S^f{}_{sg} \\ 5 &= -\tfrac{1}{2} V_c\, T_b{}^g\, T_a{}^s \quad T_f{}^c\, S^f{}_{sg} \end{aligned}$$

$$6 = +\tfrac{1}{2} V^{g} T_{bc} T_{a}{}^{s} \quad T_{f}{}^{c} S^{f}{}_{sg}$$

Now it became clear that terms 1 and 6 cancel, and also terms 2 and 4 cancel, and terms 3 and 5 are equal, so our total result is this:

$$LHS_2 = 3+5 = -V_c T_b{}^{g} T_a{}^{s} \quad T_f{}^{c} S^{f}{}_{sg} = RHS_2 \quad !$$

and this concluded the proof.

(7) Among the "lessons learned" by doing the above, I realized that you cannot willy-nilly raise and lower indices on tensors *that are acted upon by a derivative*. For example, consider this simple example

$$P_{sc} = A^{d} (\partial_s T_{dc}) \neq A_d (\partial_s T^{d}{}_{c}) \qquad \text{// where I am doing a "tilt change"}$$

In fact, calculation shows that

$$P_{sc} = A^{d} (\partial_s T_{dc}) = A^{d} (\partial_s g_{de} T^{e}{}_{c}) = A^{d} g_{de} (\partial_s T^{e}{}_{c}) + A^{d} (\partial_s g_{de}) T^{e}{}_{c}$$

$$= A_e (\partial_s T^{e}{}_{c}) + A^{d} (\partial_s g_{de}) T^{e}{}_{c} = A_d (\partial_s T^{d}{}_{c}) + \text{extra term.}$$

Also, note that, just as $\partial_s V_d$ is not a tensor (our main proof!) , so also $(\partial_s T_{dc})$ is not a tensor. Thus, even though T and A are tensors, Psc is not a tensor.

Another more obvious example is this.

$$A_c = (\partial_s g_{sc}) \quad \not\Rightarrow \quad A^{c} = (\partial_s g_s{}^{c})$$

If it did, we would conclude from the right equation that $A^{c} = 0$ since $\partial_s(\delta_s{}^{c}) = 0$ and we would then think that $A_c = g_{cd} A^{d} = 0$, which is not true. Also, the fragment A_c is not a tensor!

Mainline raw proof of covariance.

Now suppose I just try my own brute force on this, where I copy paste and edit above for the first term:

$$V'_{b,a} = [(\partial'_a V'_b) - \{ab,c\} V'_c]$$

$$= [(T_a{}^{d} \partial_d (T_b{}^{c} V_c)) - \{ab,c\} V'_c]$$

$$= T_a{}^{d} T_b{}^{c} \partial_d V_c + T_a{}^{d} (\partial_d T_b{}^{c}) V_c - \{ab,c\} V'_c$$

We want this to come out being the following

$$T_a{}^{d} T_b{}^{c} V_{d,c} = T_a{}^{d} T_b{}^{c} [\partial_d V_c - \{dc,e\} V_e]$$

$$= T_a{}^d\, T_b{}^c\ \partial_d V_c - T_a{}^d\, T_b{}^c\ \{dc,e\}V_e$$

The first terms are the same, so we need to show this:

$$T_a{}^d(\partial_d\, T_b{}^c)V_c \ - \{ab,c\}'V'_c \ = - T_a{}^d\, T_b{}^c\{dc,e\}V_e$$

Write the above as

$$T_a{}^d(\partial_d\, T_b{}^c)V_c \ = \ \{ab,c\}'V'_c \ - T_a{}^d\, T_b{}^c\{dc,e\}V_e \qquad\qquad **$$

STOP!!!!! Right here we have a "symmetry problem". I claim the RHS is symmetrical under a↔b, but the LHS does not seem to have this symmetry, so something is wrong right at this point, I have made some kind of mistake? The last term is symmetric as follows:

$$T_a{}^d\, T_b{}^c\{dc,e\}V_e \ = T_a{}^c\, T_b{}^d\{cd,e\}V_e = T_b{}^d\, T_a{}^c\ \{dc,e\}V_e$$

So there is no point in my continuing this calculation until I resolve this confusion.

Let's look more closely at the LHS of ** to see if it might be symmetric in a hidden way! Omit the V_c

$$Q_{ab} = T_a{}^d(\partial_d\, T_b{}^c)$$

At the end of the previous section, we showed that

$$(\partial_s\, T_c{}^e) \ = \ - T_g{}^e\, T_c{}^f\, S^g{}_{sf}$$

or

$$(\partial_d\, T_b{}^c) \ = \ - T_g{}^c\, T_b{}^f\, S^g{}_{df}$$

Thus we have that

$$Q_{ab} = T_a{}^d(\partial_d\, T_b{}^c) \ = - T_a{}^d\, T_g{}^c\, T_b{}^f\, S^g{}_{df} \ = - T_g{}^c\, T_a{}^d\, T_b{}^f\, S^g{}_{df}$$

and now, finally, **we see that in fact Q_{ab} really is symmetric**!

So we can continue our calculation. We had

$$T_a{}^d(\partial_d\, T_b{}^c)V_c \ = \ \{ab,c\}'V'_c \ - T_a{}^d\, T_b{}^c\{dc,e\}V_e$$

which we rewrite as

$$\{ab,c\}'V'_c = T_a{}^d(\partial_d\, T_b{}^c)V_c + T_a{}^d\, T_b{}^c\{dc,e\}V_e \qquad = \ Q_{ab}\, V_c + T_a{}^d\, T_b{}^c\{dc,e\}V_e$$

or the following, where all terms are manifestly symmetric in a,b :

$$\{ab,c\}'V'_c \ = - T_g{}^c\, T_a{}^d\, T_b{}^f\, S^g{}_{df}\, V_c + T_a{}^d\, T_b{}^c\{dc,e\}V_e$$

LHS = RHS

To see if this equation is true, write out the LHS:

$$\text{LHS} = \{ab,c\}'V'_c = g'^{cd}\,[ab,d]'\,V'_c =$$

$$\tfrac{1}{2}\, g'^{cd}\,(\ \partial'_a g'_{bd} + \partial'_b g'_{ad} - \partial'_d g'_{ab}\ )\, V'_c$$

$$= \tfrac{1}{2} T^c{}_f T^d{}_h\ g^{fh} *$$
$$\{\ T_a{}^{a'}\partial_{a'}(T_b{}^{b'}T_d{}^{c'}g_{b'c'})\ + T_b{}^{a'}\partial_{a'}(T_a{}^{b'}T_d{}^{c'}g_{b'c'}) - T_d{}^{a'}(\partial_{a'}T_a{}^{b'}T_b{}^{c'}g_{b'c'})\ \}\ T_c{}^e V_e$$

which is an amazing mess. We can contract the first and last T at least to get something a little simpler:

$$\text{LHS} = \tfrac{1}{2} T^d{}_h\ g^{eh} *$$
$$\{\ T_a{}^{a'}\partial_{a'}(T_b{}^{b'}T_d{}^{c'}g_{b'c'})\ + T_b{}^{a'}\partial_{a'}(T_a{}^{b'}T_d{}^{c'}g_{b'c'}) - T_d{}^{a'}(\partial_{a'}T_a{}^{b'}T_b{}^{c'}g_{b'c'})\ \}\ V_e$$

Each of the derivatives creates 3 terms so we have a total of 9 terms. I can always get the rightmost two objects by permutations, so write the above as

$$\text{LHS} = \tfrac{1}{2} T^d{}_h\ g^{eh} *$$
$$\{\ T_a{}^{a'}\partial_{a'}(T_b{}^{b'}T_d{}^{c'}g_{b'c'})\ + P(abd\rightarrow bad) - P(abd\rightarrow dab\ \}\ V_e$$

// OK to here 10 AM

So expand this first term in {} to get

$$T_a{}^{a'}\partial_{a'}(T_b{}^{b'}T_d{}^{c'}g_{b'c'}) =$$

$$T_a{}^{a'}(\partial_{a'}T_b{}^{b'})T_d{}^{c'}g_{b'c'} + T_a{}^{a'}T_b{}^{b'}(\partial_{a'}T_d{}^{c'})g_{b'c'})\ + T_a{}^{a'}T_b{}^{b'}T_d{}^{c'}(\partial_{a'}g_{b'c'})$$

So we now have

$$\text{LHS} = \tfrac{1}{2} T^d{}_h\ g^{eh} * V_e$$
$$\{\ T_a{}^{a'}(\partial_{a'}T_b{}^{b'})T_d{}^{c'}g_{b'c'} + T_a{}^{a'}T_b{}^{b'}(\partial_{a'}Td_c{}^{c'})g_{b'c'})\ + T_a{}^{a'}T_b{}^{b'}T_d{}^{c'}(\partial_{a'}g_{b'c'})$$
$$+ P(abd\rightarrow bad)$$
$$- P(abd\rightarrow dab\ \}$$

Now let's write it all out:

$$\text{LHS} = \tfrac{1}{2} T^d{}_h\ g^{eh} * V_e$$
$$\{\ \ T_a{}^{a'}(\partial_{a'}T_b{}^{b'})T_d{}^{c'}g_{b'c'} + T_a{}^{a'}T_b{}^{b'}(\partial_{a'}T_d{}^{c'})g_{b'c'})\ + T_a{}^{a'}T_b{}^{b'}T_d{}^{c'}(\partial_{a'}g_{b'c'})$$
$$+ T_b{}^{a'}(\partial_{a'}T_a{}^{b'})T_d{}^{c'}g_{b'c'} + T_b{}^{a'}T_a{}^{b'}(\partial_{a'}T_d{}^{c'})g_{b'c'})\ + T_b{}^{a'}T_a{}^{b'}T_d{}^{c'}(\partial_{a'}g_{b'c'})$$
$$- T_d{}^{a'}(\partial_{a'}T_a{}^{b'})T_b{}^{c'}g_{b'c'} - \ T_d{}^{a'}T_a{}^{b'}(\partial_{a'}T_b{}^{c'})g_{b'c'})\ - T_d{}^{a'}T_a{}^{b'}T_b{}^{c'}(\partial_{a'}g_{b'c'})$$
$$\}\ \ \text{// OK to here 10 AM}$$

Now, we can contract the opening $T^d{}_h$ in 7 of the 9 terms to get

$$\text{LHS} = \tfrac{1}{2} T^{d}{}_{h}\; g^{eh} * V_e$$
$$\{\;\; T_a{}^{a'}(\partial_{a'} T_b{}^{b'}) T_d{}^{c'} g_{b'c'} + T_a{}^{a'} T_b{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'}) \;+ T_a{}^{a'} T_b{}^{b'} T_d{}^{c'} (\partial_{a'} g_{b'c'})$$
$$+ T_b{}^{a'}(\partial_{a'} T_a{}^{b'}) T_d{}^{c'} g_{b'c'} + T_b{}^{a'} T_a{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'}) \;+ T_b{}^{a'} T_a{}^{b'} T_d{}^{c'} (\partial_{a'} g_{b'c'})$$
$$- T_d{}^{a'}(\partial_{a'} T_a{}^{b'}) T_b{}^{c'} g_{b'c'} - \; T_d{}^{a'} T_a{}^{b'} (\partial_{a'} T_b{}^{c'}) g_{b'c'}) \;- T_d{}^{a'} T_a{}^{b'} T_b{}^{c'} (\partial_{a'} g_{b'c'})$$
$$\}$$

$$= \;+\; \tfrac{1}{2} T^{d}{}_{h}\; g^{eh} * V_e \{\; T_a{}^{a'} T_b{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} + T_b{}^{a'} T_a{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} \;\}$$

$$+\; \tfrac{1}{2} g^{eh} * V_e$$
$$\{\;\; T_a{}^{a'}(\partial_{a'} T_b{}^{b'})\; g_{b'h} + T_a{}^{a'} T_b{}^{b'}\; (\partial_{a'} g_{b'h})$$
$$+ T_b{}^{a'}(\partial_{a'} T_a{}^{b'})\; g_{b'h} + T_b{}^{a'} T_a{}^{b'}\; (\partial_{a'} g_{b'h})$$
$$- (\partial_h T_a{}^{b'}) T_b{}^{c'} g_{b'c'} \; - T_a{}^{b'} T_b{}^{c'} (\partial_h g_{b'c'}) \; - T_a{}^{b'} (\partial_h T_b{}^{c'}) g_{b'c'} \} \quad \text{// OK to here 10 AM}$$

Now just for the moment, let's keep only the $(\partial_{a'} g_{b'h})$ type terms and call them LHS_1

$$\text{LHS}_1 \;= \tfrac{1}{2} g^{eh} * V_e \{\; T_a{}^{a'} T_b{}^{b'}\; (\partial_{a'} g_{b'h}) + T_b{}^{a'} T_a{}^{b'}\; (\partial_{a'} g_{b'h}) \;- T_a{}^{b'} T_b{}^{c'} (\partial_h g_{b'c'}) \;\}$$

Meanwhile, here is where we are trying to get to:

$$\text{RHS} = - T_g{}^{c}\; T_a{}^{d}\; T_b{}^{f}\; S^{g}{}_{df}\; V_c + T_a{}^{d}\; T_b{}^{c} \{dc,e\} V_e$$

Now go off and do these things

$$\{ab,c\} \equiv g^{cs}\; [ab,s]$$
$$\Rightarrow \{dc,e\} \equiv g^{es}\; [dc,s]$$

$$[ab,c] \equiv \tfrac{1}{2} \left(\; \partial_a g_{bc} + \partial_b g_{ac} \; - \partial_c g_{ab} \;\right)$$
$$\Rightarrow [dc,s] \equiv \tfrac{1}{2} \left(\; \partial_d g_{cs} + \partial_c g_{ds} \; - \partial_s g_{dc} \;\right)$$

$$\Rightarrow \;\; \{dc,e\} \;= \tfrac{1}{2}\; g^{es} \left(\; \partial_d g_{cs} + \partial_c g_{ds} \; - \partial_s g_{dc} \;\right) \qquad \text{// OK to here 10 AM}$$

so we have

$$\text{RHS} = - T_g{}^{c}\; T_a{}^{d}\; T_b{}^{f}\; S^{g}{}_{df}\; V_c \; + \tfrac{1}{2}\; T_a{}^{d}\; T_b{}^{c}\; g^{es} \left(\; \partial_d g_{cs} + \partial_c g_{ds} \; - \partial_s g_{dc} \;\right) V_e$$

If this is to work out, it is pretty clear that the $\partial_d g_{cs}$ terms have to match, so we need

$$\text{RHS}_1 = \; \tfrac{1}{2}\; T_a{}^{d}\; T_b{}^{c}\; g^{es} \left(\; \partial_d g_{cs} + \partial_c g_{ds} \; - \partial_s g_{dc} \;\right) V_e$$

$$\text{LHS}_1 \;= \tfrac{1}{2} g^{eh} * \{\; T_a{}^{a'} T_b{}^{b'}\; (\partial_{a'} g_{b'h}) + T_b{}^{a'} T_a{}^{b'}\; (\partial_{a'} g_{b'h}) \;- T_a{}^{b'} T_b{}^{c'} (\partial_h g_{b'c'}) \;\} V_e$$

At least we have the right number of factors everywhere. We now need to do some index renaming:

$$\text{LHS}_1 \;= \tfrac{1}{2} g^{eh} * \{\; T_a{}^{d} T_b{}^{c}\; (\partial_d g_{ch}) + T_b{}^{c} T_a{}^{d}\; (\partial_c g_{dh}) \;- T_a{}^{d} T_b{}^{c} (\partial_h g_{dc}) \;\} V_e$$

which I just did live. Now all three T factors are the same so pull them out to get (and h→s)

$$LHS_1 = \tfrac{1}{2} T_a{}^d T_b{}^c \quad g^{es} \quad [(\partial_d g_{cs}) + (\partial_c g_{ds}) - (\partial_s g_{dc})]\, V_e$$

$$RHS_1 = +\tfrac{1}{2}\, T_a{}^d\, T_b{}^c\, g^{es} \quad (\, \partial_d g_{cs} + \partial_c g_{ds} - \partial_s g_{dc}\,) V_e$$

and **we have a match on this set of terms**. // OK to here 10 AM

It then remains to show that all the *other* terms match. On the LHS we have these terms:

$$\begin{aligned} LHS_2 \;=\; & + \tfrac{1}{2} T^d{}_h\, g^{eh} * V_e \{\, T_a{}^{a'} T_b{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} + T_b{}^{a'} T_a{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} \,\} \\ & + \tfrac{1}{2} g^{eh} * V_e \; * \\ & \{\; T_a{}^{a'} (\partial_{a'} T_b{}^{b'})\, g_{b'h} + T_b{}^{a'} (\partial_{a'} T_a{}^{b'})\, g_{b'h} \\ & - (\partial_h T_a{}^{b'}) T_b{}^{c'} g_{b'c'} - T_a{}^{b'} (\partial_h T_b{}^{c'}) g_{b'c'} \} \end{aligned}$$

and on the RHS we have these terms

$$RHS_2 = -\, T_g{}^c\, T_a{}^d\, T_b{}^f\, S^g{}_{df}\, V_c$$ // OK to here 10 AM

Now some live index renaming on the LHS terms:

$$+ \tfrac{1}{2} T^d{}_h\, g^{eh} * V_e \{\, T_a{}^s T_b{}^t (\partial_s T_d{}^c) g_{tc} + T_b{}^t T_a{}^s (\partial_t T_d{}^c) g_{sc} \,\}$$

$$\begin{aligned} & + \tfrac{1}{2} g^{eh} * V_e \\ & \quad \{\; T_a{}^s (\partial_s T_b{}^c)\, g_{ch} + T_b{}^t (\partial_t T_a{}^c)\, g_{ch} \\ & \quad - (\partial_h T_a{}^c) T_b{}^t g_{ct} - T_a{}^s (\partial_h T_b{}^c) g_{sc} \} \end{aligned}$$ // OK to here 10 AM

Separate out the last terms like this:

$$+ \tfrac{1}{2} T^d{}_h\, g^{eh} * V_e \{\, T_a{}^s T_b{}^t (\partial_s T_d{}^c) g_{tc} + T_b{}^t T_a{}^s (\partial_t T_d{}^c) g_{sc} \,\}$$

$$+ \tfrac{1}{2} g^{eh} * V_e \;\{\; T_a{}^s (\partial_s T_b{}^c)\, g_{ch} + T_b{}^t (\partial_t T_a{}^c)\, g_{ch} \,\}$$

$$- \tfrac{1}{2} g^{eh} * V_e \;\{ (\partial_h T_a{}^c) T_b{}^t g_{ct} + T_a{}^s (\partial_h T_b{}^c) g_{sc} \}$$

Now rename summed indices to try and reduce the different forms of $(\partial_s T_d{}^c)$:

$$+ \tfrac{1}{2} T^d{}_h\, g^{eh} * V_e \{\, T_a{}^s T_b{}^t (\partial_s T_d{}^c) g_{tc} + T_b{}^s T_a{}^t (\partial_s T_d{}^c) g_{tc} \,\}$$

$$+ \tfrac{1}{2} g^{eh} * V_e \;\{ T_a{}^s (\partial_s T_b{}^c)\, g_{ch} + T_b{}^s (\partial_s T_a{}^c)\, g_{ch} \,\}$$

$$- \tfrac{1}{2} g^{es} * V_e \;\{ (\partial_s T_a{}^c) T_b{}^t g_{ct} + T_a{}^t (\partial_s T_b{}^c) g_{tc} \}$$

where now we have only three different "forms" like $(\partial_s T_d{}^c)$. Group by these forms:

$$+ \tfrac{1}{2} T^d{}_h \; g^{eh} * V_e \; \{ T_a{}^s T_b{}^t + T_b{}^s T_a{}^t \} (\partial_s T_d{}^c) g_{tc}$$

$$+ \tfrac{1}{2} g^{eh} * V_e \; \{ T_a{}^s (\partial_s T_b{}^c) \, g_{ch} \; - \tfrac{1}{2} g^{es} * V_e \; T_a{}^t (\partial_s T_b{}^c) g_{tc}$$

$$+ \tfrac{1}{2} g^{eh} * V_e \; T_b{}^s (\partial_s T_a{}^c) \, g_{ch} \; - \tfrac{1}{2} g^{es} * V_e \; (\partial_s T_a{}^c) T_b{}^t g_{ct}$$

And write again factoring from last lines:

$$+ \tfrac{1}{2} T^d{}_h \; g^{eh} \, V_e \; \{ T_a{}^s T_b{}^t + T_b{}^s T_a{}^t \} (\partial_s T_d{}^c) g_{tc}$$

$$+ [\tfrac{1}{2} g^{eh} \, V_e \; T_a{}^s g_{ch} \; - \tfrac{1}{2} g^{es} \, V_e \; T_a{}^t g_{tc}] \, (\partial_s T_b{}^c)$$

$$+ [\tfrac{1}{2} g^{eh} \, V_e \; T_b{}^s g_{ch} \; - \tfrac{1}{2} g^{es} \, V_e \; T_b{}^t g_{ct}] \, (\partial_s T_a{}^c)$$

Now use up as many g_{ab} as possible just to get less symbols showing:

$$+ \tfrac{1}{2} T^d{}_h \; V^h \; \{ T_a{}^s T_{bc} + T_b{}^s T_{ac} \} (\partial_s T_d{}^c)$$

$$+ \tfrac{1}{2} [V_c \; T_a{}^s \; - V^s \, T_{ac}] \, (\partial_s T_b{}^c)$$

$$+ \tfrac{1}{2} [\; V_c \, T_b{}^s \; - V^s \, T_{bc}] \, (\partial_s T_a{}^c)$$

and now all the g_{ab} are gone. Now, finally, we want to use our Fact 2 rule:

$$(\partial_s \, T_c{}^e) = - T_a{}^e \, T_c{}^b \, S^a{}_{sb}$$

We have to install this in 3 locations in the above.

$$(\partial_s T_d{}^c) \; = \; - T_f{}^c \, T_d{}^g \, S^f{}_{sg}$$

$$(\partial_s T_b{}^c) \; = - T_f{}^c \, T_b{}^g \, S^f{}_{sg}$$

$$(\partial_s T_a{}^c) = - T_f{}^c \, T_a{}^g \, S^f{}_{sg}$$

We then have:

$$- \tfrac{1}{2} T^d{}_h \; V^h \; \{ T_a{}^s T_{bc} + T_b{}^s T_{ac} \} \, T_f{}^c \, T_d{}^g \, S^f{}_{sg}$$

$$- \tfrac{1}{2} [V_c \; T_a{}^s \; - V^s \, T_{ac}] \, T_f{}^c \, T_b{}^g \, S^f{}_{sg}$$

$$- \tfrac{1}{2} [\; V_c \, T_b{}^s \; - V^s \, T_{bc}] \, T_f{}^c \, T_a{}^g \, S^f{}_{sg}$$

Now in the first line, write $T^{d}{}_{h}\, T_{d}{}^{g} = \delta_{h}{}^{g}$ and rewrite all

$$-\tfrac{1}{2}V^{g}\{\, T_{a}{}^{s}T_{bc} + T_{b}{}^{s}T_{ac}\,\}\, T_{f}{}^{c}\, S^{f}{}_{sg} \qquad = 1 + 2$$

$$-\tfrac{1}{2}\,[V_{c}\; T_{a}{}^{s} \;-\; V^{s}\, T_{ac}\,]\, T_{f}{}^{c}\, T_{b}{}^{g}\, S^{f}{}_{sg} \qquad = 3 + 4$$

$$-\tfrac{1}{2}\,[\;\; V_{c}\, T_{b}{}^{s} \;-\; V^{s}\, T_{bc}\,]\, T_{f}{}^{c}\, T_{a}{}^{g}\, S^{f}{}_{sg} \qquad = 5 + 6$$

Recall that $RHS_2 = -V_{c}\, T_{a}{}^{s}\, T_{b}{}^{g}\; T_{f}{}^{c}\, S^{f}{}_{sg}$ is where we are trying to arrive. Write the terms out:

$$1 = -\tfrac{1}{2}V^{g}\; T_{a}{}^{s}T_{bc} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$2 = -\tfrac{1}{2}V^{g}\, T_{b}{}^{s}T_{ac} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$3 = -\tfrac{1}{2}\, V_{c}\; T_{a}{}^{s}\, T_{b}{}^{g}\; T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$4 = +\tfrac{1}{2}\, V^{s}\, T_{ac}\, T_{b}{}^{g} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$5 = -\tfrac{1}{2}\, V_{c}\, T_{b}{}^{s}\, T_{a}{}^{g} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$6 = +\tfrac{1}{2}\, V^{s}\, T_{bc}\, T_{a}{}^{g} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

Now finally use symmetry of S to change some terms swapping s and g

$$1 = -\tfrac{1}{2}V^{g}\; T_{a}{}^{s}T_{bc} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$2 = -\tfrac{1}{2}V^{g}\, T_{b}{}^{s}T_{ac} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$3 = -\tfrac{1}{2}\, V_{c}\; T_{a}{}^{s}\, T_{b}{}^{g}\; T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$4 = +\tfrac{1}{2}\, V^{g}\, T_{ac}\, T_{b}{}^{s} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$5 = -\tfrac{1}{2}\, V_{c}\, T_{b}{}^{g}\, T_{a}{}^{s} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$6 = +\tfrac{1}{2}\, V^{g}\, T_{bc}\, T_{a}{}^{s} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

Now we see that terms 1 and 6 cancel, and also terms 2 and 4 cancel, and terms 3 and 5 are equal, so our total result is this:

$$3{+}5 \quad = -V_{c}\, T_{b}{}^{g}\, T_{a}{}^{s} \quad T_{f}{}^{c}\, S^{f}{}_{sg}$$

$$RHS_2 = -V_{c}\, T_{a}{}^{s}\, T_{b}{}^{g}\; T_{f}{}^{c}\, S^{f}{}_{sg}$$

so finally we have our proof!!! What a horrendous mess.

2. Proof that jac = $\sqrt{g}$

Oddly, the Jacobian is not even mentioned in M&M's discussion of transformations. Here is some info from the Wolfram site:

$$\mathbf{J}(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}. \qquad J = \left|\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}\right|. \qquad dy_1 \cdots dy_n = \left|\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}\right| dx_1 \cdots dx_n.$$

If I make the change x→ x' and y→ x in the above, we get

$$\text{jac} = \det\left(\frac{\partial x^b}{\partial x'^a}\right) = \det(T_a{}^b) \qquad \text{and} \qquad dV = \text{jac}\, dV'$$

On page 195 we get from M&M that

$$dV = \sqrt{g}\, dV'$$

so it must be true that

$$\text{jac} = \sqrt{g}$$

which is what we shall prove here, hopefully.

Meanwhile, we know these facts

$$g = \det(g_{ij})$$

$$g_{km} = \Sigma_i\, T_k{}^i\, T_m{}^i = \Sigma_i\, (\partial x^i/\partial x'^k)\, (\partial x^i/\partial x'^m) \qquad \text{// page 192 5-4a}$$

For use below write

$$g_{1i} = \Sigma_s\, T_1{}^s\, T_i{}^s$$

The context here is that $T_a{}^b$ transforms us from a Cartesian system x^i with $g_{ij} = \delta_{ij}$ to a new system x'^i with some new g'_{ij}. However, we shall refer to this simply as g_{ij} (instead of g'_{ij}) and det = g (instead of g'). This new x' world is that of the curvilinear coordinates where q = x'.

So we will really prove that $\text{jac}^2 = g$.

We can start writing things out:

$$\text{jac} = \varepsilon_{stu} T_1^s T_2^t T_3^u \qquad = \text{jac1}$$

We can do this another way which is this (rows swap columns)

$$\text{jac} = \varepsilon_{ijk} T_i^1 T_j^2 T_k^3 \qquad = \text{jac2}$$

These will both come in very handy very soon!

Meanwhile, we have to pick one way or the other for the g determinant

$$g = \varepsilon_{ijk}\, g_{1i} g_{2j} g_{3k} = \varepsilon_{ijk}\, T_1^s\, T_i^s\, T_2^t\, T_j^t\, T_3^u\, T_k^u \qquad \text{//6 sums and 6 T's}$$

which we can just reorder to get

$$g = (\, \varepsilon_{ijk}\, T_i^s\, T_j^t\, T_k^u\,) T_1^s\, T_2^t\, T_3^u$$

Now define a rank 3 tensor from the first terms of g

$$F^{stu} \equiv \varepsilon_{ijk}\, T_i^s\, T_j^t\, T_k^u$$

This F^{stu} is completely antisymmetric. For example,

$$F^{stu} = \varepsilon_{ijk}\, T_i^s\, T_j^t\, T_k^u = \varepsilon_{jik}\, T_j^s\, T_i^t\, T_k^u = \varepsilon_{jik}\, T_i^t\, T_j^s\, T_k^u = -\varepsilon_{ijk}\, T_i^t\, T_j^s\, T_k^u = -F^{tsu}$$

That fact means that F^{stu} is zero if any two indices are equal. Thus, although you might think this tensor had 3x3x3 = 27 terms, there are only 6 non-vanishing terms, and all those are equal to $\pm F^{123}$ according to ε_{stu}. Thus we have this pretty amazing and powerful claim

$$F^{stu} = \varepsilon_{stu} F^{123} = \varepsilon_{stu}\, \varepsilon_{ijk}\, T_i^1\, T_j^2\, T_k^3$$

This allows us to rewrite g as follows:

$$g = (\, \varepsilon_{ijk}\, T_i^s\, T_j^t\, T_k^u\,) T_1^s\, T_2^t\, T_3^u = F^{stu}\, T_1^s\, T_2^t\, T_3^u$$

$$= (\varepsilon_{stu}\, \varepsilon_{ijk}\, T_i^1\, T_j^2\, T_k^3)\, T_1^s\, T_2^t\, T_3^u$$

$$= (\varepsilon_{ijk}\, T_i^1\, T_j^2\, T_k^3)(\, \varepsilon_{stu}\, T_1^s\, T_2^t\, T_3^u)$$

$$= \text{jac2} * \text{jac1} = \text{jac} * \text{jac} = \text{jac}^2$$

Very spiffy I would say, offhand. QED,

Now add three more sums to j^2

$$j^2 = (\varepsilon_{ijk} T_i^s T_j^t T_k^u)\ \delta_{s1}\ \delta_{t2}\ \delta_{s3}\ (\varepsilon_{abc} T_a^s T_b^t T_c^u)$$

Let's look at only the abc = 123 term in j^2 "

$$j^2_{abc=123} = (\varepsilon_{ijk} T_i^s T_j^t T_k^u)\ \delta_{s1}\ \delta_{t2}\ \delta_{s3}\ (T_1^s T_2^t T_3^u)$$

$$= \delta_{s1}\ \delta_{t2}\ \delta_{s3}\ \varepsilon_{ijk} T_i^s T_j^t T_k^u\ T_1^s\ T_2^t\ T_3^u$$

g $$= \varepsilon_{ijk}\ T_i^s\ T_j^t T_k^u\ T_1^s\ T_2^t\ T_3^u$$

Thus, our selected term from j^2 matches one of 27 terms in the g sum, in terms of stu sums. Interesting. We need some symmetry help! Rewrite

$$g = \varepsilon_{ijk}\ T_i^s\ T_1^s\ T_j^t\ T_2^t\ T_k^u\ T_3^u$$

$j^2 = \varepsilon_{ijk} T_1^i T_2^j T_3^k\ \varepsilon_{abc} T_1^a T_2^b T_3^c$ // 6 sums and 6 T's

$g = \varepsilon_{ijk}\ T_1^s\ T_2^t\ T_3^u\ T_i^s\ T_j^t\ T_k^u$

$j^2 = \varepsilon_{ijk}\ \varepsilon_{abc}\ T_1^i\ T_2^j\ T_3^k\ T_1^a\ T_2^b\ T_3^c$

Let's just "look at" the ijk = 123 term in g

$g = \varepsilon_{123}\ T_1^s\ T_2^t\ T_3^u\ T_1^s\ T_2^t\ T_3^u$

$j^2 = \varepsilon_{123}\ \varepsilon_{abc}\ T_1^i\ T_2^j\ T_3^k\ T_1^a\ T_2^b\ T_3^c$