Margenau and Murphy Chapters 4 and 5 PhL 8.29.08

These are "raw notes" from reading sections of Chapters 4 and 5. See nearby documents for special topics related to these chapters.

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Selected Sections from Chapter 4:

Section 4.20 Tensors (161)

I read these tensor sections through today 8.20.08. I can see I read this earlier, but don't have any notes and don't know when I was here last. M&M first try to motivate why tensors show up in nature. One idea is that if a tensor is needed, and since it describes a physical system, and since you can transform frames, probably it will "transform as a tensor" and not just be a "dead matrix" in math space. Goldstein (which I have now finished) talks about tensors being defined as such by their transformation properties, and we have some mention of them in other parts of the book, but not much. He uses the imaginary time idea which I am sure I can fit into a metric tensor viewpoint when I am done here, but he does not get into up and down indices and all that stuff.

So OK, in this chapter he will use overbars for the "overbarred" reference frame, compared to the "starting" reference frame. I prefer primes since much easier to type.

Let's start with the **contravariant vector** idea. It is something that transforms like this:

$$A'^{a} = \frac{\partial x'^{a}}{\partial x^{b}} A^{b}$$

$$A'^{a} = T^{a}{}_{b}\ A^{b} \qquad \text{where } T^{a}{}_{b} \equiv \frac{\partial x'^{a}}{\partial x^{b}} \qquad \text{note "a" in upper partial, "b" in lower partial}$$

Notice that ALL indices are up here everywhere except in our shorthand T second index. Something that does the above is a "contra variant vector". We are allowed to define a shorthand in any way we want, so the up/down of its indices are arbitrary right now.

(Section corrected 11.7.09.) M&M make a misleading comment on page 162 under equation (4-51a) which had me fooled. They say that the coordinate vector itself x^{a} transforms as a contravariant vector and we write

$$x'^{a} = T^{a}{}_{b}\ x^{b} = \frac{\partial x'^{a}}{\partial x^{b}}\ x^{b}$$

But this is only true if $T^{a}{}_{b}$ is not a function of position !!! In general, this is not true. So in the general case, all we can say is that a *differential* coordinate vector dx^{a} transforms as a tensor,

$$dx'^{a} = T^{a}{}_{b}\ dx^{b} = \frac{\partial x'^{a}}{\partial x^{b}}\ dx^{b}$$

This is always true. Imagine that x^{a} is a vector joining two points in space (these two points could be the origin and the point x, or two points r1 and r2). In general $T^{a}{}_{b}(x)$ will vary as you move along this vector and will be different for example at the two ends! As an example, suppose (r,θ,φ) are x_{i}'. We know we cannot write $\theta = Ax + By + cz$ so $x'^{a} = T^{a}{}_{b}\ x^{b}$ cannot possibly be true.

Now it is true that much tensor analysis is done with *linear* transformations which would mean for example that $x_{1}' = Ax_{1} + Bx_{2} + Cx_{3}$ so that in fact there is a matrix $x' = M\ x$. In this situation, the matrix M is exactly our matrix T and indeed we can say $x'^{a} = T^{a}{}_{b}\ x^{b}$. In effect we are integrating $dx'^{a} = T^{a}{}_{b}$

dx^b since T is constant in space. Better, we start with our linear transformation x' = M x and differentiate it, and that lets us identify M = T.

Examples of linear transformations are rotations or Lorentz transformations where T is a function of some angles or a boost parameter, so T is constant in space. In the case of a Lorentz Transformation we might write $T^{\mu}{}_{\nu} = a^{\mu}{}_{\nu}$ as in BD vol 1 early chapters. In general relativity, however, I think T is going to vary with space so we must be careful. I think in this case of space-dependent T, when we talk about a things that transforms as a vector, it will be a vector field, and we will have $A'^a = T^a{}_b\ A^b$ be something that is true at each point in space separately. Perhaps $A'^a(x') = T^a{}_b(x)\ A^b(x)$.

Since I am trying to apply all this tensor stuff even to curvilinear coordinates, such as sphericals, we have to realize that T is a function of space.

I hope this realization does not invalidate anything major below. M&M are quite aware that T might depend on x, and so also then will g_{ij} and g etc. It is exactly this space dependence that causes the extra term in the covariant derivative studied below, and that is why the Christoffel symbols only appear in general relativity, not special relativity.

In the rest of this document, I will continue to use x^a as if it were a true tensor vector. The reader needs to think of this as dx^a to keep general, but I will not edit everything. And of course things are true as stated for linear transformations T. At some point, I need to purge this entire directory to get this misunderstanding removed from all files.

So, how does a **covariant vector** transform? According to 4-51, it is this:

$$A'_a = \frac{\partial x^b}{\partial x'^a} A_b$$

$$= T_a{}^b A_b \qquad \text{where } T_a{}^b \equiv \frac{\partial x^b}{\partial x'^a} \qquad \text{note "a" in lower partial, "b" in upper partial}$$

The partial derivative is exactly upside down. So the summation tilt is \ instead of /, and the prime is now on the bottom. You exactly swap the partials top and bottom. But the indices within the partial derivative are still both "up". In our shorthand notation, we could have used a different letter, but we instead stick with the letter T, but we just regard $T^a{}_b$ and $T_a{}^b$ as *different transformations*. The notational idea is to have the first index match the thing on the left of the equation, the second index goes with the thing on the right, and you have a diagonal tilt on the sum.

Now here are four properties of the various T elements. In the first row, we are summing on the second index, either sum tilt up or sum tilt down. In the second row, we sum on first index. In all four cases, the two T objects are not the same object -- they always have different tilts and that is how our shorthand notation shows that they are not the same object.

$$T^a{}_b\ T_c{}^b = \frac{\partial x'^a}{\partial x^b}\frac{\partial x^b}{\partial x'^c} = \delta^a{}_c = T_c{}^b\ T^a{}_b \qquad \text{sum on second index of T}$$

$$T^a{}_b\ T_a{}^c = \frac{\partial x'^a}{\partial x^b}\frac{\partial x^c}{\partial x'^a} = \delta_b{}^c = T_a{}^c\ T^a{}_b \qquad \text{sum on first index of T}$$

where $\delta^{i}{}_{j} = \delta_{i}{}^{j}$ = good old $\delta_{i,j}$, but we choose to write them with up/down sense that sort of matches the product. Later we will see this has more significance. Here is a way to remember the above rules. (1) choose to sum on either first or second index; (2) have the tilts of the two T's be opposite.

Note Added. Go back up to

$$x'^{a} = T^{a}{}_{b}\ x^{b} = \frac{\partial x'^{a}}{\partial x^{b}}\ x^{b}$$

We know also that

$$x^{a} = (T^{-1})^{a}{}_{b}\ x'^{b} \qquad // = \left(\frac{\partial x^{b}}{\partial x'^{a}}\ x'^{b} = T_{a}{}^{b}\ x'^{b} \right) \text{ as we shall confirm below}$$

Notice that this gives

$$x^{a} = (T^{-1})^{a}{}_{b}\ T^{b}{}_{c}\ x^{c}$$

which says

$$(T^{-1})^{a}{}_{b}\ T^{b}{}_{c} = \delta^{a}{}_{c}$$

Now apply $T_{d}{}^{c}$ to the right on both sides to get

$$(T^{-1})^{a}{}_{b}\ T^{b}{}_{c}\ T_{d}{}^{c} = \delta^{a}{}_{c}\ T_{d}{}^{c} \quad => \quad (T^{-1})^{a}{}_{d} = T_{d}{}^{a}$$

So we find these facts:

$$T_{a}{}^{b} = (T^{-1})^{b}{}_{a}$$
$$T^{a}{}_{b} = (T^{-1})_{b}{}^{a}$$

Notice that the tilt changes AND the indices are in reverse order!

And of course then we can write:

$$x^{a} = (T^{-1})^{a}{}_{b}\ x'^{b} = T_{b}{}^{a}\ x'^{b}$$

So here is the comparison of the forward and backward transformations:

$$x'^{a} = T^{a}{}_{b}\ x^{b} \qquad x'_{a} = T_{a}{}^{b}\ x_{b}$$
$$x^{a} = T_{b}{}^{a}\ x'^{b} = x'^{b}\ T_{b}{}^{a} \qquad x_{a} = T^{b}{}_{a}\ x'_{b} = x'_{b}\ T^{b}{}_{a}$$

Similarly we have

$Q'^{ab} = T^a{}_c\ T^b{}_d\ Q^{cd}$ $\qquad$ $Q'_{ab} = T_a{}^c\ T_b{}^d\ Q_{cd}$

$Q^{ab} = T_c{}^a\ T_d{}^b\ Q'^{cd}$ $\qquad$ $Q_{ab} = T^c{}_a\ T^d{}_b\ Q'_{cd}$

and so on for any sized tensor. You apply the inversion rule to each index.

So far, A^a and A_b are completely different animals and we don't have a way to relate them to each other.

Now here are some contravariant tensor transformation rules:

$S' = S$ $\qquad$ contravariant rank 0

$V'^a = T^a{}_{a'}\ V^{a'}$ $\qquad$ contravariant rank 1

$X'^{ab} = T^a{}_{a'}\ T^b{}_{b'}\ X^{a'b'}$ $\qquad$ contravariant rank 2

$U'^{abc} = T^a{}_{a'}\ T^b{}_{b'}\ T^c{}_{c'}\underline{U}^{a'b'c'}$ $\qquad$ contravariant rank 3

Here we have introduced some slightly dangerous notation. The prime was originally used to denote something in the "primed frame" like S', but now we start priming indices so we can see how the indices are matching up. So index a' could have been called α or s, it is just a summation index. So these two uses of the "prime" character ' are completely unrelated!

Here are the corresponding covariant tensor transformation rules. All we do is take the above rules and change the up-down-ness of every single index:

$S' = S$ $\qquad$ covariant rank 0

$V'_a = T_a{}^{a'}\ V_{a'}$ $\qquad$ covariant rank 1

$X'_{ab} = T_a{}^{a'}\ T_b{}^{b'}\ X_{a'b'}$ $\qquad$ covariant rank 2

$U'_{abc} = T_a{}^{a'}\ T_b{}^{b'}\ T_c{}^{c'}\underline{U}_{a'b'c'}$ $\qquad$ covariant rank 3

The next idea is that you can define a "mixed tensor" with some indices up (contra) and some indices down (co), and you adjust the T's appropriately. We are just defining tensor types. For example:

$X'^a{}_b = T^a{}_{a'}\ T_b{}^{b'}\ X^{a'}{}_{b'}$ $\qquad$ mixed high-low rank 2 $\qquad$ *

$U'^a{}_b{}^c = T^a{}_{a'}\ T_b{}^{b'} T^c{}_{c'}\ U^{a'}{}_{b'}{}^{,c'}$ $\qquad$ mixed high-low-high rank 3

It is quite easy to keep track of things using the tilts and the primed indices.

Question: is $T_a{}^b$ itself a mixed tensor?

See Question #8 in "ponderings" doc. The answer is that $T_{\mathbf{a}}{}^{\mathbf{b}}$ is NOT a rank-2 tensor, mixed or otherwise. It is "mixed" in a completely different sense because it has the first index in the x' world and the second index in the x world. Consider the transformation of two true rank 2 tensors:

$X'^{\mathbf{ab}} = T^{\mathbf{a}}{}_{\mathbf{i}}\ T^{\mathbf{b}}{}_{\mathbf{j}}\ X^{\mathbf{ij}}$ // an arbitrary tensor

$g'^{\mathbf{ab}} = T^{\mathbf{a}}{}_{\mathbf{i}}\ T^{\mathbf{b}}{}_{\mathbf{j}}\ g^{\mathbf{ij}}$ // the metric tensor

The rules are these:

Use $g'^{\mathbf{ab}}$ to raise and $g'_{\mathbf{ab}}$ to lower either index on $X'^{\mathbf{ab}}$ in any of its index position forms.
Use $g^{\mathbf{ab}}$ to raise and $g_{\mathbf{ab}}$ to lower either index on $X^{\mathbf{ab}}$ in any of its index position forms.
[if x-space happens to be Cartesian, then raising and lowering an index on $X^{\mathbf{ab}}$ is free.]

But for T the rules are different:

Use $g'^{\mathbf{ab}}$ to raise and $g'_{\mathbf{ab}}$ to lower the *first* index of $T^{\mathbf{a}}{}_{\mathbf{b}}$ or $T_{\mathbf{a}}{}^{\mathbf{b}}$.
Use $g^{\mathbf{ab}}$ to raise and $g_{\mathbf{ab}}$ to lower the *second* index of $T^{\mathbf{a}}{}_{\mathbf{b}}$ or $T_{\mathbf{a}}{}^{\mathbf{b}}$.

This follows from the definitions

$$T^{\mathbf{a}}{}_{\mathbf{b}} \equiv \frac{\partial x'^{\mathbf{a}}}{\partial x^{\mathbf{b}}} \qquad T_{\mathbf{a}}{}^{\mathbf{b}} \equiv \frac{\partial x^{\mathbf{b}}}{\partial x'^{\mathbf{a}}}$$

since in both cases the second index is in the x-space and the first index is in the x' space.

Question: is $\delta_a{}^b$ a mixed tensor?

Yes, it is, provided you think of tensor δ being the same as the metric tensor g. Explanation follows:

Copy * from above, then replace X with δ :

$\delta'^{\mathbf{a}}{}_{\mathbf{b}} = T^{\mathbf{a}}{}_{\mathbf{a}'}\ T_{\mathbf{b}}{}^{\mathbf{b}'}\ \delta^{\mathbf{a}'}{}_{\mathbf{b}'} = T^{\mathbf{a}}{}_{\mathbf{a}'}\ T_{\mathbf{b}}{}^{\mathbf{a}'} = \delta^{\mathbf{a}}{}_{\mathbf{b}}$

I think the answer here is that yes, δ is a tensor, but in this sense

$\delta_{\mathbf{ij}} = g_{\mathbf{ij}}$ in the x-space
$\delta'_{\mathbf{ij}} = g'_{\mathbf{ij}}$ in the x'-space

We know that, $g^{\mathbf{i}}{}_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}}$ and $g'^{\mathbf{i}}{}_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}}$ where $\delta_{\mathbf{i},\mathbf{j}}$ is not the same as $\delta'_{\mathbf{ij}}$! Really, this δ tensor is identical to the metric tensor. You have to be careful because, for example $\delta'_{\mathbf{ij}} \neq \delta_{\mathbf{i},\mathbf{j}}$. In general, people

often use $\delta^{i}{}_{j}$ as synonym for $g^{i}{}_{j}$ but I would recommend never using δ'_{ij} . For example, when we are working in q-space, we don't show this prime, and we would then have $\delta_{ij} = g_{ij} \neq \delta_{i,j}$!!!!!

Extra Facts 1,2 and 3 Added:

These "facts"

Fact 1: Recall that

$$T^{b}{}_{c} \equiv \frac{\partial x'^{b}}{\partial x^{c}}$$

Therefore we have

$$(\partial_a T^{b}{}_{c}) = \partial_a \frac{\partial x'^{b}}{\partial x^{c}} = \frac{\partial^2 x'^{b}}{\partial x^{a} \partial x^{c}} = S^{b}{}_{ac} \qquad \text{// } \textit{symmetric} \text{ on a}\leftrightarrow\text{c !}$$

This symmetry is not very apparent when you write $(\partial_a T^{b}{}_{c})$.

Fact 2: Let's differentiate our two orthogonality conditions, might be useful later:

$\delta^{a}{}_{c} = T^{a}{}_{b}\, T_{c}{}^{b}$ sum on second index of T

$\delta_{b}{}^{c} = T^{a}{}_{b}\, T_{a}{}^{c}$ sum on first index of T

Therefore we apply ∂_s to get

$$0 = \partial_s(T^{a}{}_{b}\, T_{c}{}^{b}) = (\partial_s T^{a}{}_{b})\, T_{c}{}^{b} + T^{a}{}_{b}\, (\partial_s\, T_{c}{}^{b})$$

$$0 = \partial_s(T^{a}{}_{b}\, T_{a}{}^{c}) = (\partial_s T^{a}{}_{b})\, T_{a}{}^{c} + T^{a}{}_{b}\, (\partial_s\, T_{a}{}^{c})$$

which we can write as

$$S^{a}{}_{sb}\, T_{c}{}^{b} + T^{a}{}_{b}\, (\partial_s\, T_{c}{}^{b}) = 0 \qquad (\partial_s T^{a}{}_{b}) = S^{a}{}_{sb}$$

$$S^{a}{}_{sb}\, T_{a}{}^{c} + T^{a}{}_{b}\, (\partial_s\, T_{a}{}^{c}) = 0$$

where we have put in our visibly-symmetric object $S^{a}{}_{sb} = S^{a}{}_{bs}$. The two results above are different because in the first, we sum on one of the symmetric indices of S, b, while in the second we sum on the other index. Now apply $T_{a}{}^{e}$ to the first equation, and apply $T_{e}{}^{b}$ to the second equation to get

$$T_{a}{}^{e}\, S^{a}{}_{sb}\, T_{c}{}^{b} + T_{a}{}^{e}\, T^{a}{}_{b}\, (\partial_s\, T_{c}{}^{b}) = 0 \quad \Rightarrow \quad T_{a}{}^{e}\, S^{a}{}_{sb}\, T_{c}{}^{b} + \delta^{e}{}_{b}\, (\partial_s\, T_{c}{}^{b}) = 0$$

$$T_{e}{}^{b}\, S^{a}{}_{sb}\, T_{a}{}^{c} + T_{e}{}^{b}\, T^{a}{}_{b}\, (\partial_s\, T_{a}{}^{c}) = 0 \quad \Rightarrow \quad T_{e}{}^{b}\, S^{a}{}_{sb}\, T_{a}{}^{c} + \delta_{e}{}^{a}\, (\partial_s\, T_{a}{}^{c}) = 0$$

So we get these two results:

$$T_{a}^{e}\, S^{a}_{sb}\, T_{c}^{b} + (\partial_s\, T_{c}^{e}) = 0 \qquad \Rightarrow\ (\partial_s\, T_{c}^{e}) = -\, T_{a}^{e}\, S^{a}_{sb}\, T_{c}^{b}$$

$$T_{e}^{b}\, S^{a}_{sb}\, T_{a}^{c} + (\partial_s\, T_{e}^{c}) = 0 \qquad \Rightarrow\ (\partial_s\, T_{e}^{c}) = -\, T_{e}^{b}\, S^{a}_{sb}\, T_{a}^{c}$$

Now redo indices so both are in the same form

$$(\partial_s\, T_{c}^{e}) = -\, T_{a}^{e}\, S^{a}_{sb}\, T_{c}^{b} = -\, T_{a}^{e}\, T_{c}^{b}\, S^{a}_{sb}$$

$$(\partial_s\, T_{c}^{e}) = -\, T_{c}^{b}\, S^{a}_{sb}\, T_{a}^{e} = -\, T_{a}^{e}\, T_{c}^{b}\, S^{a}_{sb} \qquad \text{where } (\partial_s T^{a}_{b}) = S^{a}_{sb}$$

so these results are identical, and we end up with this possibly useful fact:

$$(\partial_s\, T_{c}^{e}) = -\, T_{a}^{e}\, T_{c}^{b}\, S^{a}_{sb} \qquad \text{where recall } S^{a}_{sb} = (\partial_s T^{a}_{b}) = \text{symmetric } s\leftrightarrow b$$

So, if we have a derivative of T^{*}_{*} , we replace with S at once. But derivative of T_{*}^{*} gives the messier result on the left.

Fact 3: Let's try a similar thing on $g_{ab}\, g^{ac} = \delta_{b}^{c}$ which we derive in the next section. Apply ∂_s :

$$\partial_s\, (g_{ab}\, g^{ac}) = 0 \qquad \Rightarrow \qquad (\partial_s\, g_{ab})\, g^{ac} + g_{ab}(\partial_s\, g^{ac}) = 0$$

Now apply g^{bd} to both terms to get

$$g^{bd}\, (\partial_s\, g_{ab})\, g^{ac} + g^{bd}\, g_{ab}(\partial_s\, g^{ac}) = 0 \qquad \Rightarrow \qquad g^{bd}\, (\partial_s\, g_{ab})\, g^{ac} + (\partial_s\, g^{dc}) = 0$$

so we find that

$$(\partial_s\, g^{dc}) = -\, g^{db}\, g^{ca}(\partial_s\, g_{ab})$$

which says we can "raise" both indices of the $(\partial_s\, g_{ab})$ in this manner. Note that you cannot willy-nilly raise and lower indices *through a derivative* ∂_s . For example, if you could the above would say

$$(\partial_s\, g^{dc}) = -\, g^{db}\, g^{c}_{\ a}(\partial_s\, g^{a}_{\ b})$$

So our result is very special and says you can do what it says, and notice the minus sign to boot!

Section 4.21 Addition, Multiplication and Contraction (164)

He notes that a rank 2 tensor can be split into symmetric and antisymmetric terms, but the phrase "skew symmetric" is used instead of the now more commonplace antisymmetric.

Now they implicitly start talking about **parity**, but it is poorly done and illogical to me. You cannot represent a parity transformation by the matrix $T^{a}{}_{b} = -\delta^{a}{}_{b}$. Such a transformation would change the sign of any vector.

Question: Is there a transformation $T^{a}{}_{b}$ that takes a vector A^{b} from a right handed to a left handed coordinate system? Note that $T^{a}{}_{b} = -\delta^{a}{}_{b}$ would negate *all* contravariant vectors, so that is not the one. The answer is that there is no matrix T that does this action!

pseudoscalars, etc

I think in order to deal with parity, you have to have a separate type of transformation that I will just indicate by an arrow. Thus, IF a tensor has the following transformation properties under the parity transformation, THEN it has the name shown. A tensor might be neither regular nor pseudo, but you could probably express it as the sum of two such tensors.

$S \rightarrow +S$	scalar
$S \rightarrow -S$	pseudoscalar
$V \rightarrow -V$	vector (normal polar)
$V \rightarrow +V$	pseudovector (axial)

This parity transformation applies to the tensor as a whole, and has nothing to do with contra or covariant indices on the tensor.

Now we can consider the cross product of two polar vectors

$$C = A \times B \rightarrow (-A) \times (-B) = A \times B = +C$$

so the result is an axial vector C.

Outer products, contraction, and inner products.

The next topic is ways to build a larger tensor from a smaller one. Consider

$$M^{ab} = A^{a}B^{b}$$

$$M'^{ab} = A'^{a}B'^{b} = (T^{a}{}_{a'}\ A^{a'})(\ T^{b}{}_{b'}\ B^{b'}) = T^{a}{}_{a'}\ T^{b}{}_{b'}\ A^{a'}\ B^{b'} = T^{a}{}_{a'}\ T^{b}{}_{b'}\ M^{a'b'}$$

This shows that the thing so assembled transforms as a second rank tensor. These assemblies in general are called **outer products**. You can imagine assembling two huge arbitrary mixed tensors together in this way, and the result is a mixed tensor.

Next, suppose we have a large mixed tensor and we sum on one (or more) of the index pairs up/down. Each such summation is called a **contraction**. The resulting object has lower rank by 2 for each contraction, and transforms as a tensor of that lower rank! Here is an example: recall from above we had

$$U'^{a}{}_{b}{}^{c} = T^{a}{}_{a'}\ T_{b}{}^{b'} T^{c}{}_{c'} U^{a'}{}_{b'}{}^{c'}$$

Now contract the a and b indices to get

$$U'^{a}{}_{a}{}^{c} = (T^{a}{}_{a'}\ T_{a}{}^{b'})T^{c}{}_{c'}\underline{U}^{a'}{}_{b}{}^{,c'} = \delta_{a}{}^{,b'}\ T^{c}{}_{c'}U^{a'}{}_{b}{}^{,c'} = T^{c}{}_{c'}U^{a'}{}_{a}{}^{,c'} = T^{c}{}_{c'}\ (U^{a}{}_{a}{}^{c'})$$

so the result is a rank one tensor we could call $Q^{c} = U'^{a}{}_{a}{}^{c}$.

This contraction can happen between indices of two separate tensors which are assembled together, for example

$A^{abc}B_{ade}$ = $Q^{bc}{}_{de}$ = a mixed tensor of rank 4.

In this case, we have an "**inner product**" of the two tensors with respect to the contracted index or indices.

The prototype example of this is the following. Start with

$$M^{m}{}_{n} = A^{m}B_{n}$$

which is a rank two mixed tensor. Contract the only index pair to get

$S = M^{m}{}_{m} = A^{m}B_{m}$ = a tensor of rank 2-2 = 0, a scalar.

Notice that we can NOT say this is the same as $A_{m}B^{m}$ because at this point, we have no connection between up and down indexed objects. Still, it looks like (A,B) inner product, and (A,A) looks like a norm, and we might start talking Hilbert Space at some point. Authors even go so far as to suggest a cosθ dot product thing.

Suppose an index-lowering tensor exists g_{ab}: associated components of a vector

Now finally we come to the idea of an **index-lowering tensor g_{ab}**. We are told to *imagine* that some symmetric rank 2 tensor g_{ab} exists such that $A_{a} = g_{ab}A^{b}$. If such a tensor exists, then the two kinds of A vectors are said to be "**associated**" with each other by this tensor g_{ab}. We are not calling this the metric tensor, just some tensor.

Now comes the big question. If we have such a g_{ab} that lowers an index, can we find a matrix g^{ab} that raises an index? As usual, we imply no connection between g with the lower and g with the upper indices. We have g_{ab} and we try to construct g^{ab}. This suggestion is made:

$$g^{ab} = \text{cofactor}(g_{ab})/\det(g_{ab})$$

Now, luckily, in my matrix binder in section 2 page 3 I have this theorem (from MM Chapter 12)

$$\Sigma_{p}\ A_{pq}\ \text{cof}(A_{pq'}) = \delta_{q,q'}\ \det(A)$$

which I derive right in those notes. In our current context, we would say

$$\Sigma_{p}\ g_{ab}\ \text{cof}(A_{ab'}) = \delta_{bb'}\ \det(g)$$

so let's define

$g^{ab} \equiv \text{cof}(g_{ab})/\det(g)$

If g_{ab} is symmetric, then g^{ab} is also symmetric from definition of a cofactor, since of course detA = $\det A^{T}$. We know from the above quoted theorem then that

$g_{ab}g^{ab'} = \delta_{b}{}^{b'}$ and g indices in any order since

Certainly g^{ab} looks like a symmetric contravariant rank 2 tensor, but a brute force proof would be involved, since det(g') is a function of the transformation T, etc.

Now consider

$g^{ab}A_{b} = g^{ab}(g_{bc}A^{c}) = (g^{ab}g_{bc})A^{c} = A^{a}$

so we now learn that g^{ab} raises indices. If we really have $g^{ab}A_{b} = A^{a}$, for any A, then this is a contraction rule, and M&M conclude that therefore g^{ab} must be a tensor.

So IF an index-lowering tensor g_{ab} exists, then of course g^{ab} also exists (assume detg ≠ 0 of course) and we can now "reverse tilt" anywhere we like: for example

$A\cdots^{a}{}_{\cdots}....B\cdots_{\cdots a\cdots} = g^{ab} A\cdots^{b}{}_{\cdots}....B\cdots_{\cdots a\cdots}$

$= g^{ab} g_{ab'} A\cdots^{b}{}_{\cdots}....B\cdots_{\cdots}{}^{b'}{}_{\cdots} = \delta^{b}{}_{b'} A\cdots^{b}{}_{\cdots}....B\cdots_{\cdots}{}^{b'}{}_{\cdots}$

$= A\cdots^{b}{}_{\cdots}....B\cdots_{\cdots}{}^{b}{}_{\cdots} = A\cdots^{a}{}_{\cdots}....B\cdots_{\cdots}{}^{a}{}_{\cdots}$

so we have shown that your can pick any contracted index pair and reverse the tilt by raising one and lowering the other, and the result is the same. Clearly you could do this on many at once. All ASSUMING that a lowering g_{ab} tensor exists.

So, case of interest in this context is

$A_{m}A^{m} = A^{m}A_{m}$ to be true, only requires that a symmetric lowering tensor $g_{\mu\nu}$ exist

so this looks even more like a scalar product and norm. Notice that one application would be

$T^{a}{}_{b} = g^{aa'}g_{bb'}T_{a'}{}^{b'}$

which means if you know $T_{a}{}^{b}$ you can compute $T^{a}{}_{b}$. Recall that in general these are unrelated objects.

Section 4.22 Differentiation of Tensors (167)

You might think that the following would be a tensor

$\partial_{a}V_{b}$

Let's just test it:

$$(\partial'_a V'_b) = T_a^{\ d}\partial_d (T_b^{\ c} V_c) = T_a^{\ d}\, T_b^{\ c}\, (\partial_d V_c) + T_a^{\ d}(\partial_d\, T_b^{\ c}) V_c$$

so it goes like a tensor, but there is an extra term! Now for the first time, we are taking note of this fact:

$$T^a_{\ b} = T^a_{\ b}(x^\mu)$$

that is, the transformation can be a function of the coordinates. This is not the case for special relativity say where T might be a rotation or a boost, but it is the case in **general relativity**. Up to now, we have not cared about this x^μ dependence, but when we talk differentiation, we have to worry about it.

Christoffel symbols and comma notation

Now let's digress to define the **Christoffel symbols**

$$[ab,c] \equiv \tfrac{1}{2}\,(\ \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\ ) \qquad \text{// symmetric in ab}$$

$$\{ab,c\} \equiv g^{cd}\,[ab,d] \qquad \text{// symmetric in ab}$$

Thus, the {...} thing is a "raise" on the last argument of [...], but neither of these things is a tensor, so be careful. In any event,

$$\{ab,c\} = \tfrac{1}{2}\, g^{cd}(\ \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}\ ) \ \neq\ \tfrac{1}{2}\,(\ \partial_a g_b^{\ d} + \partial_b g_a^{\ d} - \partial^d g_{ab}\ )$$

M&M then go through a huge mess of math to arrive at this result: the following object transforms as a tensor:

$$\partial_a V_b - \{ab,c\} V_c \ \equiv\ V_{b,a}$$

where we now see "**the comma notation**" where the index after the comma says what you are differentiating with respect to in the first term. This {....} stuff is what you have to add to $\partial_a V_b$ to make it become a tensor, so neither $\partial_a V_b$ nor $\{ab,c\}V_c$ is a tensor, only the difference.

Summary of Proof of Covariance of $V_{b,a}$ (raw details in separate document)

I did a brute force proof on my own to show that $V_{b,a}$ transforms like a tensor (see M&M Chapter 5 "support" doc). The proof was very messy, but I just wanted to try it without looking at the M&M solution. I learned many useful things along the way. Here is an outline of my proof;

(1) By requiring the tensor transformation, I was quickly led to the need for the following to be true:

$$\{ab,c\}'V'_c = T_a^{\ d}(\partial_d\, T_b^{\ c}) V_c - T_a^{\ d}\, T_b^{\ c}\{dc,e\} V_e$$

(2) I was disturbed by the fact that $T_a{}^d(\partial_d T_b{}^c)V_c$ did not appear symmetric in a,b, so this led me off on a digression where I found that I could rewrite this term in a manifestly symmetric form:

$$T_a{}^d(\partial_d T_b{}^c)V_c = -T_a{}^s T_b{}^g T_f{}^c S^f{}_{sg} V_c \qquad \text{where } S^f{}_{sg} = (\partial_s T^f{}_g) = \frac{\partial^2 x'^f}{\partial x^s \partial x^g} = \text{sym(s,g)}$$

Showing this required me to derive and then use a little "fact #2" based on $\partial_s(T^a{}_b\, T_c{}^b) = 0$, namely,

$$(\partial_s T_c{}^e) = -T_a{}^e T_c{}^b\ (\partial_s T^a{}_b) = -T_a{}^e T_c{}^b S^a{}_{sb} \qquad \text{"Fact 2"}$$

This "fact" showed how to relate the derivative of $T_*{}^*$ to that of $T^*{}_*$. So the problem then became to show this to be true:

$$\{ab,c\}'V'_c = -T_a{}^s T_b{}^g T_f{}^c S^f{}_{sg} V_c + T_a{}^d T_b{}^c \{dc,e\} V_e$$
$$\text{LHS} = \text{RHS}_2 + \text{RHS}_1$$

(3) I then expanded the LHS and ended up with two big terms $\text{LHS}_1 + \text{LHS}_2$. LHS_1 contained the three terms having factors of the form $(\partial_* g_{**})$. I then compared this to RHS_1,

$$\text{LHS}_1 = \tfrac{1}{2} g^{eh} * V_e \{ T_a{}^{a'} T_b{}^{b'} (\partial_{a'} g_{b'h}) + T_b{}^{a'} T_a{}^{b'} (\partial_{a'} g_{b'h}) - T_a{}^{b'} T_b{}^{c'} (\partial_h g_{b'c'}) \}$$

$$\text{RHS}_1 = \tfrac{1}{2} T_a{}^d T_b{}^c\ g^{es} (\partial_d g_{cs} + \partial_c g_{ds} - \partial_s g_{dc}) V_e$$

and, by renaming indices, it was easy to show these terms were the same. So what remained to show was that $\text{LHS}_2 = \text{RHS}_2$. At this point then I had:

$$\text{RHS}_2 = -T_a{}^s T_b{}^g T_f{}^c S^f{}_{sg} V_c$$

$$\begin{aligned}\text{LHS}_2 = {} & + \tfrac{1}{2} T^d{}_h\ g^{eh} * V_e \{ T_a{}^{a'} T_b{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} + T_b{}^{a'} T_a{}^{b'} (\partial_{a'} T_d{}^{c'}) g_{b'c'} \} \\ & + \tfrac{1}{2} g^{eh} * V_e * \\ & \{ T_a{}^{a'} (\partial_{a'} T_b{}^{b'})\, g_{b'h} + T_b{}^{a'} (\partial_{a'} T_a{}^{b'})\, g_{b'h} \\ & - (\partial_h T_a{}^{b'}) T_b{}^{c'} g_{b'c'} - T_a{}^{b'} (\partial_h T_b{}^{c'}) g_{b'c'} \}\end{aligned}$$

where it took quite a bit of routine but detailed work to get this form for LHS_2. There are 6 terms here, and each pair of terms shows the a,b symmetry. BUT, it was hard for me to see how this was going to equal RHS_2 because I could see no cancellations of terms.

(4) I then processed LHS_2 into the following form, exposing the $(\partial_s T_d{}^c)$ type factors to which I knew I could apply Fact #2. This gave:

$$\text{LHS}_2 = + \tfrac{1}{2} T^d{}_h\, V^h \{ T_a{}^s T_{bc} + T_b{}^s T_{ac} \} (\partial_s T_d{}^c) \qquad (\partial_s T_d{}^c) = -T_f{}^c T_d{}^g S^f{}_{sg}$$

$$+ \tfrac{1}{2} [V_c\, T_a{}^s - V^s T_{ac}] (\partial_s T_b{}^c) \qquad (\partial_s T_b{}^c) = -T_f{}^c T_b{}^g S^f{}_{sg}$$

$$+ \tfrac{1}{2}\,[\;\; V_c\, T_b{}^s \;\; - \;\; V^s\, T_{bc}\;]\,(\partial_s T_a{}^c) \qquad\qquad (\partial_s T_a{}^c) = -\, T_f{}^c\, T_a{}^g\, S^f{}_{sg}$$

(5) I then installed Fact #2 three times on the right, and was able the express LHS_2 as the sum of 6 terms:

$$1 = -\,\tfrac{1}{2} V^g\; T_a{}^s T_{bc} \quad T_f{}^c\, S^f{}_{sg} \qquad\qquad T_f{}^c\, S^f{}_{sg} \;=\; \frac{\partial x^c}{\partial x'^f}\;\frac{\partial^2 x'^f}{\partial x^s \partial x^g} \;=\; \text{not simple}$$

$$2 = -\,\tfrac{1}{2} V^g\; T_b{}^s T_{ac} \quad T_f{}^c\, S^f{}_{sg}$$

$$3 = -\,\tfrac{1}{2}\, V_c\;\; T_a{}^s\, T_b{}^g \;\; T_f{}^c\, S^f{}_{sg}$$

$$4 = +\;\tfrac{1}{2}\, V^s\, T_{ac}\, T_b{}^g \;\; T_f{}^c\, S^f{}_{sg}$$

$$5 = -\,\tfrac{1}{2}\, V_c\, T_b{}^s\, T_a{}^g \;\; T_f{}^c\, S^f{}_{sg}$$

$$6 = \;+\tfrac{1}{2}\, V^s\, T_{bc}\, T_a{}^g \;\; T_f{}^c\, S^f{}_{sg}$$

(6) At this point, I made use of the symmetry of $S^f{}_{sg}$ to rewrite terms 4,5 and 6, then getting

$$1 = -\,\tfrac{1}{2} V^g\; T_a{}^s T_{bc} \quad T_f{}^c\, S^f{}_{sg}$$

$$2 = -\,\tfrac{1}{2} V^g\; T_b{}^s T_{ac} \quad T_f{}^c\, S^f{}_{sg}$$

$$3 = -\,\tfrac{1}{2}\, V_c\;\; T_a{}^s\, T_b{}^g \;\; T_f{}^c\, S^f{}_{sg}$$

$$4 = +\;\tfrac{1}{2}\, V^g\, T_{ac}\, T_b{}^s \;\; T_f{}^c\, S^f{}_{sg}$$

$$5 = -\,\tfrac{1}{2}\, V_c\, T_b{}^g\, T_a{}^s \;\; T_f{}^c\, S^f{}_{sg}$$

$$6 = \;+\tfrac{1}{2}\, V^g\, T_{bc}\, T_a{}^s \;\; T_f{}^c\, S^f{}_{sg}$$

Now it became clear that terms 1 and 6 cancel, and also terms 2 and 4 cancel, and terms 3 and 5 are equal, so our total result is this:

$$LHS_2 = \;\; 3{+}5 \;\; = -\, V_c\, T_b{}^g\, T_a{}^s \;\; T_f{}^c\, S^f{}_{sg} \;\; = \;\; RHS_2 \;\; !$$

and this concluded the proof.

(7) Among the "lessons learned" by doing the above, I realized that you cannot willy-nilly raise and lower indices on tensors *that are acted upon by a derivative*. For example, consider this simple example

$$P_{sc} = A^d\,(\partial_s T_{dc}) \;\neq\; A_d\,(\partial_s T^d{}_c) \qquad\qquad \text{// where I am doing a "tilt change"}$$

In fact, calculation shows that

$$P_{sc} = A^d\,(\partial_s T_{dc}) = A^d\,(\partial_s g_{de} T^e{}_c) = A^d\;\, g_{de}\,(\partial_s T^e{}_c) + A^d (\partial_s g_{de})\, T^e{}_c$$

$$= \;\; A_e\,(\partial_s T^e{}_c) + A^d (\partial_s g_{de})\, T^e{}_c \;\; = A_d\,(\partial_s T^d{}_c) \;\; + \text{extra term.}$$

Also, not that, just as $\partial_s V_d$ is not a tensor (our main proof!) , so also $(\partial_s T_{dc})$ is not a tensor. Thus, even though T and A are tensors, Psc is not a tensor.

Another more obvious example is this.

$$A_c \;=\; (\partial_s g_{sc}) \qquad \not\Rightarrow \qquad A^c = (\partial_s g_s{}^c)$$

If it did, we would conclude from the right equation that $A^c = 0$ since $\partial_s(\delta_s{}^c) = 0$ and we would then think that $A_c = g_{cd}A^d = 0$, which is not true. Also, the fragment A_c is not a tensor!

Selected Sections from Chapter 5:

Section 5.1. Curvilinear Coordinates.

Repairs shown in red done 9/5/08

This section is the main act of this chapter, it is all on one page 173, and sadly it is presented poorly so I need all these notes to clean it up. I am changing notation to everything is contravariant upper index.

Imagine we have our starting Cartesian coordinates x^i and we want to use some q^i which are related as shown, that is, $q^i = q^i(r^j)$. { later we think $q_i = x'_i$) Then of course we know that

$dx^i = \Sigma_k(\partial x^i/\partial q^k)\, dq^k = \Sigma_k(\partial x^i/\partial x'^k)\, dx'^k$ // page 5.1 equation A

which we can write in matrix form as (I don't know the official name for the T_{ik} object, even in Weinberg there is no name, so lets just call it the Partial Derivative Tensor or the Coordinate Transformation Tensor , T for Transformation). Recall our forward and inverse transforms on coordinates, as noted above,

$x'^a = T^a{}_b\, x^b$ transform // see correction note earlier!
$x^a = T_b{}^a\, x'^b$ inverse transform

Thus we have

$dx^i = \Sigma_k\, T_k{}^i\, dq^k$ $T_k{}^i = (\partial x^i/\partial q^k) = (\partial x^i/\partial x'^k)$ = the covariant transformation tensor

The distance between two close points is given by

$ds^2 = \Sigma_i\, (dx^i)^2 = \Sigma_i\, (\Sigma_k\, T_k{}^i\, dq^k)(\Sigma_m\, T_m{}^i\, dq^m) = \Sigma_k\, \Sigma_m\, (\Sigma_i\, T_k{}^i\, T_m{}^i\,)dq^k\, dq^m = \Sigma_k\, \Sigma_m\, G_{km}\, dq^k\, dq^m$

The metric tensor in a space transformed to from Cartesian space, relation between g'_{ij} and $T_a{}^b$

where we have defined our **"metric tensor" G** (same as g) according to

$G_{km} = \Sigma_i\, T_k{}^i\, T_m{}^i = \Sigma_i\, (\partial x^i/\partial q^k)\, (\partial x^i/\partial q^m)$ // page 192 5-4a

Notice the G is symmetric, so if we separate the diagonal terms we can write,

$ds^2 = \Sigma_k\, G_{kk}\, dq^k\, dq^k + \Sigma_{k\neq m}\, G_{km}\, dq^k\, dq^m = \Sigma_k\, G_{kk}\, dq^k\, dq^k + 2\, \Sigma_{k<m}\, G_{km}\, dq^k\, dq^m$

We could also write ds^2 in dyadic or vector notation in this way

$ds^2 = d\mathbf{q} \bullet G \bullet d\mathbf{q} = d\mathbf{q}^T G\, d\mathbf{q}$

Now we pause to comment on an *immense confusion* M&M have caused. They use these symbols:

$(Q_{ii})^2$ [written also as Q^2_{ii}] to represent the quantity G_{ii}
but
Q_{ij} to represent the quantity G_{ij} $\quad i \neq j$

This is completely illogical to anyone who uses matrices, because if $Q_{ij} = G_{ij}$, we expect $Q_{ii} = G_{ii}$. I cannot explain why they did this stupid thing. But given it, we can then say

$$ds^2 = \Sigma_k\ Q_{kk}^2\ (dq^k)^2\ + \Sigma_{k\neq m}\ Q_{km}\ dq^k\ dq^m\ = \Sigma_k\ Q_{kk}^2\ (dq^k)^2\ + 2\ \Sigma_{k<m}\ Q_{km}\ dq^k\ dq^m$$

and this is what you see in equation 5.3. Now to add more to the confusion, they define

$Q_i = Q_{ii}$ by which they mean $Q_i = \sqrt{Q^2_{ii}} = \sqrt{G_{ii}}$

So now at least we get rid of their horrible Q^2_{ii} thing, and we now have

$$ds^2 = \Sigma_k\ Q_k^2\ (dq^k)^2\ + \Sigma_{k\neq m}\ G_{km}\ dq^k\ dq^m\ = \Sigma_k\ Q_k^2\ (dq^k)^2\ + 2\ \Sigma_{k<m}\ G_{km}\ dq^k\ dq^m$$

It seems pretty clear that if you only allow one dq^k to vary, you will have

$$ds^2 = Q_k^2\ (dq^k)^2\ = G_{kk}(dq^k)^2$$

so we can talk about a distance without regard to sign as

$ds^k = Q_k\ dq^k\ = \sqrt{G_{kk}}\ dq^k$ // equation **5.5** and 5-5a page 192

For example, in spherical coordinates, if we start at some point (r,θ,φ) and allow only θ to vary, the ds_θ is a small distance down along the great circle of radius r in the plane φ = constant. This line is an example of a "**coordinate line**" and is a curve in the Cartesian space along which only one q_i coordinate varies. This curve is of course the intersection between the sphere r = constant, and the plane φ = constant. These surfaces are called **coordinate surfaces**.

Now we come to the next huge confusion. Let's first go back to where we started,

$dx^i = \Sigma_k\ T_k^i\ dq^k$ $\qquad T_k^i = (\partial x^i/\partial q^k)$

and write as a vector equation

$d\mathbf{x} = T\, d\mathbf{q}$

In our variation above, let's rewrite and assume that $\delta q_k > 0$ for a single coordinate only, so

$$d\mathbf{q}^{(k)} = (0,0,....\delta q^k....0) \qquad \text{or } dq^{(k)}{}_i = \delta_{i,k}\,\delta q^k$$

In this case we find that

$$d\mathbf{x}^{(k)} = T\, d\mathbf{q}^{(k)}$$

or

$$dx^{(k)m} = \Sigma_i\, T_i{}^m\, dq^{(k)i} = \Sigma_i\, T_i{}^m\, (\delta_{i,k}\,\delta q^k) = T_k{}^m\,\delta q^k$$

What is the length of this vector?

$$|\, d\mathbf{x}^{(k)}\,|^2 = \Sigma_m\, (T_k{}^m\,\delta q^k)^2 = (\delta q^k)^2\, \Sigma_m\, (T_k{}^m)^2 = (\delta q^k)^2\, G_{kk} = (\delta q^k)^2\, Q_k{}^2$$

since $G_{kk} = \Sigma_m\, T_k{}^m\, T_k{}^m$. So we learn that $|\, d\mathbf{x}^{(k)}\,| = \delta q^k\, Q_k = \delta q^k \sqrt{G_{kk}}$.

Now, consider doing two separate variations of the form

$$dq^{(i)m} = \delta^m{}_i\,\delta q^i$$
$$dq^{(j)m} = \delta^m{}_j\,\delta q^j$$

where in the first case we vary only q_i, and in the second only q_j . What is $d\mathbf{x}^{(i)} \bullet d\mathbf{x}^{(j)}$?

$$d\mathbf{x}^{(i)} \bullet d\mathbf{x}^{(j)} = \Sigma_m\, dx^{(i)m}\, dx^{(j)m} = \Sigma_m\, T^m{}_i\,\delta q^i\, T^m{}_j\,\delta q^j = \Sigma_m\, T^m{}_i\, T^m{}_j\,\delta q^i\,\delta q^j = G_{ij}\,\delta q^i\,\delta q^j$$

But we know that

$$|\, d\mathbf{x}^{(j)}\,| = \delta q^j\, Q_j$$
$$|\, d\mathbf{x}^{(i)}\,| = \delta q^i\, Q_i$$

Therefore, we can compute the cosine of the angle between our two vectors as

$$\cos\theta_{ij} = d\mathbf{x}^{(i)} \bullet d\mathbf{x}^{(j)} / (|\, d\mathbf{x}^{(j)}\,|\,|\, d\mathbf{x}^{(i)}\,|) = G_{ij}\,\delta q^i\,\delta q^j / (\delta q^j\, Q_j\,\delta q^i\, Q_i) = G_{ij}/(Q_j\, Q_i)$$

and FINALLY we have derived page 173 equation E, since $G_{ij} = Q_{ij}$.

Now that we have this result, let's try to parse their enigmatic text. They say,

" the cosine of the angle between ds_1 and dz is $(dz/dq_1)(dq_1/ds_1)$ ..."

In this phrase, there is the implication that ds_1 and dz are vectors, otherwise you cannot have an angle between them. But the authors have not written any vectors anywhere for these things. I have to conclude this is just junk text, garbage!

Orthogonal Systems: We want to consider systems in which $G_{ij} = 0$ for off diagonal elements. In other words, a system with a diagonal metric tensor G. Fine. Yes, that means $\theta_{ij} = 0$. This means that the angles between the coordinate lines at any given point are 90 degrees. For example, in spherical coordinates, our three little directions $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ are perpendicular for any point we pick in the space.

Now, in an orthogonal system we have this for distance,

$$ds^2 = \Sigma_k \; G_{kk}\, dq^k\, dq^k$$

What can we say about **surface elements**? We know that $d\mathbf{x}^{(i)}$ and $d\mathbf{x}^{(j)}$ are perpendicular, so my inclination would be to define

$$d\mathbf{S}_{(ij)} = d\mathbf{x}^{(i)} \times d\mathbf{x}^{(j)} \qquad \text{// a vector equation}$$

but of course the angle is 90 degrees, so we know that

$$|\, d\mathbf{S}_{(ij)} \,| = |\, d\mathbf{x}^{(i)} \,|\, |\, d\mathbf{x}^{(j)} \,| \;\; = \delta q^i \;\; Q_i \, \delta q^j \;\; Q_j = Q_i \, Q_j \, \delta q^i \, \delta q^j$$

which we can write in magnitudes as

$$dS_{(ij)} \;\; = Q_i \, Q_j \, dq^i \, dq^j \qquad \text{// equation 5.6}$$

Finally, a little **volume element** would be three of these multiplied together

$$|\, d\mathbf{x}^{(i)} \,|\, |\, d\mathbf{x}^{(j)} \,|\, |\, d\mathbf{x}^{(k)} \,| \;\; = Q_i \, Q_j \, Q_k \, \delta q^i \, \delta q^j \, \delta q^k \qquad \text{// equation 5.7}$$

What a battle just parsing a single page of M&M, page 173 !

Notice that if things are not orthogonal, then the area and volume have to have extra factors in them.

At this point, I am going to skip to Section 5.16 because it continues on the above discussion.

Section 5.16. Non-orthogonal Curvilinear Coordinates: tensor relations

OK, they now do what I did above, and they refer to my G_{ij} as g_{ij}. Notice that all coordinates now have upper indices, so we are doing **contravariant notation** as usual, and that word does in fact appear.

The two sets of basis vectors $\mathbf{e}_i$ and $\mathbf{e}^i$ and their dot product relations

Now our first action is to define some "base *vectors*" like so:

$$\mathbf{e}_{(i)} \equiv \partial \mathbf{r} / \partial q^i$$

Note that these are in general **not** unit vectors, and they are **not** orthogonal! The direction of $\mathbf{e}_{(i)}$ is along the curve where q^i varies. That is, we are changing only dq^i and leaving all other dq_j fixed, so we move

in 3D space where this change takes us. For example, we might have $\mathbf{e}_{(i)} = \hat{\theta}$ and it points in the direction of increasing θ such that r and φ remain constant. That is what **base vectors** are (although not normalized).

And then according to page 192 D we have

$d\mathbf{r} = \mathbf{e}_{(i)} dq^i$ // bold are the vectors, implied sum

Now define ***upper label*** **basis vectors** like so

$\mathbf{e}^{(i)} \equiv \frac{1}{2} \varepsilon_{ijk} \mathbf{e}_{(j)} \times \mathbf{e}_{(k)} / v$ // sums implied on j and k

where

$v = \mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)}$ = cyclic = -anticyclic

so can write

$\mathbf{e}_{(i)} \bullet \mathbf{e}_{(j)} \times \mathbf{e}_{(k)} = \varepsilon_{ijk} v.$

Then examine:

$\mathbf{e}_{(n)} \bullet \mathbf{e}^{(m)} = \mathbf{e}_{(n)} \bullet [\frac{1}{2} \varepsilon_{mjk} \mathbf{e}_{(j)} \times \mathbf{e}_{(k)}] / v = (\frac{1}{2} \varepsilon_{mjk} / v) \{ \mathbf{e}_{(n)} \bullet \mathbf{e}_{(j)} \times \mathbf{e}_{(k)} \}$

$= (\frac{1}{2} \varepsilon_{mjk} / v) \{ \varepsilon_{njk} v \} = \frac{1}{2} \varepsilon_{mjk} \varepsilon_{njk} = \frac{1}{2} * 2\delta_{m,n} = \delta_{m,n}$

where we use a reduction from our Levi-Civita page. Thus is proved 5.79 on page 193.

So what is the point here? The covariant basis vectors are orthogonal to the contravariant ones. How does that help me?

Now, let us conjecture the existence of some *covariant* vector components dq_i such that 5-81 is true

$d\mathbf{r} = \mathbf{e}^{(i)} dq_i$ earlier we had $d\mathbf{r} = \mathbf{e}_{(i)} dq^i$

It must then be true that

$\mathbf{e}^{(i)} dq_i = \mathbf{e}_{(i)} dq^i$ (1)

Now dot equation (1) on the left with $\mathbf{e}_{(j)}$ and, using 5-79 above, we find

$dq_j = [\mathbf{e}_{(j)} \bullet \mathbf{e}_{(i)}] dq^i = G_{ji} dq^i$ where $G_{ij} \equiv [\mathbf{e}_{(i)} \bullet \mathbf{e}_{(j)}]$

Or, dot equation (1) from the right with $\mathbf{e}^{(j)}$ to get

$[\mathbf{e}^{(j)} \bullet \mathbf{e}^{(i)}] dq_i = dq^j = G^{ji} dq_i$ where $G^{ij} \equiv [\mathbf{e}^{(i)} \bullet \mathbf{e}^{(j)}]$

Think about what these things are saying! Suppose you vary only dq^i (for some specific i). Then it turns out you will have varied probably *all* the dq_j's. The amount each dq_j (covariant!) varies is given by G_{ji} times the initial dq^i. So this G_{ji} is a cross effect, and of course we see it is $\mathbf{e}_{(j)} \bullet \mathbf{e}_{(i)}$.

So several things have happened all at once here. We have defined two matrices G_{ij} and G^{ij} as shown, and as far as we know they are different. Second, we conjectured the existence of the components dq_i, but now we know exactly how to compute them , ie, $dq_j = G_{ji}\, dq^i$, so they must exist.

Nothing in the previous paragraph requires our vector to be d**r,** it could be any vector **A**, and we would then say

$\mathbf{A} = \mathbf{e}^{(i)} a_i = \mathbf{e}_{(i)} a^i$ and $a_i = G_{ij}\, a^j$ and $a^i = G^{ij}\, a_j$ // 5-87, 5-88

and $\mathbf{A} \bullet \mathbf{e}_{(i)} = a_i$ and $\mathbf{A} \bullet \mathbf{e}^{(i)} = a^i$

and $\mathbf{A} = a_i\, \mathbf{e}^{(i)} = (\mathbf{A} \bullet \mathbf{e}_{(i)})\, \mathbf{e}^{(i)} = a^i\, \mathbf{e}_{(i)} = (\mathbf{A} \bullet \mathbf{e}^{(i)})\, \mathbf{e}_{(i)}$ // 5-89

just to write down all the things M&M are writing down on page 194. Notice that we are using component names a_i and a^i rather than A_i and A^i because our basis vectors are not unit vectors, and we want to save the component names A_i and A^i for components against unit vectors!

But now let's return to the differential vectors where we have:

Note Added: What about **derivatives of the basis vectors**?

$\mathbf{e}_i \equiv \partial \mathbf{r} / \partial q^i$

$\partial \mathbf{e}_i / \partial q^j = \partial^2 \mathbf{r} / \partial q^j \partial q^i$

and I don't have a good formula for this. Besides, I am more interested in $\partial \hat{\mathbf{e}}_i / \partial q^j$, so let's quietly drop this subject.

Linear, Area and Volume Elements

$d\mathbf{r} = \mathbf{e}^{(i)}\, dq_i = \mathbf{e}_{(i)}\, dq^i$

we can compute:

$ds^2 = d\mathbf{r} \bullet d\mathbf{r} = [\mathbf{e}_{(i)}\, dq^i] \bullet [\mathbf{e}^{(j)} dq_j] = \delta_{ij}\, dq^i\, dq_j = dq^i\, dq_i = dq^i\, (G_{ij}\, dq^j) = G_{ij}\, dq^i\, dq^j$

If we compare this to our definition of the metric tensor g_{ij} in 5-3a, **we discover that**

$G_{ij} = g_{ij}$ = the metric tensor (with both lower indices)

Similarly, we can write

$\mathbf{A}\bullet\mathbf{A} = \mathbf{e}^{(i)} a_i \bullet \mathbf{e}_{(j)} a^j = a_i a^i = g_{ij}\, a^i\, a^j = g^{ij}\, a_i\, a_j$

In particular, notice that

$\mathbf{e}_{(i)} \bullet \mathbf{e}_{(i)} = g_{ii}$ // no sum implied

$|\mathbf{e}_{(i)}| = \sqrt{g_{ii}}$

$\mathbf{u}_i = \mathbf{e}_{(i)} / \sqrt{g_{ii}}$ = lower index unit vector

We can normalize either set of "basis vectors" as shown in page 194 B and. In terms of these unit vectors, we can write

$A = \mathbf{e}_{(i)} a^i = \mathbf{u}_i (a^i \sqrt{g_{ii}}) = \mathbf{u}_i A^i$ and similarly for unit vectors $\mathbf{u}^i$.

Comments: In this section above, M&M define a set of contravariant basis vectors, and a set of covariant ones as well. You can "expand" a vector onto either set, but your expansion coefficients are the opposite type to the basis vectors. That is to say

$\mathbf{A} = a_i\, \mathbf{e}^{(i)} = a^i\, \mathbf{e}_{(i)}$

Because the contra and co basis vectors are orthogonal, you can isolate one of the components in the obvious manner, all as shown above, such as $\mathbf{A} \bullet \mathbf{e}_{(i)} = a_i$.

I am not used to having two sets of basis vectors! To which set does $\hat{r}$ belong in sphericals?

Surface Area Element:

Above we said $d\mathbf{r} = \mathbf{e}^{(i)} dq_i$ for some arbitrary $d\mathbf{r}$. Now define:

$d\mathbf{s}_i = \mathbf{e}_{(i)}\, dq^i$ // no sum implied

These are small vectors ALONG the non-orthogonal non-unit basis vectors. Then define the vector differential area to be

$d\mathbf{S}_1 = d\mathbf{s}_2 \times d\mathbf{s}_3$ or $d\mathbf{S}_i = \tfrac{1}{2}\, \varepsilon_{ijk}\, d\mathbf{s}_j \times d\mathbf{s}_k$ // sums implied

We know this is a parallelepiped with area dS_1 which is

$dS_1 = |\mathbf{e}_{(2)} \times \mathbf{e}_{(3)}|\, dq^2\, dq^3$

But

$$\mathbf{e}_{(2)} \times \mathbf{e}_{(3)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)} = (\mathbf{e}_{(2)} \bullet \mathbf{e}_{(2)})\ (\mathbf{e}_{(3)} \bullet \mathbf{e}_{(3)}) - (\mathbf{e}_{(2)} \bullet \mathbf{e}_{(3)})\ (\mathbf{e}_{(3)} \bullet \mathbf{e}_{(2)})$$

$= g_{22}g_{33} - g_{23}^2$

so we conclude that

$$dS_1 = \sqrt{g_{22}g_{33} - g_{23}^2}\ dq^2\, dq^3 \qquad \text{// 5-6a}$$

and in general

$$dS_i = \tfrac{1}{2}\, \varepsilon_{ijk}\ \sqrt{g_{jj}g_{kk} - g_{jk}^2}\ dq^j\, dq^k \qquad \text{// sums implied}$$

Volume Element.

$$dV = d\mathbf{s}_1 \bullet d\mathbf{S}_1 = d\mathbf{s}_1 \bullet d\mathbf{s}_2 \times d\mathbf{s}_3 = \mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)}\ dq^1\, dq^2\, dq^3 = v\, dq^1\, dq^2\, dq^3$$

where v is the volume of the parallelepiped. The claim is that $v^2 = \det(g)$ so we get

$$dV = \sqrt{\det(g)}\ dq^1\, dq^2\, dq^3$$

I prove this in the next section, and have just ignored the book's proof because I wanted to do it my own way. Trying to sharpen the tools which have become very dull.

Here is my proof that det(g) = v^2

First, we know that

$$v^2 = [\mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)}]^2$$

and we know that

$$\det(g) = \varepsilon_{ijk}\, g_{1i}g_{2j}g_{3k} \qquad \text{// sums implied}$$

$$g_{ij} \equiv \mathbf{e}_{(i)} \bullet \mathbf{e}_{(j)}$$

So we can write

$$\det(g) = \varepsilon_{ijk}\, g_{1i}g_{2j}g_{3k} = \varepsilon_{ijk}\, (\mathbf{e}_{(1)} \bullet \mathbf{e}_{(i)})\, (\mathbf{e}_{(2)} \bullet \mathbf{e}_{(j)})\, (\mathbf{e}_{(3)} \bullet \mathbf{e}_{(k)})$$

so **we want to show that**

$$\det(g) = v^2$$

or

$$\varepsilon_{ijk}\, (\mathbf{e}_{(1)} \bullet \mathbf{e}_{(i)})\, (\mathbf{e}_{(2)} \bullet \mathbf{e}_{(j)})\, (\mathbf{e}_{(3)} \bullet \mathbf{e}_{(k)}) = [\mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)}]^2$$

or

LHS = RHS

OK, here comes the brute force method that has to work! For a shorthand notation, let's write

$\mathbf{e}_{(1)} = \mathbf{1}$ and $\mathbf{e}_{(j)} = \mathbf{j}$, etc. Then we want to show that LHS = RHS or

$$\varepsilon_{ijk}\,(\mathbf{1}\bullet \mathbf{i})\,(\mathbf{2}\bullet \mathbf{j})\,(\mathbf{3}\bullet \mathbf{k}) \;=\; [\mathbf{1}\bullet \mathbf{2} \times \mathbf{3}]^2$$

or, showing all the vector components,

$$\varepsilon_{ijk}\,(\mathbf{1}_m\,\mathbf{i}_m)\,(\mathbf{2}_n\,\mathbf{j}_n)\,(\mathbf{3}_s\,\mathbf{k}_s) \;=\; [\mathbf{1}_m\varepsilon_{mns}\mathbf{2}_n\,\mathbf{3}_s]\;\;[\mathbf{1}_a\varepsilon_{abc}\mathbf{2}_b\,\mathbf{3}_c]$$

Now rewrite the second factor on the right as

$$[\mathbf{1}_a\varepsilon_{abc}\mathbf{2}_b\,\mathbf{3}_c] \;=\; 1/6 * \varepsilon_{ijk}\;\;[\mathbf{i}_a\,\varepsilon_{abc}\;\mathbf{j}_b\,\mathbf{k}_c]$$

The reason this is right is that $[\mathbf{i}_a\,\varepsilon_{abc}\;\mathbf{j}_b\,\mathbf{k}_c] = F_{ijk}$ is a totally antisymmetric rank-3 tensor. For example, if we swap i and j, we compensate by swapping a and b, and then we undo that by doing a swap on the ab indices of the ε tensor, but that makes a minus sign. So, the idea is this

$$F_{123} = (1/6)\,\varepsilon_{ijk}F_{ijk}$$

because all 6 non-zero terms in the sum are identical. So we now have to prove the following:

$$\varepsilon_{ijk}\,(\mathbf{1}_m\,\mathbf{i}_m)\,(\mathbf{2}_n\,\mathbf{j}_n)\,(\mathbf{3}_s\,\mathbf{k}_s) = \varepsilon_{ijk}\,(\mathbf{1}_m\,\mathbf{i}_a)\,(\mathbf{2}_n\,\mathbf{j}_b)\,(\mathbf{3}_s\,\mathbf{k}_c)\;\{\,1/6 * \varepsilon_{mns}\,\varepsilon_{abc}\}$$

where the LHS has sum on mns and ijk, and RHS side has these sums plus an abc sum.

Let's now write the following grouping of terms which appears on the RHS

$$A_{abc}^{\;mns} \equiv \varepsilon_{ijk}\,(\mathbf{1}_m\,\mathbf{i}_a)\,(\mathbf{2}_n\,\mathbf{j}_b)\,(\mathbf{3}_s\,\mathbf{k}_c) \quad \text{// implied sum on ijk}$$

This is a totally antisymmetric rank-3 tensor in abc indices for exactly the reason given above for tensor F. In terms of this A_{abc} object, we have

$$\text{RHS} = A_{abc}^{\;mns}\;\{\,1/6 * \varepsilon_{mns}\,\varepsilon_{abc}\} \quad \text{which means}$$

$$= \Sigma_{mns}\,\varepsilon_{mns}\,[\,1/6 * \Sigma_{abc}\;(\,A_{abc}^{\;mns}\,\varepsilon_{abc})\;\;]$$

The outer mns sum has exactly 6 non-vanishing terms due to the ε_{mns} factor. Thus, contributing values of mns must be permutations of 123. For a particular contributing set of values mns, look at the inner sum. We know that each of the 6 non-vanishing terms in the inner abc sum is identical, because the summand is the product of two antisymmetric tensors on the abc indices. Therefore, let's pick one term in the sum and multiply by 6. Thus, we pick the term abc = mns where, as just noted, mns is a contributing combination of indices. Then , using the fact that $\varepsilon_{mns}^2 = 1$ for a contributing term, we have

RHS = $\Sigma_{mns}\, \varepsilon_{mns}$ [1/6 * 6 * (A_{mns}^{mns} { ε_{mns}})] = $\Sigma_{mns}\, A_{mns}^{mns}$

= $\Sigma_{mns}\, \varepsilon_{ijk}$ $(\mathbf{1}_m\, \mathbf{i}_m)$ $(\mathbf{2}_n\, \mathbf{j}_n)$ $(\mathbf{3}_s\, \mathbf{k}_s)$

= LHS ! QED.

Question: Does the above presentation apply to a 4D space? Assume dq^μ in such a space. We can still write out these things: (all contravariant to start with)

$ds^2 = g_{\mu\nu} dq^\mu dq^\nu$ $\quad g_{\mu\nu} = \Sigma_i\, T_\mu^\kappa\, T_\nu^\kappa$ $\quad T_{\mu\nu} = (\partial x^\mu / \partial q^\nu)$

$ds^\mu = \sqrt{g_{\mu\mu}}\, dq^\mu$ for distance along the μ curve

Then for p192 D we would have

$dx^\mu = (\partial x^\mu / \partial q^\kappa)\, dq^\kappa = T_{\mu\kappa}\, dq^\kappa$

For p193 A we might try, exactly as before,

$e_{(\mu)\kappa} = T_{\mu\kappa}$ $\quad \mathbf{A} = a^\mu\, \mathbf{e}_{(\mu)}$

Now comes the first sign of trouble. Before we said

$\mathbf{e}^{(i)} \equiv \tfrac{1}{2}\, \varepsilon_{ijk}\, \mathbf{e}_{(j)} \times \mathbf{e}_{(k)} / v$

But we could generalize this by saying

$\mathbf{e}^{(\mu\nu)} \equiv \tfrac{1}{2}\, \varepsilon_{\mu\nu\kappa\sigma}\, \mathbf{e}_{(\kappa)} \times \mathbf{e}_{(\sigma)} / v$

but now we are in trouble because there seem to be 6 of these. You need to always use ε_{abcd} in this world, so axb is hard to define. Here is another possibility:

$\mathbf{e}^{(i)}{}_a \equiv \tfrac{1}{2}\, \varepsilon_{ijk}\, \varepsilon_{abc} \mathbf{e}_{(j)b}\, \mathbf{e}_{(k)c} / v$ // 3D one in components

$\mathbf{e}^{(\alpha)}{}_a \equiv (1/6)\, \varepsilon_{\alpha\beta\gamma\delta}\, \varepsilon_{abcd} \mathbf{e}_{(\beta)}{}^b\, \mathbf{e}_{(\gamma)}{}^c\, \mathbf{e}_{(\delta)}{}^d / v$

where we now combine three base vectors to make the contravariant base vector. The v would be

$v = \mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)} = \mathbf{e}_{(1)n} \varepsilon_{nms} \mathbf{e}_{(2)m} \mathbf{e}_{(3)s}$

$\Rightarrow$ $v = \mathbf{e}_{(1)}{}^\mu \varepsilon_{\mu\nu\alpha\beta} \mathbf{e}_{(2)}{}^\nu \mathbf{e}_{(3)}{}^\alpha\, \mathbf{e}_{(4)}{}^\beta = \varepsilon_{\mu\nu\alpha\beta}\, \mathbf{e}_{(1)}{}^\mu \mathbf{e}_{(2)}{}^\nu \mathbf{e}_{(3)}{}^\alpha\, \mathbf{e}_{(4)}{}^\beta$

which seems the right thing and generalizable to more dimensions as well. Now let's check on the all-important rule 5-79:

$$\mathbf{e}^{(m)} \bullet \mathbf{e}_{(n)} = \mathbf{e}^{(m)\,a}\,\mathbf{e}_{(n)\,a} = (1/6)\,\varepsilon_{m\beta\gamma\delta}\,\varepsilon_{abcd}\,\mathbf{e}_{(\beta)}{}^{b}\,\mathbf{e}_{(\gamma)}{}^{c}\,\mathbf{e}_{(\delta)}{}^{d}\,/v\;\mathbf{e}_{(n)\,a}$$

$$= \varepsilon_{m123}\,\varepsilon_{abcd}\,\mathbf{e}_{(1)}{}^{b}\,\mathbf{e}_{(2)}{}^{c}\,\mathbf{e}_{(3)}{}^{d}\,/v\;\mathbf{e}_{(n)\,a}$$

If m and n are different, we get zero unless m = 4. Assuming m = 4 we get

$$\mathbf{e}^{(4)} \bullet \mathbf{e}_{(n)} = \varepsilon_{4123}\,\varepsilon_{abcd}\,\mathbf{e}_{(1)}{}^{b}\,\mathbf{e}_{(2)}{}^{c}\,\mathbf{e}_{(3)}{}^{d}\,/v\;\mathbf{e}_{(n)\,a}$$

Suppose n = 3. Then we have this grouping $\mathbf{e}_{(3)}{}^{d}\,\mathbf{e}_{(3)\,a}\,\varepsilon_{abcd} = 0$. So we recover our desired result if m and n are different. If they are the same we get

$$\mathbf{e}^{(m)} \bullet \mathbf{e}_{(m)} = (1/6)\,\varepsilon_{m\beta\gamma\delta}\,\varepsilon_{abcd}\,\mathbf{e}_{(\beta)}{}^{b}\,\mathbf{e}_{(\gamma)}{}^{c}\,\mathbf{e}_{(\delta)}{}^{d}\,/v\;\mathbf{e}_{(m)\,a}$$

$$= \varepsilon_{1234}\,\varepsilon_{abcd}\,\mathbf{e}_{(2)}{}^{b}\,\mathbf{e}_{(3)}{}^{c}\,\mathbf{e}_{(4)}{}^{d}\,/v\;\mathbf{e}_{(1)\,a}\;\text{// where we pick a term and x6}$$

$$= v/v = 1$$

so this does work, very good. I think everything else will go through now, and we will have

$$g_{\mu\nu} = \mathbf{e}_{(\mu)} \bullet \mathbf{e}_{(\nu)} = \mathbf{e}_{(\mu)}{}^{\kappa}\,\mathbf{e}_{(\nu)\,\kappa}$$

A **surface element** would now be a tensor instead of a vector since 12 kinds of areas = 4x3

$$dS_{\mu\nu}{}^{,\rho\sigma} = \varepsilon_{\mu\nu\alpha\beta}(ds^{\alpha})^{\rho}(ds^{\beta})^{\sigma} \qquad d\mathbf{s}^{\alpha} = \mathbf{e}_{(\alpha)}dq^{\alpha} \quad \text{no sum } \alpha \text{ as in 5-76}$$

$$= \varepsilon_{\mu\nu\alpha\beta}\,e_{(\alpha)}{}^{\rho}\,e_{(\beta)}{}^{\sigma}\,dq^{\alpha}\,dq^{\beta}$$

What is the magnitude of our "area" with label μν ? It is probably this:

$$|dS_{\mu\nu}|^2 = dS_{\mu\nu}{}^{,\rho\sigma}\,dS_{\mu\nu,\rho\sigma} = \varepsilon_{\mu\nu\alpha\beta}\,e_{(\alpha)}{}^{\rho}\,e_{(\beta)}{}^{\sigma}\,dq^{\alpha}\,dq^{\beta}\,\varepsilon_{\mu\nu\alpha'\beta'}\,e_{(\alpha')}{}^{\rho}\,e_{(\beta')}{}^{\sigma}\,dq^{\alpha'}\,dq^{\beta'}$$

$$= \varepsilon_{\mu\nu\alpha\beta}\,\varepsilon_{\mu\nu\alpha'\beta'}\,e_{(\alpha)}{}^{\rho}\,e_{(\beta)}{}^{\sigma}\,e_{(\alpha')}{}^{\rho}\,e_{(\beta')}{}^{\sigma}\,dq^{\alpha}\,dq^{\beta}\,dq^{\alpha'}\,dq^{\beta'}$$

Now remember that

$$g_{\alpha\beta} = e_{(\alpha)}{}^{\rho}\,e_{(\beta)}{}^{\rho} = \mathbf{e}_{(\alpha)} \bullet \mathbf{e}_{(\beta)}$$

so we can write the above as (ignoring combinatoric factors now!)

$$|dS_{\mu\nu}|^2 = \varepsilon_{\mu\nu\alpha\beta}\,\varepsilon_{\mu\nu\alpha'\beta'}\,g_{\alpha\alpha'}\,g_{\beta\beta'}\,dq^{\alpha}\,dq^{\beta}\,dq^{\alpha'}dq^{\beta'}$$

There are only four terms in the sum, here is an example

$$|dS_{12}|^2 = \varepsilon_{12\alpha\beta}\,\varepsilon_{12\alpha'\beta'}\,g_{\alpha\alpha'}\,g_{\beta\beta'}\,dq^{\alpha}\,dq^{\beta}\,dq^{\alpha'}dq^{\beta'}$$

and here is one of the four terms probably all the same

$$|dS_{12}|^2 = \varepsilon_{1234}\,\varepsilon_{1234}\,g_{33}g_{44}\,dq^3\,dq^4\,dq^3\,dq^4 = g_{33}g_{44}\,dq^3\,dq^4\,dq^3\,dq^4$$

so you would say then that

$$|dS_{12}| = \sqrt{g_{33}g_{44}}\;\,dq^3\,dq^4\,dq^3\,dq^4$$

A **3-volume element** might be this:

$$dA_{\alpha}{}^{\mu\nu\kappa} = \varepsilon_{\alpha\beta\delta\gamma}(ds^{\beta})^{\mu}\,(ds^{\delta})^{\nu}(ds^{\beta})^{\kappa} = \varepsilon_{\alpha\beta\delta\gamma}\,(e_{\beta})^{\mu}\,(e_{\delta})^{\nu}(e_{\gamma})^{\kappa}\,dq^{\beta}\,dq^{\delta}\,dq^{\gamma}$$

I would guess that the magnitude of this 3-area is going to be

$$|dA_{\alpha}|^2 = dA_{\alpha}{}^{\mu\nu\kappa}\,dA_{\alpha}{}^{\mu\nu\kappa}$$

$$= \varepsilon_{\alpha\beta\delta\gamma}\,(e_{\beta})^{\mu}\,(e_{\delta})^{\nu}(e_{\gamma})^{\kappa}\,dq^{\beta}\,dq^{\delta}\,dq^{\gamma}\;\;\varepsilon_{\alpha\beta'\delta'\gamma'}\,(e_{\beta'})^{\mu}\,(e_{\delta'})^{\nu}(e_{\gamma'})^{\kappa}\,dq^{\beta'}\,dq^{\delta'}\,dq^{\gamma'}$$

$$= \varepsilon_{\alpha\beta\delta\gamma}\,\varepsilon_{\alpha\beta'\delta'\gamma'}\,(e_{\beta})^{\mu}\,(e_{\beta'})^{\mu}(e_{\delta})^{\nu}(e_{\delta'})^{\nu}(e_{\gamma})^{\kappa}(e_{\gamma'})^{\kappa}\,dq^{\beta}\,dq^{\delta}\,dq^{\gamma}\,dq^{\beta'}\,dq^{\delta'}\,dq^{\gamma'}$$

$$= \varepsilon_{\alpha\beta\delta\gamma}\,\varepsilon_{\alpha\beta'\delta'\gamma'}g_{\beta\beta'}\,g_{\delta\delta'}g_{\gamma\gamma'}\,dq^{\beta}\,dq^{\delta}\,dq^{\gamma}\,dq^{\beta'}\,dq^{\delta'}\,dq^{\gamma'}$$

Now there are 6 terms on the right and they are not all the same. Here is one term

$$|dA_1|^2 = \varepsilon_{1234}\,\varepsilon_{123}\,g_{22}g_{33}g_{44}dq^2\,dq^3\,dq^4\,dq^2\,dq^3\,dq^4 = g_{22}g_{33}g_{44}dq^2\,dq^3\,dq^4\,dq^2\,dq^3\,dq^4$$

Perhaps the sum of all terms is the determinant of the 1-cofactor of the $g_{\mu\nu}$ matrix

And my guess for the **4-volume** is that you get $dV = \sqrt{\det(g)}\;\,dq^1\,dq^2\,dq^3\,dq^4$

OK, enough of that. This was just an exercise, and someday I will find the above all written up somewhere, perhaps in Weinberg or perhaps elsewhere. I think the whole idea of covariant and contravariant can be extended to any number of dimensions N, with the "basis vectors" and the metric tensor matrix being an N x N matrix. I don't really know what this subject is called. It should have the words "tensor" and "coordinate systems" or something like that.

Obviously the Lorentz world would be a simple case N = 4. So for now, let's move on to the next section. Probably you should learn about better "tools" for doing all this tensor stuff before blindly tackling problems in the area.

Comments

The material of the above sections needs some serious "integration". I have attempted to do this in a separate document MM Chap 4-5 integration.

Section 5.17. The Differential Operators in Tensor Notation

We are in the q-space which I called x' earlier and I could put primes on everything, but let's not.

NOTE: I have detailed out this section in a separate document "divergence etc.doc"

(1) Gradient: Nothing very exciting here, we have

$$\nabla\varphi = \mathbf{e}^{i}\,\partial_{i}\varphi$$

This is the change of a function φ with respect to a change in coordinate q^{i} , summed. This is the rule we have always used for gradient. Were we to move the q^{i} coordinate by δq^{i}, function changes by $\partial_{i}\varphi$ and we weight this by the basis vector in this direction. M&M sometimes are not to helpful.

(2) Divergence. We have to contract the Christoffel symbol on two indices and we get to the left equality in equation D on page 196. But then we need a trick that I had to go look up here

http://books.google.com/books?id=8vlGhlxqZjsC&pg=PA57&lpg=PA57&dq=Christoffel+symbol+contracted&source=web&ots=8rH9dL-iqO&sig=sh5L0ixGcpTJA_8iwfJlrMo3vdA#PPA32,M1

Here is the trick:

$$\partial_{j}\ln(g) = [\,\partial\ln(g)/\partial g_{ik}][\partial g_{ik}/\partial q^{j}\,]$$

Remember that we have $g_{ij}(q^{j})$, the metric tensor in general is a function of the coordinates. But the determinant g is a function of all the elements of g_{ij}. So, determinant g is a function of the g_{ik} which are functions of the q_{j} so we are just using the chain rule above. I always forget to think about such things.

Now look at the first bracket

$$[\,\partial\ln(g)/\partial g_{ik}] = (1/g)\,(\partial g/\partial g_{ik})$$

But this is the derivative they got in equation C where we learned $(\partial g/\partial g_{ik}) = g g^{ik}$, so

$$[\,\partial\ln(g)/\partial g_{ik}] = (1/g)\,(\partial g/\partial g_{ik}) = (1/g)(\,g g^{ik}) = g^{ik}$$

So we then have

$$\partial_{j}\ln(g) = g^{ik}\,[\partial g_{ik}/\partial q^{j}\,] \;\Rightarrow\; \tfrac{1}{2}\,\partial_{j}\ln(g) = \tfrac{1}{2}\,g^{ik}\,[\partial g_{ik}/\partial q^{j}\,]$$

But this rightmost combination is {ij,i} from the left equality in D which I am happy with. So this shows that

$$\{ij,i\} = \tfrac{1}{2}\, g^{ik} [\partial g_{ik}/\partial q^j] = \tfrac{1}{2}\, \partial_j \ln(g) = \tfrac{1}{2}\,(1/g)\, \partial_j g$$

But we can also write this as

$$= \tfrac{1}{2}\, \partial_j \ln(g) = \partial_j(\tfrac{1}{2} \ln(g)) = \partial_j \ln\sqrt{g} = (1/\sqrt{g})(\partial_j \sqrt{g})$$

So thanks to this result, we end up with the **divergence** as shown in 5-20a. An interesting formula:

$$\nabla \bullet \mathbf{V} = (1/\sqrt{g})\, \partial_i [\sqrt{g}\, V^i]$$

so all you need to know here is $\sqrt{g}$ and you have it. Since this is a scalar, we don't have those basis vectors flying around at all.

(3) The Laplacian is much messier, 5-21a. You now have three things inside the derivative:

$$\nabla^2 \varphi = (1/\sqrt{g})\, \partial_i [\, g^{ij} \sqrt{g}\, \partial_j \varphi\,]$$

(4) The curl discussion is less clear. First, the usual difference thing DOES transform as a true tensor because the {..} symbols cancel out between these two terms. But then they say that some tensor is "invariant to such a transformation". I think they just mean that it keeps its form because it is a true tensor, lousy wording. And yes, it is antisymmetric so there are only three components of V_{ij}.

I will try this one on my own. Let unprimed be Cartesian, let primed be some new frame with q_i. Then define:

$$V_{ij} = \partial_i V_j - \partial_j V_i$$

In the new primed frame this is

$$V'_{ij} = \partial'_i V'_j - \partial'_j V'_i$$

But we know that $\partial'_i = T_i^{\,m}\, \partial_m$ and $V'_j = T_j^{\,n}\, V_n$, so we find that

$$\partial'_i V'_j = T_i^{\,m}\, \partial_m\, (T_j^{\,n}\, V_n) = T_i^{\,m}\, (\partial_m\, T_j^{\,n})\, V_n + T_i^{\,m}\, T_j^{\,n} (\partial_m\, V_n)$$

Luckily, I can now quote my "extra facts added" where

$$(\partial_s\, T_c^{\,e}) = -\, T_a^{\,e}\, S^a_{\ sb}\, T_c^{\,b} \qquad \text{where } (\partial_s T^a_{\ b}) = S^a_{\ sb}$$

or

$$(\partial_m\, T_j^{\,n}) = -\, T_a^{\,n}\, S^a_{\ mb}\, T_j^{\,b}$$

I can install this above to get

$$\partial'_i V'_j = T_i^m (\partial_m T_j^n) V_n + T_i^m T_j^n (\partial_m V_n)$$

$$= - T_i^m T_a^n S^a_{mb} T_j^b V_n + T_i^m T_j^n (\partial_m V_n)$$

$$= - T_i^a T_j^b T_t^n S^t_{ab} V_n + T_i^a T_j^b (\partial_a V_b)$$

Well, all I am doing here is showing that these first terms will cancel and the result transforms like a tensor, but I already know that. That is, I know this:

$$V'_{ij} = (\partial'_i V'_j - \partial'_j V'_i) = T_i^a T_j^b (\partial_a V_b - \partial_b V_a)$$

So, this is how you compute the curl in the primed system.

Except I have not yet defined the curl. I know that in the unprimed Cartesian system we have

$$\text{curl } \mathbf{V} = \hat{\mathbf{x}}^i \varepsilon_{ijk} (\partial_j V_k) = \hat{\mathbf{x}}^1 \varepsilon_{123} (\partial_2 V_3 - \partial_3 V_2) + \ldots = \hat{\mathbf{x}}^1 V_{32} + \ldots \qquad \text{// analysis curl}$$

Well, this is really what equation C says, except for an overall minus sign noted in the footnote. So we are doing the "tensorial curl" here, and we will have to add a minus sign to get the "analysis curl".. If we had a unit transformation, we would have our curl as

$$\nabla \times \mathbf{V} = V_{12}\mathbf{e}^{(3)} + V_{23}\mathbf{e}^{(1)} + V_{31}\mathbf{e}^{(2)} = C_3\,\mathbf{e}^{(3)} + V_1\mathbf{e}^{(1)} + V_2\mathbf{e}^{(2)} \qquad \text{//tensor curl}$$

and here we have covariant vector against contravariant base vectors. So in Cartesian what I have just written is the same as their equation C because $\mathbf{e}^{(3)} = \hat{\mathbf{z}}$ and so on. But I guess they are telling us to start with the other form

$$\nabla \times \mathbf{V} = V_{12} (\mathbf{e}^1 \times \mathbf{e}^2) + V_{23} (\mathbf{e}^2 \times \mathbf{e}^3) + V_{31} (\mathbf{e}^3 \times \mathbf{e}^1)$$

$$= \tfrac{1}{2} V_{ij} (\mathbf{e}^i \times \mathbf{e}^j) \qquad \text{since V is asymmetric}$$

We use covariant V and contravariant base vectors. I give up, I cannot justify using this form versus the other form, perhaps I need to think more about what "curl" means. But let's accept this for the moment. It does agree with Cartesian in the unprimed frame, and in the primed frame = the q frame, it is just what it says. If we now replace with the covariant base vectors, we have things like this

$$(\mathbf{e}^1 \times \mathbf{e}^2) = \mathbf{e}_3/v$$

from 5-80 where v' = 1/v. Then in terms of these basis vectors we pick up a $v = \sqrt{g}$ in the denominator and we get

$$\nabla \times \mathbf{V} = (1/\sqrt{g}) [V_{12}\,\mathbf{e}_3 + V_{23}\,\mathbf{e}_1 + V_{31}\,\mathbf{e}_2]$$

and recall that these lower case basis vectors are the ones aligned with the q curves like $\hat{r}$ or $\hat{\varphi}$. In general these are not normalized and we know that $\mathbf{e_3 \bullet e_3} = g_{33}$ for example, so I guess you need extra factors in the above if you want to use normalized vectors.

Note added: What about my friend the $A \bullet \nabla$ operator that I think started all this reading? From above we have these two facts:

$$\nabla\varphi = \mathbf{e}^i \partial_i \varphi \qquad\qquad \mathbf{A} = A^k \mathbf{e}_k$$

Thus

$$A \bullet \nabla\varphi = \mathbf{A} = A^k \mathbf{e}_k \bullet \mathbf{e}^i \partial_i \varphi = A^k \delta_k{}^i \partial_i \varphi = A^i \partial_i \varphi$$

Seems pretty simple which is why M&M never mentioned it. But see below:

Now the grand finale: we consider "orthogonal" coordinates

so the $\mathbf{e}_i$ are perpendicular (claim 1). I agree that in this case, $\mathbf{e}^3 = \mathbf{e}_1 \times \mathbf{e}_2 / v$ must be parallel to $\mathbf{e}_3$. (claim 2).

Since $e^i \bullet e_i = 1$, we know $| e^i| = 1/| e_i|$ which leads to equation D whose mag says exactly this. We already know the dot products are 0 because this is what "orthogonal" means. Now,

$g^{ij} = (1/g)\, G^{ij}$ but $G^{ij} = \text{cof}(g_{ij}) = \delta_{ij}\, G^{ii}$ because g is diagonal so

only diagonal elements will have non-zero subdeterminants, ie, cofactors. So consider

$G^{11} = \text{cof}(g_{11}) = g_{22}g_{33} = g/g_{11}$. Therefore

$g^{11} = (1/g)\, G^{11} = (1/g)\,(g/g_{11}) = g_{11}$ (claim 4). that is $g_{ii} = g^{ii}$ i = 1,2,3

In this case, what to our various formulas say? On the left I put the full result, then simplify for orthogonal systems if possible.

$$\nabla\varphi = \mathbf{e}^i \partial_i \varphi = (\mathbf{e}_i / g_{ii})\, \partial_i \varphi = (\, 1/\sqrt{g_{ii}}\,)\, \hat{\mathbf{e}}_i\, \partial_i \varphi$$

$$\nabla \bullet \mathbf{V} = (1/\sqrt{g})\, \partial_i [\sqrt{g}\, V^i] \qquad \text{// no simplification} \qquad g = g_{11}g_{22}g_{33} = \Pi_i\, g_{ii}$$

$$\nabla^2\varphi = (1/\sqrt{g})\, \partial_i [\, g^{ij} \sqrt{g}\, \partial_j \varphi\,] = (1/\sqrt{g})\, \partial_i [g^{ii} \sqrt{g}\, \partial_i \varphi\,]$$

$$\nabla \times \mathbf{V} = (1/\sqrt{g})\, [\, V_{12}\, \mathbf{e}_3 + V_{23}\, \mathbf{e}_1 + V_{31}\, \mathbf{e}_2\,] \qquad \text{// no simplification}$$

And so concludes section 5.17 which is showing us the "differential operators" in "q coordinates" in the general case of non-orthogonal coordinates! Note that $\sqrt{g_{ii}} = Q_i$ of section 5., claim 5. And,

$\sqrt{g} = Q_1Q_2Q_3$

Well, let's go a little more right in this section to anticipate what 5.2 will say. We want unit vectors, and we now $\mathbf{u}_i = \mathbf{e}_i/Q_i$ This for **gradient** we get

$\nabla\varphi = (1/\sqrt{g_{ii}})\ \hat{\mathbf{u}}_i\ \partial_i\varphi\ = \Sigma_i\ (1/Q_i)\ (\partial_i\varphi)\ \hat{\mathbf{u}}_i$ will be 5.9

Now look at **divergence**.

$\nabla \bullet \mathbf{V} = (1/\sqrt{g})\ \partial_i[\sqrt{g}\ V'^i]$ $\mathbf{V} = \Sigma_i\ \mathbf{e}_i\ V'^i\ = \Sigma_i\ \mathbf{u}_i\ (Q_iV'^i)\ = \Sigma_i\ \mathbf{u}_i\ V^i$

Although it was not stated on page 196, the authors assumed that the components of V are defined as I show above and which I now call V'^i. In practice, we really want to define "V^i" to be projections onto unit vectors, so you see how I have done this above (see page 194 C). So we have to replace

$V'^i = V^i/\ Q_i$

so I then get

$\nabla \bullet \mathbf{V} = (1/Q_1Q_2Q_3)\ \Sigma_i\ \partial_i[Q_1Q_2Q_3V^{i\prime}]\ =$

$(1/Q_1Q_2Q_3)\{\ \partial_1(Q_1Q_2Q_3V^1/Q_1) + ...$

$= (1/Q_1Q_2Q_3)\{\ \partial_1(Q_2Q_3V^1) + \partial_2(Q_3Q_1V^2) + \partial_3(Q_1Q_2V^3)\ \}$

which now agrees with page 175 5-20! On page 175, all contravariant indices are written lower as we normally do when not worrying about up and down details.

Our orthogonally simplified **Laplacian** was this:

$\nabla^2\varphi\ = (1/\sqrt{g})\ \partial_i[g^{ii}\ \sqrt{g}\ \partial_i\varphi\ ]\ = (1/Q_1Q_2Q_3)\ \partial_i[\ Q_i^{-2}\ Q_1Q_2Q_3\ \partial_i\varphi]$

$= (1/Q_1Q_2Q_3)\{\ \partial_1[Q_2Q_3/Q_1\ *\ \partial_1\varphi] + \partial_2[Q_3Q_1/Q_2\ *\ \partial_2\varphi] + \partial_3[Q_1Q_2/Q_3\ *\ \partial_3\varphi]\ \}$

and this agrees with page 175 5-21.

Finally we come to the **curl** which we show above as

$\nabla \times \mathbf{V} = (1/\sqrt{g})\ [\ V'_{12}\ \mathbf{e}_3 + \text{cyclic}\ ]$

But again the V's inside the V_{ij} are really the primed ones as discussed in the divergence above, so I put primes above. So consider: (the V component goes with the first index!)

$V'_{12} = \partial_2V_1 - \partial_1V_2$

We need first to get these V' components to be contravariant, so get

$$V'_i = g_{ii} V'^i = Q_i^2 V'^i$$

Then we have

$$V'_{12} = \partial_2(Q_1^2 V'^1) - \partial_1 (Q_2^2 V'^2)$$

But now we have to get into the unit vector components, as above, so we use

$$V'^i = V^i / Q_i$$

and this gives

$$V'_{12} = \partial_2(Q_1 V^1) - \partial_1 (Q_2 V^2)$$

But $\mathbf{u}_3 = \mathbf{e}_3/Q_3$, so our curl equation becomes

$$\nabla \times \mathbf{V} = (1/\sqrt{g}\,) [\, V'_{12}\, \mathbf{e}_3 + \text{cyclic} \,]$$

$$= (1/Q_1Q_2Q_3)[\, \{ \partial_2(Q_1 V^1) - \partial_1 (Q_2 V^2)\}\, Q_3 \mathbf{u}_3 + \text{cyclic} \,]$$

Now lower the V indices to go back to conventional non-tensor notation to get

$$\nabla \times \mathbf{V} = (1/Q_1Q_2Q_3)[\, \{ \partial_2(Q_1 V_1) - \partial_1 (Q_2 V_2)\}\, Q_3 \mathbf{u}_3 + \text{cyclic} \,]$$

(they are really still contravariant in meaning). Now rewrite as

$$\nabla \times \mathbf{V} = (1/Q_1Q_2Q_3)[\, \{ \partial_2(Q_1 V_1) - \partial_1 (Q_2 V_2)\}\, Q_3 \mathbf{u}_3 + \text{cyclic} \,]$$

If you define a matrix in this way

$M_{1i} = Q_i \mathbf{u}_i$	first row
$M_{2i} = \partial_i$	second row
$M_{3i} = Q_i V_i$	third row

and if you notice that there are 6 terms in [..] above which can be written as

$$-[...] = \varepsilon_{ijk} M_{1i} M_{2j} M_{3k} = \varepsilon_{321} M_{13} M_{22} M_{31} + ...$$
$$= \varepsilon_{321} Q_3 \mathbf{u}_3 \partial_2 Q_1 V_1 + ...$$
$$= - Q_3 \mathbf{u}_3 \partial_2 Q_1 V_1 + ...$$

so we get

$$\det(M) = -[...] = - Q_3 \mathbf{u}_3 \partial_2 Q_1 V_1 + ...$$

and we end up with

$$\nabla \text{ x } \mathbf{V}_{\texttt{tensor convention}} = (1/Q_1Q_2Q_3)[\ -\det(M)\ ]$$

which gives our desired result

$$\nabla \text{ x } \mathbf{V}_{\texttt{analysis convention}} = (1/Q_1Q_2Q_3)\det(M)$$

where M is the matrix shown on page 176 which is the same as what I wrote above.

And noted added on A•∇φ : Above we had

$$A\bullet\nabla\varphi = \mathbf{A} = A^{\mathbf{k}}\mathbf{e}_k\bullet\ \mathbf{e}^{\mathbf{i}}\ \partial_{\mathbf{i}}\varphi = A^{\mathbf{k}}\delta_{\mathbf{k}}{}^{\mathbf{i}}\ \partial_{\mathbf{i}}\varphi = A'^{\mathbf{i}}\partial_{\mathbf{i}}\varphi$$

But once again, I think we have to remember that the A' are not the A's, just as in the various cases above. So we would write

$$A'^{\mathbf{i}} = A^{\mathbf{i}}/\ Q_{\mathbf{i}}$$

and then we get

$$\mathbf{A}\bullet\nabla\varphi = \Sigma_{\mathbf{i}}\ (1/Q_{\mathbf{i}})\ (\partial_{\mathbf{i}}\varphi)$$ / I am pretty sure this is correct!

Can I find this stated somewhere? It is just omitted from M&M's list, so look on the web. No one on the web mentions this operator combination! It does show up in the A x (∇ x B) identity and in ∇(**A**•**B**) . How about ∇(A•A) where my identity says this:

$$\nabla(\mathbf{A}\bullet\mathbf{A}) = 2\ (\mathbf{A}\bullet\nabla)A\ + 2\ \mathbf{A}\ \text{x}\ (\nabla\ \text{x}\ A)$$

But I don't even know if this identity is true in curvilinear coordinates.

General rule: I suppose in any identity, you need to think about where new Q factors might go!

Review of the above: In this section we developed formulas for gradient, divergence, Laplacian and curl based on general non-orthogonal coordinates. Then we specialized these general formulas for orthogonal coordinates. Then we used the $Q_{\mathbf{i}}$ notation, made all vector components lower although we mean them as contravariant, and we thus derived the four major results of Section 5.2 which section we have not even read yet! I think it is good to get these main results from the general tensor formalism. Tomorrow I will read 5.2 and some of the following sections!

Section 5.2. Vector Relations in Curvilinear Coordinates (174)

gaga