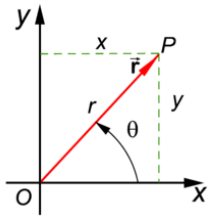
Polar coordinates PhL 6.25.09

1. Compute partial derivatives of Cartesian coordinates: 1
2. Compute partial derivatives of Polar coordinates: 1
3. Compute the Laplacian. 2
4. Relation between unit vectors: 4
5. Time derivatives of the unit vectors 4
6. The gradient in polar coordinates: 4
7. The divergence in polar coordinates: 5

Added section 5 on 12.14.16.

In retrospect, all these results can be obtained from cylindrical coordinates if you ignore z.
But let's go ahead anyway and do all the details as if we didn't know that fact.

Here we go:



$x = r\cos\theta$ $\quad r^2 = x^2 + y^2$

$y = r\sin\theta$ $\quad y/x = \tan\theta$

1. Compute partial derivatives of Cartesian coordinates:

$dx = dr\cos\theta - r\sin\theta d\theta$
$dy = dr\sin\theta + r\cos\theta d\theta$

$\partial x/\partial r = \cos\theta$ $\quad \partial x/\partial\theta = -r\sin\theta$
$\partial y/\partial r = \sin\theta$ $\quad \partial y/\partial\theta = r\cos\theta$

2. Compute partial derivatives of Polar coordinates:

$rdr = xdx + ydy$
$(xdy-ydx)/x^2 = \sec^2\theta\, d\theta$

$\partial r/\partial x = x/r = \cos\theta$ $\quad \partial r/\partial y = y/r = \sin\theta$ $\quad$ // web verified

$\partial\theta/\partial x = (-y/\sec^2\theta\, x^2) = (-r\sin\theta / [\sec^2\theta\, r^2\cos^2\theta]) = -\sin\theta/r$ $\quad$ // web verified

$\partial\theta/\partial y = (1/\sec^2\theta\, x) = (1/[\sec^2\theta\, r\cos\theta]) = \cos\theta/r$ $\quad$ // web verified

Summary:

$\partial r/\partial x = \cos\theta$ $\quad \partial r/\partial y = \sin\theta$

$\partial\theta/\partial x = -\sin\theta/r$ $\quad \partial\theta/\partial y = \cos\theta/r$

3. Compute the Laplacian.

(a) Compute ∂_x and then ∂_x^2

$\partial_x = \partial/\partial x = \partial\theta/\partial x * \partial/\partial\theta + \partial r/\partial x * \partial/\partial r = (-\sin\theta/r)\partial_\theta + \cos\theta\,\partial_r$

$\partial_x^2 = [(-\sin\theta/r)\partial_\theta + \cos\theta\,\partial_r]^2 = [(\sin\theta/r)\partial_\theta - \cos\theta\,\partial_r]^2$

$= [(\sin\theta/r)\partial_\theta - \cos\theta\,\partial_r]\,[(\sin\theta/r)\partial_\theta - \cos\theta\,\partial_r]$

Have to compute out all four terms (put in signs later)

$(\sin\theta/r)\partial_\theta(\sin\theta/r)\partial_\theta = (\sin\theta/r)\{(\cos\theta/r)\partial_\theta + (\sin\theta/r)\,\partial_\theta^2\} = \sin\theta\cos\theta/r^2\partial_\theta + \sin^2\theta/r^2\,\partial_\theta^2$

$(\sin\theta/r)\partial_\theta \cos\theta\,\partial_r = (\sin\theta/r)\{-\sin\theta\,\partial_r + \cos\theta\,\partial_\theta\partial_r\} = -\sin^2\theta/r\,\partial_r + \sin\theta\cos\theta/r\,\partial_\theta\partial_r$

$\cos\theta\,\partial_r(\sin\theta/r)\partial_\theta = \cos\theta\{-\sin\theta/r^2\,\partial_\theta + \sin\theta\,\partial_r\partial_\theta\} = -\sin\theta\cos\theta/r^2\,\partial_\theta + \sin\theta\cos\theta\,\partial_r\partial_\theta$

$\cos\theta\,\partial_r(\cos\theta\,\partial_r) = \cos^2\theta\,\partial_r^2$

Add up the four terms to get

$\partial_x^2 = \sin\theta\cos\theta/r^2\partial_\theta + \sin^2\theta/r^2\,\partial_\theta^2 + \sin^2\theta/r\,\partial_r - \sin\theta\cos\theta/r\,\partial_\theta\partial_r$

$+ \sin\theta\cos\theta/r^2\,\partial_\theta - \sin\theta\cos\theta\,\partial_r\partial_\theta + \cos^2\theta\,\partial_r^2$

$\partial_x^2 = 2\sin\theta\cos\theta/r^2\,\partial_\theta + \sin^2\theta/r^2\,\partial_\theta^2 + \sin^2\theta/r\,\partial_r - 2\sin\theta\cos\theta/r\,\partial_\theta\partial_r + \cos^2\theta\,\partial_r^2$

This painful result agrees with a website calculation I found which I quote here:

$$= \cos^2(\theta)\cdot\frac{\partial^2 w}{\partial r^2} + \frac{\sin^2(\theta)}{r^2}\cdot\frac{\partial^2 w}{\partial\theta^2} + \frac{\sin^2(\theta)}{r}\cdot\frac{\partial w}{\partial r}$$
$$+ 2\cdot\frac{\sin(\theta)\cos(\theta)}{r^2}\cdot\frac{\partial w}{\partial\theta} - 2\cdot\frac{\sin(\theta)\cos(\theta)}{r}\cdot\frac{\partial^2 w}{\partial r\partial\theta}.$$

(a) Compute ∂_y and then ∂_y^2

$\partial_y = \partial/\partial y = \partial\theta/\partial y * \partial/\partial\theta + \partial r/\partial y * \partial/\partial r = (\cos\theta/r)\partial_\theta + \sin\theta\,\partial_r$

$\partial_y^2 = [(\cos\theta/r)\partial_\theta + \sin\theta\,\partial_r]^2 = [(\cos\theta/r)\partial_\theta + \sin\theta\,\partial_r]^2$

$= [(\cos\theta/r)\partial_\theta + \sin\theta\,\partial_r]\,[(\cos\theta/r)\partial_\theta + \sin\theta\,\partial_r]$

Have to compute out all four terms (put in signs later)

$(\cos\theta/r)\partial_\theta(\cos\theta/r)\partial_\theta = (\cos\theta/r)\{(-\sin\theta/r)\partial_\theta + (\cos\theta/r)\,\partial_\theta^2\} = -\sin\theta\cos\theta/r^2\partial_\theta + \cos^2\theta/r^2\,\partial_\theta^2$

$(\cos\theta/r)\partial_\theta \sin\theta\,\partial_r = (\cos\theta/r)\{+\cos\theta\,\partial_r + \sin\theta\,\partial_\theta\partial_r\} = \cos^2\theta/r\,\partial_r + \sin\theta\cos\theta/r\,\partial_\theta\partial_r$

$\sin\theta\,\partial_r(\cos\theta/r)\partial_\theta = \sin\theta\{-\cos\theta/r^2\,\partial_\theta + \cos\theta/r\,\partial_r\partial_\theta\} = -\sin\theta\cos\theta/r^2\,\partial_\theta + \sin\theta\cos\theta/r\,\partial_r\partial_\theta$

$\sin\theta\,\partial_r(\sin\theta\,\partial_r) = \sin^2\theta\,\partial_r^2$

Add up the four terms to get

$\partial_y^2 = -\sin\theta\cos\theta/r^2\partial_\theta + \cos^2\theta/r^2\,\partial_\theta^2 + \cos^2\theta/r\,\partial_r + \sin\theta\cos\theta/r\,\partial_\theta\partial_r$

$- \sin\theta\cos\theta/r^2\,\partial_\theta + \sin\theta\cos\theta/r\,\partial_r\partial_\theta + \sin^2\theta\,\partial_r^2$

$\partial_y^2 = \cos^2\theta/r^2\,\partial_\theta^2 + \cos^2\theta/r\,\partial_r + 2\sin\theta\cos\theta/r\,\partial_r\partial_\theta + \sin^2\theta\,\partial_r^2 - 2\sin\theta\cos\theta/r^2\,\partial_\theta$

Summarize to this point:

$\partial_x^2 = 2\sin\theta\cos\theta/r^2\,\partial_\theta + \sin^2\theta/r^2\,\partial_\theta^2 + \sin^2\theta/r\,\partial_r - 2\sin\theta\cos\theta/r\,\partial_\theta\partial_r + \cos^2\theta\,\partial_r^2$

$\partial_y^2 = -2\sin\theta\cos\theta/r^2\,\partial_\theta + \cos^2\theta/r^2\,\partial_\theta^2 + \cos^2\theta/r\,\partial_r + 2\sin\theta\cos\theta/r\,\partial_r\partial_\theta + \sin^2\theta\,\partial_r^2$

All four $\sin\theta\cos\theta$ terms cancel, and we are left with

$\partial_x^2 + \partial_y^2 = \partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 = (1/r)\partial_r(r\partial_r) + (1/r^2)\,\partial_\theta^2$

$\nabla^2 = \partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 = (1/r)\partial_r(r\partial_r) + (1/r^2)\,\partial_\theta^2$

Summary:

$\partial_x = (-\sin\theta/r)\partial_\theta + \cos\theta\,\partial_r$

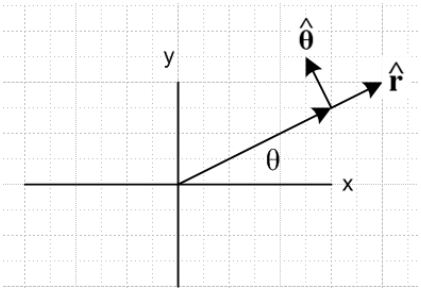
$\partial_y = (\cos\theta/r)\partial_\theta + \sin\theta\,\partial_r$

$\partial_x^2 = 2\sin\theta\cos\theta/r^2\,\partial_\theta + \sin^2\theta/r^2\,\partial_\theta^2 + \sin^2\theta/r\,\partial_r - 2\sin\theta\cos\theta/r\,\partial_\theta\partial_r + \cos^2\theta\,\partial_r^2$

$\partial_y^2 = -2\sin\theta\cos\theta/r^2\,\partial_\theta + \cos^2\theta/r^2\,\partial_\theta^2 + \cos^2\theta/r\,\partial_r + 2\sin\theta\cos\theta/r\,\partial_r\partial_\theta + \sin^2\theta\,\partial_r^2$

$$\partial_x^2 + \partial_y^2 = \partial_r^2 + (1/r)\,\partial_r + (1/r^2)\,\partial_\theta^2 = (1/r)\partial_r(r\partial_r) + (1/r^2)\,\partial_\theta^2$$

4. Relation between unit vectors:



$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$ $\qquad$ $\hat{r} = R_z(\theta)\,\hat{x}$

$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$ $\qquad$ $\hat{\theta} = R_z(\theta)\,\hat{y}$

$\hat{x} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ $\qquad$ $\hat{x} = R_z(-\theta)\,\hat{r}$

$\hat{y} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$ $\qquad$ $\hat{y} = R_z(-\theta)\,\hat{\theta}$

$\hat{r}\bullet\hat{x} = \cos\theta$ $\qquad$ $\hat{r}\bullet\hat{y} = \sin\theta$

$\hat{\theta}\bullet\hat{x} = -\sin\theta$ $\qquad$ $\hat{\theta}\bullet\hat{y} = \cos\theta$

5. Time derivatives of the unit vectors

$$d\hat{r}/dt = d/dt(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) = -\sin\theta\,\dot{\theta}\,\hat{x} + \cos\theta\,\dot{\theta}\,\hat{y}$$
$$= -\sin\theta\,\dot{\theta}\,[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\,] + \cos\theta\,\dot{\theta}\,[\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}]$$
$$= \dot{\theta}\,\hat{\theta}$$

$$d\hat{\theta}/dt = d/dt(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) = -\cos\theta\,\dot{\theta}\,\hat{x} - \sin\theta\,\dot{\theta}\,\hat{y}$$
$$= -\cos\theta\,\dot{\theta}[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\,] - \sin\theta\,\dot{\theta}\,[\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}]$$
$$-\dot{\theta}\,\hat{r}$$

6. The gradient in polar coordinates:

$$\nabla = \hat{x}\,\partial_x + \hat{y}\,\partial_y$$

$$= (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\,[(-\sin\theta/r)\partial_\theta + \cos\theta\,\partial_r\,]$$
$$+ (\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})\,[(\cos\theta/r)\partial_\theta + \sin\theta\,\partial_r]$$

$$= \hat{r}\,\{\,(-\sin\theta\cos\theta/r)\partial_\theta + \cos^2\theta\,\partial_r + \sin\theta(\cos\theta/r)\partial_\theta + \sin^2\theta\,\partial_r\,\}$$

$+ \hat{\boldsymbol{\theta}}\ \{\ \sin^2\theta/r\ \partial_{\theta} - \sin\theta\cos\theta\ \partial_r + \cos^2\theta/r\ \partial_{\theta} + \sin\theta\cos\theta\ \partial_r\}$

$$\nabla = \hat{r}\partial_{\mathbf{r}} + \hat{\theta}\ (1/r)\ \partial_{\boldsymbol{\theta}}$$

7. The divergence in polar coordinates:

$\nabla\bullet\mathbf{f} = \partial_x\ (\mathbf{f}\bullet\hat{\mathbf{x}}) + \partial_y(\mathbf{f}\bullet\hat{\mathbf{y}}) =$

$= [(-\sin\theta/r)\partial_{\theta} + \cos\theta\ \partial_r\ ]\ \mathbf{f}\bullet(\cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}})$
$+ [(\cos\theta/r)\partial_{\theta} + \sin\theta\ \partial_r]\ \mathbf{f}\bullet(\sin\theta\ \hat{\mathbf{r}} + \cos\theta\ \hat{\boldsymbol{\theta}})$

$= [(-\sin\theta/r)\partial_{\theta} + \cos\theta\ \partial_r\ ]\ (\cos\theta\ f_r - \sin\theta\ f_{\theta})$
$+ [(\cos\theta/r)\partial_{\theta} + \sin\theta\ \partial_r]\ (\sin\theta\ f_r + \cos\theta\ f_{\theta})$

$= -\sin\theta/r\ \partial_{\theta}(\cos\theta\ f_r) + \sin\theta/r\ \partial_{\theta}(\sin\theta\ f_{\theta}) + \cos^2\theta\ \partial_r\ f_r - \sin\theta\cos\theta\ \partial_r\ f_{\theta}$
$+ \cos\theta/r\ \partial_{\theta}(\sin\theta\ f_r) + \cos\theta/r\ \partial_{\theta}(\cos\theta\ f_{\theta}) + \sin^2\theta\ \partial_r\ f_r + \sin\theta\cos\theta\ \partial_r\ f_{\theta}$

$= -\sin\theta/r\ \partial_{\theta}(\cos\theta\ f_r) + \sin\theta/r\ \partial_{\theta}(\sin\theta\ f_{\theta}) + \partial_r\ f_r$
$+ \cos\theta/r\ \partial_{\theta}(\sin\theta\ f_r) + \cos\theta/r\ \partial_{\theta}(\cos\theta\ f_{\theta})$

$= -\sin\theta/r\ [\ \cos\theta\ \partial_{\theta}\ f_r - \sin\theta\ f_r] + \sin\theta/r\ [\ \sin\theta\ \partial_{\theta}f_{\theta} + \cos\theta\ f_{\theta}] + \partial_r\ f_r$
$+ \cos\theta/r\ [\ \sin\theta\ \partial_{\theta}f_r + \cos\theta\ f_r] + \cos\theta/r\ [\ \cos\theta\ \partial_{\theta}\ f_{\theta} - \sin\theta\ f_{\theta}]$

$$\nabla\bullet\mathbf{f} = (1/r)f_r + \partial_r\ f_r + (1/r)\ \partial_{\theta}f_{\theta}$$

8. Particle Motion in Polar Coordinates

$\mathbf{r} = r\ \hat{\mathbf{r}}$

$\mathbf{v} = \dot{r}\ \hat{\mathbf{r}} + r\ \partial_t\hat{\mathbf{r}} = \dot{r}\ \hat{\mathbf{r}} + r\ \dot{\theta}\ \hat{\boldsymbol{\theta}}$ // agrees with Marion p 31 (1.97)

$\mathbf{a} = \ddot{r}\ \hat{\mathbf{r}} + \dot{r}\ \partial_t\hat{\mathbf{r}} + \dot{r}\ \dot{\theta}\ \hat{\boldsymbol{\theta}} + r\ \ddot{\theta}\ \hat{\boldsymbol{\theta}} + r\ \dot{\theta}\ \partial_t\hat{\boldsymbol{\theta}}$
$= \ddot{r}\ \hat{\mathbf{r}} + \dot{r}\ (\dot{\theta}\ \hat{\boldsymbol{\theta}}) + \dot{r}\ \dot{\theta}\ \hat{\boldsymbol{\theta}} + r\ \ddot{\theta}\ \hat{\boldsymbol{\theta}} + r\ \dot{\theta}\ (-\dot{\theta}\ \hat{\mathbf{r}})$
$= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\ \dot{r}\ \dot{\theta} + r\ \ddot{\theta}\ )\ \hat{\boldsymbol{\theta}}$ // agrees with Marion p 31 (1.98)

Rewrite as

$\mathbf{v} = v_r\ \hat{\mathbf{r}} + v_{\theta}\ \hat{\boldsymbol{\theta}}$ $\quad v_r = \dot{r}$ $\quad v_{\theta} = r\ \dot{\theta}$

$\mathbf{a} = a_r\ \hat{\mathbf{r}} + a_{\theta}\ \hat{\boldsymbol{\theta}}$ $\quad a_r = \ddot{r} - r\dot{\theta}^2$ $\quad a_{\theta} = 2\ \dot{r}\ \dot{\theta} + r\ \ddot{\theta}$

If motion is constrained to a circle of radius r, set $\dot{r} = \ddot{r} = 0$ in the above. Then

$\mathbf{v} = v_\theta \hat{\boldsymbol{\theta}}$ $\qquad v_\theta = r\dot{\theta}$

$\mathbf{a} = a_r \hat{\mathbf{r}} + a_\theta \hat{\boldsymbol{\theta}}$ $\qquad a_r = -r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta}$