A study of 1/R expansions in spherical coordinates PhL 5.3.10

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Overview (added 8.22.10, 5 pages)

0. Here I write the **free-space Green's 1/R** using spherical atoms and A_{mn} , just a Smythian form:

$$1/R = f(\mathbf{r} \mid \mathbf{r}') = \Sigma_{mn} A_{mn} (r_</r')^n (r_>/r')^{-n-1} P_n^m(\cos\theta) e^{im\varphi} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

1. Since (θ,φ) are oscillatory, I use the **pillbox** method to find A_{mn}, and I install into the above to get

$$A_{mn} = q\,(1/r')\; f(n,-m) e^{-im\varphi'} P_n^m(z')$$

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} f(n,-m)\,(r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z')\, e^{im\varphi}\, e^{-im\varphi'} \qquad (*)$$

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} \varepsilon_m\, f(n,-m)\,(r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z') \cos[m(\varphi-\varphi')] \qquad (**)$$

and I am able to verify these results against other sources. I did not derive alternative forms for 1/R based on different pairs of oscillatory coordinates, though I could have.

2. I derive the following **theorem**:

Theorem 1. If f(**r**,**r'**) is a rotational scalar, and if we know this is true,

$$f(\mathbf{r},r'\,\hat{\mathbf{z}}) = \Sigma_n\, k_n(r,r')\, \varphi_n(\cos\theta)$$

then it follows that

$$f(\mathbf{r},\mathbf{r'}) = \Sigma_n\, k_n(r,r')\, \varphi_n(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}')$$

3. I then apply this theorem as follows. I first take (*) above and set $\mathbf{r}' = r'\,\hat{\mathbf{z}}$ so z'=cosθ'=1 and $P_n^m(1) = \delta_m^0$, therefore we have [beware that symbol z is overloaded]

$$1/\,|\mathbf{r} - r'\hat{\mathbf{z}}| \;=\; \Sigma_{n=0}^{\infty}\; (r_<)^n (r_>)^{-n-1}\; P_n(\cos\theta)$$

The theorem then states that (where $\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi') = \cos\gamma$)

$$1/R = 1/\,|\mathbf{r} - \mathbf{r}'| \;=\; \Sigma_{n=0}^{\infty}\; (r_<)^n (r_>)^{-n-1}\; P_n(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}')$$

Comparing this to (**) we get

$$P_n[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] \;= \Sigma_{m=0}^{\infty}\, \varepsilon_m\; f(n,-m)\, P_n^m(\cos\theta)\, P_n^m(\cos\theta')\, \cos[m(\varphi-\varphi')]$$

which is the famous "first kind Legendre addition theorem". It has the SO(3) group representation interpretation $D^n(g_1g_2)_{00} = \Sigma_m\, D^n(g_1)_{0m}\, D^n(g_2)_{m0}$ mentioned below in section 10.

4. I **verify** from our 1/R expansion that, as r'→0, 1/R → 1/r.

5. Here I rewrite the **1/R expansion using Jackson's Y's** (and verify it)

$$1/R = (4\pi)\, \Sigma_{mn}\, (2n+1)^{-1}\; (r_<)^n (r_>)^{-n-1}\, Y_{nm}(\theta',\varphi')^*\, Y_{nm}(\theta,\varphi)$$

6. The point of this section is that you can **almost** (up to a constant C_{nm}) determine the form of the general 1/R expansion just by knowledge that the result must be *symmetric* under $\mathbf{r} \leftrightarrow \mathbf{r}'$.

7. Here I prove the obvious fact that the **imaginary** part of 1/R as in (*) above gives 0.

8. Here I ask if there is a **connection** between our 1/R expansion (*) above, and the general Stakgold "full eigenfunction expansion" for a Green's Function g. I start with a **grounded metal sphere** and write an expression for the eigenfunctions of said sphere, which includes of course Y functions and J radial Bessel functions. I then use said eigenfunctions in the fancy full eigenfunction sum that gives g. To wit:

$$g(\mathbf{x}|x') = (4\pi)\, \Sigma_{nmk}\, u_{nmk}(\mathbf{x})u^*_{nmk}(\mathbf{x}')/\lambda_{kn}$$

where

$$u_k(\mathbf{x}) = u_{nmk}(r,\theta.\varphi) = C_{nmk}\ (\sqrt{a/r}\ )\ J_{n+1/2}(\beta_k{}^{n+1/2} r/a)\ Y_{nm}(\theta,\varphi)$$

with $\quad C_{nmk} = \sqrt{2}\ a^{-3/2} / J_{n+3/2}(\beta_k{}^{n+1/2}) \qquad \lambda_{kn} = (\beta_k{}^{n+1/2}/a)^2$

I wanted to show that, as the sphere radius a → ∞, this Green's Function g approaches my free-space 1/R expansion. I was unable to do a proof, but I am sure one can do it. You have to show this to be true:

$$\Sigma_k\ C_{nmk}{}^2(\beta_k{}^{n+1/2}/a)^{-2}\ \sqrt{a/r}\ J_{n+1/2}(\beta_k{}^{n+1/2} r/a)\ \sqrt{a/r'}\ J_{n+1/2}(\beta_k{}^{n+1/2} r'/a)\ \rightarrow\ (2n+1)^{-1}\ (r_<)^n (r_>)^{-n-1}$$

As a consolation prize, I was able to show that the above huge sum for g does in fact approach 1/r in the limit that r' → 0. Notice in the above line that Σ_k is an infinite sum over zeros, $J_{n+1/2}(\beta_k{}^{n+1/2}) = 0$.

9. In 8 above I tried to obtain the 1/R expansion using an **"expanding metal sphere"** and the "full eigenfunction expansion" method, a 3D problem. Here I consider the analogous 1D problem of finding the 1D Green's function using an **"expanding tied-down-at-the-ends string"** and the same "full eigenfunction expansion" method. I am able to show that you can do this and you get the correct g = -(1/2)| x |. This required adding up some tricky series and the details were put off into a separate document "Question about the 1D string Green's function.doc" which lives in the Stak folder.

10. This is a long and complex section. The opening observation is that in section 1 I obtained the A_{mn} of the 1/R expansion without doing any real integrations, because I used the shortcut "pillbox method". I review in 10 numbered steps exactly how this pillbox thing works. Next, I reformulate the same problem in a more **"standard method"** where you really have to do a "projection" integral to obtain the expansion coefficients. We can then recharacterize our 1/R expansion in this general Smythian form framework:

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ a_{nm}(r)\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$a_{nm}(r)\ =\ (2n+1)\ f(n,-m)\ (1/4\pi)\ \int d\Omega\ V(r,\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

which you would use, for example, in a Dirichlet problem with V prescribed on a sphere. The point is that you have to do an integral to get the coefficients, here called $a_{nm}(r)$. I then consider the above pair of equations (really a transform) in a series of examples, each a bit more complex than the previous one.

In Example 1, I put a point charge at **r'** = 0 so we know V(r,θ,φ) = q/r. This is an example of a central potential of course. If we first assume a little more generally that V(r,θ,φ) = V(r), we can do the dΩ integral on the second line above and we find that $a_{nm}(r) = V(r)\ \delta_{m0}\ \delta_{n0}$. In our particular case we get $a_{nm}(r) = \delta_{m0}\ \delta_{n0}\ q/r$. It is then trivial to show that the first line above then reproduces V(r,θ,φ) = q/r. We are just calibrating our equipment.

In Example 2, I put a point charge at $\mathbf{r'} = r'\hat{\mathbf{z}}$. We write $V(r,\theta; r') = q/\sqrt{r^2 + r'^2 - 2rr'\cos\theta}$ which is of course just q/R in this situation. We install this into our second line's dΩ integral and the dφ part is just 2π and we have this intermediate result:

$$a_{nm}(r)\ = \delta_{m0}\ (2n+1)\ (q/2)\ \int_{-1}^{1} dz\ P_n(z)\ 1/\sqrt{r^2 + r'^2 - 2rr'\ z}$$

I then show in full detail how you expand the radical in P functions using the "generating function" expansion. This expansion has two different forms depending on whether $\alpha = (r'/r)$ is > 1 or < 1. I then show that, when you do the dz integral, you get this result:

$$a_{nm}(r) = = \delta_{m0}\ (q/r')\ (r_</r')^n (r_>/r')^{-n-1} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

When this is then installed into our first line, we get

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ a_{nm}(r)\ P_n^m(\cos\theta)\ e^{im\varphi} = (q/r')\ \Sigma_{n=0}^{\infty}\ (r_</r')^n (r_>/r')^{-n-1}\ P_n(\cos\theta)$$

which is really the same as $\Sigma_{n=0}^{\infty}\ (r_<)^n (r_>)^{-n-1}\ P_n(\cos\theta)$ and thus agrees with our earlier section 3 calculation for this example's situation.

In Example 3 I consider the point charge to be at some arbitrary location **r'**. This example is somewhat off the track of the general flow. I first use Theorem 1 noted above applied to our Example 2 result above, getting then $V(r,\theta,\varphi) = (q/r')\ \Sigma_{n=0}^{\infty}\ (r_</r')^n (r_>/r')^{-n-1}\ P_n(\cos\gamma)$. In a definite off the track move, I then write out the SO(3) addition theorem $\Sigma_b\ D^\sigma(g_1)_{0b}\ D^\sigma(g_2)_{b0} = D^\sigma(g_1 g_2)_{00}$ as

$$P_n(\cos\gamma) = \Sigma_m\ f(n,-m)\ P_n^m(z)\ P_n^m(z')\ e^{im\varphi}\ e^{-im\varphi'}$$

I then install this into $V(r,\theta,\varphi)$ just stated and I obtain the 1/R expansion derived earlier in this doc. I say this Example 3 is off track because my theme was doing projection integrals and verifying the expansion first line with the projections from the second line. So, ending this diversion, we get to Example 4.

In Example 4 I again consider the point charge to be at some arbitrary location **r'**. I write the two lines from above in more detail now as

$$V(r,\theta,\varphi\ ;\ r',\theta',\varphi') = \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ a_{nm}(r;\ r',\theta',\varphi')\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$a_{nm}(r;\ r',\theta',\varphi') = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\ V(r,\theta,\varphi\ ;\ r',\theta',\varphi')\ e^{-im\varphi}\ P_n^m(z)$$

My goal is to take V = q/R in this form

$$V(r,\theta,\varphi\ ;\ r',\theta',\varphi') = q/\sqrt{r^2 + r'^2 - 2rr'\cos\gamma} = q\ f(\varphi-\varphi') \qquad \text{// see below}$$

$$\cos(\gamma) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

and install it into the second line above and do the dΩ integral. We get then this intermediate result:

$$a_{nm}(r;\ r',\theta',\varphi') = (2n+1)\ f(n,-m)\ (q/4\pi) \int dz\ P_n^m(z) \int_{-\pi}^{\pi} d\varphi\ \ 1/\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}\ \ e^{-im\varphi}$$

The imaginary part somehow vanishes we know, so we focus on the cos(mφ) term and we set φ' = 0 "without loss of generality", as the saying goes. The dφ integral is then my infamous and now well-studied integral which gives a Q "ring function" :

$$f(\varphi) = 1/\sqrt{a - b\cos(\varphi)} \qquad a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta' \qquad b = 2rr'\sin\theta\sin\theta'$$

$$\int_{-\pi}^{\pi} d\varphi\ f(\varphi)\cos(m\varphi) \;= 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)$$

We are then faced with this remaining dz integral

$$a_{nm}(r;\, r',\theta',\varphi'=0) = (2n+1)\;\; f(n,-m)\;(q/4\pi)\int dz\; P_n^m(z)\{\, 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\}$$

where the dz integral is over (-1,1) and of course a and b are functions of $z = \cos\theta$. This is at first blush a very formidable integral, but I am able to evaluate it in Appendix A with this result:

$$\int dz\; P_n^m(z)\{\, 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\} \;= [(2n+1)/(4\pi)]^{-1}\;\; (1/r')\,(r_</r')^n (r_>/r')^{-n-1}\, P_n^m(z')$$

which then tells us

$$a_{nm}(r;\, r',\theta',\varphi'=0) = \;(q/r')\; f(n,-m)\;(r_</r')^n (r_>/r')^{-n-1}\, P_n^m(z')$$

and finally, when we put this into our first line, we get our usual 1/R expansion! So this example shows that the general case projection integral is in fact non-trivial and the pillbox method really is a much more efficient way to obtain the 1/R expansion!

11. This section considers a **half-baked idea** I had at Cape Cod in May 2010. I imagined constructing a table of 3D transforms of the following form

$$V(r,\theta,\varphi\,;\, r',\theta',\varphi') \;= \Sigma_{n=0}^{\infty}\,\Sigma_{m=-n}^{n}\, a_{nm}(r;\, r',\theta',\varphi')\;\; P_n^m(\cos\theta)\; e^{im\varphi}$$

$$a_{nm}(r;\, r',\theta',\varphi') \;=\; (2n+1)\;\; f(n,-m)\;(1/4\pi)\int d\Omega\; V(r,\theta,\varphi\,;\, r',\theta',\varphi')\; e^{-im\varphi}\; P_n^m(z)$$

in which table the 1/R expansion would be but one entry, and there might be many other entries. This was a vague notion only. At first I was planning to have as entries only functions which satisfied the Laplace equation, but then I realized that even simple functions like $1/R^2$ did not satisfy Laplace, I was confusing this with $f(z) = 1/z^2$ satisfying Laplace. I then redirected the idea to apply to arbitrary functions without regard to Laplace. My dim hope was that somehow in my table I might have an expansion which would allow me to "simplify" the charge density on a disk and other objects that I got from oblate coordinates work. Of course that would require an "oblates" version of the above expansion, no big problem. After a small amount of fiddling, such as differentiating the 1/R expansion, I put the idea to rest.

Appendix A: The purpose of this appendix is to show that

$$\int dz\; P_n^m(z)\{\, 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\} \;= [(2n+1)/(4\pi)]^{-1}\;\; (1/r')\,(r_</r')^n (r_>/r')^{-n-1}\, P_n^m(z')$$

where $a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta'$ $\qquad b = 2rr'\sin\theta\sin\theta'$

$z = \cos\theta$ $\qquad z' = \cos\theta'$

a result needed in Section 10, Example 4 above. Plan A was to use Whipple, it failed. Plan B was to use a second kind Q addition theorem I spotted in GR7, but that led nowhere as well. Finally in Plan C I write the Selvaggi/Chester Snow cylindrical 1/R expansion which involves ring Q functions, I set this equal to the normal 1/R expansion in spherical coordinates, which tells me that this must be true:

$$\sqrt{2/b}\ Q_{m-1/2}(a/b) = (\pi/r')\ \Sigma_n f(n,-m)\ (r_</r')^n (r_>/r')^{-n-1}\ P_n^m(\cos\theta)\ P_n^m(\cos\theta')$$

$$a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta' \qquad b = 2rr'\sin\theta\sin\theta'$$

I then multiply both sides by $P_n^m(z)$ and integrate dz, and the above result then falls out. I might add that I fully derived the ring Q function 1/R expansion in "cos(nx) over.... doc" stored in integrals + GR.

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0. Introduction

The potential of a point charge at the origin is q/R with R = r. If we move the point charge to some other origin **r'**, the potential is q/R where R = |**r** - **r'**|. In either case, the potential is a solution of the Laplace equation (away from the point charge) and can therefore be written in terms of the Laplace "atoms" of whatever coordinate system we want to use. For example, if we think of R as f(**r**) we know we can expand it this way in sphericals (the notation here means that **r'** is a parameter, **r** is the variable)

$$f(\mathbf{r} \mid \mathbf{r}') = \Sigma_{mn} A_{mn} (r_</r')^n (r_>/r')^{-n-1}\ P_n^m(\cos\theta)\ e^{im\varphi} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

We know this r stuff as follows. Draw a sphere centered on the origin of radius r' which passes through our point charge. Inside this sphere we must have a uniform solution everywhere (this sphere is, after all, completely empty) and we know the atomic form must be r^n in that sphere. Similarly we know we must have r^{-n-1} outside that same sphere. These then satisfy the requirement that f be finite at r = 0 and r = ∞.

Notice that we are parameterizing our two points **r** and **r'** this way

$$\mathbf{r} = r\ (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \quad \Rightarrow \quad \hat{\mathbf{r}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$
$$\mathbf{r}' = r'(\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta') \quad \Rightarrow \quad \hat{\mathbf{r}}' = (\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta')$$

$$\Rightarrow\ \hat{\mathbf{r}} \bullet \hat{\mathbf{r}}' = \sin\theta\sin\theta'(\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi') + \cos\theta\cos\theta' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi') = \cos(\gamma)$$

so that γ is the angle directly between our two vectors **r** and **r'**.

At this point there are several different things we can do.

1. Use the pillbox condition to find the A_{mn} coefficients.

I need to get this pillbox stuff into its own document somewhere, DONE! It is in physics/ electrostatics/the pillbox condition.doc. So imagine we have our pillbox centered on the point **r'** where our charge is, with its thin face 1 pointing to the origin. From our pillbox doc we have:

$$\partial_r V_i - \partial_r V_o = 4\pi q\,(1/r'^2)\,\delta(z-z')\,\delta(\varphi-\varphi') \qquad \text{since } r = r' \text{ at our pillbox}$$

where we have // note that $A_{mn} = A_{mn}(\mathbf{r}') = A_{mn}(r',\theta',\varphi')$

$$V_i = \Sigma_{mn}\, A_{mn}\,(r/r')^n\, P_n^m(\cos\theta)\, e^{im\varphi}$$
$$V_o = \Sigma_{mn}\, A_{mn}\,(r/r')^{-n-1}\, P_n^m(\cos\theta)\, e^{im\varphi}$$

so

$$\partial_r V_i - \partial_r V_o = \Sigma_{mn}\, A_{mn}\,[\, n(r/r')^{n-1}/r' \;-\; (-n-1)(r/r')^{-n-2}/r'\,]\; P_n^m(\cos\theta)\, e^{im\varphi}$$

$$= \Sigma_{mn}\, A_{mn}\,[\, n(r/r')^{n-1} \;+\; (n+1)(r/r')^{-n-2}\,]\; P_n^m(\cos\theta)\, e^{im\varphi} / r'$$

But we evaluate this now at r = r' which is on the sphere containing the point charge

$$\partial_r V_i - \partial_r V_o = \Sigma_{mn}\, A_{mn}\,(2n+1)\, P_n^m(z)\, e^{im\varphi} / r'$$

$$= 4\pi q\,(1/r'^2)\,\delta(z-z')\,\delta(\varphi-\varphi')$$

or

$$\Sigma_{mn}\, A_{mn}\,(2n+1)\, P_n^m(z)\, e^{im\varphi} = 4\pi q\,(1/r')\,\delta(z-z')\,\delta(\varphi-\varphi')$$

We will use two orthogonalities to find the A_{mn}. First, apply $\int d\varphi\, e^{-im'\varphi}$ to both sides and use

$$(1/2\pi)\int_{-\pi}^{\pi} d\varphi\, e^{im\varphi}\, e^{-im'\varphi} = \delta_{mm'} \qquad \text{// orthogonality}$$

The LHS becomes

$$\int d\varphi\, e^{-im'\varphi}\, \Sigma_{mn}\, A_{mn}\,(2n+1)\, P_n^m(z)\, e^{im\varphi} = \Sigma_{mn}\, A_{mn}\,(2n+1)\, P_n^m(z) \int d\varphi\, e^{-im'\varphi}\, e^{im\varphi}$$

$$= 2\pi\, \Sigma_{mn}\, A_{mn}\,(2n+1)\, P_n^m(z)\, \delta_{mm'} = 2\pi\, \Sigma_n\, A_{m'n}\,(2n+1)\, P_n^{m'}(z)$$

and the RHS becomes $4\pi q\,(1/r')\,\delta(z-z')\,e^{-im'\varphi'}$ so we then have (replace m' with m)

$$2\pi\, \Sigma_n\, A_{mn}\,(2n+1)\, P_n^m(z) = 4\pi q\,(1/r')\,\delta(z-z')\,e^{-im\varphi'}$$

Next, apply $\int dz\, P_{n'}^m(z)$ to both sides and we get

$$2\pi\, \Sigma_n\, A_{mn}\,(2n+1) \int dz\, P_{n'}^m(z)\, P_n^m(z) = 2\pi\, \Sigma_n\, A_{m'n}\,(2n+1)\,\{\, \delta_{n,n'}\, K_n^m \,\}$$

$$= 2\pi\, A_{mn'}\,(2n'+1)\, K_{n'}^m \qquad K_n^m = (n+1/2)^{-1}\, f(n,m) = 2(2n+1)^{-1}\, f(n,m)$$

which then says

$$2\pi\, A_{mn}\,(2n+1)\,K_n^m \;=\; 4\pi q\,(1/r')\;e^{-im\varphi'}\,P_n^m(z')$$

or

$$A_{mn}\,(2n+1)\,K_n^m \;=\; 2q\,(1/r')\;e^{-im\varphi'}\,P_n^m(z')$$

or

$$A_{mn}\,(2n+1)\,2(2n+1)^{-1}\,f(n,m) \;=\; 2q\,(1/r')\;e^{-im\varphi'}\,P_n^m(z')$$

or

$$A_{mn}\,f(n,m) \;=\; q\,(1/r')\;e^{-im\varphi'}\,P_n^m(z')$$

or

$$A_{mn} \;=\; q\,(1/r')\;f(n,-m)e^{-im\varphi'}\,P_n^m(z') \qquad *******$$

for our final answer. We can then install this back into our expansion with q = 1 for f = 1/R :

$$f(\mathbf{r}\mid\mathbf{r}') \;=\Sigma_{mn}\,A_{mn}\,(r_</r')^n(r_>/r')^{-n-1}\;P_n^m(\cos\theta)\,e^{im\varphi} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

$$1/R \;=(1/r')\;\Sigma_{mn}\,f(n,-m)\,(r_</r')^n(r_>/r')^{-n-1}\;P_n^m(z)\,P_n^m(z')\,e^{im\varphi}\,e^{-im\varphi'}$$

The powers of r' above are

$$r'^{-1}\,r'^{-n}\,r'^{n+1} \;= r'^0$$

so our result simplifies to,

$$1/R \;=\;\Sigma_{n=0}^{\infty}\,\Sigma_{m=-n}^{n}\,f(n,-m)\,(r_<)^n(r_>)^{-n-1}\;P_n^m(z)\,P_n^m(z')\,e^{im\varphi}\,e^{-im\varphi'} \qquad (*)$$

Below I will rewrite this in Jackson Y functions and obtain verification. Assuming that (*) is correct as shown above, this is a good time to restate the result using non-negative m in the sum. We do this in detail:

$$1/R \;=\;\Sigma_{n=0}^{\infty}\,\Sigma_{m=-n}^{n}\,f(n,-m)\,(r_<)^n(r_>)^{-n-1}\;P_n^m(z)\,P_n^m(z')\,e^{im\varphi}\,e^{-im\varphi'}$$

$$=\;\Sigma_{n=0}^{\infty}\,\Sigma_{m=-\infty}^{-1}\,f(n,-m)\,(r_<)^n(r_>)^{-n-1}\;P_n^m(z)\,P_n^m(z')\,e^{im\varphi}\,e^{-im\varphi'}$$
$$+\;\Sigma_{n=0}^{\infty}\,\Sigma_{m=1}^{\infty}\,f(n,-m)\,(r_<)^n(r_>)^{-n-1}\;P_n^m(z)\,P_n^m(z')\,e^{im\varphi}\,e^{-im\varphi'} \;+\;\Sigma_{n=0}\,(r_<)^n(r_>)^{-n-1}\,P_n(z)\,P_n(z')$$

$$=\;\Sigma_{n=0}^{\infty}\,\Sigma_{m=1}^{\infty}\,f(n,m)\,(r_<)^n(r_>)^{-n-1}\;P_n^{-m}(z)\,P_n^{-m}(z')\,e^{-im\varphi}\,e^{+im\varphi'} \qquad \text{// folding}$$
$$+\;\Sigma_{n=0}^{\infty}\,\Sigma_{m=1}^{\infty}\,f(n,-m)\,(r_<)^n(r_>)^{-n-1}\;P_n^m(z)\,P_n^m(z')\,e^{im\varphi}\,e^{-im\varphi'} \;+\;\Sigma_{n=0}\,(r_<)^n(r_>)^{-n-1}\,P_n(z)\,P_n(z')$$

But we know that

$$P_n^m(z) = f(n,m)\,P_n^{-m}(z) \qquad \text{// Bateman p 140 (7)}$$

so that $\qquad$ // note that $f(n,m) = \Gamma(n+m+1)/\Gamma(n-m+1)$

$P_n^{-m}(z)\, P_n^{-m}(z') = f(n,-m)\, f(n,-m)\, P_n^m(z)\, P_n^m(z')$

so we then get for our 1/R business,

$$= \Sigma_{n=0}^{\infty} \Sigma_{m=1}^{\infty} f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z')\, e^{-im\varphi}\, e^{+im\varphi'}$$
$$+ \Sigma_{n=0}^{\infty} \Sigma_{m=1}^{\infty} f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z')\, e^{im\varphi}\, e^{-im\varphi'} \;+\; \Sigma_{n=0} (r_<)^n (r_>)^{-n-1}\, P_n(z)\, P_n(z')$$

When we now combine the two sums, we have $e^{ix} + e^{-ix} = 2\cos(x)$, so

$$= 2\, \Sigma_{n=0}^{\infty} \Sigma_{m=1}^{\infty} f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z') \cos[m(\varphi-\varphi')] + \Sigma_{n=0} (r_<)^n (r_>)^{-n-1}\, P_n(z)\, P_n(z')$$

and at this point we define ε_m in the usual way to get // $\varepsilon_m = 2 - \delta_{m,0}$?

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} \varepsilon_m\; f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z') \cos[m(\varphi-\varphi')] \qquad (**)$$

which we can compare with our earlier result

$$1/R \;=\; \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z')\, e^{im\varphi}\, e^{-im\varphi'} \qquad (*)$$

2. Derivation of a Theorem

Suppose you know that some function $f(\mathbf{r},\mathbf{r}')$ is a rotational scalar. Then you know you can write $f(\mathbf{r},\mathbf{r}') = F(r,r',\cos\gamma)$ where $\cos\gamma = \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}'$. You then expand F on some complete set of basis functions $\varphi_n(\cos\gamma)$ for the interval $\gamma = (0,2\pi)$ so you then have

$$f(\mathbf{r},\mathbf{r}') = F(r,r',\cos\gamma) = \Sigma_n\, k_n(r,r')\, \varphi_n(\cos\gamma) \;= \Sigma_n\, k_n(r,r')\, \varphi_n(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}') \qquad (*)$$

You know this expansion exists, and you know that everything in this equation is a rotational scalar. You can evaluate it in a frame of reference in which $\hat{\mathbf{r}}' = \hat{\mathbf{z}}$ in which case $\cos\gamma = \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = \hat{\mathbf{r}}\bullet\hat{\mathbf{z}} = \cos\theta$ where θ is the polar angle for $\mathbf{r}$. You then get

$$f(\mathbf{r},r'\hat{\mathbf{z}}) = F(r,r',\cos\theta) = \Sigma_n\, k_n(r,r')\, \varphi_n(\cos\theta) \qquad (**)$$

Now we start our Theorem Development. Suppose someone tells you that $f(\mathbf{r},\mathbf{r}')$ is a rotational scalar, and they tell you that (**) is true where k_n is some function as shown, and where $\varphi_n(\cos\theta)$ is a set of complete basis functions for the interval $\theta = (0,2\pi)$. You first "go off" and expand $f(\mathbf{r},\mathbf{r}')$ on these same basis functions, so you know that the following is true for some unknown functions h_n

$$f(\mathbf{r},\mathbf{r}') = F(r,r',\cos\gamma) = \Sigma_n\, h_n(r,r')\, \varphi_n(\cos\gamma) \;= \Sigma_n\, h_n(r,r')\, \varphi_n(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}') \qquad (***)$$

You know that you can evaluate this using $\hat{\mathbf{r}}' = \hat{\mathbf{z}}$ and you will get

$$f(\mathbf{r},r'\,\hat{\mathbf{z}}) = F(r,r',\cos\theta) = \Sigma_{\mathbf{n}}\, h_{\mathbf{n}}(r,r')\, \varphi_{\mathbf{n}}(\cos\theta)$$

But you know that (**) is true because that was handed to you. Thus, you know that

$$f(\mathbf{r},r'\,\hat{\mathbf{z}}) = \Sigma_{\mathbf{n}}\, k_{\mathbf{n}}(r,r')\, \varphi_{\mathbf{n}}(\cos\theta) = \Sigma_{\mathbf{n}}\, h_{\mathbf{n}}(r,r')\, \varphi_{\mathbf{n}}(\cos\theta)$$

Since the $\varphi_{\mathbf{n}}(\cos\theta)$ are a basis, you conclude that

$$k_{\mathbf{n}}(r,r') = h_{\mathbf{n}}(r,r')$$

You then look back at your expansion (***) and you arrive at this conclusion

$$f(\mathbf{r},\mathbf{r}') = F(r,r',\cos\gamma) = \Sigma_{\mathbf{n}}\, k_{\mathbf{n}}(r,r')\, \varphi_{\mathbf{n}}(\cos\gamma) = \Sigma_{\mathbf{n}}\, k_{\mathbf{n}}(r,r')\, \varphi_{\mathbf{n}}(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}')$$

So we have proven this theorem:

Theorem 1. If $f(\mathbf{r},\mathbf{r}')$ is a rotational scalar, and if we know this is true,

$$f(\mathbf{r},r'\,\hat{\mathbf{z}}) = \Sigma_{\mathbf{n}}\, k_{\mathbf{n}}(r,r')\, \varphi_{\mathbf{n}}(\cos\theta)$$

then it follows that

$$f(\mathbf{r},\mathbf{r}') = \Sigma_{\mathbf{n}}\, k_{\mathbf{n}}(r,r')\, \varphi_{\mathbf{n}}(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}')$$

3. Derivation of the Legendre 1/R expansion and Addition Theorem

Above we found this expansion for 1/R which we take to be our $f(\mathbf{r},\mathbf{r}')$,

$$1/R \quad = \Sigma_{\mathbf{mn}}\, f(n,-m)\, (r_<)^{\mathbf{n}}(r_>)^{-\mathbf{n}-1}\; P_{\mathbf{n}}^{\mathbf{m}}(z)\, P_{\mathbf{n}}^{\mathbf{m}}(z')\, e^{\mathbf{i}\mathbf{m}\varphi}\, e^{-\mathbf{i}\mathbf{m}\varphi'} \qquad (*)$$

If we put $\mathbf{r}'$ on the z axis, then $P_{\mathbf{n}}^{\mathbf{m}}(1) = \delta_{\mathbf{m}}^{\mathbf{0}}$ and we get a simplified result

$$1/\,|\mathbf{r} - r'\hat{\mathbf{z}}| \;=\; \Sigma_{\mathbf{n}}\; (r_<)^{\mathbf{n}}(r_>)^{-\mathbf{n}-1}\; P_{\mathbf{n}}(\cos\theta) \qquad (**)$$

Since $P_{\mathbf{n}}(\cos\theta)$ is a basis for θ in $(0,2\pi)$, we apply our theorem and it tells us that

$$1/R = 1/\,|\mathbf{r} - \mathbf{r}'| \;=\; \Sigma_{\mathbf{n}}\; (r_<)^{\mathbf{n}}(r_>)^{-\mathbf{n}-1}\; P_{\mathbf{n}}(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}')$$

Earlier we showed that

$$\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi') = \cos\gamma$$

so we conclude that

$$1/|\mathbf{r} - \mathbf{r}'| = \Sigma_n \ (r_<)^n (r_>)^{-n-1} \ P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))$$

This appears in Jackson on page 62. So now we have these two results:

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} \varepsilon_m \ f(n,-m) \ (r_<)^n (r_>)^{-n-1} \ P_n^m(z) \ P_n^m(z') \cos[m(\varphi-\varphi')] \qquad (**)$$

$$1/R = \Sigma_n \ (r_<)^n (r_>)^{-n-1} \ P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))$$

Although the functions $(r_<)^n (r_>)^{-n-1}$ are not really a complete set, we suspect that the two expressions can only be equal for all values or r and r' if the following is true:

$$P_n[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] = \Sigma_{m=0}^{\infty} \varepsilon_m \ f(n,-m) \ P_n^m(\cos\theta) \ P_n^m(\cos\theta') \cos[m(\varphi-\varphi')]$$

This is the famous "addition theorem" which I know has a group theory basis, but here we just ignore that fact. If we had to PROVE this thing, we would resort to the group theory. Verification comes from several places. First, here is GR

Addition theorems

8.794

$$1.^{11} \quad P_\nu(\cos\psi_1\cos\psi_2 + \sin\psi_1\sin\psi_2\cos\varphi)$$
$$= P_\nu(\cos\psi_1)\, P_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}(-1)^k P_\nu^{-k}(\cos\psi_1)\, P_\nu^k(\cos\psi_2)\cos k\varphi$$
$$= P_\nu(\cos\psi_1)\, P_\nu(\cos\psi_2) + 2\sum_{k=1}^{\infty}\frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)} P_\nu^k(\cos\psi_1)\, P_\nu^k(\cos\psi_2)\cos k\varphi$$
$$[0 \le \psi_1 < \pi, \quad 0 \le \psi_2 < \pi, \quad \psi_1 + \psi_2 < \pi, \quad \varphi \text{ real}] \qquad (\text{cf. } \mathbf{8.814}, \mathbf{8.844}\ 1) \quad \text{MO } 90$$

where it is the second form that matches ours. Notice the ε_m factor of 2, and the f(ν,-k). This same result appears in Bateman page 169. It also appears page 69 of Jackson.

4. Limit of 1/R as r'→0 gives 1/r

If we take **r'** → 0 we examine our expansion above

$$1/R = \Sigma_{mn} \ f(n,-m) \ (r')^n (r)^{-n-1} \ P_n^m(z) \ P_n^m(z') \ e^{im\varphi} \ e^{-im\varphi'}$$

and we see that only the n = 0 term survives (which means only m = 0 as well) and we get 1/R → 1/r, as expected.

5. Write 1/R using Jackson's Y_{nm}

You don't see our 1/R expansion written very often in this manner,

$$1/R \quad = \Sigma_{mn}\, f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z')\, e^{im\varphi}\, e^{-im\varphi'} \qquad (*)$$

I don't have a reference right now that verifies it. But to verify it, let's first go with Jackson Y functions page 65

$$Y_{nm} \equiv (2n+1)^{1/2}\, (1/\sqrt{4\pi}\,)\; (f(n,-m))^{1/2}\, P_n^m(z)\, e^{im\varphi}$$

Then we have

$$P_n^m(z)\, e^{im\varphi} = (2n+1)^{-1/2}\sqrt{4\pi}\; (f(n,-m))^{-1/2}\, Y_{nm}(\theta,\varphi)$$
$$P_n^m(z')\, e^{im\varphi'} = (2n+1)^{-1/2}\sqrt{4\pi}\; (f(n,-m))^{-1/2}\, Y_{nm}(\theta',\varphi')$$

$$P_n^m(z)\, P_n^m(z')\, e^{im\varphi}\, e^{-im\varphi'} = (2n+1)^{-1}(4\pi)\, (f(n,-m))^{-1}\, Y_{nm}(\theta',\varphi')^*\, Y_{nm}(\theta,\varphi)$$

$$\Sigma_{mn}\, f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; P_n^m(z)\, P_n^m(z')\, e^{im\varphi}\, e^{-im\varphi'}$$

$$= \Sigma_{mn}\, f(n,-m)\, (r_<)^n (r_>)^{-n-1}\; (2n+1)^{-1}(4\pi)\, (f(n,-m))^{-1}\, Y_{nm}(\theta',\varphi')^*\, Y_{nm}(\theta,\varphi)$$

$$= (4\pi)\, \Sigma_{mn}\, (2n+1)^{-1}\; (r_<)^n (r_>)^{-n-1}\, Y_{nm}(\theta',\varphi')^*\, Y_{nm}(\theta,\varphi) = 1/R = 1/\,|\mathbf{r} - \mathbf{r}'|$$

and this is exactly the way it appears on Jackson page 69, so we have confirmed our result (*) above.

6. Another way derive the 1/R expansion (almost)

We started with the potential of our point charge this way above

$$1/|\mathbf{r} - \mathbf{r}'| = \Sigma_{mn}\, A_{mn}'\, (r_</r')^n (r_>/r')^{-n-1}\; P_n^m(\cos\theta)\, e^{im\varphi} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

We can see from the LHS that things must be **symmetric** in terms or $\mathbf{r} \leftrightarrow \mathbf{r}'$ so we can also write

$$1/|\mathbf{r} - \mathbf{r}'| = \Sigma_{mn}\, A_{mn}\, (r_</r)^n (r_>/r)^{-n-1}\; P_n^m(\cos\theta')\, e^{im\varphi'} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

Here we use the notation $A_{mn}' = A_{mn}(\mathbf{r}')$ and $A_{mn} = A_{mn}(\mathbf{r})$. We could at this point change the sign of the expo exponent in either of both equations since we know that $\mathrm{Im}(\Sigma_{nm} \ldots) = 0$ so the sign cannot possibly matter. At the end below I will change both signs.

Equate to get

$$\Sigma_{mn}\, A_{mn}'\, (r_</r')^n (r_>/r')^{-n-1}\; P_n^m(\cos\theta)\, e^{im\varphi} = \Sigma_{mn}\, A_{mn}\, (r_</r)^n (r_>/r)^{-n-1}\; P_n^m(\cos\theta')\, e^{im\varphi'}$$

Let's select $r < r'$ to be specific, then we have

$$\Sigma_{mn}\, A_{mn}'\, (r/r')^n (r'/r')^{-n-1}\; P_n^m(\cos\theta)\, e^{im\varphi} = \Sigma_{mn}\, A_{mn}\, (r/r)^n (r'/r)^{-n-1}\; P_n^m(\cos\theta')\, e^{im\varphi'}$$

$$\Sigma_{mn} A_{mn}' (r/r')^n P_n^m(\cos\theta) e^{im\varphi} = \Sigma_{mn} A_{mn} (r'/r)^{-n-1} P_n^m(\cos\theta') e^{im\varphi'}$$

$$\Sigma_{mn} A_{mn}' (r/r')^n P_n^m(\cos\theta) e^{im\varphi} = \Sigma_{mn} A_{mn} (r'/r)^{-n-1} P_n^m(\cos\theta') e^{im\varphi'}$$

We expect a term by term equality since functions are complete in either variables set, so

$$A_{mn}' (r/r')^n P_n^m(\cos\theta) e^{im\varphi} = A_{mn} (r'/r)^{-n-1} P_n^m(\cos\theta') e^{im\varphi'}$$

Multiply both sides by $(r/r')^{-n}$ to get

$$A_{mn}' P_n^m(\cos\theta) e^{im\varphi} = A_{mn} (r'/r)^{-1} P_n^m(\cos\theta') e^{im\varphi'}$$

$$A_{mn}' P_n^m(\cos\theta) e^{im\varphi} = A_{mn} (r/r') P_n^m(\cos\theta') e^{im\varphi'}$$

$$A_{mn}(r',\theta',\varphi') [(1/r) P_n^m(\cos\theta) e^{im\varphi}] = A_{mn}(r,\theta,\varphi) [(1/r') P_n^m(\cos\theta') e^{im\varphi'}]$$

And now, as predicted above, I will imagine we redid the above with the expo signs both changed,

$$A_{mn}(r',\theta',\varphi') [(1/r) P_n^m(\cos\theta) e^{-im\varphi}] = A_{mn}(r,\theta,\varphi) [(1/r') P_n^m(\cos\theta') e^{-im\varphi'}]$$

One obvious solution to this equation would be

$$A_{mn}(r,\theta,\varphi) = C_{nm} * [(1/r) P_n^m(\cos\theta) e^{-im\varphi}]$$

which we compare to our result above

$$A_{mn} = q\, f(n,-m) [(1/r') e^{-im\varphi'} P_n^m(z')]$$

Thus, we have arrived at the right answer except for the constant for each nm term. I don't think we can determine this constant just from our symmetry argument above. We need that pillbox condition to nail it down.

So if one asks "why does the 1/R expansion always seem to have this multiplicative form

$$1/R \sim P_n^m(z) P_n^m(z') e^{im\varphi} e^{-im\varphi'}$$

the answer is (1) mostly this is implied by the fact that R is symmetric. Once you expand in z,φ

$$A_{nm} P_n^m(z) e^{\pm im\varphi}$$

you then know that part of the constant of your expansion must be

$$A_{nm} \sim P_n^m(z') e^{\pm im\varphi'}$$

The thing we showed above was that if one phase is plus, the other must be minus, just because of the fake complex form we chose to use. (2) the other answer is that if we assume $A_{nm}\ P_n^m(z)\ e^{\pm im\varphi}$, then we know that the orthogonalities will cause $A_{nm} \sim\ P_n^m(z')\ e^{\pm im\varphi'}$.

7. Comments on symmetry and reality.

Let's go back to our final result

$$1/R \quad = \Sigma_{mn}\ f(n,-m)\ (r_<)^n (r_>)^{-n-1}\ P_n^m(z)\ P_n^m(z')\ e^{im\varphi}\ e^{-im\varphi'} \qquad (*)$$

I said above that we expect the expression on the RHS to be symmetric under swap of the coordinates, but it does not look symmetric because the phases have opposite signs. This could be explained away of we could show that the sine term vanishes, that is, if we could show that

$$\Sigma_{mn}\ f(n,-m)\ (r_<)^n (r_>)^{-n-1}\ P_n^m(z)\ P_n^m(z')\ \sin[m(\varphi-\varphi')] = 0 \quad = \Sigma_n\ (r_<)^n (r_>)^{-n-1}\ \Sigma_m\ S_{nm}$$

Remember that for each n, the sum on m is symmetric about m = 0 and the m=0 term obviously makes no contribution since sin(0) = 0. We know for on-cut Legendre functions that

$$f(n,m)\ P_n^{-m}(x) = (-1)^m\ P_n^m(x)$$

We have defined a "summand" S_{nm} above and we have

$$S_{nm} = f(n,-m)\ P_n^m(z)\ P_n^m(z')\ \sin[m(\varphi-\varphi')]$$

Therefore

$$S_{n,-m} = -\ f(n,m)\ P_n^{-m}(z)\ P_n^{-m}(z')\ \sin[m(\varphi-\varphi')]$$

$$= -\ f(n,m)\ \{P_n^m(z)\ P_n^m(z')\ f(n,-m)\ f(n,-m)\}\ \sin[m(\varphi-\varphi')]$$

$$= -\ f(n,-m)\ \{P_n^m(z)\ P_n^m(z')\ \}\ \sin[m(\varphi-\varphi')]$$

$$= -\ S_{nm}$$

Since the summand is odd in m and vanishes for m=0, we conclude that $\Sigma_m\ S_{nm} = 0$. Then our true result is this

$$1/R \quad = \Sigma_{mn}\ f(n,-m)\ (r_<)^n (r_>)^{-n-1}\ P_n^m(z)\ P_n^m(z')\ \cos[m(\varphi-\varphi')]$$

where we still have our symmetric Σ_m. The summand here, R_{nm}, is of course *even*, so we could write this by breaking out the m=0 contribution and folding over the rest, with the result (Neumann's factor ε_m)

$$1/R \quad = \Sigma_n\ \Sigma_{m=1}^{\infty} \varepsilon_m\ f(n,-m)\ (r_<)^n (r_>)^{-n-1}\ P_n^m(z)\ P_n^m(z')\ \cos[m(\varphi-\varphi')]$$

So, we have shown explicitly that the imaginary part of RHS(*) = 0. It involves a cancellation between the +m and -m values in the local m sums. But of course we already knew that the imaginary part had to be zero since we know the equation (*) is true, and the LHS is obviously real.

So in the final form above, we have our claimed result that things are fully symmetric if we exchange coordinates **r** and **r'**.

8. How does this fit into the full-eigenfunction-expansion for Green's Function formalism?

Stak page 153 points out that g(x|x') can always be written as a "full eigenfunction expansion"

$$g(x|x') = \Sigma_{\mathbf{k}}\, u_{\mathbf{k}}(x)u^*_{\mathbf{k}}(x')/\lambda_{\mathbf{k}} \qquad \text{where} \qquad -\nabla^2 u_{\mathbf{k}}(x) = \lambda_{\mathbf{k}}\, u_{\mathbf{k}}(x) \;\text{ and }\; -\nabla^2 g(x|x') = \delta(x-x')$$

Here "k" refers to all the eigenvalue labels, whatever they may be. This is in any number of dimensions.

Now, since we are going to do everything below in Jackson units, we need to re-scale the Green's function. We need to make it "bigger" by 4π so that we will then have

$$g(x|x') = (4\pi)\,\Sigma_{\mathbf{k}}\, u_{\mathbf{k}}(x)u^*_{\mathbf{k}}(x')/\lambda_{\mathbf{k}} \qquad \text{where} \qquad -\nabla^2 u_{\mathbf{k}}(x) = \lambda_{\mathbf{k}}\, u_{\mathbf{k}}(x) \;\text{ and }\; -\nabla^2 g(x|x') = 4\pi\delta(x-x')$$

where on the right we have Jackson page 78 (3.116). This is the Poisson equation in cgs units with a unit source charge of "1" unit.

Now, on page 145 H (where I fixed his error) Stak shows that for a sphere of unit radius, the eigenfunctions can be written in this manner

$$u_{\mathbf{k}}(x) \;\sim\; (\sqrt{1/r}\,)\, J_{\mathbf{n+1/2}}(\beta_{\mathbf{k}}{}^{\mathbf{n+1/2}} r)\, Y_{\mathbf{nm}}(\theta,\varphi) \qquad \lambda_{\mathbf{kn}} = (\beta_{\mathbf{k}}{}^{\mathbf{n+1/2}})^2 \;= \text{zeros squared of } J_{\mathbf{n+1/2}}(z)$$

and we learn that the eigenvalues are as shown on the right (k is overloaded, be careful).

Laplace Scaling Theorem. Suppose we have a PDE $L_{\mathbf{x}}u(x) = f(x)$. Suppose $L_{\mathbf{x}}$ has the property that $L_{\boldsymbol{\alpha x}} = \alpha^{-2}L_{\mathbf{x}}$. I don't have a name for this property, but the full Laplace operator seems to have it, and a general 2nd order or any order operators does NOT have it. Then we can say

$$L_{\mathbf{x}}u(x) = f(x) \;\Rightarrow\; L_{\boldsymbol{\alpha x}}u(\alpha x) = f(\alpha x) \;\Rightarrow\; \alpha^{-2}L_{\mathbf{x}}\, u(\alpha x) = f(\alpha x) \;\Rightarrow\; L_{\mathbf{x}}\, u(\alpha x) = \alpha^2\, f(\alpha x)$$

In particular, if $f(x) = \lambda\, u(x)$, we find that

$$L_{\mathbf{x}}u(x) = \lambda\, u(x) \qquad \Rightarrow \qquad L_{\mathbf{x}}\, u(\alpha x) = (\alpha^2\lambda)\, u(\alpha x)$$

Now, if we write $L_{\mathbf{x}}$ in spherical coordinates, we find that $L_{\boldsymbol{\alpha r,\theta,\varphi}} = \alpha^{-2}\, L_{\mathbf{r,\theta,\varphi}}$ so $L_{\mathbf{x}}$ written in sphericals has our scaling property in the r coordinate. Thus we can say:

$$L_{\mathbf{r,\theta,\varphi}}u(r,\theta,\varphi) = \lambda\, u(r,\theta,\varphi) \qquad \Rightarrow \qquad L_{\mathbf{r,\theta,\varphi}}u(\alpha r,\theta,\varphi) = (\alpha^2\lambda)\, u(\alpha r,\theta,\varphi)$$

Then, for example, we could set $\alpha = 1/a$ to get

$L_{r,\theta,\varphi}u(r,\theta,\varphi) = \lambda\, u(r,\theta,\varphi) \quad \Rightarrow \quad L_{r,\theta,\varphi}u(r/a,\theta,\varphi) = (\lambda/a^2)\, u(r/a,\theta,\varphi)$

This says: "if $u(r,\theta,\varphi)$ is an eigenfunction of $L_{r,\theta,\varphi}$ with eigenvalue λ, then $u(r/a,\theta,\varphi)$ is also an eigenfunction of $L_{r,\theta,\varphi}$, but it has eigenvalue (λ/a^2).".

As an application of this little theorem, looking at the solution u_k above for the unit sphere, we may conclude that, for a sphere of radius a, the eigenfunctions have this form

$$u_k(x) \sim (\sqrt{a/r}\,)\, J_{n+1/2}(\beta_k^{n+1/2} r/a)\, Y_{nm}(\theta,\varphi) \qquad \lambda_{kn} = (\beta_k^{n+1/2}/a)^2$$

The main point here is the scaling of the eigenvalue. If we regard z in $J_\nu(z)$ as dimensionless, then the zeros are also dimensionless, but our λ_{kn} have dimensions L^{-2}. We regard r and a as having dimensions L, and of course r/a is dimensionless. You can see that this u_k vanishes now at r = a instead of r = 1.

So, let's consider a finite metal sphere and see what happens if we then take the limit of infinite radius -- we expect to obtain just the 1/R Green's function. In the Stak raw notes we find that (Jackson Y)

$$u_k(x) = u_{nmk}(r,\theta.\varphi) = C_{nmk}\, (\sqrt{a/r}\,)\, J_{n+1/2}(\beta_k^{n+1/2} r/a)\, Y_{nm}(\theta,\varphi)$$

with $\quad C_{nmk} = \sqrt{2}\, a^{-3/2} / J_{n+3/2}(\beta_k^{n+1/2}) \qquad \lambda_{kn} = (\beta_k^{n+1/2}/a)^2$

Now, we want to use these eigenfunctions to write our Green's function (the full eigenfunction method)

$$g(x|x') = (4\pi)\, \Sigma_{nmk}\, u_{nmk}(x)\, u_{nmk}(x')/\lambda_{nmk}$$

We pause for a dimension check. we know that for large a, $g(x|x') \sim 1/R$ which has dimensions L^{-1}. We see that our $u_{nmk}(x)$ each have dimensions $L^{-3/2}$ so together dim(uu) = L^{-3}. But dim(1/λ) = L^2 and this then verifies that dim(g) = L^{-1}. I had a lot of trouble with this at an earlier stage.

Now, as we take the limit $a \to \infty$, I conjecture that $g(x|x') \to 1/|x-x'|$. But I don't really know how to show this, apart from the fact that I know what the answer is and it is unique, QED. If you tried to show it, you would have this situation

$$g(x|x') = (4\pi)\, \Sigma_{nmk}\, u_{nmk}(x)\, u_{nmk}(x')/\lambda_{nk}$$

$$=(4\pi)\, \Sigma_{nmk}\; C_{nmk}^2 (\beta_k^{n+1/2}/a)^{-2} \sqrt{a/r}\, J_{n+1/2}(\beta_k^{n+1/2} r/a)\, \sqrt{a/r'}\, J_{n+1/2}(\beta_k^{n+1/2} r'/a)\, Y_{nm}(\theta,\varphi)\, Y_{nm}(\theta',\varphi')^*$$

In the limit $a \to \infty$, somehow this is going to become our free space 1/R (from above) (all Jackson Y)

$$= (4\pi)\, \Sigma_{mn}\, (2n+1)^{-1}\; (r_<)^n (r_>)^{-n-1}\, Y_{nm}(\theta',\varphi')^*\, Y_{nm}(\theta,\varphi) \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

so it must be true that, for large a, (#)

$\Sigma_k\, C_{nmk}^2(\beta_k^{n+1/2}/a)^{-2}\sqrt{a/r}\, J_{n+1/2}(\beta_k^{n+1/2}r/a)\sqrt{a/r'}\, J_{n+1/2}(\beta_k^{n+1/2}r'/a) \;\rightarrow\; (2n+1)^{-1}\,(r_<)^n(r_>)^{-n-1}$

Since there are arbitrarily large values of $\beta_k^{n+1/2}$ in the Σ_k, you *cannot* use the small arg limit for the J functions, even when a is large and r,r' << a. [If you could, you would find that r and r' have the same power r^n which is wrong] So let's process the LHS a bit, but to save space, use $\nu = n+1/2$ for a while

LHS $= \Sigma_k\, C_{nmk}^2(\beta_k^{n+1/2}/a)^{-2}\sqrt{a/r}\, J_{n+1/2}(\beta_k^{n+1/2}r/a)\sqrt{a/r'}\, J_{n+1/2}(\beta_k^{n+1/2}r'/a)$

$= \Sigma_k\, C_{nmk}^2 \quad (\beta_k^{\nu}/a)^{-2}\sqrt{a/r}\, J_\nu(\beta_k^{\nu}r/a)\sqrt{a/r'}\, J_\nu(\beta_k^{\nu}r'/a)$ // next, install C_{nmk} :

$= \Sigma_k \quad [\sqrt{2}\, a^{-3/2} / J_{\nu+1}(\beta_k^{\nu})]^2 \quad (\beta_k^{\nu}/a)^{-2}\sqrt{a/r}\, J_\nu(\beta_k^{\nu}r/a) \quad \sqrt{a/r'}\, J_\nu(\beta_k^{\nu}r'/a)$

$= \Sigma_k \quad 2a^{-3}\,[J_{\nu+1}(\beta_k^{\nu})])^{-2} \quad (\beta_k^{\nu})^{-2}\, a^2 \quad (a/\sqrt{rr'}\,)\; J_\nu(\beta_k^{\nu}r/a) \quad J_\nu(\beta_k^{\nu}r'/a)$

$= (2/\sqrt{rr'}\,)\,\Sigma_k \quad [\beta_k^{\nu}\, J_{\nu+1}(\beta_k^{\nu})])^{-2} \quad J_\nu(\beta_k^{\nu}r/a) \quad J_\nu(\beta_k^{\nu}r'/a)$

I *do* know from transforms.doc that

$\Sigma_{k=1}^{\infty}\,[a\, J_{\nu+1}(\beta_k^{\nu})/\sqrt{2}\,]^{-2}\, J_\nu(\beta_k^{\nu}\, x/a)\, J_\nu(\beta_k^{\nu}\, x'/a) \;=\; \delta(x-x')/x$ // completeness

But this is not what we get here because we have the extra factor $(\beta_k^{\nu})^{-2}$sitting in the sum! So I leave it as an unsolved problem to show the validity of the large a limit shown above, namely:

$(2/\sqrt{rr'}\,)\,\Sigma_k \quad [\beta_k^{\nu}\, J_{\nu+1}(\beta_k^{\nu})])^{-2} \quad J_\nu(\beta_k^{\nu}r/a) \quad J_\nu(\beta_k^{\nu}r'/a) \;\rightarrow\; (2n+1)^{-1}\,(r_<)^n(r_>)^{-n-1}$ (#)

At least both sides are symmetric under r↔r' and both sides have dimensions L^{-1}.

Try taking the r'→0 limit of (#) to see if (#) is true in this limit.

For the RHS of (#) we get

RHS $= (2n+1)^{-1}\,(r')^n(r)^{-n-1} \;= \delta_{n,0}\; r^{-1}$ in limit r' → 0

For the LHS we get

LHS $= (2/\sqrt{rr'}\,)\,\Sigma_k\,[\beta_k^{\nu}\, J_{\nu+1}(\beta_k^{\nu})])^{-2}\, J_\nu(\beta_k^{\nu}r/a)\, J_\nu(\beta_k^{\nu}r'/a)$ // next, use Jackson p 72,

$\rightarrow\; (2/\sqrt{rr'}\,)\,\Sigma_k\,[\beta_k^{\nu}\, J_{\nu+1}(\beta_k^{\nu})])^{-2}\, J_\nu(\beta_k^{\nu}r/a) \quad (\beta_k^{\nu}r'/2a)^{\nu} / \Gamma(\nu+1)$

so the power of r' here is $(r')^{\nu-1/2} = (r')^n$ and in our limit, only the n = 0 term survives, just as happened on the RHS. We then have $\nu = 1/2$ so

$\mathrm{LHS}_{n=0} \rightarrow (2/\sqrt{rr'}\,)\,\Sigma_k\,[\beta_k^{1/2}\, J_{3/2}(\beta_k^{1/2})])^{-2}\, J_{1/2}(\beta_k^{1/2}r/a) \quad (\beta_k^{1/2}r'/2a)^{1/2} / \Gamma(3/2)$

We know that $J_{1/2}(z) = (2/\pi z)^{1/2} \sin(z) = (2/\pi)^{1/2} z^{-1/2} \sin(z)$ which clearly has zeros at $z = k\pi$ so we know that $\beta_k^{1/2} = k\pi$.

Question: When we talk about "the zeros of $J_\nu(x)$, we mean the ones for $x > 0$. But in this case, we have zeros at $x = k\pi$ where k = integer. So we wonder whether or not we should be including the zero located at $k = 0$? I will try to not nail this down yet by using Σ_k.

Then we have

$$LHS_{n=0} \rightarrow (2/\sqrt{rr'}) \Sigma_k [k\pi J_{3/2}(k\pi)])^{-2} J_{1/2}(k\pi r/a) \quad (k\pi r'/2a)^{1/2} / \Gamma(3/2)$$

As shown in my Stak Ch6 raw notes we have for k = 1,2,3...

$$J_{3/2}(k\pi) = - (2/\pi^2 k)^{1/2} \cos(k\pi) = - (2/\pi^2 k)^{1/2} (-1)^k$$

so we then have

$$LHS_{n=0} \rightarrow (2/\sqrt{rr'}) \Sigma_k [k\pi (2/\pi^2 k)^{1/2}])^{-2} J_{1/2}(k\pi r/a) \quad (k\pi r'/2a)^{1/2} / \Gamma(3/2)$$

$$= (2/\sqrt{rr'}) \Sigma_k (\pi k)^{-2} (\pi^2 k/2) J_{1/2}(k\pi r/a) \quad (k\pi r'/2a)^{1/2} / \Gamma(3/2)$$

$$= (1/\sqrt{rr'}) \Sigma_k (k^{-1}) J_{1/2}(k\pi r/a) \quad (k\pi r'/2a)^{1/2} (2/\sqrt{\pi})$$

$$= (2/\sqrt{rr'}) (r'/2a)^{1/2} \Sigma_k J_{1/2}(k\pi r/a)/\sqrt{k} = \sqrt{2} (1/rr')^{1/2} (r'/a)^{1/2} \Sigma_k J_{1/2}(k\pi r/a)/\sqrt{k}$$

$$= \sqrt{2} (1/ra)^{1/2} \Sigma_k J_{1/2}(k\pi r/a)/\sqrt{k} = \sqrt{2} (1/\sqrt{ar}) \Sigma_k J_{1/2}(k\pi r/a)/\sqrt{k}$$

BUT, suppose we want to include the k = 0 eigenvalue? We would then add in this amount:

$$\lim_{k\to 0} (2/\sqrt{rr'}) [k\pi J_{3/2}(k\pi)])^{-2} J_{1/2}(k\pi r/a) \quad (k\pi r'/2a)^{1/2} / \Gamma(3/2)$$

Suppose we use the small argument form everywhere for J, then this becomes

$$\lim_{k\to 0} (2/\sqrt{rr'}) [k\pi (k\pi/2)^{3/2}/\Gamma(3/2)])^{-2} (k\pi r/a2)^{1/2} / \Gamma(3/2) \quad (k\pi r'/2a)^{1/2} / \Gamma(3/2)$$

The power of k here is this:

$$k^{-2} k^{-3} k^{1/2} k^{1/2} = k^{-4}$$

so the limit does not exist. So I think we may conclude: do not include k = 0 in your spectrum!

I will get LHS = RHS if I can show that

$$\sqrt{2}\ (1/\sqrt{ar}\ )\ \Sigma_k\ J_{1/2}(k\pi\ r/a)/\sqrt{k} \quad = \quad r^{-1}$$

or

$$\Sigma_k\ J_{1/2}(k\pi\ r/a)/\sqrt{k} \quad = \quad r^{-1}(\sqrt{ar}\ /\sqrt{2}\ ) \ = (4\pi/\sqrt{2}\ )\ \sqrt{a/r} \qquad\qquad r/a < 1$$

or

$$\Sigma_k\ J_{1/2}(k\pi\ x)/\sqrt{k} \quad = (1/\sqrt{2}\ )/\sqrt{x} \qquad\qquad x < 1 \qquad\qquad x = r/a \qquad\qquad (***)$$

Is this perhaps true? We know that

$$J_{1/2}(z) = (2/\pi z)^{1/2}\ \sin(z) \ = \ (2/\pi)^{1/2}\ z^{-1/2}\ \sin(z)$$

so we have $z = k\pi\ x$ so

$$\text{LHS}(***) \quad = \Sigma_k\ J_{1/2}(k\pi\ x)/\sqrt{k} \ = \Sigma_k\ (2/\pi)^{1/2}\ (k\pi\ x)^{-1/2}\ \sin(k\pi\ x)/\sqrt{k}$$

$$= \Sigma_k \quad (k\pi\ x\pi/2)^{-1/2} \quad \sin(k\pi\ x)/\sqrt{k}$$

$$= \quad (\pi x\pi/2)^{-1/2}\ \Sigma_k \quad \sin(k\pi\ x)/k \quad = \ (1/\pi)\ (2/x)^{1/2} \quad \Sigma_k\ \sin(k\pi x)/k$$

GR page 46 tells us that

1.44–1.45 Trigonometric (Fourier) series

1.441

1. $$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad [0 < x < 2\pi] \qquad \text{FI III 539}$$

But this result seems a bit unusual. In the limit $x \to 0$, you would think LHS = 0, not $\pi/2$. I think this might apply for $f(x) = \pi/2 - x/2$ which is a periodic sawtooth shaped function, which maybe is what is implied by the range shown. Ie, if you did regular Fourier series on this sawtooth, this might be how it would look. The sawtooth has value $\pi/2$ at $x = 0$ so that would then be correct. If I try to apply this to my series, I would replace x with πx to get

$$\Sigma_k\ \sin(k\pi x)/k \quad = \ (\pi\text{-}\pi x)/2 = (\pi/2)(1\text{-}x) \qquad\qquad [0 < \pi x < 2\pi] \quad \text{or} \quad 0 < x < 2.$$

If I go ahead and apply this, I get

$$\text{LHS}(***) = \ (1/\pi)\ (2/x)^{1/2}\ [\ \pi/2\text{-}\ \pi x/2] \quad = \ (2/x)^{1/2}\ (1/2)(1\text{-}x)$$

$$= \ (1/\sqrt{2}\ )\ [\ x^{-1/2}\ \text{-}\ x^{1/2}]$$

But remember that $x = r/a$ so in our $a \to \infty$ limit, we have $x \to 0$, so our answer is then really

$$\text{LHS}(***) = \ (1/\sqrt{2}\ )\ x^{-1/2}$$

Now above we *wanted* to show (***) to be true

$$\Sigma_k\, J_{1/2}(k\pi\, x)/\sqrt{k} \quad = (1/\sqrt{2}\,)/\sqrt{x}$$

and we have thus done showed it!

Let's now just review what we did. We wanted to verify that we could write the Green's Function for a sphere of radius a in this manner:

$$g(x|x') = (4\pi)\, \Sigma_{nmk}\, u_{nmk}(x) u^*_{nmk}(x')/\lambda_{kn}$$

where

$$u_k(x) = u_{nmk}(r,\theta.\varphi) = C_{nmk}\ (\sqrt{a/r}\,)\, J_{n+1/2}(\beta_k^{\,n+1/2} r/a)\, Y_{nm}(\theta,\varphi)$$

$$\text{with} \qquad C_{nmk} = \sqrt{2}\, a^{-3/2} / J_{n+3/2}(\beta_k^{\,n+1/2}) \qquad\qquad \lambda_{kn} = (\beta_k^{\,n+1/2}/a)^2$$

We ended up with the following result for g(x|x'),

$$g(x|x')=(4\pi)\, \Sigma_{nmk} C_{nmk}^{\,2} (\beta_k^{\,n+1/2}/a)^{-2} \sqrt{a/r}\, J_{n+1/2}(\beta_k^{\,n+1/2} r/a)\, \sqrt{a/r'}\, J_{n+1/2}(\beta_k^{\,n+1/2} r'/a) Y_{nm}(\theta,\varphi)\, Y_{nm}(\theta',\varphi')^*$$

In the limit $a \to \infty$ we expected to find that $g(x|x') \to 1/|x-x'|$ since our metal sphere becomes the great sphere at infinity. But we know an expansion for $1/|x-x'|$,

$$1/|x-x'| \ = (4\pi)\, \Sigma_{mn}\, (2n+1)^{-1}\ (r_<)^n (r_>)^{-n-1}\, Y_{nm}(\theta',\varphi')^*\, Y_{nm}(\theta,\varphi) \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

We were then able to conclude that we should have this equality, where $\nu = n+1/2$,

$$\lim_{a\to\infty}(2/\sqrt{rr'}\,)\, \Sigma_k \quad [\beta_k^{\,\nu}\, J_{\nu+1}(\beta_k^{\,\nu})]^{-2}\ \ J_\nu(\beta_k^{\,\nu} r/a) \quad J_\nu(\beta_k^{\,\nu} r'/a) \ = (2n+1)^{-1}\, (r_<)^n (r_>)^{-n-1} \qquad (\#)$$

I did not and still do not know how to verify this fact. Just expanding the J_ν for small argument does not cut the mustard because you get on the LHS $r^n\, r'^n$ which is wrong. The reason for this wrongness is that there are arbitrarily large zeros $\beta_k^{\,\nu}$ and no matter how large a is, the argument like $\beta_k^{\,\nu} r'/a$ is never small for all k.

I decided instead to try to verify (#) in the case that $r' \to 0$ in which case I expect both sides to reduce to the simple result 1/r in the limit $a\to\infty$. In this case, you CAN use the small argument approximation for the function $J_\nu(\beta_k^{\,\nu} r'/a)$ and after much fiddling, the result works out right. On both sides, the only surviving term is the n = 0 term and both sides give 1/r .

I admit I am intrigued by how you would verify (#) for finite r and r' and $a \to \infty$. I suspect the method involves converting the Σ_k to a contour integral which picks up pole residues to make the sum shown on the LHS of (#). Then somehow the relative size of r and r' determines which way you can close the contour and this is what causes the RHS to have different forms for the two relative sizes. I'm sure it all works, but let's let it go, we have already spent 12 pages just getting the special case done.

9. The 1D analog to our 3D situation

In "Question about the 1D string Green's function.doc" I consider a similar situation in 1D, where things are simpler in one sense, but more complicated in terms of what happens at ∞. I talk about a string problem defined on (-ℓ/2,ℓ/2) where (here we are back in Stakgold units)

$$g(x|\xi) = \Sigma_{n=1}^{\infty} \varphi_n(x)\varphi_n(\xi)/\lambda_n = \Sigma_{n=1}^{\infty} (2/\ell) \sin(n\pi[x+\ell/2]/\ell) \sin(n\pi[\xi+\ell/2]/\ell) / [n\pi/\ell]^2$$

with $\varphi_n(x) = (2/\ell)^{1/2} \sin(n\pi[x+\ell/2]/\ell)$ // normalized

In this case, I am able to directly compute that

$$g(x|0) = -(1/2)|x| + \ell/4$$

and if we throw out the $\ell/4$ constant, then as $\ell \to \infty$ we find $g(x|0) = -(1/2)|x| = E(x)$ which is the fundamental solution in 1D, just as 1/r is in 3D. So this is more support for the idea that you can arrive at the fundamental solution if you start with some isotropic solution and take the limit as the boundary moves away to infinity. Above I did this with the sphere in 3D.

10. Yet another way to find the A_{mn} coefficients for 1/R.

In the "pillbox method" given above, things were just a bit swept under the rug! Let's try to reconstruct the logic of this method and state it explicitly:

1. Assume a point charge q located at **r'** and our observation point is **r** in spherical coordinates.
2. We know that $V(\mathbf{r}|\mathbf{r}') = q/R$ where $R = |\mathbf{r}-\mathbf{r}'|$, the potential of a point charge.
3. We define a "math surface" as the sphere of radius r' centered at **r** = 0 which divides all space into two regions which contain no charge and in which V must satisfy Laplace.
4. For these inside and outside regions, we expand $V(\mathbf{r}|\mathbf{r}')$ on appropriate spherical atoms:

$$V(\mathbf{r}|\mathbf{r}') = V_i(\mathbf{r}|\mathbf{r}') = \Sigma_{mn} A_{nm} (r/r')^n P_n^m(\cos\theta) e^{im\varphi} \quad \text{valid for } r < r'$$
$$V(\mathbf{r}|\mathbf{r}') = V_o(\mathbf{r}|\mathbf{r}') = \Sigma_{mn} B_{nm} (r/r')^{-n-1} P_n^m(\cos\theta) e^{im\varphi} \quad \text{valid for } r > r'$$

We notionally suppress the fact that $A_{nm} = A_{nm}(\mathbf{r}')$ where **r'** is a "bystander parameter".

5. We "match" these two potentials at the spherical surface r = r' (away from the point charge) , and this at once tells us that $B_{nm} = A_{nm}$ and this then lets us rewrite our expansions in this simpler unified manner

$$V(\mathbf{r}|\mathbf{r}') = q/R = \Sigma_{mn} A_{nm} (r_</r')^n (r_>/r')^{-n-1} P_n^m(\cos\theta) e^{im\varphi}$$

This is just a cosmetic form, and in practice we use the V_i and V_o forms with $B_{nm} = A_{nm}$.

6. If we can find values for the coefficients A_{nm}, then we will have obtained our desired expansion for the quantity 1/R, which is our whole purpose here.

7. We now never deal directly with the "function" 1/R itself. Instead, we make use of an integral form of Poisson's equation (ie, the pillbox) relating V to the point charge located at **r**'.

$$\partial_{\mathbf{r}} V_{\mathbf{i}} - \partial_{\mathbf{r}} V_{\mathbf{o}} = 4\pi q (1/r'^2)\delta(z-z')\delta(\varphi-\varphi')$$

8. Using the above expansions for $V_{\mathbf{i}}$ and $V_{\mathbf{o}}$ this tells us that

$$\Sigma_{\mathbf{mn}} A_{\mathbf{mn}} (2n+1) P_{\mathbf{n}}^{\mathbf{m}}(z) e^{\mathbf{i}m\varphi} / r' = 4\pi q (1/r'^2)\delta(z-z')\delta(\varphi-\varphi')$$

where the RHS vanishes almost everywhere. This is a distributional equation!

9. Because both $P_{\mathbf{n}}^{\mathbf{m}}(z)$ and $e^{\mathbf{i}m\varphi}$ are orthogonal functions sets, it is then trivial to solve this for the coefficients and we find that

$$A_{\mathbf{mn}} = A_{\mathbf{nm}}(\mathbf{r}') = q\,(1/r')\; f(n,-m) e^{-\mathbf{i}m\varphi'} P_{\mathbf{n}}^{\mathbf{m}}(z')$$

10. Installing this into our expansion we then get

$$V(\mathbf{r}|\mathbf{r}') = q/R = (q/r') \Sigma_{\mathbf{mn}} f(n,-m) (r_</r')^{\mathbf{n}} (r_>/r')^{-\mathbf{n}-1} P_{\mathbf{n}}^{\mathbf{m}}(z) P_{\mathbf{n}}^{\mathbf{m}}(z') e^{\mathbf{i}m\varphi} e^{-\mathbf{i}m\varphi'}$$

and this we have found our expansion for 1/R (just set q = 1 above).

Notice that we never did any integrals against the explicit function 1/R to get the $A_{\mathbf{mn}}$ coefficients! Somehow this information was extracted instead from the Poisson equation driven by the point charge. Instead of "working with" the function V, we "worked with" this weird subtracted derivative of V and found these delta functions. This derivative is of course just the statement that $\nabla^2 V = 4\pi\rho = 4\pi q\, \delta(\mathbf{r}-\mathbf{r}')$ (ignoring signs and constants). By assuming the potential came from an isolated point in space (**r**'), we were able to use our knowledge of spherical atoms to write forms for V inside and outside of the sphere containing the point **r**'. These forms had restrictions which then let us solve the problem.

Normally, when you write an expansion like this, as we did above,

$$f(\mathbf{r} \mid \mathbf{r}') = \Sigma_{\mathbf{mn}} A_{\mathbf{mn}} (r_</r')^{\mathbf{n}} (r_>/r')^{-\mathbf{n}-1} P_{\mathbf{n}}^{\mathbf{m}}(\cos\theta) e^{\mathbf{i}m\varphi} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

you compute the $A_{\mathbf{mn}}$ by doing some integrals against the function f, but in our trick pillbox method we completely avoided doing any such integrals -- well, we did some, but they were against delta functions. Of course the expansion shown above is very specific to our point charge problem, and does not apply for a more general f.

I would now like to "work with" a more general arbitrary function $f(\mathbf{r} \mid \mathbf{r}')$ and show these integrals, then go back and see how these integrals work for f = 1/R. This should be pretty easy to do. From our transforms.doc, we have this transform pair

$$f(\theta,\varphi) = \Sigma_{\mathbf{n}=0}^{\infty} \Sigma_{\mathbf{m}=-\mathbf{n}}^{\mathbf{n}} f_{\mathbf{nm}} P_{\mathbf{n}}^{\mathbf{m}}(\cos\theta) e^{\mathbf{i}m\varphi} \qquad \text{// expansion}$$

$$f_{\mathbf{nm}} = (2n+1)\; f(n,-m)\, (1/4\pi) \int d\Omega\, f(\theta,\varphi) e^{-\mathbf{i}m\varphi} P_{\mathbf{n}}^{\mathbf{m}}(z) \qquad \text{// projection}$$

Let's change f_{nm} to a_{nm} and add a third parameter r to get

$$f(r,\theta,\varphi) = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} a_{nm}(r)\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$a_{nm}(r) = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\ f(r,\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

That is to say, on each sphere of radius r, we are doing a θ,φ expansion using our usual complete set. In order for the expansion to apply to a potential function V, we must assume that our expansion applies in a region of space which contains no charges, so V is then a Laplace solution. So write again, making this assumption

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} a_{nm}(r)\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$a_{nm}(r) = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\ V(r,\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

Now we try some sample cases.

Example 1: Point charge at r' = 0.

Suppose there is a point charge located at our spherical coordinates origin. We know then that $V(r,\theta,\varphi) = q/r$ in Jackson units. We then have this integral to do:

$$\int d\Omega\ e^{-im\varphi}\ P_n^m(z)$$

and I obstinately am not using normalized spherical harmonics at the moment, so we need these facts,

$$(1/2\pi) \int_{-\pi}^{\pi} d\varphi\ e^{im\varphi}\ e^{-im'\varphi} = \delta_{mm'} \qquad \text{// orthogonality} \qquad K_n^m = (n+1/2)^{-1}\ f(n,m)$$

$$\int_{-1}^{1} dz\ P_n^m(z) P_{n'}^m(z) = \delta_{n,n'}\ K_n^m \qquad n,n' = |m|, |m|+1, \ldots \qquad \text{// orthogonality}$$

Then we have

$$\int d\Omega\ e^{-im\varphi}\ P_n^m(z) = \int_{-1}^{1} dz \int_{-\pi}^{\pi} d\varphi\ e^{-im\varphi}\ P_n^m(z)$$

$$= \int_{-1}^{1} dz\ P_n^m(z) \left\{ \int_{-\pi}^{\pi} d\varphi\ e^{-im\varphi} \right\} = \int_{-1}^{1} dz\ P_n^m(z) \{ 2\pi\delta_{m0} \}$$

$$= 2\pi\delta_{m0} \left\{ \int_{-1}^{1} dz\ P_n^m(z) \right\} = 2\pi\delta_{m0}\ \delta_{n,0}\ K_n^m \qquad = 2\pi\delta_{m0}\ \delta_{n0}\ K_0^0 = 4\pi(2n+1)^{-1}\ \delta_{m0}\ \delta_{n0}$$

and then we get, for any $V(r,\theta,\varphi) = V(r)$ [a central potential, shall we call it]

$$a_{nm}(r) = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\ V(r,\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

$$= (2n+1)\ f(n,-m)\ (1/4\pi)\ V(r)\{ \int d\Omega\ e^{-im\varphi}\ P_n^m(z) \quad \}$$

$$= (2n+1)\ f(n,-m)\ (1/4\pi)\ V(r)\{ 4\pi(2n+1)^{-1}\ \delta_{m0}\ \delta_{n0} \}$$

$$= V(r)\ \delta_{m0}\ \delta_{n0}$$

so for our point charge example we have V(r) = q/r so that

$$a_{nm}(r) = \delta_{m0}\ \delta_{n0}\ q/r$$

and our expansion becomes

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ a_{nm}(r)\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$= \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ \{\ \delta_{m0}\ \delta_{n0}\ q/r\}\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$= (q/r)P_0^0(\cos\theta)\ e^{i0\varphi} \qquad = (q/r)P_0(\cos\theta)\ e^{i0\varphi} = (q/r)*\ 1\ *\ 1$$

$$= (q/r)$$

and things are consistent. So we have now done the case V = q/R explicitly, where in our spherical coordinates we have R = r.

Example 2: Point charge at $\mathbf{r}' = r'\hat{\mathbf{z}}$.

In the example we just did, our point charge was located at the spherical coordinates origin. Let's now move this point charge to the point $\mathbf{r}' = r'\hat{\mathbf{z}}$, and redo the work above. We know now that

$$V = q/|\mathbf{r} - \mathbf{r}'| \quad = V(r,\theta,\varphi\ ;\ r',\theta',\varphi')$$

where the primed coordinates are bystander parameters. Again we have this expansion:

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty}\ \Sigma_{m=-n}^{n}\ a_{nm}(r)\ P_n^m(\cos\theta)\ e^{im\varphi}$$

$$a_{nm}(r) = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\ V(r,\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

which we know is valid in all space away from the point charge at $\mathbf{r}'$. We want to compute $a_{nm}(r)$. The angle between $\mathbf{r}$ and $\mathbf{r}' = r'\hat{\mathbf{z}}$ is our polar integration angle θ and $\mathbf{r} \bullet \mathbf{r}' = rr'\cos\theta$ and then

$$|\mathbf{r} - \mathbf{r}'|^2 \quad = r^2 + r'^2 - 2rr'\cos\theta$$

which does NOT depend on φ. That is to say, we have

$$V(r,\theta,\varphi\,;\,r',\theta',\varphi') = V(r,\theta,\varphi\,;\,r',0,\varphi' = \text{any}) = V(r,\theta;\,r') = q/\sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

so we can now do the φ integral part of computing a_{nm} and we get

$$a_{nm}(r) = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\ V(r,\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$$

$$= (2n+1)\ f(n,-m)\ (1/4\pi) \int_{-1}^{1} dz\ P_n^m(z)\ V(r,\theta)\ \{ \int_{-\pi}^{\pi} d\varphi\ e^{-im\varphi}\ \}$$

$$= (2n+1)\ f(n,-m)\ (1/4\pi) \int_{-1}^{1} dz\ P_n^m(z)\ V(r,\theta)\ \{ 2\pi\delta_{m0}\ \}$$

$$= \delta_{m0}\ (2n+1)\ (1/2) \int_{-1}^{1} dz\ P_n^0(z)\ V(r,\theta)$$

$$= \delta_{m0}\ (2n+1)\ (q/2) \int_{-1}^{1} dz\ P_n(z)\ 1/\sqrt{r^2 + r'^2 - 2rr'\,z}$$

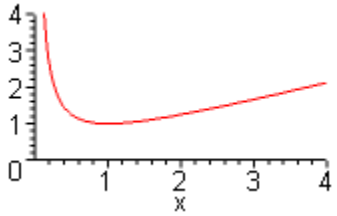
So putting $\mathbf{r}' = r'\hat{\mathbf{z}}$ forces m = 0 in the usual manner, and we are left with the z integral shown.

$$\int_{-1}^{1} dz\ P_n(z)\ 1/\sqrt{r^2 + r'^2 - 2rr'\,z} = (1/r) \int_{-1}^{1} dz\ P_n(z)\ 1/\sqrt{1 + (r'/r)^2 - 2(r'/r)\,z}$$

$$= (1/r) \int_{-1}^{1} dz\ P_n(z)\ 1/\sqrt{1 + \alpha^2 - 2\alpha\,z} \qquad \alpha = (r'/r)$$

This is always a fascinating integral. The square root vanishes when

$$z = (\alpha^2 + 1)/(2\alpha) = (1/2)\,(\alpha + \alpha^{-1})$$



As long as $\alpha \neq 1$, the value of z at which the radical vanishes is > 1. In other words, the branch point of the radical is to the right of the integration contour. At α = 1 it just touches this contour at z = 1. But this fact does not do much to tell us the integral! The big trick is to realize this generating function form, shown on GR7 p 988,

8.921 The generating function:

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{k=0}^{\infty} t^k\, P_k(z) \qquad \left[|t| < \min\left|z \pm \sqrt{z^2-1}\right|\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}}\, P_k(z) \qquad \left[|t| > \max\left|z \pm \sqrt{z^2-1}\right|\right]$$

Usually one only sees the first form which converges for small t. For z real in (0,1) the |..| thing is just 1 as you expect. Given this, you replace t' = 1/t and you quickly derive the second form. So we have these two cases based on whether t < 1 or t > 1. So assume α < 1 for the moment, then

$$\int_{-1}^{1} dz\, P_n(z)\, 1/\sqrt{1+\alpha^2-2\alpha z} \;=\; \int_{-1}^{1} dz\, P_n(z)\; \{\Sigma_k\, \alpha^k\, P_k(z)\} = \Sigma_k\, \alpha^k \int_{-1}^{1} dz\, P_n(z)\, P_k(z)$$

$$= \Sigma_k\, \alpha^k\, (n+1/2)^{-1}\, \delta_{n,k} = \; (n+1/2)^{-1}\, \alpha^n \qquad\qquad \alpha < 1$$

and of course we get $(n+1/2)^{-1}\, \alpha^{-n-1}$ for α > 1. Therefore , with α = (r'/r),

$$a_{nm}(r) = \delta_{m0}\, (2n+1)\, (q/2) \int_{-1}^{1} dz\, P_n(z)\, 1/\sqrt{r^2+r'^2-2rr'z}$$

$$= \delta_{m0}\, (2n+1)\, (q/2r)\, (n+1/2)^{-1}\, \{\; \theta(\alpha<1)\, \alpha^n \; + \theta(\alpha>1)\, \alpha^{-n-1}\}$$

$$= \delta_{m0}\, (q/r)\; \{\; \theta(\alpha<1)\, \alpha^n \; + \theta(\alpha>1)\, \alpha^{-n-1}\}$$

$$= \delta_{m0}\, (q/r')(r'/r) \;\; \{\; \theta(r'<r)\, (r'/r)^n \; + \theta(r'>r)\, (r'/r)^{-n-1}\}$$

$$= \delta_{m0}\, (q/r')\{\; \theta(r'<r)\, (r'/r)^{n+1} \; + \theta(r'>r)\, (r'/r)^{-n}\}$$

$$= \delta_{m0}\, (q/r')\{\; \theta(r>r')\, (r/r')^{-n-1} \; + \theta(r<r')\, (r/r')^n\}$$

$$= \delta_{m0}\, (q/r')\{\; \theta(r>r')\, (r/r')^{-n-1} \; + \theta(r<r')\, (r/r')^n\}$$

$$= \delta_{m0}\, (q/r')\, (r_</r')^n (r_>/r')^{-n-1} \qquad r_< = \min(r,r') \qquad r_> = \max(r,r')$$

We see that this coefficient, vanishes as r^{-n-1} for large r, and vanishes as r^n for small r, which is what we think we want. Our expansion now is this:

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty}\, \Sigma_{m=-n}^{n}\, a_{nm}(r)\; P_n^m(\cos\theta)\, e^{im\varphi}$$

$$= \; (q/r')\, \Sigma_{n=0}^{\infty} \; (r_</r')^n (r_>/r')^{-n-1}\, P_n(\cos\theta) \qquad\qquad (*)$$

Example 3: Point charge at general r' using addition theorem and Theorem 1

Now let's move the point charge q at **r'** to an arbitrary location, not just a location on the z axis. Let's recall our theorem from above:

Theorem 1. If f(**r**,**r'**) is a rotational scalar, and if we know this is true,

$$f(\mathbf{r},r'\,\hat{\mathbf{z}}) = \Sigma_n\, k_n(r,r')\, \varphi_n(\cos\theta)$$

then it follows that

$$f(\mathbf{r},\mathbf{r'}) = \Sigma_n\, k_n(r,r')\, \varphi_n(\hat{\mathbf{r}}\bullet\hat{\mathbf{r}}')$$

In Example 2 we ended up with equation (*) where $V = q/|\mathbf{r} - \mathbf{r}'| = f(\mathbf{r},\mathbf{r'})$ = a rotational scalar. We can then apply the theorem to get this expansion

$$V(r,\theta,\varphi) = (q/r')\, \Sigma_{n=0}^{\infty}\, (r_</r')^n (r_>/r')^{-n-1}\, P_n(\cos\gamma) \qquad (**)$$

where $\cos\gamma = \hat{\mathbf{r}}\bullet\hat{\mathbf{r}}'$. Now recall from group theory on SO(3) that we have the addition theorem which can be written as

$$\Sigma_b\, D^{\sigma}(g_1)_{ab}\, D^{\sigma}(g_2)_{bc} = D^{\sigma}(g_1 g_2)_{ac}$$

I have not yet updated my SO(3) group theory world for addition theorems, but it turns out there that we can do a special case of the above as follows, setting a = c = 0

$$\Sigma_b\, D^{\sigma}(g_1)_{0b}\, D^{\sigma}(g_2)_{b0} = D^{\sigma}(g_1 g_2)_{00}$$

and when this is written out in terms of the P functions, we get the addition theorem quoted for example in Jackson page 68 top which is written in terms of Y functions. I will have to do some work here to get this converted back to P functions. Jackson uses:

$$Y_{nm} = [\, f(n,-m)\, (2n+1)/(4\pi)\,]^{1/2}\, P_n^m(z) e^{im\varphi}$$

so his page 68 addition theorem then reads,

$$P_n(\cos\gamma) = [(2n+1)/(4\pi)]^{-1}\, \Sigma_m$$
$$[\, f(n,-m)\, (2n+1)/(4\pi)\,]^{1/2}\, P_n^m(z)\, e^{im\varphi}$$
$$[\, f(n,-m)\, (2n+1)/(4\pi)\,]^{1/2}\, P_n^m(z') e^{-im\varphi'}$$

$$= \Sigma_m\, f(n,-m)\, P_n^m(z)\, P_n^m(z')\, e^{im\varphi}\, e^{-im\varphi'}$$

I can then insert this result into our (**) result above to get

$$V(r,\theta,\varphi) = (q/r') \Sigma_{n=0}^{\infty} (r_</r')^n (r_>/r')^{-n-1} P_n(\cos\gamma)$$

$$= (q/r') \Sigma_{n=0}^{\infty} (r_</r')^n (r_>/r')^{-n-1} \{ \Sigma_m f(n,-m) P_n^m(z) P_n^m(z') e^{im\varphi} e^{-im\varphi'} \}$$

$$= (q/r') \Sigma_{nm} f(n,-m) (r_</r')^n (r_>/r')^{-n-1} P_n^m(z) P_n^m(z') e^{im\varphi} e^{-im\varphi'}$$

and this then replicates the result we found in the first section of this document using the pillbox method. That is to say, the point charge q is located now at a general point **r'** = (r', θ', φ').

Example 4: Point charge at general r' using brute force

Now let's attempt to redo the Example 3 case directly, without first doing the z axis trick and then using the addition theorem. We start with

$$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} a_{nm}(r) P_n^m(\cos\theta) e^{im\varphi}$$

$$a_{nm}(r) = (2n+1) f(n,-m) (1/4\pi) \int d\Omega\, V(r,\theta,\varphi) e^{-im\varphi} P_n^m(z)$$

which we now soup up, exposing the bystander parameter dependences,

$$V(r,\theta,\varphi\,;\, r',\theta',\varphi') = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} a_{nm}(r;\, r',\theta',\varphi') P_n^m(\cos\theta) e^{im\varphi}$$

$$a_{nm}(r;\, r',\theta',\varphi') = (2n+1) f(n,-m) (1/4\pi) \int d\Omega\, V(r,\theta,\varphi\,;\, r',\theta',\varphi') e^{-im\varphi} P_n^m(z)$$

where now

$$V(r,\theta,\varphi\,;\, r',\theta',\varphi') = q/\sqrt{r^2 + r'^2 - 2rr'\cos\gamma} = q\, f(\varphi-\varphi') \qquad \text{// see below}$$

$$\cos(\gamma) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

as was shown at the start of this doc. So we then have

$$V(r,\theta,\varphi\,;\, r',\theta',\varphi') = \Sigma_{nm} a_{nm}(r;\, r',\theta',\varphi') P_n^m(\cos\theta) e^{im\varphi}$$

$$a_{nm}(r;\, r',\theta',\varphi') = (2n+1) f(n,-m) (q/4\pi) \int dz\, P_n^m(z) \int_{-\pi}^{\pi} d\varphi\; 1/\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}\; e^{-im\varphi}$$

I think I have done this elsewhere, so I will just outline what happens. We usually set φ' = 0 and then reinstate it at the end. This causes the φ integrand to be even, so only the cos(mφ) part of $e^{-im\varphi}$ contributes. The dφ integral then has a certain doable form and I quote from the doc where I did this integral. Let

$$f(\varphi) = 1/\sqrt{a - b\cos(\varphi)}$$

Then

$$\int_{-\pi}^{\pi} d\varphi\, f(\varphi)\cos(m\varphi) = 2\int_{0}^{\pi} d\varphi\, f(\varphi)\cos(m\varphi) = 2\,(\pi/2)A_m = 2\,(\pi/2)\,(2/\pi)\sqrt{2/b}\;\; Q_{m-1/2}(a/b)$$

$$= 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)$$

where in our application we have

$$a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta' \qquad\qquad b = 2rr'\sin\theta\sin\theta'$$

The fact that this integral is a toroidal Q function was a surprise to me when I first "discovered it". We are then faced with the intermediate result

$$a_{nm}(r) = (2n+1)\;\; f(n,-m)\,(1/4\pi)\int d\Omega\; V(r,\theta,\varphi)\; e^{-im\varphi}\; P_n^m(z)$$

$$= (2n+1)\;\; f(n,-m)\,(q/4\pi)\int dz\; P_n^m(z)\{\; 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\}$$

This is a pretty ugly integral, but it is easier for the toroidal Q than for the general Q (I have some pre-Cod notes on this but cannot find them right now). We know that the final answer is going to be this (ie, from Example 3 above, or from the start of this doc)

$$V(r,\theta,\varphi;\, r',\theta',\varphi'=0) = (q/r')\,\Sigma_{nm}\, f(n,-m)\,(r_</r')^n(r_>/r')^{-n-1}\; P_n^m(z)\; P_n^m(z')\; e^{im\varphi}$$

which we compare to our expansion above

$$V(r,\theta,\varphi\;;\, r',\theta',\varphi') = \Sigma_{nm}\, a_{nm}(r;\, r',\theta',\varphi')\;\; P_n^m(\cos\theta)\; e^{im\varphi}$$

so we know the answer must be this:

$$a_{nm}(r;\, r',\theta',\varphi') = (q/r')\; f(n,-m)\,(r_</r')^n(r_>/r')^{-n-1}\; P_n^m(z')$$

If we equate this to the $a_{nm}(r)$ form shown above, we have

$$(2n+1)\;\; f(n,-m)\,(q/4\pi)\int dz\; P_n^m(z)\{\; 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\} = (q/r')\; f(n,-m)\,(r_</r')^n(r_>/r')^{-n-1}\; P_n^m(z')$$

$$(2n+1)/(4\pi)\;\int dz\; P_n^m(z)\{\; 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\} = (1/r')\,(r_</r')^n(r_>/r')^{-n-1}\; P_n^m(z')$$

$$\int dz\; P_n^m(z)\{\; 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\} = [(2n+1)/(4\pi)]^{-1}\;\; (1/r')\,(r_</r')^n(r_>/r')^{-n-1}\; P_n^m(z')$$

All I am saying is that the toroidal integral must somehow come out as shown here. I will attempt this integral in Appendix A below.

11. Going beyond 1/R: the Cape Cod idea

I am finally (after 22 pages) getting to the vague idea I had while on Cape Cod which was to try to generalize the 1/R expansion to more general scalar functions. Go back yet again to the expansion idea

$$V(r,\theta,\varphi\,;\,r',\theta',\varphi') = \Sigma_{n=0}^{\infty}\,\Sigma_{m=-n}^{n}\,a_{nm}(r;\,r',\theta',\varphi')\;P_n^m(\cos\theta)\,e^{im\varphi}$$

$$a_{nm}(r;\,r',\theta',\varphi') = (2n+1)\;f(n,-m)\,(1/4\pi)\int d\Omega\;V(r,\theta,\varphi\,;\,r',\theta',\varphi')\,e^{-im\varphi}\,P_n^m(z)$$

where we expand V on spherical atoms for each value of r. In order to do this, we know that V must be a solution of the Laplace equation. We know that 1/R is such a solution. Are there other interesting solutions to look at? What could we say, for example, about $1/R^2$? I don't think this function is "harmonic" in 3D. It is not the potential of a dipole, for example, or any other configuration of charges I can think of. So $1/R^2$ is not a Laplace solution. On Cape Cod, I was confusing this with the idea that both z and z^n are harmonic functions in 2D, so I was thinking $1/R^2$ would be a Laplace solution, which is wrong. Still, we could I think take $V = 1/R^2$ where $R = |\mathbf{r}-\mathbf{r}'|$ and we could in theory compute the a_{nm} and write out the above expansion.

My idea, then, was to make a table showing the expansion of various functions like $1/R^2$ in spherical coordinates. One could then make similar expansions using other curvilinear coordinate systems. The hope then was that I might be able to simplify the horrible expansion I got solving the disk problem in oblate spheroidal coordinates. I got a double sum over various P and Q functions there, but perhaps with my little table of function expansions I might "look up" the expansion and find it to be a simple result. In particular, the surface charge σ for the disk Green's function is purported by Smythe to have a certain simple form, but all I could get was a messy double summation of P and Q functions.

Just as a passing idea, you might differentiate the 1/R expansion perhaps with respect to r or some other parameter and thereby obtain a set of expansion formulas. For example, we know that

$$1/R = (1/r')\;\Sigma_{mn}\,f(n,-m)\,(r_</r')^n(r_>/r')^{-n-1}\;P_n^m(z)\,P_n^m(z')\,e^{im\varphi}\,e^{-im\varphi'}$$

and it would not be hard to differentiate both sides with respect to r, say.

$$R^2 = |\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2rr'\cos\gamma \qquad \cos(\gamma) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

$$\partial_r R^2 = 2r - 2r'\cos\gamma = 2R\partial_r R \qquad \Rightarrow \qquad \partial_r R = (r-r'\cos\gamma)/R$$

$$\partial_r(R^{-1}) = -R^{-2}\partial_r R = -R^{-2}\,(r-r'\cos\gamma)/R = -\,(r-r'\cos\gamma)/R^3$$

So this tells us the LHS if we do this differentiation. The RHS is just powers. There are issues with what happens at r = 0 and with convergence of the series.

Notice that the potential for the disk must be a Laplace solution, and of course cannot then be something like $1/R^2$. But I suppose the charge density on the disk need NOT be a Laplace solution, and then could be some function like this (ie, like the one Smythe states).

Note Added June 25 2010. My Cape Cod discussion above, as first stated,

$V(r,\theta,\varphi\,;\,r',\theta',\varphi') = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} a_{nm}(r;\,r',\theta',\varphi')\; P_n^m(\cos\theta)\, e^{im\varphi}$

$a_{nm}(r;\,r',\theta',\varphi') = (2n+1)\; f(n,-m)\,(1/4\pi) \int d\Omega\; V(r,\theta,\varphi\,;\,r',\theta',\varphi')\, e^{-im\varphi}\, P_n^m(z)$

may be regarded as the first step of doing a "reduced Green's function" problem, or what Stak calls the "partial eigenfunction expansion". If V is supposed to satisfy some ODE LV = 0, then, when we insert the expansion for V into LV=0, we end up with a 1D problem for a_{nm} in the radial r coordinate which might be 0 at both ends of an interval like r in (0,a) where A = radius of some sphere. In this case, you solve the 1D Green's problem and find $a_{nm}(r;\,r',\theta',\varphi')$. For L = Laplace, we expect $a_{nm} \sim (r_<\!/a)^n (r_>\!/a)^{-n-1}$ where our Green's point charge is at r = a, which is a product of first kind and second kind functions, and this is what we see in our (Laplace-satisfying and symmetric) 1/R expansion above. On the other hand, for L = Helmholtz with some k, we might expect $a_{nm} \sim j_\ell(kr_<) h_\ell(kr_>)$ if our r interval is (A,∞) where one of the r-arguments gives the atomic form [for example, $j_\ell(kr)\, P_n^m(\cos\theta)\, e^{im\varphi}$]. If the function we are expanding is *symmetric* in **r** and **r'** , we expect that $a_{nm}(r;\,r',\theta',\varphi') \sim j_\ell(kr_<)\, h_\ell(kr_>)\, P_n^m(\cos\theta')\, e^{im\varphi'}$ as argued earlier. This little discussion I think goes far to explaining the general appearance of the following expansion I happened to find of the web:

where $G_\infty = \frac{e^{ikr}}{4\pi r}$ is the free-space Green's function and $\vec{r} = \left\{ \begin{array}{c} x_o - x_s \\ y_o - y_s \\ z_o - z_s \end{array} \right\}; r = \|\vec{r}\|.$

The spherical wave is impinging on the sphere centered at the origin.
In order to solve the scattering problem we need to expand the free-space Green's function in spherical harmonics.

$$G_\infty = \frac{ik}{2\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{+l} \varepsilon_m \cos m(\phi - \phi_s) P_l^m(\cos\theta_s) P_l^m(\cos\theta) j_l(kr_<) h_l(kr_>) \qquad (2)$$

Here the function G_∞ is really what I would call $G_\infty = e^{ikR}/R$ which is symmetric in **r** and **r'** = $\mathbf{r_s}$ since R is, so we see all the features just outlined above. Presumably G is a solution of the Helmholtz with k.

So in our Cape Cod "table" we now have two entries: one for Laplace, and one for Helmholtz.

Comments:

Notice how much *simpler* the pillbox method of Section (a) is than any other method presented above to obtain an expression for 1/R.

Appendix A: A certain integral

We want to show that

$$\int dz\, P_n^m(z)\{\, 2\sqrt{2/b}\;\; Q_{m-1/2}(a/b)\} = [(2n+1)/(4\pi)]^{-1}\;\; (1/r')\,(r_</r')^n (r_>/r')^{-n-1}\, P_n^m(z')$$

$a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta'$ $\qquad$ $b = 2rr'\sin\theta\sin\theta'$

$z = \cos\theta$ $\qquad$ $z' = \cos\theta'$

I think I could show this quickly using properties of the toroidal harmonics, but I have not really written that up yet, so let's instead use brute force.

Plan A. I suspect that the secret here is to use (the first time ever for me) the famous Whipple's formula which appears in Bateman p 141. It says this, for $\mu = 0$

$$Q_\nu(z) = \sqrt{\pi/2}\ \Gamma(\nu+1)\ (z^2-1)^{-1/4}\ P_{-1/2}^{-\nu-1/2}\ [\ z(z^2-1)^{-1/2}]$$

We now set $\nu = m-1/2$ so that $\nu+1 = m+1/2$, and $\nu+1/2 = m$

$$Q_{m-1/2}(z) = \sqrt{\pi/2}\ \Gamma(m+1/2)\ (z^2-1)^{-1/4}\ P_{-1/2}^{-m}\ [\ z(z^2-1)^{-1/2}]$$

This is slightly promising because we might then end up with $\int dz\ P_n^m(z)\ P_{-1/2}^{-m}(z)$ which I know how to do. Whipple converts from a Q to a P, and also changes the argument. The only question is to see whether it does this in the way we need.

First, look at the argument:

$$a = r^2 + r'^2 - 2rr'zz' \qquad\qquad b = 2rr' \sqrt{1-z^2}\ \sqrt{1-z'^2}$$

Let's rewrite our Whipple above replacing its z with ξ

$$Q_{m-1/2}(\xi) = \sqrt{\pi/2}\ \Gamma(m+1/2)\ (\xi^2-1)^{-1/4}\ P_{-1/2}^{-m}\ [\ \xi(\xi^2-1)^{-1/2}]$$

We then have $\xi = a/b$ to deal with. So

$$\xi^2-1 \ = (a/b)^2 - 1 \ = (\ a^2 - b^2)/b^2$$

But we have'

$$a^2 - b^2 = [r^2 + r'^2 - 2rr'zz']^2 - [2rr' \sqrt{1-z^2}\ \sqrt{1-z'^2}\ ]^2$$

$$= (r^2+ r'^2)^2 + 4\ (rr'zz')^2 - 4(r^2+ r'^2)\ (rr'zz') \ - 4r^2r'^2(1-z^2)(1-z'^2)$$

$$= (r^2+ r'^2)^2 + 4\ (rr'zz')^2 - 4\ (r^2+ r'^2)\ (rr'zz') \ - 4r^2r'^2(\ 1 - z^2 - z'^2 + z^2z'^2)$$

$$= (r^2+ r'^2)^2 - 4\ (r^2+ r'^2)\ (rr'zz') \ - 4r^2r'^2(\ 1 - z^2 - z'^2)$$

Well, this is not panning out. I was hoping this would be a perfect square, and then $\xi(\xi^2-1)^{-1/2} = z$, but that is not working. Also, if it did work, I would net get P function for the integral.

Plan B. Let's try using a second-kind addition theorem from GR7 page 973

$$2. \qquad Q_\nu\left(x_1x_2-\sqrt{x_1^2-1}\sqrt{x_2^2-1}\cos\varphi\right)=P_\nu(x_1)\,Q_\nu(x_2)+2\sum_{k=1}^{\infty}(-1)^k\,P_\nu^{-k}(x_1)\,Q_\nu^k(x_2)\cos k\varphi$$
$$[1<x_1<x_2, \quad \nu\neq -1,-2,-3,\ldots, \quad \varphi \text{ real}] \quad \text{MO 91}$$

I have not derived this recently, so I don't know its group-theoretic connection, but I did write a whole thesis on this subject so I probably could find where this came from. In our case, however, we want to use $Q_{m-1/2}(a/b)$ and this a/b argument does not mesh well with what I see above.

$a = r^2 + r'^2 - 2rr'zz' \qquad b = 2rr'\sqrt{1-z^2}\sqrt{1-z'^2}$

So this is not going to pan out either. It might if we did the integrals in the other order.

Neither Bateman nor GR have much to say specifically about the toroidal Q function.

Plan C. I think I will hold on this subject until I can find my notes for the little toroidal ring paper PDF I read and annotated. // OK, I found my minimal notes in the same file with the famous integral. This points out a connection to cylindrical coordinates. Let's rewrite

$a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta' \qquad b = 2rr'\sin\theta\sin\theta'$

The cylindricals here would be

$\rho = r\sin\theta \qquad r^2 = \rho^2 + z^2$
$z = r\cos\theta$

so we then have

$a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta' = \rho^2 + z^2 + \rho'^2 + z'^2 - 2\,zz'$

$b = 2rr'\sin\theta\sin\theta' = 2\,\rho\rho' \quad \Rightarrow \quad \sqrt{\rho\rho'} = \sqrt{b/2}$

Then

$a/b = [\rho^2 + z^2 + \rho'^2 + z'^2 - 2\,zz']/(2\rho\rho')$

$= [\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho') = \beta$

and this is the " β" object of the Selvaggi paper. Suppose we then use (5) from that paper, which I derive in my "integral" paper, where I will set φ' = 0

$1/R = (1/\pi)\,(1/\sqrt{\rho\rho'})\,\Sigma_{m=0}^{\infty}\,\varepsilon_m\,Q_{m-1/2}(\beta)\cos(m\varphi)$

Now we can write this same 1/R in spherical coordinates from above as (φ' = 0 again)

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} \varepsilon_m \; f(n,-m)\, (r_<)^n (r_>)^{-n-1} \; P_n^m(z)\, P_n^m(z')\, \cos[m(\varphi-\varphi')] \qquad (**)$$

Now we do a PWA in cos(mφ) and it must then be true that

$$(1/\pi)\,(1/\sqrt{\rho\rho'})\, Q_{m-1/2}(\beta) \;=\; (1/r')\; \Sigma_n\, f(n,-m)\, (r_</r')^n (r_>/r')^{-n-1} \; P_n^m(\cos\theta)\, P_n^m(\cos\theta')$$

where I am constantly being careful about overloading z. It is pretty hard to write the RHS in terms of cylindrical coordinates, but we can write the LHS in terms of sphericals:

$$\sqrt{2/b}\; Q_{m-1/2}(a/b) \;=\; (\pi/r')\; \Sigma_n\, f(n,-m)\, (r_</r')^n (r_>/r')^{-n-1} \; P_n^m(\cos\theta)\, P_n^m(\cos\theta')$$

$$a = r^2 + r'^2 - 2rr'\cos\theta\cos\theta' \qquad\qquad b = 2rr'\,\sin\theta\sin\theta'$$

This is a *very strange* expansion for a toroidal Q function, and it is of course the crucial expansion we need in order to be able to do our integral. In general, if we compare any pair of curvilinear coordinate systems which both have an azimuthal coordinate, we are going to get something like the result above! The two systems we compared here are spherical and cylindrical (not toroidal!), although I keep referring to this type of Q function as a toroidal Q function. It just happens that $\sqrt{2/b}\, Q_{m-1/2}(a/b)$ is the Fourier Series coefficient of the function $1/\sqrt{a - b\cos(\theta)}$. But I do think there is a toroidal coordinates connection, though I don't know yet what it is.

We should now be able to derive our *desired* integral result which recall was this:

$$\int dz\, P_n^m(z)\{\, 2\sqrt{2/b}\; Q_{m-1/2}(a/b)\} \;=\; [(2n+1)/(4\pi)]^{-1} \;\; (1/r')\, (r_</r')^n (r_>/r')^{-n-1}\, P_n^m(z') \qquad (*)$$

Evaluating, we get

$$\int dz\, P_n^m(z)\{\, 2\sqrt{2/b}\; Q_{m-1/2}(a/b)\}$$

$$= 2 \int dz\, P_n^m(z)\{\, (\pi/r')\; \Sigma_n\, f(n,-m)\, (r_</r')^n (r_>/r')^{-n-1} \; P_n^m(z)\, P_n^m(\cos\theta')\}$$

where I am now back to z = cosθ instead of cylindrical. Continuing along,

$$= \; (2\pi/r')\, \Sigma_n\, (r_</r')^n (r_>/r')^{-n-1}\, f(n,-m) \; P_n^m(\cos\theta') \int dz\, P_n^m(z) P_n^m(z)$$

We can of course now do the normalization integral which is

$$\int dz\, P_n^m(z) P_n^m(z) \;= K_n^m = \; 2(2n+1)^{-1}\, f(n,m)$$

so continuing some more,

$$=(2\pi/r')\,\Sigma_n\,(r_<\!/r')^n(r_>\!/r')^{-n-1}\,f(n,-m)\;\;P_n^m(\cos\theta')\;\{\,2(2n+1)^{-1}\,f(n,m)\}$$

$$=(2\pi/r')\,\Sigma_n\,(r_<\!/r')^n(r_>\!/r')^{-n-1}P_n^m(\cos\theta')\;\{\,2(2n+1)^{-1}\}$$

$$=(4\pi/(2n+1))\;(1/r')\,\Sigma_n\,(r_<\!/r')^n(r_>\!/r')^{-n-1}P_n^m(\cos\theta')$$

and this is our desired result (*).