Coordinate changes in Hilbert Space PhL 11.13.10

1. Here is our first example. We can represent the same point by x,y,z or r,θ,φ. In ket notation, this must mean that

$$| x,y,z\rangle = \alpha\, |r,\theta,\varphi\rangle$$

where there might be some α which relates to the respective normalizations. The usual normalizations however are these

$$\langle x',y',z'|x,y,z\rangle = \delta(x'-x)\,\delta(y'-y)\,\delta(z'-z) = \delta^3(\mathbf{r'}-\mathbf{r})$$

$$\langle r',\theta,\varphi'|r,\theta,\varphi\rangle = \delta(r'-r)/r^2\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi') = \delta^3(\mathbf{r'}-\mathbf{r}) = \delta(r'-r)\delta(\theta-\theta')\delta(\varphi-\varphi')/[\, r^2\sin\theta]$$

where $r^2\sin\theta$ is the Jacobian thing we sometimes call v

$$v = \sqrt{\det(g')} = \det(T_a{}^b) = \det\left(\frac{\partial x^b}{\partial x'^a}\right) = J(\mathbf{x},\mathbf{x'}) = \text{the Jacobian}$$

For an orthogonal system, $\det(g') = Q_1^2Q_2^2Q_3^2$ so $v = Q_1Q_2Q_3$. For sphericals we have

$$1,2,3 = r,\theta,\varphi \qquad Q_1 = Q_r = 1 \qquad Q_2 = Q_\theta = r \qquad Q_3 = Q_\varphi = r\sin\theta$$

so we get $v = r^2\sin\theta$.

So, given these normalizations, we can just say

$$| x,y,z\rangle = |r,\theta,\varphi\rangle$$

where it is understood that on the left, x,y,z are given by

$$x = r\sin\theta\sin\varphi$$
$$y = r\sin\theta\cos\varphi$$
$$z = r\cos\theta$$

2. Consider:

$$\langle r_1,\theta_1,\varphi_1|\, x,y,z\rangle = \langle r_1,\theta_1,\varphi_1|\, r,\theta,\varphi\rangle$$

$$= \delta(r_1-r)\delta(\cos\theta_1-\cos\theta)\delta(\varphi_1-\varphi)/r^2$$

But we know that

$$r = \sqrt{x^2+y^2+z^2}$$
$$\cos\theta = z/\sqrt{x^2+y^2+z^2}$$

$\tan\varphi = x/y$

Thus we can say

$$\langle r_1,\theta_1,\varphi_1|\, x,y,z\rangle = \delta(r_1-\sqrt{x^2+y^2+z^2}\,)\delta(\cos\theta_1 - z/\sqrt{x^2+y^2+z^2}\,)\delta(\varphi_1 - \tan^{-2}(x/y)) \,/\, \sqrt{x^2+y^2+z^2}$$

This is a matrix element you don't see very often.

3. Here is our second example. We can represent the same point in spherical coordinates r,θ,φ or in conical coordinates r,μ,ν. We write

$$|\, r,\mu,\nu\,\rangle = \alpha\, |r,\theta,\varphi\rangle$$

For this system we find in our conical notes that

$1,2,3 = r,\theta,\varphi$ $\qquad Q_1 = Q_r = 1 \qquad Q_2 = Q_\theta = r \qquad Q_3 = Q_\varphi = r\sin\theta$

$h_{rr}^2 = 1 \qquad Q_1 = 1 \qquad r, \eta, \zeta = 1,2,3$

$h_{\eta\eta}^2 = h_{\zeta\zeta}^2 = [\mu(\eta)^2 - \nu(\zeta)^2]\, r^2 \qquad Q_2 = Q_3 = r\sqrt{\mu^2 - \nu^2}$

Note: This document got aborted, but I will park this piece in sphericals.