Separation of Laplace and Helmholtz in spherical coordinates PhL 3.26.05

1. Laplace. $\nabla^2 u = 0$.

This is a famous and trivial result, but I seem to not have it written down anywhere. We can write the 3D Laplace operator in this manner

$$\nabla^2 = \nabla_r^2 - (1/r^2)\, L^2_{\theta,\varphi}$$

from my TK page and memory of what L^2 is . We then try a Laplace solution of the form

$$u(r,\theta,\varphi) = R_n(r)\, Y_{nm}(\theta,\varphi)$$

which gives

$$[\, \nabla_r^2 - (1/r^2)\, L^2_{\theta,\varphi}]\, R_n(r)\, Y_{nm}(\theta,\varphi) = 0$$

$$[\, \nabla_r^2 - (1/r^2) n(n+1)]\, R_n(r)\, Y_{nm}(\theta,\varphi) = 0$$

$$[\, \nabla_r^2 - (1/r^2) n(n+1)]\, R_n(r) = 0$$

$$[\, 1/r^2\, \partial_r(r^2\, \partial_r) - (1/r^2) n(n+1)]\, R_n(r) = 0$$

$$[\, \partial_r(r^2\, \partial_r) - n(n+1)]\, R_n(r) = 0$$

We try a solution of the form r^α and find then that

$$\partial_r(r^2\, \partial_r) r^\alpha = \partial_r(r^2\, \alpha\, r^{\alpha-1}) = \alpha\, \partial_r\, r^{\alpha+1} = \alpha(\alpha+1) r^\alpha$$

We will have a solution if $\alpha(\alpha+1) = n(n+1)$. Thus

$$\alpha^2 + \alpha - n(n+1) = 0$$

$$\alpha = [\, -1 \pm \sqrt{1 + 4n(n+1)}\,]/2 = [\, -1 \pm \sqrt{1 + 4n^2+4n)}\,]/2$$

$$= [\, -1 \pm \sqrt{(2n+1)^2)}\,]/2 = [\, -1 \pm (2n+1)]/2$$

$$\alpha_+ = [\, -1 + 2n+1]/2 = n$$

$$\alpha_- = [\, -1 - 2n-1]/2 = -(n+1)$$

So, the solutions of $\alpha(\alpha+1) = n(n+1)$ are $\alpha = n$ and $\alpha = -n-1$ and we have thus found our two radial solutions

$$R_n(r) = Ar^n + Br^{-n-1}$$

2. Helmholtz: $(\nabla^2+k^2)\,u = 0$.

We repeat most of the above with k^2 sitting in there, so we get to

$$[\ 1/r^2\ \partial_r(r^2\ \partial_r) - (1/r^2)n(n+1) + k^2]\ R_n(r) = 0$$

$$[\ \partial_r(r^2\ \partial_r) - n(n+1) + r^2k^2]\ R_n(r) = 0$$

Well, sneaking a peak at page 27 of Moon and Spencer we see the solution is

$$R_n(r) = r^{-1/2}\ [\ A\ J_{n+1/2}(kr) + B\ J_{-n-1/2}(kr)\ ]$$

If we have an isotropic problem, then n = 0 and we have

$$R_0(r) = r^{-1/2}\ [\ A\ J_{1/2}(kr) + B\ J_{-1/2}(kr)\ ]$$

$$= r^{-1/2}\ [\ A\sqrt{2/\pi}\ \ r^{-1/2}\ \ \sin(kr) + B\sqrt{2/\pi}\ \ r^{-1/2}\ \ \cos(kr)\ ]$$

$$=\ [\ a\ \sin(kr) + b\ \cos(kr)\ ]\ /\ r$$

$$=\ [\ \alpha\ \ e^{-ikr}\ + \beta\ e^{-ikr}\ ]\ /\ r$$

and we arrive at the famous result that the solutions for n = 0 are simply $e^{\pm ikr}/r$.