Adding two vectors in spherical coordinates PhL 10.26.12

The transform is given by

$x = r\sin\theta\cos\varphi$ $\rho^2 = x^2 + y^2 = r^2\sin^2\theta$ $\rho = r\sin\theta$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$ $r^2 = x^2 + y^2 + z^2$ $y/x = \tan\varphi$

The inverse transformation is given by

$r = [x^2 + y^2 + z^2]^{1/2}$ $r \geq 0$

$\rho = [x^2 + y^2]^{1/2}$ $\rho \geq 0$

$\cos\theta = z/r$

$\sin\theta = \rho/r$

$\cos\varphi = x/(r\sin\theta) = x/\rho$

$\sin\varphi = y/(r\sin\theta) = y/\rho$

Now consider two vectors

$\mathbf{r}_1 = (r_1, \theta_1, \varphi_1)_{\mathbf{sph}} = (x_1, y_1, z_1)$

$\mathbf{r}_2 = (r_2, \theta_2, \varphi_2)_{\mathbf{sph}} = (x_2, y_2, z_2)$

$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 = (x,y,z) = (r,\theta,\varphi)_{\mathbf{sph}}$

Adding in Cartesian space,

$x = r_1\sin\theta_1\cos\varphi_1 + r_2\sin\theta_2\cos\varphi_2$

$y = r_1\sin\theta_1\sin\varphi_1 + r_2\sin\theta_2\sin\varphi_2$

$z = r_1\cos\theta_1 + r_2\cos\theta_2$

Then the inverse transformation maybe be used to find (r,θ,φ) as follows. We first find from Maple,

$$r^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2$$

$$= r_1^2 + r_2^2 + 2r_1r_2\sin\theta_1\sin\theta_2\cos\varphi_1\cos\varphi_2 + 2r_1r_2\sin\theta_1\sin\theta_2\sin\varphi_1\sin\varphi_2 + 2r_1r_2\cos\theta_1\cos\theta_2$$

$$= r_1^2 + r_2^2 + 2r_1r_2\sin\theta_1\sin\theta_2\,[\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2] + 2r_1r_2\cos\theta_1\cos\theta_2$$

$$= r_1^2 + r_2^2 + 2r_1r_2\sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2) + 2r_1r_2\cos\theta_1\cos\theta_2$$

$$= r_1^2 + r_2^2 + 2r_1r_2\{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2)\}$$

$= r_1^2 + r_2^2 + 2r_1r_2\cos\gamma \qquad \cos\gamma \equiv \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)$

where γ is then the angle between the two vectors. That is to say

$\mathbf{r}_1 \bullet \mathbf{r}_2 = r_1r_2\cos\gamma$

as again Maple confirms. These results are old friends but I don't know where to find these friends: Jackson, Sneddon, Goldstein, particle physics books, somewhere. Meanwhile we take a piece of the above to get

$\rho^2 = (x_1+x_2)^2 + (y_1+y_2)^2$

$= r_1^2 + r_2^2 + 2r_1r_2\sin\theta_1\sin\theta_2\cos\varphi_1\cos\varphi_2 + 2r_1r_2\sin\theta_1\sin\theta_2\sin\varphi_1\sin\varphi_2 - r_1^2\cos\theta_1^2 - r_2^2\cos\theta_2^2$

$= r_1^2 + r_2^2 + 2r_1r_2\sin\theta_1\sin\theta_2\,[\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2] - r_1^2\cos\theta_1^2 - r_2^2\cos\theta_2^2$

$= r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 + 2r_1r_2\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)$

So we summarize these two results

$r^2 = r_1^2 + r_2^2 + 2r_1r_2\cos\gamma \qquad \cos\gamma \equiv \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) + \cos\theta_1\cos\theta_2$

$\rho^2 = r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 + 2r_1r_2\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)$

The sum of the two vectors in sphericals is then given by

$r = [r_1^2 + r_2^2 + 2r_1r_2\cos\gamma]^{1/2} \qquad r \geq 0$

$\rho = [r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 + 2r_1r_2\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]^{1/2} \qquad \rho \geq 0$

$\cos\theta = z/r$
$\sin\theta = \rho/r$

$\cos\varphi = x/\rho$
$\sin\varphi = y/\rho$

$x = r_1\sin\theta_1\cos\varphi_1 + r_2\sin\theta_2\cos\varphi_2$
$y = r_1\sin\theta_1\sin\varphi_1 + r_2\sin\theta_2\sin\varphi_2$
$z = r_1\cos\theta_1 + r_2\cos\theta_2$

Now let's define

$\alpha \equiv r_1/r_2 \qquad r_2 = \alpha\, r_1$

and then the above becomes

$$r/r_1 = [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2} \qquad r \geq 0$$

$$\rho/r_1 = [\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]^{1/2} \qquad \rho \geq 0$$

$$x/r_1 = \sin\theta_1\cos\varphi_1 + \alpha\sin\theta_2\cos\varphi_2$$
$$y/r_1 = \sin\theta_1\sin\varphi_1 + \alpha\sin\theta_2\sin\varphi_2$$
$$z/r_1 = \cos\theta_1 + \alpha\cos\theta_2$$

$$\cos\theta = z/r = (\cos\theta_1 + \alpha\cos\theta_2)/ [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2}$$
$$\sin\theta = \rho/r = [\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]^{1/2} / [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2}$$

$$\cos\varphi = x/\rho = (\sin\theta_1\cos\varphi_1 + \alpha\sin\theta_2\cos\varphi_2)/ [\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2]^{1/2}$$
$$\sin\varphi = y/\rho = (\sin\theta_1\sin\varphi_1 + \alpha\sin\theta_2\sin\varphi_2) / [\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2]^{1/2}$$

So here is the answer to the question asked: Here is the r,θ,φ for the sum of two vectors.

$$r = [r_1^2 + r_2^2 + 2r_1r_2\cos\gamma]^{1/2} \qquad \cos\gamma \equiv \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) + \cos\theta_1\cos\theta_2$$

$$\cos\theta = z/r = (\cos\theta_1 + \alpha\cos\theta_2)/ [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2}$$
$$\sin\theta = \rho/r = [\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]^{1/2} / [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2}$$

$$\cos\varphi = x/\rho = (\sin\theta_1\cos\varphi_1 + \alpha\sin\theta_2\cos\varphi_2)/ [\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]^{1/2}$$
$$\sin\varphi = y/\rho = (\sin\theta_1\sin\varphi_1 + \alpha\sin\theta_2\sin\varphi_2) / [\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]^{1/2}$$

Small α approximation

Now rewrite the above to make it suitable for expanding for small α, but don't assume small α.

$$[\sin^2\theta_1 + \alpha^2\sin^2\theta_2 + 2\alpha\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]^{1/2}$$

$$= \sin\theta_1 [1 + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2) + \alpha^2(\sin\theta_2/\sin\theta_1)^2]^{1/2}$$

Case that $\alpha << 1$. We then get

$$\cos\theta = (\cos\theta_1 + \alpha\cos\theta_2)/ [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2}$$

$$\approx (\cos\theta_1 + \alpha\cos\theta_2)[1 - \alpha\cos\gamma] = \cos\theta_1 + \alpha [\cos\theta_2 - \cos\theta_1\cos\gamma]$$

$$\approx \cos\theta_1 + \alpha [\cos\theta_2 - \cos\theta_1(\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) + \cos\theta_1\cos\theta_2)]$$

$$\approx \cos\theta_1 + \alpha [\cos\theta_2 - \cos\theta_1\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) - \cos^2\theta_1\cos\theta_2)]$$

$$\approx \cos\theta_1 + \alpha [\cos\theta_2(1-\cos^2\theta_1) - \cos\theta_1\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)]$$

$$\approx \cos\theta_1 + \alpha\,[\cos\theta_2\sin^2\theta_1 - \cos\theta_1\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)\,]$$

$$\approx \cos\theta_1 + \alpha\,\sin\theta_1\,[\cos\theta_2\sin\theta_1 - \cos\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)\,]$$

$$\approx \cos\theta_1 - \alpha\,\sin\theta_1\,[\cos\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) - \cos\theta_2\sin\theta_1\,]$$

$$\sin\theta = \sin\theta_1\,[1 + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2) + \alpha^2(\sin\theta_2/\sin\theta_1)^2]^{1/2} / [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2}$$

$$\approx \sin\theta_1\,[1 + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]^{1/2} / [1 + 2\alpha\cos\gamma]^{1/2}$$

$$\approx \sin\theta_1\,[1 + \alpha\,(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]\,[1 - \alpha\cos\gamma]$$

$$\approx \sin\theta_1\,[\,1 + \alpha\,\{\,(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2) - \cos\gamma\,\}$$

$$\approx \sin\theta_1 + \alpha\,\{\,\sin\theta_2\cos(\varphi_1-\varphi_2) - \sin\theta_1\cos\gamma\,\}$$

$$\approx \sin\theta_1 + \alpha\,\{\,\sin\theta_2\cos(\varphi_1-\varphi_2) - \sin\theta_1(\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) + \cos\theta_1\cos\theta_2)\,\}$$

$$\approx \sin\theta_1 + \alpha\,\{\,\sin\theta_2\cos(\varphi_1-\varphi_2) - (\sin^2\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) + \sin\theta_1\cos\theta_1\cos\theta_2)\,\}$$

$$\approx \sin\theta_1 + \alpha\,\{\,\sin\theta_2\cos(\varphi_1-\varphi_2) - \sin^2\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) - \sin\theta_1\cos\theta_1\cos\theta_2\,\}$$

$$\approx \sin\theta_1 + \alpha\,\{\,\sin\theta_2\,[\cos(\varphi_1-\varphi_2) - \sin^2\theta_1\cos(\varphi_1-\varphi_2)] - \sin\theta_1\cos\theta_1\cos\theta_2\,\}$$

$$\approx \sin\theta_1 + \alpha\,\{\,\sin\theta_2\cos(\varphi_1-\varphi_2)\,[1 - \sin^2\theta_1] - \sin\theta_1\cos\theta_1\cos\theta_2\,\}$$

$$\approx \sin\theta_1 + \alpha\,\{\,\sin\theta_2\cos(\varphi_1-\varphi_2)\cos^2\theta_1 - \sin\theta_1\cos\theta_1\cos\theta_2\,\}$$

$$\approx \sin\theta_1 + \alpha\,\cos\theta_1\,[\,\sin\theta_2\cos\theta_1\,\cos(\varphi_1-\varphi_2) - \sin\theta_1\cos\theta_2\,]$$

$$\cos\varphi = (\sin\theta_1\cos\varphi_1 + \alpha\sin\theta_2\cos\varphi_2) / \{\,\sin\theta_1\,[1 + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2) + \alpha^2(\sin\theta_2/\sin\theta_1)^2]^{1/2}\}$$

$$\approx (\sin\theta_1\cos\varphi_1 + \alpha\sin\theta_2\cos\varphi_2) / \{\,\sin\theta_1\,[1 + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]^{1/2}\}$$

$$\approx (\cos\varphi_1 + \alpha(\sin\theta_2/\sin\theta_1)\cos\varphi_2) / [1 + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]^{1/2}$$

$$\approx (\cos\varphi_1 + \alpha(\sin\theta_2/\sin\theta_1)\cos\varphi_2)\,[1 - \alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]$$

$$\approx \cos\varphi_1 + \alpha\,\{\,(\sin\theta_2/\sin\theta_1)\cos\varphi_2 - \cos\varphi_1\,(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)\}$$

$$\approx \cos\varphi_1 + \alpha\,(\sin\theta_2/\sin\theta_1)\{\cos\varphi_2 - \cos\varphi_1\cos(\varphi_1-\varphi_2)\}$$

$$\sin\varphi = (\sin\theta_1\sin\varphi_1 + \alpha\sin\theta_2\sin\varphi_2)/\ \{\ \sin\theta_1\ [1\ + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2) + \alpha^2(\sin\theta_2/\sin\theta_1)^2]^{1/2}\}$$

$$\approx (\sin\theta_1\sin\varphi_1 + \alpha\sin\theta_2\sin\varphi_2)/\ \{\ \sin\theta_1\ [1\ + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]^{1/2}\}$$

$$\approx (\sin\varphi_1 + \alpha(\sin\theta_2/\sin\theta_1)\sin\varphi_2)/\ [1\ + 2\alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]^{1/2}$$

$$\approx (\sin\varphi_1 + \alpha(\sin\theta_2/\sin\theta_1)\sin\varphi_2)\ [1\ - \alpha(\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)]$$

$$\approx \sin\varphi_1 + \alpha\ \{\ (\sin\theta_2/\sin\theta_1)\sin\varphi_2 - \sin\varphi_1\ (\sin\theta_2/\sin\theta_1)\cos(\varphi_1-\varphi_2)\}$$

$$\approx \cos\varphi_1 + \alpha\ \ (\sin\theta_2/\sin\theta_1)\{\sin\varphi_2 - \sin\varphi_1\ \cos(\varphi_1-\varphi_2)\}$$

And don't forget

$$r/r_1 = [1 + \alpha^2 + 2\alpha\cos\gamma]^{1/2} \quad \approx [1 + 2\alpha\cos\gamma]^{1/2} \quad \approx [1 + \alpha\cos\gamma]$$

Summary of the case $\alpha = (r_1/r_2) << 1$:

$$r \approx r_1(1 + \alpha\cos\gamma)$$

$$\cos\theta \approx \cos\theta_1 - \alpha \sin\theta_1\ [\cos\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) - \cos\theta_2\sin\theta_1\ ]$$

$$\sin\theta \approx \sin\theta_1 + \alpha\ \cos\theta_1\ [\ \sin\theta_2\ \cos\theta_1\ \cos(\varphi_1-\varphi_2)\ - \sin\theta_1\cos\theta_2\ ]$$

$$\cos\varphi \approx \cos\varphi_1 + \alpha\ \ (\sin\theta_2/\sin\theta_1)[\cos\varphi_2 - \cos\varphi_1\ \cos(\varphi_1-\varphi_2)]$$

$$\sin\varphi \approx \cos\varphi_1 + \alpha\ \ (\sin\theta_2/\sin\theta_1)[\sin\varphi_2 - \sin\varphi_1\ \cos(\varphi_1-\varphi_2)]$$

In the case of the sun-earth system we have

$M_S = 1.99 \times 10^{30}$ kg $\quad R_S = 696{,}000$ km

$M_E = 5.97 \times 10^{24}$ kg $\quad R_E = 6371$ km (8.25)

$M_M = 7.35 \times 10^{22}$kg $\quad R_M = 1737$ km $\quad T_M = 27.3$ days

$r_{S-E} = 1.50 \times 10^8$ km // average of aphelion and perihelion and perigee e = .016

$r_{M-E} = 386{,}000$ km // average of apogee and perigee tilt = 1.45° e = .05

So

$$\alpha = r_1/r_2 = R_E/\ r_{S-E} = 6371\ /\ 1.50 \times 10^8 = 6.371\ /\ 1.50 \times 10^5 = 4.25 \times 10^{-5} \sim 10^{-4}$$

So in this case we essentially have $\mathbf{r} = \mathbf{r}_1$.