Application of Curvilinear Theory to Spherical Coordinates PhL 9.7.08

Here we treat this Transformation T blindly and just carry out the steps of the M&M curvilinear program as outlined in various other nearby documents. Results are summarized in the last section. This doc is sort of "everything you ever wanted to know about spherical coordinates". Note **summary** at the end.

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1. The Inverse Transformation.

The inverse coordinate transformation is this: $\mathbf{x} = \mathbf{x}(\mathbf{q}) = \mathbf{x}(\mathbf{x}')$.

$x = rS_\theta C_\varphi$
$y = rS_\theta S_\varphi$
$z = rC_\theta$

2. Compute the inverse T matrix.

We use the following ordering for the q-space variables,

$q^1 = \theta \qquad q^2 = \varphi \qquad q^3 = r$

The following derivatives are obtained and we know $T_a{}^b \equiv \frac{\partial x^b}{\partial x'^a}$, so

$T_\theta{}^1 = \partial x/\partial\theta = rC_\theta C_\varphi$
$T_\theta{}^2 = \partial y/\partial\theta = rC_\theta S_\varphi$
$T_\theta{}^3 = \partial z/\partial\theta = -rS_\theta$

$T_\varphi{}^1 = \partial x/\partial\varphi = -rS_\theta S_\varphi$
$T_\varphi{}^2 = \partial y/\partial\varphi = rS_\theta C_\varphi$
$T_\varphi{}^3 = \partial z/\partial\varphi = 0$

$T_r{}^1 = \partial x/\partial r = S_\theta C_\varphi$
$T_r{}^2 = \partial y/\partial r = S_\theta S_\varphi$
$T_r{}^3 = \partial z/\partial r = C_\theta$

And we get this entered into Maple: , where the columns are second index x,y,z
and the rows are first index θ,φ,r, same order as presented above.

$$T := \begin{bmatrix} r\cos(\theta)\cos(\phi) & r\cos(\theta)\sin(\phi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{bmatrix}$$

3. Compute the forward T matrix

The other direction is much harder. We know $T^a{}_b \equiv \frac{\partial x'^a}{\partial x^b}$ so we can start computing things.

$r^2 = x^2 + y^2 + z^2 \Rightarrow rdr = xdx + ydy + zdz$

$T^1{}_r = \partial r/\partial x = x/r = rS_\theta C_\varphi/r = S_\theta C_\varphi$
$T^2{}_r = \partial r/\partial x = y/r = rS_\theta S_\varphi/r = S_\theta S_\varphi$
$T^3{}_r = \partial r/\partial x = z/r = rC_\theta/r = C_\theta$

The angle ones are not so easy. We can solve our three opening equations for tanθ to get

$\tan^2\theta = (x^2 + y^2)/z^2$

Then differentiate to get (cancel 2's, fiddle around)

$$\tan\theta \sec^2\theta \, d\theta = (x/z^2)dx + (y/z^2)dy - [(x^2+y^2)/z^3]dz$$

$$S_\theta/C_\theta^3 \, d\theta = (S_\theta C_\varphi/r\, C_\theta^2)dx + (S_\theta S_\varphi/r\, C_\theta^2)dy - (S_\theta^2/r\, C_\theta^3)\, dz$$

and then multiply both sides by C_θ^3/ S_θ to get

$$d\theta = (C_\theta\, C_\varphi/r)dx + (C_\theta\, S_\varphi/r)dy - (S_\theta/r)\, dz$$

from which we can read off

$$T^1{}_\theta = \partial\theta/\partial x = C_\theta\, C_\varphi/r$$
$$T^2{}_\theta = \partial\theta/\partial y = C_\theta\, S_\varphi/r$$
$$T^3{}_\theta = \partial\theta/\partial z = -\, S_\theta/r$$

In a similar manner we can solve for $\tan\varphi$ as

$$\tan\varphi = y/x \Rightarrow \sec^2\varphi \, d\varphi = dy/x - (y/x^2)dx$$

$$d\varphi = (C_\varphi/rC_\theta)dy - (S_\varphi/rS_\theta)dx$$

from which we can read off

$$T^1{}_\varphi = \partial\varphi/\partial x = -\, S_\varphi/rS_\theta$$
$$T^2{}_\varphi = \partial\varphi/\partial y = C_\varphi/rC_\theta$$
$$T^3{}_\varphi = \partial\varphi/\partial z = 0$$

So we did all that by brute force. To verify the results, we can take a different tack and use

$$T^a{}_b = (T^{-1})_b{}^a$$

So let's enter our matrix T_*^* into Maple and have it compute the inverse TINV

$$T := \begin{bmatrix} r\cos(\theta)\cos(\phi) & r\cos(\theta)\sin(\phi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{bmatrix}$$

```
TINV := simplify(inverse(T));
```

$$TINV := \begin{bmatrix} \frac{\cos(\phi)\cos(\theta)}{r} & -\frac{\sin(\phi)}{r\sin(\theta)} & \sin(\theta)\cos(\phi) \\ \frac{\sin(\phi)\cos(\theta)}{r} & \frac{\cos(\phi)}{r\sin(\theta)} & \sin(\theta)\sin(\phi) \\ -\frac{\sin(\theta)}{r} & 0 & \cos(\theta) \end{bmatrix}$$

where now first index = rows = x,y,z and second index = columns = θ,φ,r. We can quote our manually calculated results from above and paste them into a table, and we see it agrees with TINV.

$T^1_\theta = C\theta\, C\varphi/r$	$T^1_\varphi = -S\varphi/rS\theta$	$T^1_r = S\theta C\varphi$
$T^2_\theta = C\theta\, S\varphi/r$	$T^2_\varphi = C\varphi/rC\theta$	$T^2_r = S\theta S\varphi$
$T^3_\theta = -S\theta/r$	$T^3_\varphi = 0$	$T^3_r = C\theta$

So we can regard Maple as confirming our calculations.

Status so far: We now know the following information:

The inverse and forward transform are $(q^1 = \theta \quad q^2 = \varphi \quad q^3 = r)$

$x = rS\theta C\varphi$	$r^2 = x^2 + y^2 + z^2$
$y = rS\theta S\varphi$	$\tan^2\theta = (x^2 + y^2)/z^2$
$z = rC\theta$	$\tan\varphi = y/x$

The following matrix has as its first column $T\theta^1\ T\varphi^1\ Tr^1$, then other columns are 2 and 3

$$\begin{bmatrix} r\cos(\theta)\cos(\phi) & r\cos(\theta)\sin(\phi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{bmatrix}$$

The following matrix has as its first row $T^1_\theta\ T^1_\varphi\ T^1_r$, then other rows are 2 and 3

$$\begin{bmatrix} \frac{\cos(\phi)\cos(\theta)}{r} & -\frac{\sin(\phi)}{r\sin(\theta)} & \sin(\theta)\cos(\phi) \\ \frac{\sin(\phi)\cos(\theta)}{r} & \frac{\cos(\phi)}{r\sin(\theta)} & \sin(\theta)\sin(\phi) \\ -\frac{\sin(\theta)}{r} & 0 & \cos(\theta) \end{bmatrix}$$

4. What is the Jacobian?

According to our support doc notes we have

$$J = \det\left(\frac{\partial x^b}{\partial x'^a}\right) = \det(T_a{}^b) \qquad \text{and} \qquad dV = J\,dV' \qquad dxdydz = J\,d\theta d\varphi dr$$

So we will have Maple compute $\det(T_a{}^b)$ where T is the matrix we already entered there. We get:

```
simplify(det(T));
```

$$r^2 \sin(\theta)$$

Therefore $J = r^2\sin\theta$ and we then have the very famous result

$$dxdydz = r^2\sin\theta\, d\theta d\varphi dr$$

5. What are the basis vectors e_i and e^i in Cartesian coordinates ?

Recall these "pondering doc" results for Cartesian components of these basis vectors:

$$[e_i]_n = T_i{}^n$$
$$[e^i]_n = T^i{}_n$$

We can then just read them out of our matrices above:

$$[e\theta]_1 = T\theta^1 = rC\theta C\varphi \qquad \text{etc}$$

So we get

$$[e\theta] = \begin{pmatrix} rC\theta C\varphi \\ rC\theta S\varphi \\ -rS\theta \end{pmatrix}$$ which is the **first row of matrix T** = $T_\star{}^k$. Write it this way

$$\mathbf{e}_\theta = rC\theta C\varphi\,\hat{\mathbf{x}} + rC\theta S\varphi\,\hat{\mathbf{y}} - rS\theta\,\hat{\mathbf{z}}$$
$$\mathbf{e}_\varphi = -rS\theta S\varphi\,\hat{\mathbf{x}} + rS\theta C\varphi\,\hat{\mathbf{y}}$$
$$\mathbf{e}_r = S\theta C\varphi\,\hat{\mathbf{x}} + S\theta S\varphi\,\hat{\mathbf{y}} + C\theta\,\hat{\mathbf{z}}$$

Similarly, $\mathbf{e}_\varphi$ is the second row, and $\mathbf{e}_r$ is the third row and we wrote these in above

$$\begin{bmatrix} r\cos(\theta)\cos(\phi) & r\cos(\theta)\sin(\phi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{bmatrix}$$

Now what about the other ones e^i ?

$$[e^\theta]_n = T^\theta{}_n = (T^T)^n{}_\theta$$

From above and Maple we had:

$$[e^\theta]_n = T^\theta{}_n = (T^{-1})_n{}^\theta = TINV_n{}^\theta = TINVT\theta^n$$

This just means that the e^θ are the **columns of our matrix TINV** which was this

```
TINV := simplify(inverse(T));
```

$$TINV := \begin{bmatrix} \frac{\cos(\phi)\cos(\theta)}{r} & -\frac{\sin(\phi)}{r\sin(\theta)} & \sin(\theta)\cos(\phi) \\ \frac{\sin(\phi)\cos(\theta)}{r} & \frac{\cos(\phi)}{r\sin(\theta)} & \sin(\theta)\sin(\phi) \\ -\frac{\sin(\theta)}{r} & 0 & \cos(\theta) \end{bmatrix}$$

$$e^\theta = C\theta C\varphi/r\ \hat{x} + C\theta S\varphi/r\ \hat{y} - S\theta/r\ \hat{z}$$
$$e^\varphi = -S\varphi/rS\theta\ \hat{x} + C\varphi/rS\theta\ \hat{y}$$
$$e^r = S\theta C\varphi\hat{x} + S\theta S\varphi\hat{y} + C\theta\hat{z}$$

compare to our previous results

$$e_\theta = rC\theta C\varphi\hat{x} + rC\theta S\varphi\hat{y} - rS\theta\hat{z}$$
$$e_\varphi = -rS\theta S\varphi\hat{x} + rS\theta C\varphi\hat{y}$$
$$e_r = S\theta C\varphi\hat{x} + S\theta S\varphi\hat{y} + C\theta\hat{z}$$

By observation we see that

$$e^\theta = e_\theta / r^2$$
$$e^\varphi = e_\varphi / r^2$$
$$e^r = e_r$$

and this the linear relationship between e^i and e_i is what we expect in an orthogonal system.

Even were this system non-orthogonal, we would expect things like this to be true:

$\mathbf{e}^{\theta} \bullet \mathbf{e}_{\varphi} = 0$

Let's check this one case

$\mathbf{e}^{\theta} \bullet \mathbf{e}_{\varphi} = (rC_\theta C_\varphi/r\ \hat{\mathbf{x}} + rC_\theta S_\varphi/r\ \hat{\mathbf{y}} - S_\theta/r\ \hat{\mathbf{z}}) \bullet (-rS_\theta S_\varphi \hat{\mathbf{x}} + rS_\theta C_\varphi \hat{\mathbf{y}})$

$= rC_\theta C_\varphi/r * -rS_\theta S_\varphi + rC_\theta S_\varphi/r * rS_\theta C_\varphi = -C_\theta C_\varphi S_\theta S_\varphi + C_\theta S_\varphi S_\theta C_\varphi = 0$

But since we have an orthogonal system, we expect this *also* to be true

$\mathbf{e}_\theta \bullet \mathbf{e}_\varphi = 0$ but obvious because $\mathbf{e}_\theta \bullet \mathbf{e}_\varphi = \mathbf{e}^{\theta} \bullet \mathbf{e}_\varphi / r^2 = 0$

6. What is the metric tensor, what are the Q_i, and what is $g = \det(g_{ij})$?

The metric tensor is $\mathbf{g_{ij}} = \mathbf{e_i} \bullet \mathbf{e_i}$ and we know it is diagonal. We just showed above that the off-diagonal element $g_{\theta\varphi} = \mathbf{e}_\theta \bullet \mathbf{e}_\varphi = 0$. So let's compute the diagonal elements:

$g_{\theta\theta} = \mathbf{e}_\theta \bullet \mathbf{e}_\theta = (rC_\theta C_\varphi \hat{\mathbf{x}} + rC_\theta S_\varphi \hat{\mathbf{y}} - rS_\theta \hat{\mathbf{z}})^2$

$= (rC_\theta C_\varphi)^2 + (rC_\theta S_\varphi)^2 + (-rS_\theta)^2 = = r^2 (1) = r^2 = Q_\theta^2 = g_{\theta\theta}$

$g_{\varphi\varphi} = \mathbf{e}_\varphi \bullet \mathbf{e}_\varphi = (-rS_\theta S_\varphi \hat{\mathbf{x}} + rS_\theta C_\varphi \hat{\mathbf{y}})^2$

$= (-rS_\theta S_\varphi)^2 + (rS_\theta C_\varphi)^2 = r^2 S_\theta^2 = Q_\varphi^2 = g_{\varphi\varphi}$

$g_{rr} = \mathbf{e}_r \bullet \mathbf{e}_r = (S_\theta C_\varphi \hat{\mathbf{x}} + S_\theta S_\varphi \hat{\mathbf{y}} - C_\theta \hat{\mathbf{3}})^2$

$= (S_\theta C_\varphi)^2 + (S_\theta S_\varphi)^2 + (-C_\theta)^2 = 1 = Q_r^2 = g_{rr}$

Because in all three cases $\mathbf{e_i} = \text{const} * \mathbf{e^i}$ (as observed above), we know at once that all off-diagonal elements are zero, so we know this is an "orthogonal" curvilinear coordinate system.

So, the metric tensor is this:

$g = \text{diag}(g_{\theta\theta}, g_{\varphi\varphi}, g_{rr}) = \text{diag}(Q_\theta^2, Q_\varphi^2, Q_r^2) = \text{diag}(r^2, r^2 S_\theta^2, 1)$

and we have these all-important results

$Q_\theta = r$ $Q_\varphi = rS_\theta$ $Q_r = 1$

which agrees with M&M page 178 equation 5-27. All we have to now is insert these into our various differential operator formulas for orthogonal systems.

While here we can compute that

$$g = \det(g) = Q_\theta^2 Q_\varphi^2 Q_r^2 = (r^2 S\theta)^2 \qquad \text{so } \sqrt{g} = r^2 \sin\theta = J \text{ as computed in Sec 4 above!}$$

7. What are the mag's of the basis vectors, and what are the unit vectors?

In general we know this

$$\mathbf{e}_i \bullet \mathbf{e}_i = g_{ii} = Q_i^2 = |\mathbf{e}_i|^2 \quad \Rightarrow \quad |\mathbf{e}_i| = Q_i$$

so we know

$\lvert\mathbf{e}_\theta\rvert = Q_\theta = r$	$\hat{\mathbf{e}}_\theta = \hat{\boldsymbol{\theta}} = \mathbf{e}_\theta / r$	$\mathbf{e}_\theta = r\hat{\mathbf{e}}_\theta = r\,\hat{\boldsymbol{\theta}}$
$\lvert\mathbf{e}_\varphi\rvert = Q_\varphi = rS\theta$	$\hat{\mathbf{e}}_\varphi = \hat{\boldsymbol{\varphi}} = \mathbf{e}_\varphi / rS\theta$	$\mathbf{e}_\varphi = rS\theta\,\hat{\mathbf{e}}_\varphi = rS\theta\,\hat{\boldsymbol{\varphi}}$
$\lvert\mathbf{e}_r\rvert = Q_r = 1$	$\hat{\mathbf{e}}_r = \hat{\mathbf{r}} = \mathbf{e}_r$	$\mathbf{e}_r = \hat{\mathbf{e}}_r = \hat{\mathbf{r}}$

so we have

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{e}}_\theta = \mathbf{e}_\theta/Q_\theta = \mathbf{e}_\theta / r = C\theta C\varphi\hat{\mathbf{x}} + C\theta S\varphi\hat{\mathbf{y}} - S\theta\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\varphi}} = \hat{\mathbf{e}}_\varphi = \mathbf{e}_\varphi/Q_\varphi = \mathbf{e}_\varphi / (rS\theta) = -S\varphi\hat{\mathbf{x}} + C\varphi\hat{\mathbf{y}}$$
$$\hat{\mathbf{r}} = \hat{\mathbf{e}}_r = \mathbf{e}_r/Q_r = \mathbf{e}_r = S\theta C\varphi\hat{\mathbf{x}} + S\theta S\varphi\hat{\mathbf{y}} + C\theta\hat{\mathbf{z}}$$

8. What are the cross products of the basis vectors (and what is v) ?

From $\mathbf{e}^3 \equiv \mathbf{e}_1 \times \mathbf{e}_2 / v$ we know that

$$\mathbf{e}_\varphi \times \mathbf{e}_r = v\,\mathbf{e}^\theta = v\,\mathbf{e}_\theta / r^2$$

which says

$$(\mathbf{e}_\varphi) \times (\mathbf{e}_r) = v/r^{2*}(\mathbf{e}_\theta)$$

$$(rS\theta\,\hat{\boldsymbol{\varphi}}) \times (\hat{\mathbf{r}}) = v/r^2\,(r\,\hat{\boldsymbol{\theta}}) = v\,\hat{\boldsymbol{\theta}}/r$$

$$\Rightarrow \quad \hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = v/(r^2 S\theta)\,\hat{\theta}$$

But since we know the q-space unit vectors are orthogonal, we conclude that

$$v = r^2 S_\theta$$

Earlier we computed $\sqrt{g} = r^2 \sin\theta = J$ and now we can add v, so we have shown that

$$v = \sqrt{g} = J = r^2 \sin\theta$$

by explicitly computing each quantity.

We can finish out our cross product rules by doing cyclic order on the q^i as we have set them up:

$$\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = \hat{\theta}$$
$$\hat{\mathbf{r}} \times \hat{\theta} = \hat{\boldsymbol{\varphi}}$$
$$\hat{\theta} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}}$$

and this agrees with my old hand-written spherical unit vectors page.

9. Differential displacements and areas.

The differential displacements are

$$d\mathbf{s}_\theta = d\theta\, \hat{\boldsymbol{\theta}}$$
$$d\mathbf{s}_\varphi = d\varphi\, \hat{\boldsymbol{\varphi}}$$
$$d\mathbf{s}_r = dr\, \hat{r}$$

and the differential "areas" are the following and its cyclics

$$dA_\theta = d\mathbf{A}_{\varphi r} = d\mathbf{s}_\varphi \times d\mathbf{s}_r = (d\varphi\, \hat{\varphi}) \times (dr\, \hat{r}) = d\varphi\, dr\, \hat{\theta}$$

so get

$$d\mathbf{A}_\theta = d\varphi\, dr\, \hat{\theta}$$
$$d\mathbf{A}_\varphi = dr\, d\theta\, \hat{\varphi}$$
$$d\mathbf{A}_r = d\theta\, d\varphi\, \hat{r}$$

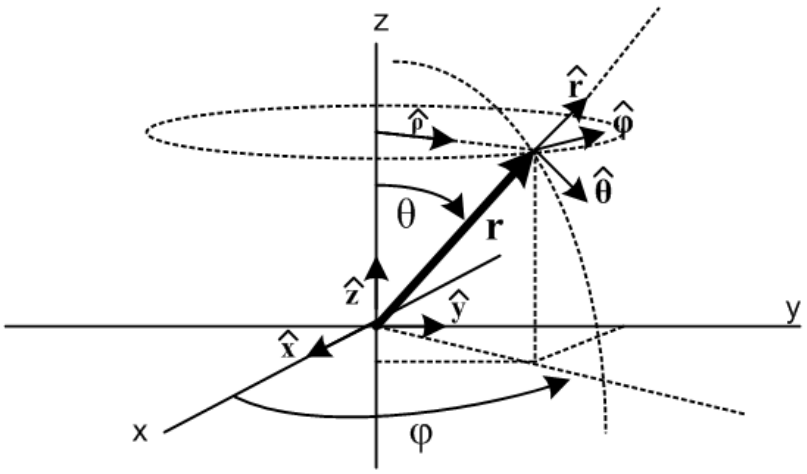
10. The rotation matrix point of view.

The triad of unit vectors ($\hat{\theta}\ \hat{\boldsymbol{\varphi}}\ \hat{\mathbf{r}}$) *at the location* (θ,φ,r) in q-space can be obtained by doing a specific active rotation $R = R_z(\varphi)\, R_y(\theta)$ on the Cartesian unit vectors. That is to say, we have

$$\hat{\boldsymbol{\theta}} = R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{x}}$$

$$\hat{\boldsymbol{\varphi}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{y}}$$
$$\hat{\mathbf{r}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{z}}$$

By operating one's right hand and staring at the picture below, one can easily convince oneself that the above claim is true. Notice that $\hat{\boldsymbol{\theta}}$ points slightly downward and is tangent to the "only change θ" curve which is a circle but we only show part of this circle.



And $\hat{\boldsymbol{\varphi}}$ points right and rear and is tangent to the "only change φ" curve which is a circle. And finally, $\hat{\mathbf{r}}$ points in the direction of the "only change r" curve which is a line. These curves are called **coordinate lines** in M&M Section 5.1. Notice that where they intersect --at the tip of vector **r** -- they are orthogonal.

We can regard this rotation as having the form $R = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, R_{\mathbf{z}}(0)$ so the y-style Euler angles of the rotation are (φ,θ,0).

The fact claimed above allows the following rather elegant statement of what these three unit vectors and their dot products are:

$$(\hat{\boldsymbol{\theta}}\ \hat{\boldsymbol{\varphi}}\ \hat{\mathbf{r}}) = \begin{pmatrix} \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{x}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{y}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\bullet\hat{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)$$

In other words, the unit vectors are just the columns of the rotation matrix, and the dot products are the matrix elements, as shown. The reader can compare this to our previous computations above, which were

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= C\theta C\varphi\hat{\mathbf{x}} + C\theta S\varphi\hat{\mathbf{y}} - S\theta\hat{\mathbf{z}} &&= (\hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}})\,\hat{\mathbf{x}} + (\hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}})\,\hat{\mathbf{y}} + (\hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}})\,\hat{\mathbf{z}} \\
\hat{\boldsymbol{\varphi}} &= -S\varphi\hat{\mathbf{x}} + C\varphi\hat{\mathbf{y}} &&= (\hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}})\,\hat{\mathbf{x}} + (\hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}})\,\hat{\mathbf{y}} + (\hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}})\,\hat{\mathbf{z}} \\
\hat{\mathbf{r}} &= S\theta C\varphi\hat{\mathbf{x}} + S\theta S\varphi\hat{\mathbf{y}} + C\theta\hat{\mathbf{z}} &&= (\hat{\mathbf{x}}\bullet\hat{\mathbf{r}})\,\hat{\mathbf{x}} + (\hat{\mathbf{y}}\bullet\hat{\mathbf{r}})\,\hat{\mathbf{y}} + (\hat{\mathbf{z}}\bullet\hat{\mathbf{r}})\,\hat{\mathbf{z}}
\end{aligned}$$

Because the matrix is real orthogonal, $R^{-1} = R^{\mathbf{T}}$, we get the inverse of these equations by reading off the rows instead of the columns:

$$\hat{\mathbf{x}} = C_\theta C_\varphi \hat{\boldsymbol{\theta}} - S_\varphi \hat{\boldsymbol{\varphi}} + S_\theta C_\varphi \hat{\mathbf{r}}$$
$$\hat{\mathbf{y}} = C_\theta S_\varphi \hat{\boldsymbol{\theta}} + C_\varphi \hat{\boldsymbol{\varphi}} + S_\theta S_\varphi \hat{\mathbf{r}}$$
$$\hat{\mathbf{z}} = - S_\theta \hat{\boldsymbol{\theta}} + C_\theta \hat{\mathbf{r}}$$

Lest there be any confusion about our active rotation matrices, here they are all with angle θ:

$$R_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad R_{\mathbf{y}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

12. Derivatives of the unit vectors with respect to coordinates

Derivatives of the unit vectors with respect to angles.

The reader needs to constantly stare at the spherical coordinates diagram below while reading this section. Notice that we have included in this figure the $\hat{\boldsymbol{\rho}}$ unit vector normally associated with cylindrical coordinates. This unit vector lies in the z=constant plane and points from the z axis to the tip of **r**.

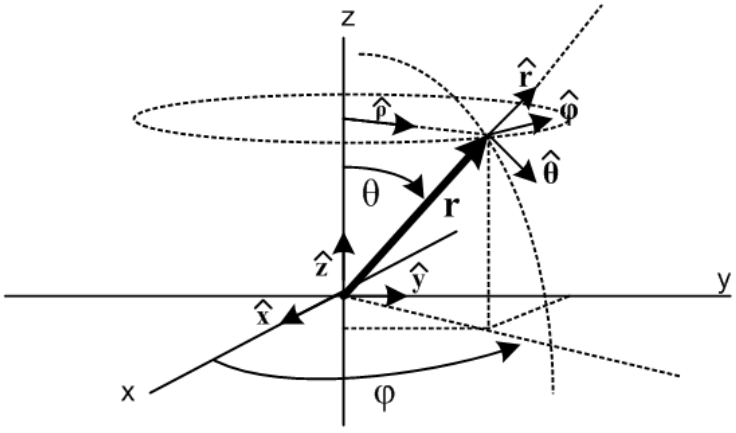
Certain unit vector derivatives vanish and we shall get them out of the way. First, if we lengthen the **r** vector by amount $dr\hat{\mathbf{r}}$, the coordinate triad stays exactly the same, we just "go out to a larger sphere" of radius r+dr, so we have these derivatives at once:

$$\partial\hat{\boldsymbol{\theta}}/\partial r = 0 \qquad \partial\hat{\boldsymbol{\varphi}}/\partial r = 0 \qquad \partial\hat{\mathbf{r}}/\partial r = 0 \qquad \partial\hat{\boldsymbol{\rho}}/\partial r = 0$$

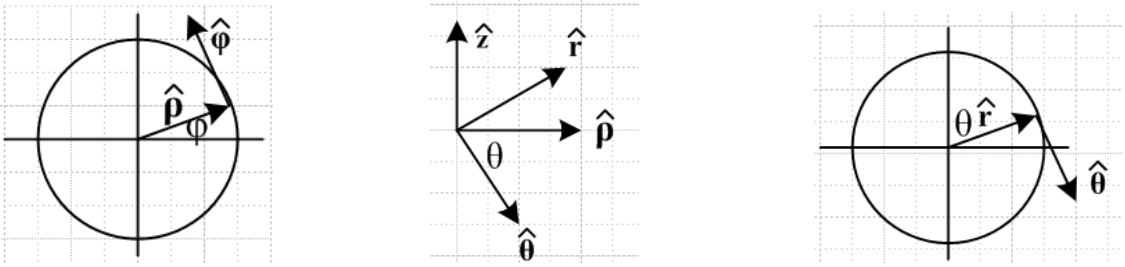
The following two derivatives are also zero as one can see just looking at the diagram:

$$\partial\hat{\boldsymbol{\varphi}}/\partial\theta = 0 \qquad \partial\hat{\boldsymbol{\rho}}/\partial\theta = 0$$

That is, increasing θ moves the dotted circle down and makes it larger, but $\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\rho}}$ stay the same.



Now consider the following detail views. The leftmost view is from the top. The middle view is "from the side" so that all four unit vectors shown are in a plane. The rightmost view just repeats the middle view with a more conventional circle drawing.



From the middle figure, we see that $\hat{\boldsymbol{\rho}}$ is just $\hat{\boldsymbol{\theta}}$ that has been *rotated* amount θ. From our experience with rotations, we don't need to go look it all up, we know that

$$\hat{\boldsymbol{\rho}} = \cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\hat{\mathbf{r}} \qquad\qquad \hat{\mathbf{z}} = -\sin\theta\,\hat{\boldsymbol{\theta}} + \cos\theta\,\hat{\mathbf{r}}$$

We know both signs are plus for $\hat{\boldsymbol{\rho}}$ from the middle figure, and we know that for $\mathbf{r}$ at the equator $\theta = \pi/2$ we will have $\hat{\boldsymbol{\rho}} = \hat{\mathbf{r}}$, so the $\hat{\mathbf{r}}$ coefficient must be sinθ. Then for $\hat{\mathbf{z}}$ we see $\hat{\mathbf{r}}$ contributes positively, but $\hat{\boldsymbol{\theta}}$ contributes negatively, and we know we have to swap the locations of the trig functions, so we get the relation on the right above for $\hat{\mathbf{z}}$.

Now getting back to the picture on the left. We know that if we change the azimuthal angle φ by dφ, then we have the following pair of differential relations

$$d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\varphi}}\,d\varphi \quad \text{and} \quad d\hat{\boldsymbol{\varphi}} = -\,\hat{\boldsymbol{\rho}}\,d\varphi \quad \Rightarrow \quad \partial\hat{\boldsymbol{\rho}}/\partial\varphi = \hat{\boldsymbol{\varphi}} \quad \text{and} \quad \partial\hat{\boldsymbol{\varphi}}/\partial\varphi = -\,\hat{\boldsymbol{\rho}}$$

Normally with the $\boldsymbol{\rho}$ vector going in a circle, we would have $d\boldsymbol{\rho} = \rho d\varphi\,\hat{\boldsymbol{\varphi}}$ but since $\hat{\boldsymbol{\rho}}$ is a *unit* vector, ρ=1 and we get the form shown. The second equation of the pair has the reverse sign since a change in $\hat{\boldsymbol{\varphi}}$ points to the center of the circle.

A similar situation occurs with the angle θ measured down from vertical. As shown in the right view, although $\hat{\boldsymbol{\theta}}$ is "reversed", so is the location of the angle, so the equations are exactly the same.

$$d\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}\,d\theta \quad \text{and} \quad d\hat{\boldsymbol{\theta}} = -\,\hat{\mathbf{r}}\,d\theta \quad \Rightarrow \quad \partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}} \quad \text{and} \quad \partial\hat{\boldsymbol{\theta}}/\partial\theta = -\,\hat{\mathbf{r}}$$

Two derivatives are still missing from our list, they are $\partial\hat{\boldsymbol{\theta}}/\partial\varphi$ and $\partial\hat{\mathbf{r}}/\partial\varphi$. To figure these out, we have to do a small amount of work. Recall three of our equations from above:

$$\hat{\boldsymbol{\rho}} = \cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\hat{\mathbf{r}} \qquad \hat{\mathbf{z}} = -\sin\theta\,\hat{\boldsymbol{\theta}} + \cos\theta\,\hat{\mathbf{r}} \qquad d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\varphi}}\,d\varphi$$

If we change only φ and change it by δφ, we can say

$$d\hat{\boldsymbol{\rho}} = \cos\theta\,d\hat{\boldsymbol{\theta}} + \sin\theta\,\mathbf{d}\hat{\mathbf{r}} = d\varphi\,\hat{\boldsymbol{\varphi}}$$
$$d\hat{\mathbf{z}} = -\sin\theta\,d\hat{\boldsymbol{\theta}} + \cos\theta\,\mathbf{d}\hat{\mathbf{r}} = 0 \qquad \text{// 0 since } \hat{\mathbf{z}} \text{ never moves!}$$

Multiply the first equation by sinθ and the second by cosθ

$$\sin\theta\, d\hat{\boldsymbol{\rho}} = \sin\theta\cos\theta\, d\hat{\boldsymbol{\theta}} + \sin^2\theta\, d\hat{\mathbf{r}} = \sin\theta\, d\varphi\, \hat{\boldsymbol{\varphi}}$$
$$\cos\theta\, d\hat{\mathbf{z}} = -\sin\theta\cos\theta\, d\hat{\boldsymbol{\theta}} + \cos^2\theta\, d\hat{\mathbf{r}} = 0$$

Add the two equations to find that

$$d\hat{\mathbf{r}} = \sin\theta\, d\varphi\, \hat{\boldsymbol{\varphi}} \quad \Rightarrow \quad \partial\hat{\mathbf{r}}/\partial\varphi = \sin\theta\, \hat{\boldsymbol{\varphi}}$$

We can guess the other equation, but let's just do. Start with our first pair of equations and this time multiply the first by cosθ and the second by sinθ

$$\cos\theta\, d\hat{\boldsymbol{\rho}} = \cos^2\theta\, d\hat{\boldsymbol{\theta}} + \sin\theta\cos\theta\, d\hat{\mathbf{r}} = \cos\theta\, d\varphi\, \hat{\boldsymbol{\varphi}}$$
$$\sin\theta\, d\hat{\mathbf{z}} = -\sin^2\theta\, d\hat{\boldsymbol{\theta}} + \sin\theta\cos\theta\, d\hat{\mathbf{r}} = 0$$

Subtract these to find that

$$d\hat{\boldsymbol{\theta}} = \cos\theta\, d\varphi\, \hat{\boldsymbol{\varphi}} \quad \Rightarrow \quad d\hat{\boldsymbol{\theta}}/\partial\varphi = \cos\theta\, \hat{\boldsymbol{\varphi}}$$

All done! We present all the above results in θ,φ,r order

$\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}}$	$\partial\hat{\boldsymbol{\varphi}}/\partial\theta = 0$	$\partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$	$\partial\hat{\boldsymbol{\rho}}/\partial\theta = 0$
$d\hat{\boldsymbol{\theta}}/\partial\varphi = C_\theta\, \hat{\boldsymbol{\varphi}}$	$\partial\hat{\boldsymbol{\varphi}}/\partial\varphi = -\hat{\boldsymbol{\rho}} = -C_\theta\, \hat{\boldsymbol{\theta}} - S_\theta\, \hat{\mathbf{r}}$	$\partial\hat{\mathbf{r}}/\partial\varphi = S_\theta\, \hat{\boldsymbol{\varphi}}$	$\partial\hat{\boldsymbol{\rho}}/\partial\varphi = \hat{\boldsymbol{\varphi}}$
$\partial\hat{\boldsymbol{\theta}}/\partial r = 0$	$\partial\hat{\boldsymbol{\varphi}}/\partial r = 0$	$\partial\hat{\mathbf{r}}/\partial r = 0$	$\partial\hat{\boldsymbol{\rho}}/\partial r = 0$

As with curvilinear coordinates in general, the orthogonal triad of unit vectors for spherical coordinates *changes* as the coordinates change, so many unit vector derivatives are non-zero as shown above. With Cartesian coordinates of course all derivatives are zero, for example, $\partial\hat{\mathbf{x}}/\partial y = 0$. The triad is in the same orientation at all points in space.

13. The differential operators.

We quote the "analysis" set of operators from our "divergence doc" which are all based on having an orthogonal system with orthonormal unit vectors

$$\nabla\bullet\mathbf{A} = \frac{1}{Q_1Q_2Q_3}\,[\,\partial_1(Q_2Q_3A^1) + \text{cyclic}] \quad \text{where } \mathbf{A} = A^i\hat{\mathbf{e}}_i \qquad \textbf{analysis}$$

$$\nabla(f) = (1/Q_i)\ (\partial_i f)\ \hat{\mathbf{e}}_i \qquad \textbf{analysis}$$

$\mathbf{A}\bullet\,\nabla(f) = (A^i/Q_i)\,(\partial_i f)$ where $\mathbf{A} = A^i\hat{\mathbf{e}}_i$ **analysis**

$$\nabla^2(f) = \frac{1}{Q_1Q_2Q_3}\{\,\partial_1\{(Q_2Q_3)/Q_1\,*(\partial_1 f)\} + \text{cyclic}\,\}$$ **analysis**

$$\nabla \times \mathbf{B} = \frac{1}{Q_1Q_2Q_3}\,\varepsilon^{ijk}\,\partial_j(Q_kB_k)\,Q_i\,\hat{\mathbf{e}}_i$$ **analysis**

$$= \frac{1}{Q_1Q_2Q_3}\begin{vmatrix} Q_1\hat{\mathbf{e}}_1 & Q_2\hat{\mathbf{e}}_2 & Q_3\hat{\mathbf{e}}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ Q_1B^1 & Q_2B^2 & Q_3B^3 \end{vmatrix}$$

So now we can fill in using

$1 = \theta$	$2 = \varphi$	$3 = r$	
$Q_1 = r$	$Q_2 = rS_\theta$	$Q_3 = 1$	$\sqrt{g} = Q_1Q_2Q_3 = r^2S_\theta$

$$\nabla\bullet\mathbf{A} = \frac{1}{Q_1Q_2Q_3}\,[\,\partial_1(Q_2Q_3A^1) + \partial_2(Q_3Q_1A^2) + \partial_3(Q_1Q_2A^3)]$$

$$= \frac{1}{r^2S_\theta}\,[\,\partial_\theta(Q_2Q_3A^\theta) + \partial_\varphi(Q_3Q_1A^\varphi) + \partial_r(Q_1Q_2A^r)]$$

$$= \frac{1}{r^2S_\theta}\,[\,\partial_\theta(rS_\theta\,A^\theta) + \partial_\varphi(rA^\varphi) + \partial_r(r^2S_\theta\,A^r)]$$

$$= \frac{1}{r^2S_\theta}\,[r\,\partial_\theta(S_\theta\,A^\theta) + r\partial_\varphi(A^\varphi) + S_\theta\,\partial_r(r^2A^r)]$$

$$= \left[\,\frac{1}{rS_\theta}\partial_\theta(S_\theta\,A^\theta) + \frac{1}{rS_\theta}\partial_\varphi(A^\varphi) + \frac{1}{r^2}\,\partial_r(r^2A^r)\,\right]$$

or

$$\nabla\bullet\mathbf{A} = \frac{1}{r^2}\,\partial_r(r^2A^r) + \frac{1}{rS_\theta}\partial_\theta(S_\theta\,A^\theta) + \frac{1}{rS_\theta}\partial_\varphi A^\varphi$$

where we put things in the conventional r,θ,φ order. This agrees with my old copied sheet.

$$\nabla(f) = (1/Q_i)\,(\partial_i f)\,\hat{\mathbf{e}}_i = (1/Q_1)\,(\partial_1 f)\,\hat{\mathbf{e}}_1 + (1/Q_2)\,(\partial_2 f)\,\hat{\mathbf{e}}_2 + (1/Q_3)\,(\partial_3 f)\,\hat{\mathbf{e}}_3$$

$$= (1/r)\,(\partial_\theta f)\,\hat{\mathbf{e}}_\theta + (1/rS_\theta)\,(\partial_\varphi f)\,\hat{\mathbf{e}}_\varphi + (1/1)\,(\partial_r f)\,\hat{\mathbf{e}}_r$$

$$= (1/r)\,(\partial_\theta f)\,\hat{\boldsymbol{\theta}} + (1/rS_\theta)\,(\partial_\varphi f)\,\hat{\boldsymbol{\varphi}} + (1/1)\,(\partial_r f)\,\hat{\mathbf{r}}$$

$$= (\partial_r f)\,\hat{\mathbf{r}} + (1/r)\,(\partial_\theta f)\,\hat{\boldsymbol{\theta}} + (1/rS_\theta)\,(\partial_\varphi f)\,\hat{\boldsymbol{\varphi}}$$

and this agrees as well. Now repeat

$1 = \theta$	$2 = \varphi$	$3 = r$

$$Q_1 = r \qquad Q_2 = rS_\theta \qquad Q_3 = 1 \qquad \sqrt{g} = Q_1Q_2Q_3 = r^2S_\theta$$

$$\begin{aligned}\mathbf{A}\bullet\nabla(f) = (A^i/Q_i)(\partial_i f) &= (A^1/Q_1)(\partial_1 f) + (A^2/Q_2)(\partial_2 f) + (A^3/Q_3)(\partial_3 f)\\ &= (A^1/r)(\partial_1 f) + (A^2/rS_\theta)(\partial_2 f) + (A^3)(\partial_3 f)\\ &= (A^\theta/r)(\partial_\theta f) + (A^\varphi/rS_\theta)(\partial_\varphi f) + (A^r)(\partial_r f)\\ &= A^r(\partial_r f) + (A^\theta/r)(\partial_\theta f) + (A^\varphi/rS_\theta)(\partial_\varphi f)\end{aligned}$$

and this is one that is not listed on my page, but which started me off on all this stuff! Next,

$$\nabla^2(f) = \frac{1}{Q_1Q_2Q_3}\left[\ \partial_1\{(Q_2Q_3)/Q_1 *(\partial_1 f)\} + \partial_2\{(Q_3Q_1)/Q_2 *(\partial_2 f)\} + \partial_3\{(Q_1Q_2)/Q_3 *(\partial_3 f)\}\ \right]$$

$$= \frac{1}{r^2S_\theta}\left[\ \partial_\theta\{(rS_\theta)/r *(\partial_\theta f)\} + \partial_\varphi\{(r)/rS_\theta *(\partial_\varphi f)\} + \partial_r\{(r\,rS_\theta) *(\partial_r f)\}\ \right]$$

$$= \frac{1}{r^2S_\theta}\left[\ \partial_\theta\{(S_\theta)*(\partial_\theta f)\} + (1/S_\theta)\,\partial_\varphi\{*(\partial_\varphi f)\} + S_\theta\,\partial_r\{(r^2) *(\partial_r f)\}\ \right]$$

$$= \frac{1}{r^2S_\theta}\left[\ \partial_\theta\{(S_\theta)*(\partial_\theta f)\} + (1/S_\theta)\,\partial_\varphi\{*(\partial_2 f)\} + S_\theta\,\partial_r\{(r^2) *(\partial_r f)\}\ \right]$$

$$= \frac{1}{r^2S_\theta}\left[\ \partial_\theta\{S_\theta\,\partial_\theta f\} + (1/S_\theta)\,\partial_\varphi\{\partial_\varphi f\} + S_\theta\,\partial_r\{r^2\,\partial_r f\}\ \right]$$

$$= \frac{1}{r^2S_\theta}\,\partial_\theta(S_\theta\,\partial_\theta f) + \frac{1}{r^2S_\theta^2}\,\partial_\varphi^2 f + \frac{1}{r^2}\,\partial_r(r^2\,\partial_r f)$$

$$= \frac{1}{r^2}\,\partial_r(r^2\,\partial_r f) + \frac{1}{r^2S_\theta}\,\partial_\theta(S_\theta\,\partial_\theta f) + \frac{1}{r^2S_\theta^2}\,\partial_\varphi^2 f$$

and this agrees as well with the sheet. Next,

$$\nabla \times \mathbf{B} = \frac{1}{Q_1Q_2Q_3}\,\varepsilon^{ijk}\,\partial_j(Q_kB_k)\,Q_i\,\hat{\mathbf{e}}_i$$

$$= \frac{1}{Q_1Q_2Q_3}\left[Q_1\,\hat{\mathbf{e}}_1\,(\,\varepsilon^{1jk}\,\partial_j(Q_kB_k)) + Q_2\,\hat{\mathbf{e}}_2\,(\,\varepsilon^{2jk}\,\partial_j(Q_kB_k)) + Q_3\,\hat{\mathbf{e}}_3\,(\,\varepsilon^{3jk}\,\partial_j(Q_kB_k))\ \right]$$

$$\begin{aligned}= \frac{1}{Q_1Q_2Q_3}\ [\quad & Q_1\,\hat{\mathbf{e}}_1\,\{\ (\,\varepsilon^{123}\,\partial_2(Q_3B_3)) + (\,\varepsilon^{132}\,\partial_3(Q_2B_2))\ \}\\ &+ Q_2\,\hat{\mathbf{e}}_2\,\{\ (\,\varepsilon^{231}\,\partial_3(Q_1B_1)) + (\,\varepsilon^{213}\,\partial_1(Q_3B_3))\ \}\\ &+ Q_3\,\hat{\mathbf{e}}_3\,\{\ (\,\varepsilon^{312}\,\partial_1(Q_2B_2)) + (\,\varepsilon^{321}\,\partial_2(Q_1B_1))\ \}\ \ ]\end{aligned}$$

$$= \frac{1}{Q_1Q_2Q_3}\ [\quad Q_1\,\hat{\mathbf{e}}_1\,\{\ (\partial_2(Q_3B_3)) - \partial_3(Q_2B_2))\ \}$$

$$+\ Q_2\,\hat{\mathbf{e}}_2\,\{\ (\partial_3(Q_1B_1))\ -\partial_1(Q_3B_3))\ \}$$
$$+\ Q_3\,\hat{\mathbf{e}}_3\,\{\ (\partial_1(Q_2B_2))\ -\partial_2(Q_1B_1))\ \}\ \ ]$$

$1 = \theta \qquad 2 = \varphi \qquad 3 = r$

$Q_1 = r \qquad Q_2 = rS_\theta \qquad Q_3 = 1 \qquad \sqrt{g}\ = Q_1Q_2Q_3\ = r^2S_\theta$

$$= \frac{1}{r^2S_\theta}\quad [\quad r\hat{\mathbf{e}}_\theta\,\{\ (\partial_\varphi(1B_r))\ -\partial_r(rS_\theta\,B_\varphi))\ \}$$
$$+\ rS_\theta\,\hat{\mathbf{e}}_\varphi\,\{\ (\partial_r(rB_\theta))\ -\partial_\theta(1B_r))\ \}$$
$$+\ 1\,\hat{\mathbf{e}}_r\,\{\ (\partial_\theta(rS_\theta\,B_\varphi))\ -\partial_\varphi(rB_\theta))\ \}\ \ ] \qquad \text{OK}$$

$$= \frac{1}{r^2S_\theta}\quad [\quad r\hat{\boldsymbol{\theta}}\,\{\ (\partial_\varphi(B_r))\ -S_\theta\,\partial_r(rB_\varphi))\ \}$$
$$+\ rS_\theta\,\hat{\boldsymbol{\varphi}}\,\{\ (\partial_r(rB_\theta))\ -\ \partial_\theta(B_r))\ \}$$
$$+\ \ r\ \hat{\mathbf{r}}\,\{(\partial_\theta(S_\theta\,B_\varphi))\ -\ \partial_\varphi(B_\theta))\ \}\ \ ]$$

$$= \frac{1}{rS_\theta}\,\hat{\boldsymbol{\theta}}\,\{\ (\partial_\varphi(B_r))\ -S_\theta\,\partial_r(rB_\varphi))\ \}$$
$$+\ \frac{1}{r}\,\hat{\boldsymbol{\varphi}}\,\{\ (\partial_r(rB_\theta))\ -\ \partial_\theta(B_r))\ \}$$
$$\frac{1}{rS_\theta}\,\hat{\mathbf{r}}\,\{\ (\partial_\theta(S_\theta\,B_\varphi))\ -\ \partial_\varphi(B_\theta))\ \}$$

$$= \frac{1}{rS_\theta}\,\hat{\mathbf{r}}\,\{\ (\partial_\theta(S_\theta\,B_\varphi))\ -\ \partial_\varphi(B_\theta)\ \}$$
$$\frac{1}{rS_\theta}\,\hat{\boldsymbol{\theta}}\,\{\ (\partial_\varphi(B_r))\ -S_\theta\,\partial_r(rB_\varphi)\ \}$$
$$+\ \frac{1}{r}\,\hat{\boldsymbol{\varphi}}\,\{\ (\partial_r(rB_\theta))\ -\ \partial_\theta(B_r)\ \}$$

$$= \frac{1}{rS_\theta}\,\hat{\mathbf{r}}\,\{\ \partial_\theta(S_\theta\,B_\varphi)\ -\ \partial_\varphi B_\theta\ \}$$
$$\frac{1}{r}\,\hat{\boldsymbol{\theta}}\,\{\frac{1}{S_\theta}(\partial_\varphi B_r) - \partial_r(rB_\varphi)\ \}$$
$$+\ \frac{1}{r}\,\hat{\boldsymbol{\varphi}}\,\{\ \partial_r(rB_\theta)\ -\ \partial_\theta B_r\ \}$$

giving the final result

$$\boldsymbol{\nabla}\ \mathrm{x}\ \mathbf{B} = \frac{1}{rS_\theta}\,\hat{\mathbf{r}}\,\{\partial_\theta(S_\theta\,B_\varphi) - \partial_\varphi B_\theta\} + \frac{1}{r}\,\hat{\boldsymbol{\theta}}\ \{\frac{1}{S_\theta}(\partial_\varphi B_r) - \partial_r(rB_\varphi)\} + \frac{1}{r}\,\hat{\boldsymbol{\varphi}}\,\{\partial_r(rB_\theta) - \partial_\theta B_r\ \}$$

and this too agrees with our sheet.

14. Summary:

(a) The inverse and forward transformations

$x = rS_\theta C_\varphi$ $\qquad r^2 = x^2 + y^2 + z^2$

$y = rS_\theta S_\varphi$ $\qquad \tan^2\theta = (x^2 + y^2)/z^2$

$z = rC_\theta$ $\qquad \tan\varphi = y/x$

and the q-space coordinates are

$q^1 = \theta$ $\qquad q^2 = \varphi$ $\qquad q^3 = r$

(b) The backwards T matrix $T_a{}^b \equiv \dfrac{\partial x^b}{\partial x'^a}$

$$\begin{bmatrix} r\cos(\theta)\cos(\phi) & r\cos(\theta)\sin(\phi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{bmatrix}$$

where first column = $T_\theta{}^1\ T_\varphi{}^1\ T_r{}^1$, then other columns are 2 and 3

(c) The forward T matrix $T^a{}_b \equiv \dfrac{\partial x'^a}{\partial x^b}$

$$\begin{bmatrix} \dfrac{\cos(\phi)\cos(\theta)}{r} & -\dfrac{\sin(\phi)}{r\sin(\theta)} & \sin(\theta)\cos(\phi) \\ \dfrac{\sin(\phi)\cos(\theta)}{r} & \dfrac{\cos(\phi)}{r\sin(\theta)} & \sin(\theta)\sin(\phi) \\ -\dfrac{\sin(\theta)}{r} & 0 & \cos(\theta) \end{bmatrix}$$

where first row = $T^1{}_\theta\ T^1{}_\varphi\ T^1{}_r$, then other rows are 2 and 3

(d) The Jacobian J

$J = \det\left(\dfrac{\partial x^b}{\partial x'^a}\right) = \det(T_a{}^b) = r^2 S_\theta$ $\qquad dxdydz = J\, d\theta d\varphi dr$

(e) The metric tensor and the Q_i

$$g_{ij} = \delta_{i,j}\, g_{ii}$$

where

$$g_{\theta\theta} = Q_\theta^2 = r^2 \qquad Q_\theta = r$$
$$g_{\varphi\varphi} = Q_\varphi^2 = r^2 S_\theta^2 \qquad Q_\varphi = rS_\theta$$
$$g_{rr} = Q_r^2 = 1 \qquad Q_r = 1$$

and we find that

$$g = \det(g) = Q_\theta^2 Q_\varphi^2 Q_r^2 = (r^2 S_\theta)^2 \qquad \text{so } \sqrt{g} = Q_\theta Q_\varphi Q_r = r^2 S_\theta$$

(f) The 6 basis vectors in Cartesian coordinates, relations and mag's

$$\mathbf{e}_\theta = rC_\theta C_\varphi \hat{\mathbf{1}} + rC_\theta S_\varphi \hat{\mathbf{2}} - rS_\theta \hat{\mathbf{3}} \qquad \mathbf{e}^\theta = \mathbf{e}_\theta / r^2 \qquad |\mathbf{e}_\theta| = Q_\theta = r$$
$$\mathbf{e}_\varphi = -rS_\theta S_\varphi \hat{\mathbf{1}} + rS_\theta C_\varphi \hat{\mathbf{2}} \qquad \mathbf{e}^\varphi = \mathbf{e}_\varphi / r^2 \qquad |\mathbf{e}_\varphi| = Q_\varphi = rS_\theta$$
$$\mathbf{e}_r = S_\theta C_\varphi \hat{\mathbf{1}} + S_\theta S_\varphi \hat{\mathbf{2}} + C_\theta \hat{\mathbf{3}} \qquad \mathbf{e}^r = \mathbf{e}_r \qquad |\mathbf{e}_r| = Q_r = 1$$

and we from $\mathbf{e}_\varphi \times \mathbf{e}_r = v\, \mathbf{e}^\theta = v\, \mathbf{e}_\theta / r^2$ we find that

$$v = r^2 S_\theta$$

and thus we have shown that

$$J = \sqrt{g} = v = r^2 S_\theta \qquad \Rightarrow \qquad dV_q = d\theta d\varphi dr = dxdydz/J = dxdydz/(r^2 S_\theta)$$

(g) The unit vectors and their cross products

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{e}}_\theta = \mathbf{e}_\theta / Q_\theta = \mathbf{e}_\theta / r = C_\theta C_\varphi \hat{\mathbf{1}} + C_\theta S_\varphi \hat{\mathbf{2}} - S_\theta \hat{\mathbf{3}}$$
$$\hat{\boldsymbol{\varphi}} = \hat{\mathbf{e}}_\varphi = \mathbf{e}_\varphi / Q_\varphi = \mathbf{e}_\varphi / (rS_\theta) = -S_\varphi \hat{\mathbf{1}} + C_\varphi \hat{\mathbf{2}}$$
$$\hat{\mathbf{r}} = \hat{\mathbf{e}}_r = \mathbf{e}_r / Q_r = \mathbf{e}_r = S_\theta C_\varphi \hat{\mathbf{1}} + S_\theta S_\varphi \hat{\mathbf{2}} + C_\theta \hat{\mathbf{3}}$$

$$\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = \hat{\theta}$$
$$\hat{\mathbf{r}} \times \hat{\theta} = \hat{\boldsymbol{\varphi}}$$
$$\hat{\theta} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}}$$

(h) Explicit expressions for the unit vectors and their dot products with the Cartesian ones

$$(\hat{\boldsymbol{\theta}} \;\; \hat{\boldsymbol{\varphi}} \;\; \hat{\mathbf{r}}) = \begin{pmatrix} \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{x}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{y}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\bullet\hat{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)$$

(i) The differential displacements and areas

$d\mathbf{s}_{\boldsymbol{\theta}} = d\theta\, \hat{\boldsymbol{\theta}}$ $\qquad$ $d\mathbf{A}_{\boldsymbol{\theta}} = d\varphi\, dr\, \hat{\boldsymbol{\theta}}$

$d\mathbf{s}_{\boldsymbol{\varphi}} = d\varphi\, \hat{\boldsymbol{\varphi}}$ $\qquad$ $d\mathbf{A}_{\boldsymbol{\varphi}} = dr\, d\theta\, \hat{\boldsymbol{\varphi}}$

$d\mathbf{s}_{\mathbf{r}} = dr\, \hat{\mathbf{r}}$ $\qquad$ $d\mathbf{A}_{\mathbf{r}} = d\theta\, d\varphi\, \hat{\mathbf{r}}$

(j) derivatives of the unit vectors

$\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}}$ $\qquad$ $\partial\hat{\boldsymbol{\varphi}}/\partial\theta = 0$ $\qquad$ $\partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$ $\qquad$ $\partial\hat{\boldsymbol{\rho}}/\partial\theta = 0$

$d\hat{\boldsymbol{\theta}}/\partial\varphi = C_{\boldsymbol{\theta}}\, \hat{\boldsymbol{\varphi}}$ $\qquad$ $\partial\hat{\boldsymbol{\varphi}}/\partial\varphi = -\hat{\boldsymbol{\rho}} = -C_{\boldsymbol{\theta}}\, \hat{\boldsymbol{\theta}} - S_{\boldsymbol{\theta}}\, \hat{\mathbf{r}}$ $\qquad$ $\partial\hat{\mathbf{r}}/\partial\varphi = S_{\boldsymbol{\theta}}\, \hat{\boldsymbol{\varphi}}$ $\qquad$ $\partial\hat{\boldsymbol{\rho}}/\partial\varphi = \hat{\boldsymbol{\varphi}}$

$\partial\hat{\boldsymbol{\theta}}/\partial r = 0$ $\qquad$ $\partial\hat{\boldsymbol{\varphi}}/\partial r = 0$ $\qquad$ $\partial\hat{\mathbf{r}}/\partial r = 0$ $\qquad$ $\partial\hat{\boldsymbol{\rho}}/\partial r = 0$

(k) The differential operators, all verified against old printed sheet:

$$\boldsymbol{\nabla}(f) = (\partial_{\mathbf{r}} f)\, \hat{\mathbf{r}} + (1/r)\, (\partial_{\boldsymbol{\theta}} f)\, \hat{\boldsymbol{\theta}} + (1/rS_{\boldsymbol{\theta}})\, (\partial_{\boldsymbol{\varphi}} f)\, \hat{\boldsymbol{\varphi}}$$

$$\boldsymbol{\nabla}\bullet\mathbf{A} = \frac{1}{r^2}\, \partial_{\mathbf{r}}(r^2 A^{\mathbf{r}}) + \frac{1}{rS_{\boldsymbol{\theta}}}\, \partial_{\boldsymbol{\theta}}(S_{\boldsymbol{\theta}}\, A^{\boldsymbol{\theta}}) + \frac{1}{rS_{\boldsymbol{\theta}}}\, \partial_{\boldsymbol{\varphi}} A$$

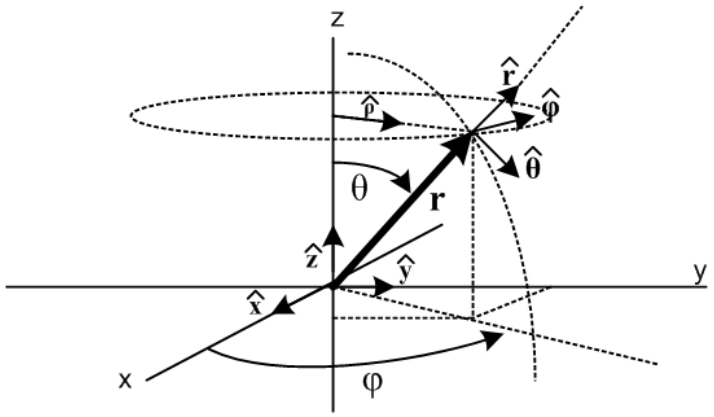
$$\nabla^2(f) = \boldsymbol{\nabla}\bullet\boldsymbol{\nabla}(f) = \frac{1}{r^2}\, \partial_{\mathbf{r}}\, (r^2\, \partial_{\mathbf{r}} f) + \frac{1}{r^2 S_{\boldsymbol{\theta}}}\, \partial_{\boldsymbol{\theta}}(S_{\boldsymbol{\theta}}\, \partial_{\boldsymbol{\theta}} f) + \frac{1}{r^2 S_{\boldsymbol{\theta}}^2}\, \partial_{\boldsymbol{\varphi}}^2 f$$

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{rS_{\boldsymbol{\theta}}}\, \hat{\mathbf{r}}\, \{\partial_{\boldsymbol{\theta}}(S_{\boldsymbol{\theta}}\, A_{\boldsymbol{\varphi}}) - \partial_{\boldsymbol{\varphi}} A_{\boldsymbol{\theta}}\} + \frac{1}{r}\, \hat{\boldsymbol{\theta}}\, \{\frac{1}{S_{\boldsymbol{\theta}}}(\partial_{\boldsymbol{\varphi}} A_{\mathbf{r}}) - \partial_{\mathbf{r}}(rA_{\boldsymbol{\varphi}})\} + \frac{1}{r}\, \hat{\boldsymbol{\varphi}}\, \{\partial_{\mathbf{r}}(rA_{\boldsymbol{\theta}}) - \partial_{\boldsymbol{\theta}} A_{\mathbf{r}}\}$$

Our one other result is no longer very interesting because it is obvious once you know the gradient.

$$\mathbf{A}\bullet\, \boldsymbol{\nabla}(f) = A^{\mathbf{r}}\, (\partial_{\mathbf{r}} f) + (A^{\boldsymbol{\theta}}/r)\, (\partial_{\boldsymbol{\theta}} f) + (A^{\boldsymbol{\varphi}}/\, rS_{\boldsymbol{\theta}})\, (\partial_{\boldsymbol{\varphi}} f)$$

Spherical Coordinates Summary Page (see application to sphericals .doc) PhL 9.9.08



$\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$ $\quad x = rS_\theta C_\varphi \quad r^2 = x^2 + y^2 + z^2$

$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}$ $\quad y = rS_\theta S_\varphi \quad \tan^2\theta = (x^2 + y^2)/z^2$

$\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}}$ $\quad z = rC_\theta \quad \tan\varphi = y/x$

$g_{\theta\theta} = Q_\theta^2 = r^2 \quad Q_\theta = r$

$g_{\varphi\varphi} = Q_\varphi^2 = r^2 S_\theta^2 \quad Q_\varphi = rS_\theta$

$g_{rr} = Q_r^2 = 1 \quad Q_r = 1$

$J = \sqrt{\det(g_{ij})} = v = r^2 S_\theta$

$dxdydz = r^2 S_\theta \, dr d\theta d\varphi = r^2 dr \, d(\cos\theta) d\varphi$

$$(\hat{\boldsymbol{\theta}} \;\; \hat{\boldsymbol{\varphi}} \;\; \hat{\mathbf{r}}) = \begin{pmatrix} \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{x}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{y}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\bullet\hat{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_z(\varphi)\, R_y(\theta)$$

$\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\mathbf{r}}$ $\quad \partial\hat{\boldsymbol{\varphi}}/\partial\theta = 0$ $\quad \partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$ $\quad \partial\hat{\boldsymbol{\rho}}/\partial\theta = 0$

$d\hat{\boldsymbol{\theta}}/\partial\varphi = C_\theta \hat{\boldsymbol{\varphi}}$ $\quad \partial\hat{\boldsymbol{\varphi}}/\partial\varphi = -\hat{\boldsymbol{\rho}} = -C_\theta \hat{\boldsymbol{\theta}} - S_\theta \hat{\mathbf{r}}$ $\quad \partial\hat{\mathbf{r}}/\partial\varphi = S_\theta \hat{\boldsymbol{\varphi}}$ $\quad \partial\hat{\boldsymbol{\rho}}/\partial\varphi = \hat{\boldsymbol{\varphi}}$

$\partial\hat{\boldsymbol{\theta}}/\partial r = 0$ $\quad \partial\hat{\boldsymbol{\varphi}}/\partial r = 0$ $\quad \partial\hat{\mathbf{r}}/\partial r = 0$ $\quad \partial\hat{\boldsymbol{\rho}}/\partial r = 0$

$$\boldsymbol{\nabla}(f) = (\partial_r f)\,\hat{\mathbf{r}} + (1/r)\,(\partial_\theta f)\,\hat{\boldsymbol{\theta}} + (1/rS_\theta)\,(\partial_\varphi f)\,\hat{\boldsymbol{\varphi}}$$

$$\boldsymbol{\nabla}\bullet\mathbf{A} = \frac{1}{r^2}\,\partial_r(r^2 A^r) + \frac{1}{rS_\theta}\partial_\theta(S_\theta A^\theta) + \frac{1}{rS_\theta}\partial_\varphi A^\varphi$$

$$\nabla^2(f) = \boldsymbol{\nabla}\bullet\boldsymbol{\nabla}(f) = \frac{1}{r^2}\,\partial_r\,(r^2\,\partial_r f) + \frac{1}{r^2 S_\theta}\,\partial_\theta(S_\theta\,\partial_\theta f) + \frac{1}{r^2 S_\theta^2}\,\partial_\varphi^2 f$$

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{rS_\theta}\,\hat{\mathbf{r}}\,\{\partial_\theta(S_\theta A_\varphi) - \partial_\varphi A_\theta\} + \frac{1}{r}\,\hat{\boldsymbol{\theta}}\,\{\frac{1}{S_\theta}(\partial_\varphi A_r) - \partial_r(rA_\varphi)\} + \frac{1}{r}\,\hat{\boldsymbol{\varphi}}\,\{\partial_r(rA_\theta) - \partial_\theta A_r\}$$