Spherical Unit Vectors PhL 8.29.08

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1. Work done August 29, 2008

Comment: In this first section, the term "old sheet" refers to the yellowed hand-written page #5 in my TK (Tool Kit) binder.

Our usual active rotation matrices are:

$$R_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_{\mathbf{y}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The spherical coordinates unit vectors

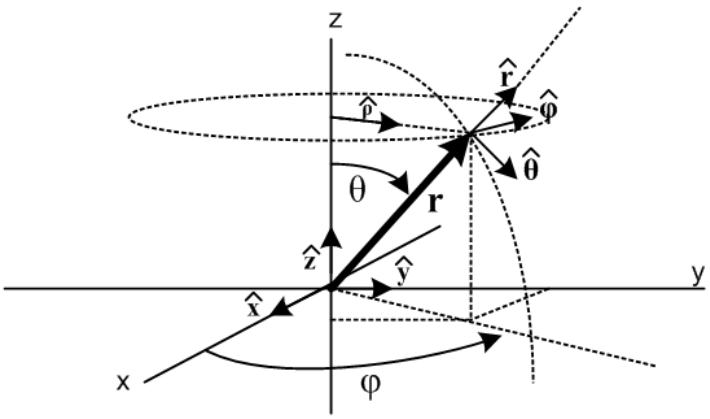
In spherical coordinates, we have three unit vectors called $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$. Here is their relation to the normal unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. We start with $\hat{\mathbf{r}}$:

$$\hat{\mathbf{r}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{z}} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi\,\sin\theta \\ \sin\varphi\,\sin\theta \\ \cos\theta \end{pmatrix}$$

$= \cos\varphi \sin\theta\, \hat{\mathbf{x}} + \sin\varphi \sin\theta\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}}$ // agrees with old sheet

$= \hat{\mathbf{x}}\bullet\hat{\mathbf{r}}\ \ \hat{\mathbf{x}} + \ \ \hat{\mathbf{y}}\bullet\hat{\mathbf{r}}\ \ \hat{\mathbf{y}} + \hat{\mathbf{z}}\bullet\hat{\mathbf{r}}\ \ \hat{\mathbf{z}}$ // another way to write it

where we have picked off the last column of the matrix for our coefficients, and we have also expressed these coefficients in terms of the usual "direction cosines". The other unit vectors are



$$\hat{\boldsymbol{\theta}} = R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{x}} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix}$$

$= \cos\varphi\cos\theta\, \hat{\mathbf{x}} + \sin\varphi\cos\theta\, \hat{\mathbf{y}} - \sin\theta\, \hat{\mathbf{z}}$ // agrees with old sheet

$$\hat{\boldsymbol{\varphi}} = R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{y}} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$= -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}}$ // agrees with old sheet

It is true that $\hat{\boldsymbol{\varphi}} = R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{y}} = R_z(\varphi)\, \hat{\mathbf{y}}$ so $R_y(\theta)$ could be omitted, but our point is the next sentence. Notice that these unit vectors $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ and $\hat{\mathbf{r}}$ are the just columns of the matrix shown above, that is,

$$(\,\hat{\boldsymbol{\theta}}\ \ \hat{\boldsymbol{\varphi}}\ \ \hat{\mathbf{r}}\,) = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_z(\varphi)\, R_y(\theta)$$

In order to paste into a Word matrix, we first list off the 9 direction cosines

$\hat{\mathbf{x}}\bullet\hat{\mathbf{r}}$ $\hat{\mathbf{y}}\bullet\hat{\mathbf{r}}$ $\hat{\mathbf{z}}\bullet\hat{\mathbf{r}}$ $\quad$ $\hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}}$ $\hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}}$ $\hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}}$ $\quad$ $\hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}}$

And then we can position these into their correct positions in the above matrix

$$(\,\hat{\boldsymbol{\theta}}\ \ \hat{\boldsymbol{\varphi}}\ \ \hat{\mathbf{r}}\,) = \begin{pmatrix} \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{x}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{y}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\bullet\hat{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_z(\varphi)\, R_y(\theta)$$

For example, we associated the middle column with $\hat{\boldsymbol{\varphi}}$, and we associate the three rows with x,y,z.

The following results are developed in sections below, but I put them here to make them findable:

$$\begin{array}{lll} d\hat{\mathbf{r}}/dr = 0 & d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}} & d\hat{\mathbf{r}}/d\varphi = \sin\theta\,\hat{\boldsymbol{\varphi}} \\ d\hat{\boldsymbol{\theta}}/dr = 0 & d\hat{\boldsymbol{\theta}}/d\theta = -\hat{\mathbf{r}} & d\hat{\boldsymbol{\theta}}/d\varphi = \cos\theta\,\hat{\boldsymbol{\varphi}} \\ d\hat{\boldsymbol{\varphi}}/dr = 0 & d\hat{\boldsymbol{\varphi}}/d\theta = 0 & d\hat{\boldsymbol{\varphi}}/d\varphi = -\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}} \end{array} \qquad \text{(A.15)}$$

2. Work done August 4, 2012

The following section is a verbatim appendix of Frames doc. I think it is somewhat better than the above since it gets the basis vectors into the "right order". It refers to certain equations inside frames doc which I will now list here so this thing can be stand alone.

$$\begin{array}{llll} \hat{\boldsymbol{e}}_i = R(\xi)\,\mathbf{e}_i & \Rightarrow & \mathbf{e}_i = R(\xi)_{ij}\hat{\boldsymbol{e}}_j & \text{or} \quad \hat{\boldsymbol{e}}_i = [R(\xi)]^{-1}{}_{ij}\,\mathbf{e}_j \\ \hat{\boldsymbol{e}}'_i = R'(\xi')\,\mathbf{e}'_i & \Rightarrow & \mathbf{e}'_i = R'(\xi')_{ij}\hat{\boldsymbol{e}}'_j & \text{or} \quad \hat{\boldsymbol{e}}'_i = [R'(\xi')]^{-1}{}_{ij}\,\mathbf{e}'_j \end{array} \qquad \text{(14.3)}$$

$$R(\xi) = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \qquad [R(\xi)]^{-1} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \qquad \text{(14.4)}$$

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad \text{(14.5)}$$

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\varphi\sin\theta\,\hat{\mathbf{x}} + \sin\varphi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\varphi\cos\theta\,\hat{\mathbf{x}} + \sin\varphi\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \end{aligned} \qquad \text{(14.6)}$$

Appendix A: Derivation of R(ξ) for Spherical Coordinates **// has been updated.**

Here we derive equations (14.4) through (14.6). First, just for reference, here are the three active rotation matrices used below :

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(A.1)}$$

These are called "active" since they rotate a vector forward (counterclockwise) relative to fixed axes by amount θ according to the right hand rule when the thumb is aligned with the axis in question.

From the usual picture of spherical coordinates,

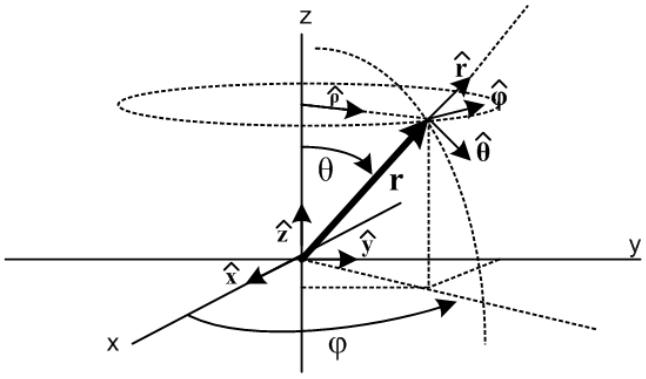


Fig A.1

one can see by staring hard enough that

$$\begin{aligned}
\hat{\mathbf{r}} &= R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{z}} &&= \mathcal{R}\hat{\mathbf{z}} && \mathcal{R} \equiv R_z(\varphi)\, R_y(\theta) \\
\hat{\boldsymbol{\theta}} &= R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{x}} &&= \mathcal{R}\hat{\mathbf{x}} \\
\hat{\boldsymbol{\varphi}} &= R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{y}} &&= \mathcal{R}\hat{\mathbf{y}}
\end{aligned} \tag{A.2}$$

which we rewrite as

$$\begin{aligned}
\hat{\boldsymbol{e}}_1 &= \mathcal{R}\, \mathbf{e}_3 \\
\hat{\boldsymbol{e}}_2 &= \mathcal{R}\, \mathbf{e}_1 \\
\hat{\boldsymbol{e}}_3 &= \mathcal{R}\, \mathbf{e}_2 \ .
\end{aligned} \tag{A.3}$$

We can repair the ordering of the basis vectors $\mathbf{e_n}$ on the right using $R_2 = (\hat{\mathbf{z}},\hat{\mathbf{x}},\hat{\mathbf{y}})$ as follows

$$\begin{aligned}
\mathbf{e}_3 = R_2\, \mathbf{e}_1 \qquad & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
\mathbf{e}_1 = R_2\, \mathbf{e}_2 \qquad & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
\mathbf{e}_2 = R_2\, \mathbf{e}_3 \qquad & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned} \tag{A.4}$$

Maple tells us that $R_2^{-1} = R_2^{T}$ and $\det(R_2) = 1$, confirming that R_2 is a rotation. So we then have

$$\begin{aligned}
\hat{\boldsymbol{e}}_1 &= \mathcal{R}\, R_2\, \mathbf{e}_1 \\
\hat{\boldsymbol{e}}_2 &= \mathcal{R}\, R_2\, \mathbf{e}_2 \\
\hat{\boldsymbol{e}}_3 &= \mathcal{R}\, R_2\, \mathbf{e}_3
\end{aligned} \tag{A.5}$$

or

$$\hat{\boldsymbol{e}}_i = R(\xi)\, \mathbf{e}_i. \tag{A.6}$$

We have therefore found the sought-after matrix R(ξ) appearing in the top left of (14.3).

$$R(\xi) = \mathcal{R}\, R_2 \ . \tag{A.7}$$

Specific evaluation gives

$$\mathcal{R} = R_z(\varphi)\, R_y(\theta) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{A8}$$

and then

$$R(\xi) = \mathcal{R}\, R_2 = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \tag{A.9}$$

from which we find

$$[R(\xi)]^{-1} = [R(\xi)]^{T} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}. \tag{A.10}$$

From the top right equation of (14.3) we can write

$$\hat{\boldsymbol{e}}_i = [R(\xi)\,]^{-1}{}_{ij}\, \mathbf{e}_j \ = \ [R(\xi)\,]^{-1}{}_{i1}\, \mathbf{e}_1 + \ [R(\xi)\,]^{-1}{}_{i2}\, \mathbf{e}_2 + \ [R(\xi)\,]^{-1}{}_{i3}\, \mathbf{e}_3 \tag{A.11}$$

which can be written in this matrix notation where the column vectors contain vectors as elements,

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{A.12a}$$

or

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \tag{A.12b}$$

or

$$\begin{aligned} \hat{\mathbf{r}} \ &= \cos\varphi\sin\theta\, \hat{\mathbf{x}} + \sin\varphi\sin\theta\, \hat{\mathbf{y}} \ + \cos\theta\ \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} \ &= \cos\varphi\cos\theta\, \hat{\mathbf{x}} + \sin\varphi\cos\theta\, \hat{\mathbf{y}} \ - \sin\theta\, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} \ &= -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}} \end{aligned} \tag{A.12c}$$

which are the well known expressions for the spherical unit vectors in terms of the Cartesian ones. One more useful result is obtained by dotting (A.11) into basis vector $\mathbf{e}_k$

$$\hat{\boldsymbol{e}}_i \bullet \mathbf{e}_k \ = [R(\xi)\,]^{-1}{}_{ij}\, \mathbf{e}_j \bullet \mathbf{e}_k = [R(\xi)\,]^{-1}{}_{ij}\, \delta_{j,k} \ = \ [R(\xi)\,]^{-1}{}_{ik}$$

and therefore we get this matrix of "direction cosines",

$$\begin{pmatrix} \hat{\mathbf{r}}\bullet\hat{\mathbf{x}} & \hat{\mathbf{r}}\bullet\hat{\mathbf{y}} & \hat{\mathbf{r}}\bullet\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{x}} & \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{y}} & \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}}\bullet\hat{\mathbf{x}} & \hat{\boldsymbol{\varphi}}\bullet\hat{\mathbf{y}} & \hat{\boldsymbol{\varphi}}\bullet\hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} = [R(\xi)]^{-1} \tag{A.13}$$

This collection of dot products is also apparent from (A.12c) upon visual inspection.

3. Brute force computation of basis vector derivatives (Sept 7, 2012)

This work is a continuation of the previous section. First,

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \tag{A.12b}$$

Then the inverse will be

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)] \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix}$$

from which we learn that

$$\hat{\mathbf{x}} = \cos\varphi\sin\theta\,\hat{\mathbf{r}} + \cos\varphi\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = \sin\varphi\sin\theta\,\hat{\mathbf{r}} + \sin\varphi\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

which is the inverse of the set provided above in the appendix. I might add this to the appendix at some point.

I then "hand computed" the following derivative using the above equations.

$$d\hat{\mathbf{r}}/d\theta = \cos\varphi\cos\theta\,\hat{\mathbf{x}} + \sin\varphi\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$

$$= \cos\varphi\cos\theta\,[\cos\varphi\sin\theta\,\hat{\mathbf{r}} + \cos\varphi\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}]$$
$$+ \sin\varphi\cos\theta\,[\sin\varphi\sin\theta\,\hat{\mathbf{r}} + \sin\varphi\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}]$$
$$- \sin\theta\,[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}]$$

$$= (\cos\varphi\cos\theta\cos\varphi\sin\theta + \sin\varphi\cos\theta\sin\varphi\sin\theta - \sin\theta\cos\theta\,\hat{\mathbf{r}}$$
$$+ (\cos\varphi\cos\theta\cos\varphi\cos\theta + \sin\varphi\cos\theta\sin\varphi\cos\theta + \sin\theta\sin\theta)\hat{\boldsymbol{\theta}}$$
$$+ (-\cos\varphi\cos\theta\sin\varphi + \sin\varphi\cos\theta\cos\varphi)\hat{\boldsymbol{\varphi}}$$

$$= (\cos^2\varphi \cos\theta \sin\theta + \sin^2\varphi \cos\theta \sin\theta - \sin\theta \cos\theta)\, \hat{\mathbf{r}}$$
$$+ (\cos^2\varphi \cos^2\theta + \sin^2\varphi \cos^2\theta + \sin^2\theta)\hat{\boldsymbol{\theta}}$$
$$+ (\quad 0 \quad)\hat{\boldsymbol{\varphi}}$$

$$= \cos\theta \sin\theta (\cos^2\varphi + \sin^2\varphi - 1)\, \hat{\mathbf{r}}$$
$$+ (\cos^2\theta + \sin^2\theta)\hat{\boldsymbol{\theta}}$$

$$= (0)\, \hat{\mathbf{r}}$$
$$+ (1)\hat{\boldsymbol{\theta}}$$

$$= \hat{\boldsymbol{\theta}}$$

I then realized that there certainly must be a better way to do these derivatives. There are 9 of them, and I think I can do them from the geometry of the above figure maybe with a few support pictures. I could do them all by brute force as I did the one above, or I could do them formally by computing the affine connection as per tensor doc. But I think "little pictures" is the most direct method.

Three simple cases: From above

$$\hat{\mathbf{r}} = \cos\varphi \sin\theta\, \hat{\mathbf{x}} + \sin\varphi \sin\theta\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\varphi \cos\theta\, \hat{\mathbf{x}} + \sin\varphi \cos\theta\, \hat{\mathbf{y}} - \sin\theta\, \hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}} \qquad \text{(A.12c)}$$

we see that none of the unit vectors is a function of variable r. So we have three of our 9 right away

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Assemble results here:

1	$d\hat{\mathbf{r}}/dr = 0$	11
2	$d\hat{\boldsymbol{\theta}}/dr = 0$	12
3	$d\hat{\boldsymbol{\varphi}}/dr = 0$	13

4. $d\hat{\boldsymbol{\varphi}}/d\theta = 0$
5. $d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}}$
6. $d\hat{\boldsymbol{\theta}}/d\varphi = \cos\theta\, \hat{\boldsymbol{\varphi}}$

7 $d\hat{\mathbf{r}}/d\varphi = \sin\theta\, \hat{\boldsymbol{\varphi}}$
8 $d\hat{\boldsymbol{\theta}}/d\theta = -\hat{\mathbf{r}}$
9 $d\hat{\boldsymbol{\varphi}}/d\varphi = -\sin\theta\, \hat{\mathbf{r}} - \cos\theta\, \hat{\boldsymbol{\theta}}$

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1 $d\hat{\mathbf{r}}/dr = 0$

2 $d\hat{\boldsymbol{\theta}}/dr = 0$

3 $d\hat{\boldsymbol{\varphi}}/dr = 0$

A fourth one is also obvious from above.

4. $d\hat{\boldsymbol{\varphi}}/d\theta = 0$

That leaves only 5 to compute by hand, and I have done one above which is this:

5. $d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}}$

This result is fairly obvious from the picture. That leaves four more and I will just do them here by brute force:

$$d\hat{\boldsymbol{\theta}}/d\varphi = d/d\varphi\,[\cos\varphi\cos\theta\,\hat{\mathbf{x}} + \sin\varphi\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}\,]$$

$$= -\sin\varphi\cos\theta\,\hat{\mathbf{x}} + \cos\varphi\cos\theta\,\hat{\mathbf{y}}$$

$$= -\sin\varphi\cos\theta\,[\cos\varphi\sin\theta\,\hat{\mathbf{r}} + \cos\varphi\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}]$$
$$+ \cos\varphi\cos\theta\,[\sin\varphi\sin\theta\,\hat{\mathbf{r}} + \sin\varphi\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}]$$

$$= (-\sin\varphi\cos\theta\cos\varphi\sin\theta + \cos\varphi\cos\theta\sin\varphi\sin\theta)\hat{\mathbf{r}}$$
$$+ (-\sin\varphi\cos\theta\cos\varphi\cos\theta + \cos\varphi\cos\theta\sin\varphi\cos\theta)\hat{\boldsymbol{\theta}}$$
$$+ (\sin\varphi\cos\theta\sin\varphi + \cos\varphi\cos\theta\cos\varphi)\hat{\boldsymbol{\varphi}}$$

$$= (0)\hat{\mathbf{r}}$$
$$+ (-\sin\varphi\cos^2\theta\cos\varphi + \cos\varphi\cos^2\theta\sin\varphi)\hat{\boldsymbol{\theta}}$$
$$+ (\sin^2\varphi\cos\theta + \cos^2\varphi\cos\theta)\hat{\boldsymbol{\varphi}}$$

$$= (0)\hat{\mathbf{r}}$$
$$+ (0)\hat{\boldsymbol{\theta}}$$
$$+ (\cos\theta)\hat{\boldsymbol{\varphi}}$$

$$= \cos\theta\,\hat{\boldsymbol{\varphi}}$$

This one is just not obvious to me staring at the picture but is correct in the obvious limits. So we have just shown that

6. $d\hat{\boldsymbol{\theta}}/d\varphi = \cos\theta\,\hat{\boldsymbol{\varphi}}$

Next:

$d\hat{\mathbf{r}}/d\varphi = d/d\varphi\ [\cos\varphi \sin\theta\ \hat{\mathbf{x}} + \sin\varphi \sin\theta\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}]$

$= -\sin\varphi\sin\theta\hat{\mathbf{x}} + \cos\varphi\sin\theta\hat{\mathbf{y}} + 0$

$= -\sin\varphi\sin\theta\ [\cos\varphi\sin\theta\ \hat{\mathbf{r}} + \cos\varphi\cos\theta\ \hat{\boldsymbol{\theta}} - \sin\varphi\ \hat{\boldsymbol{\varphi}}]$
$+ \cos\varphi\sin\theta[\sin\varphi\sin\theta\ \hat{\mathbf{r}} + \sin\varphi\cos\theta\ \hat{\boldsymbol{\theta}} + \cos\varphi\ \hat{\boldsymbol{\varphi}}]$

$= (-\sin\varphi\sin\theta\cos\varphi\sin\theta + \cos\varphi\sin\theta\sin\varphi\sin\theta)\hat{\mathbf{r}}$
$+ (-\sin\varphi\sin\theta\cos\varphi\cos\theta + \cos\varphi\sin\theta\sin\varphi\cos\theta)\hat{\boldsymbol{\theta}}$
$+ (\sin\varphi\sin\theta\sin\varphi + \cos\varphi\sin\theta\cos\varphi)\hat{\boldsymbol{\varphi}}$

$= (-\sin\varphi\sin^2\theta\cos\varphi + \cos\varphi\sin^2\theta\sin\varphi)\hat{\mathbf{r}}$
$+ (0)\hat{\boldsymbol{\theta}}$
$+ (\sin^2\varphi\sin\theta + \cos^2\varphi\sin\theta)\hat{\boldsymbol{\varphi}}$

$= (0)\hat{\mathbf{r}}$
$+ (0)\hat{\boldsymbol{\theta}}$
$+ (\sin\theta)\ \hat{\boldsymbol{\varphi}}$

so therefore

$d\hat{\mathbf{r}}/d\varphi = \sin\theta\ \hat{\boldsymbol{\varphi}}$

This one is pretty obvious looking at the picture.

Next:

$d\hat{\boldsymbol{\theta}}/d\theta = d/d\theta\ [\cos\varphi\cos\theta\ \hat{\mathbf{x}} + \sin\varphi\cos\theta\ \hat{\mathbf{y}} - \sin\theta\ \hat{\mathbf{z}}]$

$= -\cos\varphi\sin\theta\ \hat{\mathbf{x}} - \sin\varphi\sin\theta\hat{\mathbf{y}} - \cos\theta\hat{\mathbf{z}}$

$= -\cos\varphi\sin\theta[\cos\varphi\sin\theta\ \hat{\mathbf{r}} + \cos\varphi\cos\theta\ \hat{\boldsymbol{\theta}} - \sin\varphi\ \hat{\boldsymbol{\varphi}}]$
$- \sin\varphi\sin\theta[\sin\varphi\sin\theta\ \hat{\mathbf{r}} + \sin\varphi\cos\theta\ \hat{\boldsymbol{\theta}} + \cos\varphi\ \hat{\boldsymbol{\varphi}}]$
$-\cos\theta[\cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}]$

$= (-\cos\varphi\sin\theta\cos\varphi\sin\theta - \sin\varphi\sin\theta\sin\varphi\sin\theta - \cos\theta\cos\theta)\hat{\mathbf{r}}$
$+ (-\cos\varphi\sin\theta\cos\varphi\cos\theta - \sin\varphi\sin\theta\sin\varphi\cos\theta + \cos\theta\sin\theta)\hat{\boldsymbol{\theta}}$
$+ (\cos\varphi\sin\theta\sin\varphi - \sin\varphi\sin\theta\cos\varphi)\hat{\boldsymbol{\varphi}}$

$= (-\cos^2\varphi\sin^2\theta - \sin^2\varphi\sin^2\theta - \cos^2\theta)\hat{\mathbf{r}}$

$$+ (-\cos^2\varphi\sin\theta\cos\theta - \sin^2\varphi\sin\theta\cos\theta + \cos\theta\sin\theta)\hat{\boldsymbol{\theta}}$$
$$+ (0\varphi)\hat{\boldsymbol{\varphi}}$$

$$= (-\sin^2\theta - \cos^2\theta)\hat{\mathbf{r}}$$
$$+ (-\sin\theta\cos\theta + \cos\theta\sin\theta)\hat{\boldsymbol{\theta}}$$
$$+ (0\varphi)\hat{\boldsymbol{\varphi}}$$

$$= (-1)\hat{\mathbf{r}}$$
$$+ (0)\hat{\boldsymbol{\theta}}$$
$$+ (0\varphi)\hat{\boldsymbol{\varphi}}$$

$$= -\hat{\mathbf{r}}$$

so we have shown that

$$d\hat{\boldsymbol{\theta}}/d\theta = -\hat{\mathbf{r}}$$

and this seems right looking at the picture. One more to go

$$d\hat{\boldsymbol{\varphi}}/d\varphi = d/d\varphi[-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}]$$

$$= -\cos\varphi\,\hat{\mathbf{x}} - \sin\varphi\,\hat{\mathbf{y}}$$

$$= -\cos\varphi\,[\cos\varphi\sin\theta\,\hat{\mathbf{r}} + \cos\varphi\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}]$$
$$- \sin\varphi\,[\sin\varphi\sin\theta\,\hat{\mathbf{r}} + \sin\varphi\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}]$$

$$= (-\cos\varphi\cos\varphi\sin\theta - \sin\varphi\sin\varphi\sin\theta)\hat{\mathbf{r}}$$
$$+ (-\cos\varphi\cos\varphi\cos\theta - \sin\varphi\sin\varphi\cos\theta)\hat{\boldsymbol{\theta}}$$
$$+ (\cos\varphi\sin\varphi - \sin\varphi\cos\varphi)\hat{\boldsymbol{\varphi}}$$

$$= (-\cos^2\varphi\sin\theta - \sin^2\varphi\sin\theta)\hat{\mathbf{r}}$$
$$+ (-\cos^2\varphi\cos\theta - \sin^2\varphi\cos\theta)\hat{\boldsymbol{\theta}}$$
$$+ (0)\hat{\boldsymbol{\varphi}}$$

$$= (-\sin\theta)\hat{\mathbf{r}}$$
$$+ (-\cos\theta)\hat{\boldsymbol{\theta}}$$
$$+ (0)\hat{\boldsymbol{\varphi}}$$

$$= -\sin\theta\hat{\mathbf{r}} - \cos\theta\hat{\boldsymbol{\theta}}$$

so we have shown that

$d\hat{\boldsymbol{\varphi}}/d\varphi = -\sin\theta\hat{\mathbf{r}} - \cos\theta\hat{\boldsymbol{\theta}}$

This result is also not very obvious from the picture.

4. How did Lai and tensor doc deal with these derivatives?

I discuss this in "Lai's grad-v method..doc" but have no clean conclusion. I find there that

$$d\hat{\mathbf{e}}_n = \Sigma_j \{ [d(h'^{-1}_n) \delta_{j,n} + h'^{-1}_n \Sigma_i R_{ji} dS_{in}] h'_j \} \hat{\mathbf{e}}_j$$

which is at least some kind of result. In tensor doc p 98 I do it this way

$$d\mathbf{e}_n = (\partial'_j \mathbf{e}_n) dx'^j = \Gamma'^i_{jn} \mathbf{e}_i dx'^j \quad \text{// from definition of } \Gamma \text{ in Appendix F (a), but Picture A}$$

which says

$$(\partial'_j \mathbf{e}_n) = \Gamma'^i_{jn} \mathbf{e}_i$$

But of course we then have this issue

$$\mathbf{e}_n = h'_n \hat{\mathbf{e}}_n \quad \text{// tensor doc p 100}$$

so we get

$$\partial'_j \hat{\mathbf{e}}_n = \partial'_j [h'^{-1}_n \mathbf{e}_n]) = h'^{-1}_n (\partial'_j \mathbf{e}_n) + (\partial'_j [h'^{-1}_n]) \mathbf{e}_n$$

and then we can install the result above to get

$$\partial'_j \hat{\mathbf{e}}_n = h'^{-1}_n \Gamma'^i_{jn} \mathbf{e}_i + (\partial'_j [h'^{-1}_n]) \mathbf{e}_n$$

Finally, we recover the hats

$$\partial'_j \hat{\mathbf{e}}_n = h'^{-1}_n \Gamma'^i_{jn} h'_i \hat{\mathbf{e}}_i + (\partial'_j [h'^{-1}_n]) h'_n \hat{\mathbf{e}}_n$$

Maybe simplify the combination

$$(\partial'_j [h'^{-1}_n]) h'_n = - h'^{-2}_n (\partial'_j h'_n) h'_n = - h'^{-1}_n (\partial'_j h'_n)$$

Then we have

$$\partial'_j \hat{\mathbf{e}}_n = h'^{-1}_n \Gamma'^i_{jn} h'_i \hat{\mathbf{e}}_i - h'^{-1}_n (\partial'_j h'_n) \hat{\mathbf{e}}_n$$

$$= (1/h'_n) [h'_i \Gamma'^i_{jn} \hat{\mathbf{e}}_i - (\partial'_j h'_n) \hat{\mathbf{e}}_n]$$

$$= (1/h'_n)\,[h'_i\,\Gamma'^{i}_{jn}\,\hat{\mathbf{e}}_i - (\partial'_j\,h'_n)\delta_{n,i}\,\hat{\mathbf{e}}_i]$$

$$= (1/h'_n)\,[h'_i\,\Gamma'^{i}_{jn} - (\partial'_j h'_n)\delta_{n,i}]\,\hat{\mathbf{e}}_i$$

This is the general formula and I think is the simplest I can find. compare to

$$(\partial'_j \mathbf{e}_n) = \Gamma'^{i}_{jn}\,\mathbf{e}_i$$

so all that extra garbage comes from the unit vector business.

I did do Maple work with the affine connection in tensor doc App I (f). The file "sph_vec_lap.mws" has some very useful code. What I would like to make it compute is this (from above)

$$[\partial'_j \hat{\mathbf{e}}_n]_i = (1/h'_n)\,[h'_i\,\Gamma'^{i}_{jn} - (\partial'_j h'_n)\delta_{n,i}]$$

Let's try to add this to the end. Maybe define

$$de(j,n,i) = [\partial'_j \hat{\mathbf{e}}_n]_i = (1/h'_n)\,[h'_i\,\Gamma'^{i}_{jn} - (\partial'_j h'_n)\delta_{n,i}]$$

```
= (1/hp[n])* ( hp[i] * G(i,j,n) - Diff(hp[n],xp[j])*kd(n,i));
```

First, here is my claimed summary from the updated frames doc appendix

$$\begin{array}{lll} d\hat{\mathbf{r}}/dr = 0 & d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}} & d\hat{\mathbf{r}}/d\varphi = \sin\theta\,\hat{\boldsymbol{\varphi}} \\ d\hat{\boldsymbol{\theta}}/dr = 0 & d\hat{\boldsymbol{\theta}}/d\theta = -\hat{\mathbf{r}} & d\hat{\boldsymbol{\theta}}/d\varphi = \cos\theta\,\hat{\boldsymbol{\varphi}} \\ d\hat{\boldsymbol{\varphi}}/dr = 0 & d\hat{\boldsymbol{\varphi}}/d\theta = 0 & d\hat{\boldsymbol{\varphi}}/d\varphi = -\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}} \end{array} \qquad (A.15)$$

11	21	31
12	22	32
13	23	33

And now Maple confirms:

Extra code added on 9_7_12 to compute the derivatives of spherical unit vectors using

$$de(j,n,i) = [\partial'_j \hat{e}_n]_i = \ (1/h'_n)\,[h'_i\,\Gamma'^{i}_{jn} - (\partial'_j h'_n)\delta_{n,i}]$$

Kronecker Delta routine

```
> kd := proc(a,b)
  if type(a,numeric) and type(b,numeric) then
      if   (a=b) then 1;
      else 0;
      fi;
    else 'kd(a,b)';
    fi;
  end:
> de := (j,n,i) -> (1/hp[n])* ( hp[i] * G(i,j,n) - diff(hp[n],xp[j])*kd(n,i));
```

$$de := (j,n,i) \to \frac{hp_i\,\mathrm{G}(i,j,n) - \left(\frac{\partial}{\partial xp_j} hp_n\right)\mathrm{kd}(n,i)}{hp_n}$$

```
> for j from 1 to 3 do
     for n from 1 to 3 do
        print(j,n,"
  ",simplify(value(de(j,n,1))),simplify(value(de(j,n,2))),simplify(value(de(j,n,3))) );
     od;
  od;
```

1, 1, " ", 0, 0, 0
1, 2, " ", 0, 0, 0
1, 3, " ", 0, 0, 0
2, 1, " ", 0, 1, 0
2, 2, " ", -1, 0, 0
2, 3, " ", 0, 0, 0
3, 1, " ", 0, 0, sin(θ)
3, 2, " ", 0, 0, cos(θ)
3, 3, " ", −sin(θ), −cos(θ), 0

Here is a simpler Maple verification of these facts:

$$\begin{array}{lll} d\hat{\mathbf{r}}/dr = 0 & d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}} & d\hat{\mathbf{r}}/d\varphi = \sin\theta\,\hat{\boldsymbol{\varphi}} \\ d\hat{\boldsymbol{\theta}}/dr = 0 & d\hat{\boldsymbol{\theta}}/d\theta = -\,\hat{\mathbf{r}} & d\hat{\boldsymbol{\theta}}/d\varphi = \cos\theta\,\hat{\boldsymbol{\varphi}} \\ d\hat{\boldsymbol{\varphi}}/dr = 0 & d\hat{\boldsymbol{\varphi}}/d\theta = 0 & d\hat{\boldsymbol{\varphi}}/d\varphi = -\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}} \end{array} \qquad \text{(A.15)}$$

I use notation $R = \hat{\mathbf{r}}$, $T = \hat{\boldsymbol{\theta}}$ and $P = \hat{\boldsymbol{\varphi}}$:

```
R := cos(phi)*sin(theta)*X + sin(phi)*sin(theta)*Y + cos(theta)*Z;
```

$$R := \cos(\phi)\sin(\theta)X + \sin(\phi)\sin(\theta)Y + \cos(\theta)Z$$

```
T := cos(phi)*cos(theta)*X + sin(phi)*cos(theta)*Y - sin(theta)*Z;
```

$$T := \cos(\phi)\cos(\theta)X + \sin(\phi)\cos(\theta)Y - \sin(\theta)Z$$

```
P := -sin(phi)*X + cos(phi)*Y;
```

$$P := -\sin(\phi)X + \cos(\phi)Y$$

```
diff(R,r);;
diff(T,r);
diff(P,r);
```

$$0$$
$$0$$
$$0$$

```
diff(R,theta)-T;
diff(T,theta)+R;
diff(P,theta);
```

$$0$$
$$0$$
$$0$$

```
diff(R,phi)-sin(theta)*P: simplify(%);
diff(T,phi)-cos(theta)*P: simplify(%);
diff(P,phi)+sin(theta)*R+cos(theta)*T: simplify(%);
```

$$0$$
$$0$$
$$0$$

'

5. Now lets do time derivatives of the basis vectors (added 12.14.16)

Use this table:

$d\hat{\mathbf{r}}/dr = 0$	$d\hat{\mathbf{r}}/d\theta = \hat{\boldsymbol{\theta}}$	$d\hat{\mathbf{r}}/d\varphi = \sin\theta\,\hat{\boldsymbol{\varphi}}$	
$d\hat{\boldsymbol{\theta}}/dr = 0$	$d\hat{\boldsymbol{\theta}}/d\theta = -\hat{\mathbf{r}}$	$d\hat{\boldsymbol{\theta}}/d\varphi = \cos\theta\,\hat{\boldsymbol{\varphi}}$	
$d\hat{\boldsymbol{\varphi}}/dr = 0$	$d\hat{\boldsymbol{\varphi}}/d\theta = 0$	$d\hat{\boldsymbol{\varphi}}/d\varphi = -\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}$	(A.15)

which write as

$\partial_r\hat{\mathbf{r}} = 0$	$\partial_\theta\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$	$\partial_\varphi\hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\varphi}}$	
$\partial_r\hat{\boldsymbol{\theta}} = 0$	$\partial_\theta\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}$	$\partial_\varphi\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\varphi}}$	
$\partial_r\hat{\boldsymbol{\varphi}} = 0$	$\partial_\theta\hat{\boldsymbol{\varphi}} = 0$	$\partial_\varphi\hat{\boldsymbol{\varphi}} = -\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}$	(A.15)

Then write

$$\partial_t\hat{\mathbf{r}} = \partial_r\hat{\mathbf{r}}\,\dot{r} + \partial_\theta\hat{\mathbf{r}}\,\dot{\theta} + \partial_\varphi\hat{\mathbf{r}}\,\dot{\varphi}$$

$$\partial_t\hat{\boldsymbol{\theta}} = \partial_r\hat{\boldsymbol{\theta}}\,\dot{r} + \partial_\theta\hat{\boldsymbol{\theta}}\,\dot{\theta} + \partial_\varphi\hat{\boldsymbol{\theta}}\,\dot{\varphi}$$

$$\partial_t\hat{\boldsymbol{\varphi}} = \partial_r\hat{\boldsymbol{\varphi}}\,\dot{r} + \partial_\theta\hat{\boldsymbol{\varphi}}\,\dot{\theta} + \partial_\varphi\hat{\boldsymbol{\varphi}}\,\dot{\varphi}$$

Reorder

$$\partial_t\hat{\mathbf{r}} = \dot{r}\partial_r\hat{\mathbf{r}} + \dot{\theta}\partial_\theta\hat{\mathbf{r}} + \dot{\varphi}\partial_\varphi\hat{\mathbf{r}}$$

$$\partial_t\hat{\boldsymbol{\theta}} = \dot{r}\partial_r\hat{\boldsymbol{\theta}} + \dot{\theta}\partial_\theta\hat{\boldsymbol{\theta}} + \dot{\varphi}\partial_\varphi\hat{\boldsymbol{\theta}}$$

$$\partial_t\hat{\boldsymbol{\varphi}} = \dot{r}\,\partial_r\hat{\boldsymbol{\varphi}} + \dot{\theta}\partial_\theta\hat{\boldsymbol{\varphi}} + \dot{\varphi}\partial_\varphi\hat{\boldsymbol{\varphi}}$$

Then install from table

$$\partial_t\hat{\mathbf{r}} = \dot{r}0 + \dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$$

$$\partial_t\hat{\boldsymbol{\theta}} = \dot{r}0 - \dot{\theta}\,\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}}$$

$$\partial_t\hat{\boldsymbol{\varphi}} = \dot{r}0 + \dot{\theta}0 - \dot{\varphi}(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}})$$

The conclusions are :

$$\partial_t\hat{\mathbf{r}} = \dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$$

$$\partial_t\hat{\boldsymbol{\theta}} = -\dot{\theta}\,\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}}$$

$$\partial_t\hat{\boldsymbol{\varphi}} = -\dot{\varphi}\sin\theta\,\hat{\mathbf{r}} - \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\theta}}$$

I found a physics-notation source that I can use to verify the above

$$\dot{\hat{r}} = \frac{\partial\hat{r}}{\partial r}\dot{r} + \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi} = \hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta$$

$$\dot{\hat{\theta}} = \frac{\partial\hat{\theta}}{\partial r}\dot{r} + \frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\theta}}{\partial\phi}\dot{\phi} = -\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta$$

$$\dot{\hat{\phi}} = \frac{\partial\hat{\phi}}{\partial r}\dot{r} + \frac{\partial\hat{\phi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi} = -\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\dot{\phi}$$

We do agree. Source is this from Singapore,

[PDF] Spherical Coordinates - ECE@NUS

https://www.ece.nus.edu.sg/.../Spherical%20coordinate%20system%20-%20grad,%20...

The **unit vectors** in the **spherical coordinate** system are functions of position. ... for the **time derivatives** of the **unit vectors** expressed in **spherical coordinates**:.

Missing: ~~physics~~

but the author is none other than Crodt!!! Hurray!

6. Particle motion in sphericals (added 12.14.16)

$$\mathbf{r} = r\,\hat{\mathbf{r}}$$

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\,\partial_t\hat{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,(\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}) = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$$

$$\mathbf{a} = \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\partial_t\hat{\mathbf{r}} + \partial_t(r\,\dot{\theta})\,\hat{\boldsymbol{\theta}} + r\,\dot{\theta}\,\partial_t\hat{\boldsymbol{\theta}} + \partial_t(r\,\dot{\varphi}\sin\theta)\,\hat{\boldsymbol{\varphi}} + r\,\dot{\varphi}\sin\theta\,\partial_t\hat{\boldsymbol{\varphi}}$$

$$= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}(\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}) + \partial_t(r\,\dot{\theta})\,\hat{\boldsymbol{\theta}} + r\,\dot{\theta}\,(-\dot{\theta}\,\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}})$$
$$+ \partial_t(r\,\dot{\varphi}\sin\theta)\,\hat{\boldsymbol{\varphi}} + r\,\dot{\varphi}\sin\theta\,(-\dot{\varphi}\sin\theta\,\hat{\mathbf{r}} - \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\theta}})$$

$$= [\,\ddot{r} - r\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta\,]\,\hat{\mathbf{r}}$$

$$+ [\,\dot{r}\,\dot{\theta} + \partial_t(r\,\dot{\theta}) - r\,\dot{\varphi}\sin\theta\,\dot{\varphi}\cos\theta\,]\,\hat{\boldsymbol{\theta}}$$

$$+ [\,\dot{r}\,\dot{\varphi}\sin\theta + r\,\dot{\theta}\,\dot{\varphi}\cos\theta + \partial_t(r\,\dot{\varphi}\sin\theta)\,]\hat{\boldsymbol{\varphi}}$$

$$= [\,\ddot{r} - r\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta\,]\,\hat{\mathbf{r}}$$

$$+ [\,\dot{r}\,\dot{\theta} + \dot{r}\,\dot{\theta} + r\ddot{\theta} - r\,\dot{\varphi}\sin\theta\,\dot{\varphi}\cos\theta\,]\,\hat{\boldsymbol{\theta}}$$

$$+ [\,\dot{r}\,\dot{\varphi}\sin\theta + r\,\dot{\theta}\,\dot{\varphi}\cos\theta + \dot{r}\,\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + r\,\dot{\varphi}\cos\theta\,\dot{\theta}\,]\hat{\boldsymbol{\varphi}}$$

$$= [\,\ddot{r} - r\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta\,]\,\hat{\mathbf{r}}$$

$$+ [\,2\,\dot{r}\,\dot{\theta} + r\ddot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta\,]\,\hat{\boldsymbol{\theta}}$$

$$+ [\,2\,\dot{r}\,\dot{\varphi}\sin\theta + 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\,]\hat{\boldsymbol{\varphi}}$$

Where can I verify this mess? The same Crodt doc writes

W

$$\vec{v} = \dot{\vec{r}} = \dot{\hat{r}}r + \hat{r}\dot{r}$$

$$\boxed{\vec{v} = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{\phi}r\dot{\phi}\sin\theta}$$

$$\begin{aligned}\vec{a} = \dot{\vec{v}} &= \dot{\hat{r}}\dot{r} + \hat{r}\ddot{r} + \dot{\hat{\theta}}r\dot{\theta} + \hat{\theta}\dot{r}\dot{\theta} + \hat{\theta}r\ddot{\theta} + \dot{\hat{\phi}}r\dot{\phi}\sin\theta + \hat{\phi}\dot{r}\dot{\phi}\sin\theta + \hat{\phi}r\ddot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\dot{\theta}\cos\theta \\ &= \left(\hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta\right)\dot{r} + \hat{r}\ddot{r} + \left(-\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta\right)r\dot{\theta} + \hat{\theta}\dot{r}\dot{\theta} + \hat{\theta}r\ddot{\theta} \\ &\quad + \left[-\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\dot{\phi}\right]r\dot{\phi}\sin\theta + \hat{\phi}\dot{r}\dot{\phi}\sin\theta + \hat{\phi}r\ddot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\dot{\theta}\cos\theta\end{aligned}$$

$$\boxed{\vec{a} = \hat{r}\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin\theta\right) + \hat{\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right) + \hat{\phi}\left(r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta\right)}$$

We agree except in his first term he should have $\sin^2\theta$. Here is another source and I fully agree:

$$\mathbf{v} = \dot{r}\hat{\mathbf{n}}_r + r\dot{\varphi}\,\hat{\mathbf{n}}_\varphi \sin\theta + r\dot{\theta}\,\hat{\mathbf{n}}_\theta. \qquad (2.35)$$

$$\begin{aligned}\mathbf{a} = {} & \hat{\mathbf{n}}_r(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta) \\ & + \hat{\mathbf{n}}_\varphi\left(2r\dot{\varphi}\dot{\theta}\cos\theta + 2\dot{\varphi}\dot{r}\sin\theta + r\ddot{\varphi}\sin\theta\right) \\ & + \hat{\mathbf{n}}_\theta(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta). \qquad (2.37)\end{aligned}$$

Abstract Kinematics is focused on the mathematical description of motion and does not concern its dynamical causes. We shall consider the motion of particles in different coordinate systems, including Cartesian, polar (in two-dimensional 2*D*

He agrees with he exactly, source a bit hazy.

<u>Conclusions on spherical coordinate particle motion:</u>

$\mathbf{r} = r\,\hat{\mathbf{r}}$

$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\,\partial_t\hat{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$

so

$$\mathbf{v} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\boldsymbol{\varphi}}$$

where

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$v_\varphi = r\dot{\varphi}\sin\theta$$

$$\mathbf{a} = [\ \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\ ]\,\hat{\mathbf{r}} + [\ 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\ ]\,\hat{\boldsymbol{\theta}} + [\ 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\ ]\hat{\boldsymbol{\varphi}}$$

so

$$\mathbf{a} = a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}}$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta$$

$$a_\varphi = 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta$$

<u>Now suppose motion is constrained to lie on a sphere or radius r</u>. Then

$$\mathbf{v} = v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\boldsymbol{\varphi}}$$

where

$$v_\theta = r\dot{\theta}$$

$$v_\varphi = r\dot{\varphi}\sin\theta$$

$$\mathbf{a} = a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}}$$

where

$$a_r = -r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta$$

$$a_\theta = r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta$$

$$a_\varphi = +2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta$$