Spheroidal Coordinate Description and Early Notes PhL 11.6.09

These are very early notes before I learned even the spheroidal coordinates. It was not until I learned the oblate spheroidals that I went on to learn the more general ellipsoidal coordinates. I later decided to use the MF coordinate choice for oblates rather than the MM choice.

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Overview (10.4.10, 1/2 page)

In **Section 1** I ponder the notion of "atoms" for spheroidal or ellipsoidal coordinates, before I even knew what atoms or these coordinates were (see summary doc elsewhere on that whole subject!).

In an interesting comment, I thought I should try to find the solution to the **charged elliptic disk** (generalized charged round disk of Jackson) before attempting a spheroidal object. I see how that might have seemed reasonable, thinking of it as somehow a 2D problem. In fact, it is still a 3D problem, but you might use **"elliptical cylindrical coordinates"** to mimic the Jackson solution. I don't know what the atoms are for that coordinate system because I have never studied it. [It involves Chebyshev polynomials, not easy to find Laplace atoms in these coordinates on the web; see Sec 8.2 of the Canonical PDF.] We know in retrospect that we can get a solution to the charged elliptic disk in *ellipsoidal* coordinates and the result is fairly simple. [Perhaps see Moon and Spencer, recently added to library.]

In **Section 2** I make some pretty reasonable comments comparing **prolate versus oblate** spheroidal coordinates. You use the same 2D elliptic coordinates picture, and the difference is where you choose to put your azimuthal z axis, vertical or horizontal. [another Moon and Spencer idea]

In **Section 3** I take a quick look at oblate spheroidals in **M&M** for the first time.

In **Section 4** I comment that **I understand why** people like Jackson don't *do* oblate spheroidals. There is too much else going on, and one really should have the general curvilinear theory first.

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1. Pondering notion of separation of variables in spheroidal coordinates.

(1) For the sphere in spherical coordinates, we know the general form of a solution

$$\{ r^n, r^{-n-1} \} \; P_n^m(\cos\theta) \; \{ \sin(m\varphi), \cos(m\varphi) \}$$

We looked for a separable solution of Laplace in sphericals, we came up with two separation constants. We learn that m = integer from the usual φ requirement. Somehow I never clearly understood, this forces n to also be an integer, otherwise probably something does not converge. Then that is how you get these results above. If our problem has azimuthal symmetry, so we then have

$$\{ r^n, r^{-n-1} \} \; P_n(\cos\theta)$$

as our possible solution building blocks. For the charged sphere problem, we ALSO have no θ dependence, so we are then forced to have n = 0 and we end up with

$$\{ r^0, r^{-1} \} \; P_0(\cos\theta) = \{ 1, r^{-1} \}$$

Inside the sphere the solution is the first, and outside it is the second. On the sphere constant.

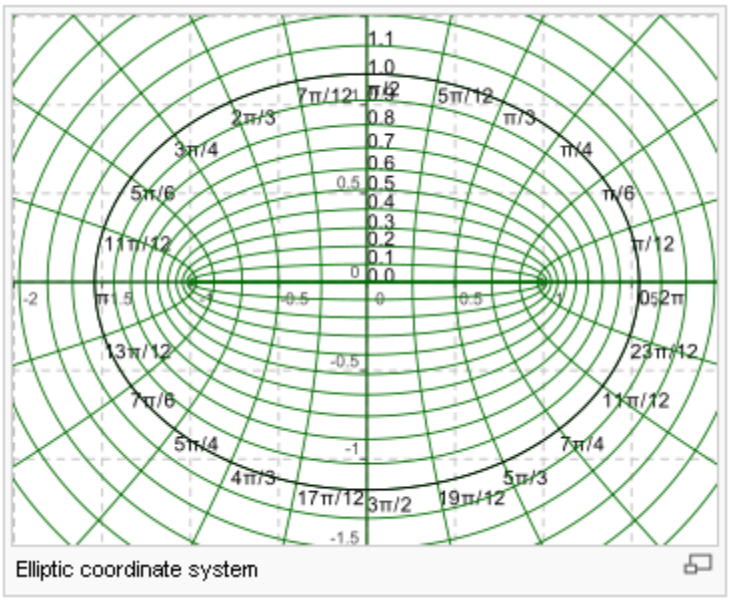
(2) So we want to somehow do this same thing in "some kind of ellipsoidal coordinates". In these coordinates, our ellipsoid will have "radius" $\tau = a$. The infinite ellipsoid might have $\tau = \infty$ and we will require that the potential vanish there. We will seek a separation of variables, we will find two separation constants m and n. As before, we will have m = integer associated with φ. As before, I think n will be forced as well to be integer. Then we will try to express our building block functions as I have done for the sphere, and somehow our answer will fall out. *[I later learned that the oblate spheroidal harmonics are also Legendre functions, sometimes of imaginary argument, and there is quantization just as presumed here in these conjecturing notes.]*

(3) It would be best to do this in 2D first to find the charge distribution on a metal ellipse! Start on the ground floor, not on the top floor. [Now σ = simple Kelvin formula]

2. Description of the prolate and oblate spheroidal coordinates. The 2D connection.

Physical Description of Coordinate Systems. There are two coordinate systems called **prolate** and **oblate spheroidal**. We are going to relate these to our usual x,y,z frame of reference. In both cases, z will be the symmetry axis so the ellipsoids will be symmetrical in azimuth φ around the z axis (the up axis). For some reason I don't yet know, they want to have this be two separate cases. In the prolate case, the ellipsoids are taller than they are fat, so they are egg-like, egg standing up. The ellipses are of course in the plane of a constant-φ slice. MM say these were called ovary ellipsoids at one time. Prolatus -- stretched out. Football is prolate on its symmetry axis. Stretched out from a sphere, that is to say. In the oblate case, the ellipsoids are wider than they are tall. They are squashed down, like a fat tomato . Oblatus means toward the sides I guess.

If you look at this picture where we have some orthogonal coordinates making ellipses and hyperbolas,



you see that the *ellipses* are all "oblate" relative to the vertical axis, and all "prolate" relative to the horizontal axis. You have your choice of symmetry axis. If you have the horizontal axis be the 3D symmetry axis, you *only* get prolate spheroids. If you have the vertical axis be the symmetry axis, you *only* get oblate ellipsoids. So **the reason for having two coordinate systems is** that you have to make this choice at some point if you want to parameterize using the above figure (which we do).

[Animation: take the above picture and pick one ellipse and squash it with both hands until it becomes round. This causes the foci to move to the center, and each full hyperbola becomes four rays out from the origin. Keep squashing and the foci then appear on the vertical axis and the hyperbolas are then rotated 90 degrees from their starting positions, so now we have the above picture rotated this amount.]

In this space above, the ellipses are curves of constant μ which ranges 0 to ∞, you see these labels on the positive vertical axis. The other coordinate in this ellipse plane (call is ν for the moment) ranges 0,2π and is "sort of" an azimuth around the perp to the plane of paper passing through the center, BUT it is really some kind of distorted azimuth. It really labels the hyperbola *quarters*. If you want to talk about hyperbola *halves*, then ν = 0,π is what you need. The metric tensor is going to be diagonal and this is an orthogonal coordinate system, which is displayed well in the figure. For our scaled picture, the foci of all ellipses are at ±1. You can see that if you go far away, the hyperbola quarters look like rays heading to the origin. If we were to put the foci at ±a and then let a→0, the ν coordinate would in fact become a regular azimuth, the quarter hyperbolas become rays, and all ellipses are circles. [ν = "cone angle" which in 2D would be a wedge angle which is the asymptote of a half hyperbola]

3. Oblate spheroidal coordinates as they appear in M&M

Since we are going to be crushing oblate spheroids into flat disks, we shall work with the oblate case at least for now. Our friends **M&M** have a very short section on page 182 where they call the coordinates by the names u, v, and φ. The three diagonal Q values are stated (which we can then use to find things like ∇^2 !!!). I have to quote my MM notes on this subject:

Now we pause to comment on an *immense confusion* M&M have caused. They use these symbols:

$(Q_{ii})^2$ [written also as Q^2_{ii}] to represent the quantity G_{ii}

but

Q_{ij} to represent the quantity G_{ij} $\quad i \neq j$

so MM give the Q_i^2 quantities, but it is the Q_i things that appear in the formulas. For our system here we have, from MM p 182,

$$Q_u^2 = Q_v^2 = a^2(\mathrm{sh}^2 u + \cos^2 v) \qquad Q_\varphi^2 = a^2\, \mathrm{ch}^2 u \sin^2 v$$

where ±a are the foci locations. He writes in fact

$$x = a\, \mathrm{ch}\, u \sin v \cos\varphi$$
$$y = a\, \mathrm{ch}\, u \sin v \sin\varphi$$
$$z = a\, \mathrm{sh}\, u \cos v$$

As we vary φ, z does not change. So varying φ at fixed z (z not too large!) describes a circle on the surface of a particular-u oblate spheroid. The symmetry axis is the z axis. MM refer to $x^2 + y^2 = r^2$, where r is the radius of this circle. In spherical coordinates we usually call this ρ.

Now browsing through my MM chap 4_5 notes, I see these nice items. First, in non-orthogonal coordinates the Laplacian is given by

$$\nabla^2\varphi = (1/\sqrt{g})\, \partial_i [\, g^{ij} \sqrt{g}\, \partial_j \varphi \,]$$

$g = \det(g_{ij})$, and in othogonals this is: $\quad g = \det(g_{ij}) = \Pi_i\, g_{ii} = Q_1^2 Q_2^2 Q_3^2$

In fact I think right now would be a good time to *reread* those MM notes because we are now all of a sudden very interested in a certain coordinate system! Those notes have been waiting for today.

4. Student Comment:

You perhaps see why this subject does not come up much in electrostatics texts for non-experts (ie, for students). Jackson has nothing on this in either green or current red book. You first should know all about general curvilinear coordinates and how and why the various tensor operators are expressed. Imagine doing all this while you are just learning potential theory for the first time, things like Green's Theorems and Green's Functions and distribution theory and on and on! Spherical and cylindrical coordinates are complicated enough for the newbie, each with its bevy of associated special functions. When I first saw "oblate spheroidal" in MM's list of systems, I had not the slightest interest, but now I am suddenly very interested.