Summary of Facts about Metal Objects PhL 11.29.09

Here we assemble already decoded material and elaborate just slightly. The "objects" are ellipsoids and spheroids, and their limits which are spheres, elliptical and round disks, and a prolate needle. We are talking only about "charged metal object" problems, not Green's function problems.

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Part I: Using Ellipsoidal Coordinates

1. First statement of the Potential of a charged metal ellipsoid.

This problem is attacked in ellipsoidal coordinates by MF, which attack is reviewed in my long doc of notes taken when reading MF. When the method of separation of variables is applied, we find that the general "solution term" for a Laplace problem in these coordinates has the form

$$[\, A_m^p \; E_m^p(\xi_1) \; + \; B_m^p \; F_m^p(\xi_1)]\, [\, A'^p_m \; E_m^p(\xi_2) \; + \; B'^p_m \; F_m^p(\xi_2)]\, [\, A''^p_m \; E_m^p(\xi_3) \; + \; B''^p_m \; F_m^p(\xi_3)]$$

where the E and F are first and second kind ellipsoidal harmonics, aka Lamé functions. The problem of the charged metal ellipsoid having focal distances a > b, label ξ = c, and potential V_0, in fact has this solution (at least outside the ellipsoidal surface ξ = c),

$$\psi = AE_0^0(\xi_3)E_0^0(\xi_2)F_0^0(\xi_1) = V_0\left[\mathrm{sn}^{-1}\left(\frac{a}{\xi_1},\frac{b}{a}\right)\Big/\mathrm{sn}^{-1}\left(\frac{a}{c},\frac{b}{a}\right)\right] \quad (10.3.91)$$

where $E_0^0(z) = 1$ so the solution is really just $\psi = A\,F_0^0(\xi_1)$ where

$$F_0^0(z) = \int_z^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a}\,\mathrm{sn}^{-1}\left(\frac{a}{z},\frac{b}{a}\right) \quad (10.3.90)$$

so we must then have $A/a = V_0/\ \mathrm{sn}^{-1}(a/c, b/a)$. The function is an "arc Jacobian sn" elliptic function. The above solution can be restated in terms of "my" standard form incomplete elliptic integral of the first kind function $F(\sin^{-1}x,k) = \mathrm{sn}^{-1}(x,k)$,

$$\mathrm{sn}^{-1}(x,k) = G(x,k) = F(\sin^{-1}x,k) = F(\sin^{-1}x \mid m) = F(\sin^{-1}x \setminus \alpha) \qquad k = \sin\alpha \qquad m = k^2$$

as follows:

$$\psi(\xi_1,\xi_2,\xi_3) = \psi(\xi_1) = V_0\,F[\sin^{-1}(a/\xi_1),b/a]\ /\ F[\sin^{-1}(a/c),b/a]$$

$C = \sqrt{c^2-a^2}$ smallest semi $\qquad$ $B = \sqrt{c^2-b^2}$ middle semi $\qquad$ A = "c" largest semi

The charged surface has ξ_1 = A which is, in fact, the largest semi-major axis of the ellipsoid. I show formulas for the other two semi-major axes. Obviously, everywhere ON the charged ellipsoidal surface which has ξ_1 = A we have $\psi = V_0$.

What do we know about a larger confocal "math" ellipsoid . It will have foci a,b and here are the semi major axes: A' > B' > C'

$C' = \sqrt{\xi_1^2-a^2}$ $\qquad$ $B' = \sqrt{\xi_1^2-b^2}$ $\qquad$ $A' = \xi_1$

The equation of the ellipsoid which is our math surface is this

$$x^2/(\xi_1^2-a^2) + y^2/(\xi_1^2-b^2) + z^2/(\xi_1^2) = 1 \qquad 0 < b < a < \xi_1$$
$$x^2/(C'^2) + y^2/(B')^2 + z^2/(A'^2) = 1 \qquad 0 < b < a < \xi_1$$

so that z is the direction of the longest semi axis, and x is the direction of the shortest. This of course applies also to the charged metal ellipsoid where we set ξ_1 = A and remove the primes.

2. Restatement of the Solution in Two Ways

We can rewrite the above solution in the following two different forms :

$\psi(\xi_1) = V_0\, F[\sin^{-1}(a/\xi_1), b/a] \;/\; F[\sin^{-1}(a/A), b/a]$ // restatement of MF above

$\psi(\xi_1) = V_0\, F[\sin^{-1}(b/\xi_1), a/b] \;/\; F[\sin^{-1}(b/A), a/b]$ // this line is "the Kelvin solution"

semis: $\xi_1 = A \geq B \geq C$ focals: $a^2 \equiv A^2 - C^2$ and $b^2 \equiv A^2 - B^2$ $a > b$

and, as above, the largest semi-major ellipsoid axis of the charged ellipsoid is $\xi_1 = A$ and the other two semis are B and C. The two ways shown correspond to B ↔ C causing a ↔ b. The equivalence of these two "ways" corresponds to a certain transformation rule of the F function, but also seems fairly logical intuitively.

The function shown is "just some function" that drops off in some way. If we set A,B,C = 5,4,3 then here is what ψ looks like, says Maple, where $V_0 = 1$:

```
> restart;
> F := EllipticF(arcsin(a/xi1),b/a);
```

$$F := \text{EllipticF}\left(\arcsin\left(\frac{a}{\xi 1}\right), \frac{b}{a}\right)$$

```
> Fref := EllipticF(arcsin(a/A),b/a);
```

$$Fref := \text{EllipticF}\left(\arcsin\left(\frac{a}{A}\right), \frac{b}{a}\right)$$

```
> b := sqrt(A^2-C^2);
```

$$b := \sqrt{A^2 - C^2}$$

```
> a := sqrt(A^2-B^2);
```

$$a := \sqrt{A^2 - B^2}$$

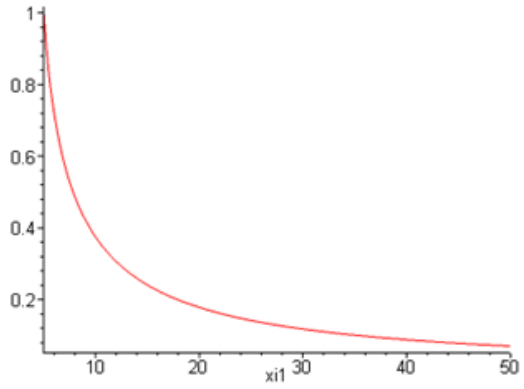
```
> A := 5; B := 4; C := 3;
```

$$A := 5$$
$$B := 4$$
$$C := 3$$

```
> psi := F/Fref;
```

$$\psi := \frac{\text{EllipticF}\left(\arcsin\left(\frac{\sqrt{9}}{\xi 1}\right), \frac{1}{9}\sqrt{16}\sqrt{9}\right)}{\text{EllipticF}\left(\arcsin\left(\frac{1}{5}\sqrt{9}\right), \frac{1}{9}\sqrt{16}\sqrt{9}\right)}$$

```
> plot(psi, xi1=A..10*A);
```



The main point really is that for each ξ_1 on this curve, we have a ellipsoidal surface in space which is confocal with the charged metal surface (not similar to it) and on which surface the potential everywhere has the constant value $\psi(\xi_1)$. Recall that in ellipsoidal coordinates, the ξ_1 are the labels of a family of confocal ellipsoids. That is why we make the claim just made.

3. What does the potential look like in Cartesian coordinates?

This seems a very natural question to ask. The first thing we need to know is how to relate ξ_1 to Cartesian coordinates. This involves the messy "inversion formula" for ξ_1. The result is so complex, that it is hard to even write it out except as a sort of "program":

$h = x^2+y^2+z^2 +(a^2+b^2)$ L^2

$f = b^2x^2+ a^2y^2+ (a^2+b^2)z^2 +a^2\, b^2$ L^4

$k = a^2\, b^2z^2$ L^6

$R = (-9hf +27k + 2h^3)/54$

$Q = (3f - h^2)/9$

$$D = Q^3 + R^2$$
$$S = [\, R + \sqrt{D}\,]^{1/3}$$
$$T = [\, R - \sqrt{D}\,]^{1/3}$$
$$\xi_1^2 = h/3 + (S+T) \qquad \Rightarrow \quad \xi_1(x,y,z) = \sqrt{h/3 + (S+T)}$$

Given any point (x,y,z), we can compute ξ_1 via this program, and thus we can compute $\psi(\xi_1)$. We need to point out that the meaning of the three Cartesian coordinates is defined by our ellipsoid equation,

$$x^2/(\xi_1^2 - a^2) + y^2/(\xi_1^2 - b^2) + z^2/(\xi_1^2) = 1 \qquad 0 < b < a < \xi_1$$

$$x^2/C^2 \quad + y^2/B^2 \quad + z^2/A^2 = 1$$

which says that the z axis goes with the longest dimension of the ellipsoid, and the x axis with the shortest. The program above is symmetrical in a ↔ b and hence B ↔ C.

A related question is this: why would you want to know ψ in Cartesian coordinates? We can imagine a family of confocal ellipsoids surrounding our charged metal one, equations as above. We could imagine a plot of 10 of these ellipsoids and for each we can compute the constant value of ψ using our little program. Basically we are just saying that one can plot a set of confocal ellipsoids in the usual sense of "ellipsoidal coordinates", and this gives an excellent "picture" of what the potential looks like. If you wanted to know what the electric field lines looked like, you could draw a family of either of the other two surfaces and superpose your 10 ellipsoids in wire frame with the 10 hyperboloids of the chosen other coordinate. Or you can just imagine those electric field lines as running roughly perpendicularly between adjacent ellipsoids.

There may be some way to write the complicated function $F[\sin^{-1}(a/\xi_1), b/a]$ in Cartesian coordinates that is relatively simple, but I don't know if such a simple expression exists or not.

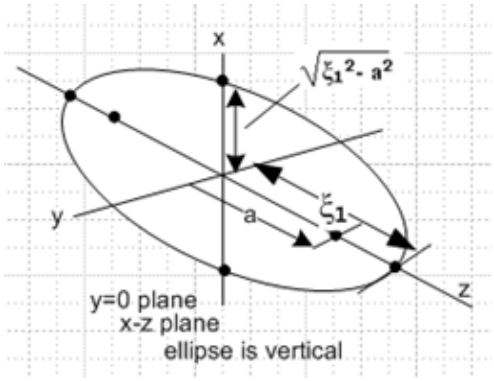
4. The Potential of a Charged Metal *elliptical* disk.

This is one of the payoffs of learning the whole elliptical world. We just want the limit as the smallest semi-major axis C goes to 0. First, copy down from above

$\psi(\xi_1) = V_0\, F[\sin^{-1}(a/\xi_1), b/a] \;/\; F[\sin^{-1}(a/A), b/a]$ // restatement of MF above
$\psi(\xi_1) = V_0\, F[\sin^{-1}(b/\xi_1), a/b] \;/\; F[\sin^{-1}(b/A), a/b]$ // this line is "the Kelvin solution"

semis: $\xi_1 = A \geq B \geq C$ foci: $a^2 \equiv A^2 - C^2$ and $b^2 \equiv A^2 - B^2$ $a > b$

In the limit C = 0 we get a = A. Interpretation: as we crush down a vertical cross section ellipse (C→0), it closes down and sucks out its two focal points (a→A) and becomes a thin loop of no height. We then end up with $\xi_1 = A = a$, as this picture suggests, if you were to crush it down.



The resulting elliptical disk will lie in the y-z plane with long axis A along z and short B along y.

Our formulas above then become

$\psi(\xi_1) = V_0\, F[\sin^{-1}(A/\xi_1), b/A] \;/\; F[\sin^{-1}(1), b/A]$ // restatement of MF above
$\psi(\xi_1) = V_0\, F[\sin^{-1}(b/\xi_1), A/b] \;/\; F[\sin^{-1}(b/A), A/b]$ // this line is "the Kelvin solution"

semis: $\xi_1 = A \geq B \geq C = 0$ foci: $a^2 \equiv A^2$ and $b^2 \equiv A^2 - B^2$ $a > b$

where A > B are now the semi's of our flat elliptical disk. The first formula looks to be the simplest in this case since $\sin^{-1}(1) = \pi/2$ and we can say

$sn^{-1}(x,k) = G(x,k) = F(\sin^{-1}x,k) = F(\sin^{-1}x \mid m) = F(\sin^{-1}x \setminus \alpha)$ $k = \sin\alpha$ $m = k^2$

$\Rightarrow F(\pi/2, k) = F(\pi/2 \mid k^2) = F(\pi/2 \mid m) = K(k)$ // not using AS argument!

so $F(\pi/2, b/A) = K(b/a)$ // see MF p 1308

so we can then write the first solution above as

$\psi(\xi_1) = V_0\, F[\sin^{-1}(A/\xi_1), b/A] \;/K(b/A)$ // $= V_0$ f(coord)

Aside: $\lim_{faraway} f(\text{coord}) = K_2 / r = (A/r)/K(b/A) \Rightarrow C = A/K(b/A)$ agrees MF p 1308.

But this hardly seems much of a simplification. So we can just take our result to be

$\psi(\xi_1) = V_0\, F[\sin^{-1}(A/\xi_1), b/A] \;/\; F[\sin^{-1}(A/A), b/A]$ $b^2 \equiv A^2 - B^2$

This is the potential of an elliptical disk. On the disk we have $\xi_1 = A$. The disk has semi major A along z, and semi minor B along y, and focal distance b. Really, this result is just as complicated as the potential of a charged metal ellipsoid. On the disk we have $\psi = V_0$. What else can be said?

Well, we might ask if ξ_1 is a simpler function of x,y,z in this case. All we get to do is replace a with A, and everything stays as complicated as it was before, so no gain here.

Note added: suppose we take the limit $b \to A$ from below, we should get a wire. We then have

$\psi(\xi_1) = V_0\, F[\sin^{-1}(A/\xi_1), 1] \;/\; F[\sin^{-1}(A/A),1]$ $\qquad b^2 \equiv A^2 - B^2$

$sn^{-1}(x,1) = G(x,1) = F(\sin^{-1}x,1) = F(\sin^{-1}x \mid 1) = F(\sin^{-1}x \setminus \pi/2)$ $\qquad k = \sin\alpha \qquad m = k^2$

$F(\sin^{-1}x \setminus \pi/2) = \ln(\sec\varphi + \tan\varphi) \quad \varphi = \sin^{-1}x$.

In the denominator, x = 1, $\varphi = \pi/2$, the denominator diverges. Something seems wrong. The answer I now know is that it really is singular, you cannot take this limit and get a finite result. The 3D wire is not like the 2D wire in this regard. See below.

5. The Potential of a Charged Metal *round* disk.

We now set A = B so that b = 0, and then our result above becomes

$\psi(\xi_1) = V_0\, F[\sin^{-1}(A/\xi_1), 0/A] \;/\; F[\sin^{-1}(A/A), 0/A]$ $\qquad b^2 \equiv A^2 - A^2 = 0$

$\psi(\xi_1) = V_0\, F[\sin^{-1}(A/\xi_1), 0] \;/\; F[\sin^{-1}(A/A), 0]$

Now we repeat our reference line just to avoid mistakes

$sn^{-1}(x,k) = G(x,k) = F(\sin^{-1}x,k) = F(\sin^{-1}x \mid m) = F(\sin^{-1}x \setminus \alpha)$ $\qquad k = \sin\alpha \qquad m = k^2$

and we set k = 0 to get

$sn^{-1}(x,0) = G(x,0) = F(\sin^{-1}x,0) = F(\sin^{-1}x \mid 0) = F(\sin^{-1}x \setminus 0)$ $\qquad k = \sin\alpha \qquad m = k^2$

$= \sin^{-1}x$ $\qquad$ // AS p 594 or just a known limit of the Jacobi sn function

So we then get

$\psi(\xi_1) = V_0 \sin^{-1}(A/\xi_1) \;/\; \sin^{-1}(A/A) \;= V_0 \sin^{-1}(A/\xi_1) \;/\; [\pi/2]$

$= (2/\pi)V_o \sin^{-1}(A/\xi_1)$ $\qquad\qquad$ // $C = (2A/\pi)$ read off

which is the potential of a charged metal disk of radius A. On this disk, we have $\xi_1 = A$ so $\psi = V_0$. This result is considerably simpler than the result for the elliptical disk! As with the elliptical disk, this round disk lies in the y-z plane.

What about our expression for general ξ_1 ? Our program now says (we set b = 0 in the program) (Also, set $y^2 + z^2 = \rho^2$, which is a cylindrical coordinate where the z-axis is the x axis.)

$h = x^2+\rho^2+(A^2)$	L^2
$f = A^2\rho^2$	L^4
$k = 0$	L^6
$R = h(-9f+2h^2)/54$	

$$Q = (3f - h^2)/9$$
$$D = Q^3 + R^2$$
$$S = [\, R + \sqrt{D}\,]^{1/3}$$
$$T = [\, R - \sqrt{D}\,]^{1/3}$$
$$\xi_1^2 = h/3 + (S+T) \qquad \Rightarrow \quad \xi_1(x,y,z) = \sqrt{h/3 + (S+T)}$$

It might be simple, but I would have to run it through Maple to see. A preliminary Maple shot is not promising. I am able to get Maple to say this, but R is a mess

$$108*D/\rho^4 = -A^4 [\, x^2 + (\rho-A)^2] [\, x^2 + (\rho+A)^2]$$

Jackson does give a Cartesian result on page 92, but it is an integral over J_0 Bessel functions 3.177. But he says this can be restated as 3.178. The tells us that we must have

$$\psi(\xi_1) = (2/\pi)V_o \sin^{-1}(A/\xi_1) = \text{const} * \sin^{-1}(2A/ [\, \sqrt{(\rho-A)^2+ x^2} + \sqrt{(\rho+A)^2+ x^2}\,) \,])$$

where $\rho = \sqrt{y^2+z^2}$. So this certainly *suggest* that

$$\xi_1 = (1/2) [\, \sqrt{(\rho-A)^2+ x^2} + \sqrt{(\rho+A)^2+ x^2}\,) \,] \qquad \rho = \sqrt{y^2+z^2}$$

which has factors like those in my D expression above. But I cannot make Maple show the correspondence, it is a huge mess. But I suspect somehow it works out. Leave it at that. Maybe in the oblate stuff below I can try again.

7. The prolate limit of the general ellipsoidal potential.

First, we copy down the general solution

$\psi(\xi_1) = V_0 F[\sin^{-1}(a/\xi_1), b/a] \,/\, F[\sin^{-1}(a/A), b/a]$ // restatement of MF above
$\psi(\xi_1) = V_0 F[\sin^{-1}(b/\xi_1), a/b] \,/\, F[\sin^{-1}(b/A), a/b]$ // this line is "the Kelvin solution"

semis: $\xi_1 = A \geq B \geq C$ focals: $a^2 \equiv A^2 - C^2$ and $b^2 \equiv A^2 - B^2$ $a > b$

The prolate limit means $A \geq B = C$ which means a = b focal distances. The two forms above are then identical and we have

$$\psi(\xi_1) = V_0 F[\sin^{-1}(a/\xi_1),1] \,/\, F[\sin^{-1}(a/A), 1]$$

Our reference line with k = k then k = 1 then AS page 594 lookup

$\text{sn}^{-1}(x,k) = G(x,k) = F(\sin^{-1}x,k) = F(\sin^{-1}x \mid m) = F(\sin^{-1}x \setminus \alpha)$ $k = \sin\alpha$ $m = k^2$
$\Rightarrow \text{sn}^{-1}(x,1) = G(x,1) = F(\sin^{-1}x,1) = F(\sin^{-1}x \mid 1) = F(\sin^{-1}x \setminus \pi/2)$ $k = \sin\alpha$ $m = k^2$
$\Rightarrow F(\sin^{-1}x,1) = F(\sin^{-1}x \setminus \pi/2) = \ln(\sec\varphi + \tan\varphi)$ $\varphi = \sin^{-1}x$ $x = a/\xi_1$

Then we have

$$1/\cos\varphi + \tan\varphi = \xi_1/\sqrt{\xi_1^2-a^2} + a/\sqrt{\xi_1^2-a^2} = (\xi_1 + a)/\sqrt{\xi_1^2-a^2}$$

so result is then

Prolate Limit Potential:

$$\psi(\xi_1) = V_0 \ln[(\xi_1 + a)/\sqrt{\xi_1^2-a^2}]/\ln[(A + a)/\sqrt{A^2-a^2}] \quad \text{// prolate limit}$$

where a = b = focal distances, and A = semimajor of the metal. I think this is right, but have not found confirmation anywhere. Most would use prolate spherical coordinates I think.

Inversion Formulas. The inversion program formulas are still a mess in this limit. If we look instead at the forward coordinate situation, we find from our raw ellipsoidal doc notes,

$$x = (1/a)\sqrt{(\xi_1^2 - a^2)(a^2 - \xi_2^2)(a^2 - \xi_3^2)/(a^2 - b^2)}$$
$$y = (1/b)\sqrt{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(b^2 - \xi_3^2)/(a^2 - b^2)}$$
$$z = \xi_1\xi_2\xi_3/ab$$

Wrong Way: If we take the limit $a \to b$ (from above of course) we know ξ_2 is bracketed between a and b. In the x form above, we might set $\xi_2 = b$ first, then take limit $a \to b$ to get cancellation of two factors. In the y equation, let's instead set $\xi_2 = a$ and take the limit $b \to a$ and we get the same cancellation. We end up with

$$x = (1/a)\sqrt{(\xi_1^2 - a^2)(a^2 - \xi_3^2)} \qquad \xi_2 = a \qquad \text{WRONG!}$$
$$y = (1/a)\sqrt{(\xi_1^2 - a^2)(a^2 - \xi_3^2)}$$
$$z = \xi_1\xi_3/a \qquad \Rightarrow \quad \xi_3 = za/\xi_1$$

Doing the limit this way gives x = y, and we find doing algebra that as $\xi_1 \to \infty$, $r^2 = \xi_1^2[2 - z^2/\xi_1^2]$ which is wrong. This is telling us that we were not careful enough doing the limit!

Right Way: Take the limit $a \to b$ of both terms, and set $\xi_2 = (a+b)/2$ in both cases. You then get a 0/0 situation in each case, and you need to do l'Hôpital's rule in each case. We then find that (a = variable)

x: $(a^2 - [(a+b)/2]^2)/(a^2 - b^2) = 0/0 = [2a - (a+b)]/2a = 0/2a = 0$

y: $([(a+b)/2]^2 - b^2)/(a^2 - b^2) = 0/0 = [a+b]/2a = 2a/2a = 1$

Therefore the correct limiting results are

$$x = 0 \qquad \xi_2 = a$$
$$y = (1/a)\sqrt{(\xi_1^2 - a^2)(a^2 - \xi_3^2)}$$
$$z = \xi_1\xi_3/a \qquad \Rightarrow \quad \xi_3 = za/\xi_1$$

Our limiting process has taken us to a particular point (x,y) with azimuth 90 degrees. That seems reasonable as I think of hyperboloids approaching their frames. BUT, let's run a sanity check. What happens as ξ_1 gets very large? I expect to get $\xi_1 \rightarrow r$. Let's see:

$$x^2 + y^2 + z^2 = \quad 0 + (\xi_1/a)^2 \ (a^2 - \xi_3^2) \ + \xi_1^2\xi_3^2 / a^2$$

$$= (\xi_1/a)^2 [(a^2 - \xi_3^2) + \xi_3^2] = (\xi_1/a)^2 [a^2] \quad = \xi_1^2 \qquad \text{// checks out}$$

Now let's try to solve the above equations for ξ_1. Set

$$\rho^2 = x^2 + y^2 = (1/a)^2(\xi_1^2 - a^2)(a^2 - \xi_3^2)$$
$$z = \xi_1\xi_3 / a \qquad \Rightarrow \qquad \xi_3 = za/\xi_1$$

$$\rho^2 = (1/a)^2(\xi_1^2 - a^2)(a^2 - a^2z^2/\xi_1^2)$$

$$\rho^2 = (\xi_1^2 - a^2)(1 - z^2/\xi_1^2)$$

$$\rho^2 \xi_1^2 = (\xi_1^2 - a^2)(\xi_1^2 - z^2)$$

This is a simple quadratic equation in ξ_1^2

$$\xi_1^4 - (a^2 + z^2 + \rho^2)\xi_1^2 + a^2z^2 = 0$$

$$\xi_1^2 = (1/2) [\ (a^2 + z^2 + \rho^2) \pm \sqrt{(a^2 + z^2 + \rho^2)^2 - 4a^2z^2} \ ] \qquad \xi_3 = za/\xi_1 \qquad (*)$$

As a check, suppose we set z = 0 and go out to large ρ.

$$\xi_1^2 = (1/2) [\ (\rho^2) \pm (\rho^2)] \qquad = \rho^2 \qquad \text{taking the + sign}$$

Probably the minus sign above tells you ξ_3^2 (just my guess), and the + sign gives you ξ_1^2. As a check on this guess, consider

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = (1/2) [\ (a^2 + z^2 + \rho^2) + \sqrt{(a^2 + z^2 + \rho^2)^2 - 4a^2z^2} \ ]$$
$$+ (1/2) [\ (a^2 + z^2 + \rho^2) - \sqrt{(a^2 + z^2 + \rho^2)^2 - 4a^2z^2} \ ] + a^2$$

$$= (a^2 + z^2 + \rho^2) + a^2$$

$$= x^2+y^2+z^2 +a^2 +a^2 \quad = h, \text{ which is correct!}$$

<u>Conclusion</u>: in the prolate limit of ellipsoidal coordinates we have a = b and the inversion formulas are

$$\xi_1^2 = (1/2) [\ (a^2 + r^2) + \sqrt{(a^2 + r^2)^2 - 4a^2z^2} \ ]$$
$$\xi_3^2 = (1/2) [\ (a^2 + r^2) - \sqrt{(a^2 + r^2)^2 - 4a^2z^2} \ ]$$

$\xi_2^2 = a^2$

The forward formulas are [where $\rho^2 = x^2 + y^2$]

$\rho = (1/a)\sqrt{(\xi_1^2 - a^2)(a^2 - \xi_3^2)}$
$z = \xi_1\xi_3 / a \qquad\qquad => \qquad \xi_3 = za/\xi_1$

8. The sphere limit of the prolate limit of the general ellipsoidal potential.

Now we want to start with the prolate results and take the limit $a \to 0$ which turns our football into a sphere by crushing the long end. Now since ξ_2 and ξ_3 must be less than a, we have $\xi_2 = \xi_3 = 0$ and the above formula tells us that $\xi_1^2 = r^2$ which is the radius of some sphere. For the charged metal sphere we have $r^2 = A^2$. The forward formulas are less obvious since we have 0/0 situations. So

sphere limit:

$a = b = 0 \quad \xi_2 = \xi_3 = 0 \quad \xi_1 = r$ // general math sphere
$a = b = 0 \quad \xi_2 = \xi_3 = 0 \quad \xi_1 = A$ // charged metal sphere of radius A.

Now consider the prolate potential formula

$$\psi(\xi_1) = V_0 \ \ln[(\xi_1 + a) / \sqrt{\xi_1^2 - a^2}\,] / \ln[(A + a) / \sqrt{A^2 - a^2}\,] \qquad \text{// prolate limit}$$

If on scratch we take the small-a limit of the top log, we can ignore the a in the denom and we get

$$\ln[(\xi_1 + a) / \sqrt{\xi_1^2 - a^2}\,] \approx \ln[(\xi_1 + a) / \xi_1] = \ln(1 + a/\xi_1) = (a/\xi_1)$$

Then the result is

$$\psi(\xi_1) = V_0 (A/\xi_1) = V_0 (A/r)$$

which is of course the correct.

9. The needle limit of the prolate limit of the general ellipsoidal potential.

This just occurred to me this moment, but I think Jim mentioned it. Here we want to take $a \to A$ so the prolate becomes a thin rotated ellipse = "a wire segment". Start with the prolate form

$$\psi(\xi_1) = V_0 \ \ln[(\xi_1 + a) / \sqrt{\xi_1^2 - a^2}\,] / \ln[(A + a) / \sqrt{A^2 - a^2}\,] \qquad \text{// prolate limit}$$

$\xi_1^2 = (1/2)\,[\ (a^2 + r^2) + \sqrt{(a^2 + r^2)^2 - 4a^2z^2}\,]$
$\xi_3^2 = (1/2)\,[\ (a^2 + r^2) - \sqrt{(a^2 + r^2)^2 - 4a^2z^2}\,]$
$\xi_2^2 = a^2$

However, in this limit, the numerator is finite but the denominator diverges as ln(∞). Now what? If we go "to the metal wire" we want x = y = 0 so $r^2 = z^2$, then we get

$$\xi_1^2 = (1/2)\left[(A^2 + z^2) + \sqrt{(A^2 + z^2)^2 - 4A^2z^2} \right] = (1/2)\left[(A^2 + z^2) + (A^2 - z^2) \right] = A^2$$
$$\xi_3^2 = (1/2)\left[(A^2 + z^2) - \sqrt{(A^2 + z^2)^2 - 4A^2z^2} \right] = 0$$
$$\xi_2^2 = A^2$$

so at least we get ξ_1 = constant on the metal wire segment which runs z = (-A,A). I know that the potential of a 2D wire is a well-defined problem with a finite potential, but something strange is happening in 3D.

We have a similar problem if we start with the flat elliptical disk result which is this:

$$\psi(\xi_1) = V_0 \, F[\sin^{-1}(A/\xi_1), b/A] \; / \; K(b/A)$$

Here the problem is that as b→A, we get K(1) = ∞.

For a point charge, we write ψ(r) = Q/r and it is true that ψ(0) = ∞. That is to say, if we start with a metal ball and shrink it down to a point, ψ = ∞ on that metal point. But anywhere away from the point we get some finite number. But for our 3D wire limit above, even away from the wire we get ∞ becomes the denominator is infinite. What is going on here?

MF index shows nothing about wires. Nothing in Smythe.

I see on the web that this "problem" has attracted the interest of Jackson himself, in some 2000 and 2001 papers! So there must indeed be something odd going on here. I have his paper. Well, it is the 1995 paper of Griffiths and Li that directly addresses my question! I have read through the two Jackson and the GL papers. They don't state my potential above, however.

So let's go ahead and take the limit in the first term above. But we did that above, so our result is

$$\psi(\xi_1) = V_0 \ln\left[(\xi_1 + A) / \sqrt{\xi_1^2 - A^2}\right] / K(b/A)$$

The capacitance limit is then

$$\psi(\xi_1) = V_0 \, A/r \; / \; K \qquad \Rightarrow \qquad C = A/K(b/A)$$

As we let this needle get thinner, K(1) = ∞ and C → 0. As you make it thinner, it can hold less and less charge for the same potential. This applies to the wire modeled as prolate in shape. Note that the capacitance of a sphere is R, so as you shrink it, C → 0 as well. This all has to do with the dimensions of space.

By the way, the general ellipsoid charge density is this

$$\sigma = \text{constant} / \sqrt{x^2/A^4 + y^2/B^4 + z^2/C^4}$$

so on the needle is must be this, since roughly B = C = 0

$$\sigma = \text{constant} \, |x|$$

which is a very simple plot, peaking at the two ends. The linear charged density along the needle however is a constant, it turns out.

Part II: Using Oblate Spheroidal Coordinates

1. Potential of a charged metal oblate spheroid.

I have a separate doc on these coordinates. The coordinates here are ξ,η,φ where this ξ is unrelated to the ξ_i of the ellipsoidals in Part I. After we write out the Laplacian in these coordinates and separate variables, we get a general solution term of this form

$$[\, A_n^m P_n^m(\eta) + B_n^m Q_n^m(\eta) \,]\, [\, C_n^m P_n^m(i\xi) + D_n^m Q_n^m(i\xi) \,]\, e^{im\varphi}$$

The forward equations are these (ignore the right side)

$$x = a\sqrt{\xi^2+1}\,\sqrt{1-\eta^2}\cos\varphi \qquad = a\sqrt{\xi^2+1}\,\sin\nu\cos\varphi \qquad \eta = \cos\nu$$
$$y = a\sqrt{\xi^2+1}\,\sqrt{1-\eta^2}\sin\varphi \qquad = a\sqrt{\xi^2+1}\,\sin\nu\sin\varphi$$
$$z = a\,\xi\,\eta \qquad = a\,\xi\cos\nu$$

Here are the full inverse formulas for our MF oblate spheroidal coordinates

$$\xi^2 = \frac{1}{2a^2}\left\{ (r^2-a^2) + \mathrm{sign}(r^2-a^2)\sqrt{(r^2-a^2)^2 + 4a^2z^2}\right\} \qquad a = \text{focus}$$
$$-\eta^2 = \frac{1}{2a^2}\left\{ (r^2-a^2) - \mathrm{sign}(r^2-a^2)\sqrt{(r^2-a^2)^2 + 4a^2z^2}\right\} \qquad r^2 = x^2+y^2+z^2$$
$$\varphi = \tan^{-1}(y/x)$$

For the charged metal spheroid we get this very simple result, which fits into the above general form,

$$V(\xi,\eta,\varphi) = V(\xi) = V_0\, Q_0(i\xi)/Q_0(i\xi_0)$$

where ξ_0 is the label of the actual charged metal spheroid.

Technical note on $Q_0(z)$:
This function in Bateman and I think everywhere has a branch cut from +1 off to the left. The actual analytic function is this

$$Q_0(z) = (1/2)\ln[\,(z+1)/(z-1)] \qquad \text{Bateman p 152 vol 2}$$

If you are interested in this function on the top side of the cut in the range (-1,1), you have to do this:

$$z-1 = |z-1|e^{\pm i\pi}$$

But $x + |z-1| = 1$ so $|z-1| = (1-x)$. Then according to Bateman page 143. you define $Q_0(x)$ as the *average* of above and below, so

$$Q_0(x) = (1/2)\,(1/2)\ \{ \ln[\,(z+1)/((1-x)e^{+i\pi})\,] + \ln\,[\,(z+1)/((1-x)e^{-i\pi})\,]$$

$$= (1/2)\,(1/2)\ \ln\{\ (z+1)/((1-x)e^{+i\pi} * (z+1)/((1-x)e^{-i\pi})\ \}$$

$$= (1/2)\,(1/2)\ \ln\{\ (z+1)^2/(1-x)^2\ \}$$

$$= (1/2)\ \ln\{\ (z+1)^1/(1-x)^1\ \}$$

$$= (1/2)\ \ln\{\ (x+1)/(1-x)\ \} \qquad = (1/2)\ln[(1+x)/(1-x)]$$

Bateman uses a script $Q_0(z)$ for the complex function, and a non-script $Q_0(x)$ for the function defined on the range (-1,1) in this way. So this explains the "anomaly" between the two forms of $Q_0(\text{arg})$ that you see in various books such as AS page 333. Clearly we want the complex form, so we have

$$Q_0(iy) = (1/2)\ln[\,(iy+1)/(iy-1)]$$

If you "do the math" you find that

$$Q_0(iy) = \tan^{-1}(y) - \pi/2 \ = \ -\tan^{-1}(1/y).$$

Therefore, our solution is this

$$\psi(\xi) = V_0\, Q_0(i\xi)/Q_0(i\xi_0) \ = V_0\, \tan^{-1}(1/\xi) \,/\, \tan^{-1}(1/\xi_0)$$

and this then is the potential of a charged oblate spheroid labeled by ξ_0. And we have an expression for ξ in Cartesian coordinates:

$$\xi^2 = \ 1/(2a^2)\,[\,(r^2 - a^2) + \text{sign}(r^2-a^2)\,\sqrt{(r^2 - a^2)^2 + 4a^2z^2}\ ] \qquad a = \text{focus}$$

The symmetry axes are x and y, while the oblate axis is z.

2. Potential of a charged metal *round disk* as oblate spheroid limit.

We plug z=0 into the above to find that $\xi_0 = 0$ on the surface of the disk where of course $\text{sign}(r^2 - a^2) = -1$. So just reminding ourselves that the disk has label $\xi = \xi_0 = 0$. The angle whose tangent is $+\infty$ is $\pi/2$, so we get [the focus a now equals the disk radius called A above]

$$\psi(\xi) = V_0\, Q_0(i\xi)/Q_0(i0) \ = V_0\, \tan^{-1}(1/\xi) \,/[\,\pi/2] = (2/\pi)V_0\, \tan^{-1}(1/\xi)$$

where

$\xi^2 = \frac{1}{2A^2}\left[(r^2 - A^2) + \text{sign}(r^2 - A^2)\sqrt{(r^2 - A^2)^2 + 4A^2z^2} \right]$ a = focus

Notice that the sign is -1 inside a sphere whose equator is the disk, and sign = +1 outside.

We get solid confirmation on our ψ result above from MF:

The potential of a disk of radius a ($\xi = 0$) held at potential V_0 with respect to infinity is, thus,

$$\psi = \left(\frac{V_0}{Q_0(0)}\right) Q_0(i\xi) = \left(\frac{V_0}{\ln(-1)}\right) \ln\left(\frac{i\xi + 1}{i\xi - 1}\right)$$
$$= \left(\frac{iV_0}{\pi}\frac{2}{i}\right) \tan^{-1}\left(\frac{1}{\xi}\right) = \left(\frac{2V_0}{\pi}\right) \tan^{-1}\left(\frac{1}{\xi}\right) \quad (10.3.56)$$

when ξ is very large, it becomes approximately equal to r/a, where r is the radial distance from the center of the disk. Therefore the asymp-

3. Reconcile Jackson's disk result with the oblate disk result

Notice that both the ellipsoidal result and the Jackson result have $\sin^{-1}$ functions, whereas the oblate result has a $\tan^{-1}$ function. I am unsure of Jackson's overall constant at the moment. I would like to be able to show this:

$\tan^{-1}(1/\xi)$ with

$\xi^2 = (1/2A^2)\left\{ (r^2 - A^2) + \sqrt{(A^2 - r^2)^2 + 4A^2z^2} \right\}$ a = focus

is same as

$\sin^{-1}(A/\xi_1)$ with Jackson

$\xi_1 = (1/2)\left[\sqrt{(\rho - A)^2 + z^2} + \sqrt{(\rho + A)^2 + z^2} \right]$ $\rho^2 = x^2 + y^2$

Before going another step, let's check x = y = z = 0. We then have

$\xi^2 = 1/(2A^2)[-A^2 + |A^2|)] = 0$ and $\tan^{-1}(\infty) = \pi/2$

$\xi_1 = (1/2)[2A] = A$, and $\sin^{-1}(A/A) = \pi/2$ so both agree at this point.

Another check: just set z = 0 but go near the origin. Then we have, for small r

$\xi^2 = (1/2A^2)\left\{ (r^2 - A^2) + (|r^2 - A^2| \right\} = 0$ and $\tan^{-1}(\infty) = \pi/2$

$\xi_1 = (1/2)[(A - \rho) + (A + \rho)] = A$ and $\sin^{-1}(A/A) = \pi/2$ so both agree again.

But now reconsider z = 0 but go far from the origin. Then

$\xi^2 = (1/2A^2)\left\{ (r^2 - A^2) + (|r^2 - A^2| \right\} = (\rho^2 - A^2)/A^2$ and $\tan^{-1}[A^2/(\rho^2 - A^2]$

$\xi_1 = (1/2)\,[\,(\rho - A) + (A + \rho)\,] = \rho$ and $\sin^{-1}(A/\rho)$ and once again they agree!

So the comparison passes three sanity checks, so let's continue. If we rewrite Jackson as a $\tan^{-1}$ it becomes

$$\tan^{-1}\left(A/\sqrt{\xi_1^2 - A^2}\right)$$

Then I need to show that

$$\left[A/\sqrt{\xi_1^2 - A^2}\right]^2 = 1/\xi^2$$

or

$$2A^2\xi^2 = (2\xi_1^2 - 2A^2)$$

Then we have from above

$$(r^2 - A^2) + \sqrt{|r^2 - A^2|^2 + 4A^2z^2} = -2A^2 + (1/2)\left[\sqrt{(\rho - A)^2 + z^2} + \sqrt{(\rho + A)^2 + z^2}\right]^2$$

which says the following is what we "want to show" :

$$(1/2)\left[\sqrt{(\rho - A)^2 + z^2} + \sqrt{(\rho + A)^2 + z^2}\right]^2 = 2A^2 + (r^2 - A^2) + \sqrt{|r^2 - A^2|^2 + 4A^2z^2}$$

or

$$(1/2)\left[\sqrt{(\rho - A)^2 + z^2} + \sqrt{(\rho + A)^2 + z^2}\right]^2 = (r^2 + A^2) + \sqrt{|r^2 - A^2|^2 + 4A^2z^2}$$

We now call upon Ellipse Theorem 3 (from ellipses.doc) which says this

$$(1/2)\left(\sqrt{(X - a)^2 + Y^2} + \sqrt{(X + a)^2 + Y^2}\right)^2 = (X^2 + Y^2 + a^2) + \sqrt{(X^2 - Y^2 - a^2)^2 + 4X^2Y^2}$$

Let's try to make these identifications

$Y = z$
$X = A$
$a = \rho$

Then our ellipse theorem says:

$$(1/2)\left(\sqrt{(A - \rho)^2 + z^2} + \sqrt{(A + \rho)^2 + z^2}\right)^2 = (A^2 + z^2 + \rho^2) + \sqrt{(A^2 - z^2 - \rho^2)^2 + 4A^2z^2}$$

Now set $z^2 + \rho^2 = r^2$ and the ellipse theorem then says

$$(1/2)\left(\sqrt{(A - \rho)^2 + z^2} + \sqrt{(A + \rho)^2 + z^2}\right)^2 = (A^2 + r^2) + \sqrt{(A^2 - r^2)^2 + 4A^2z^2}$$

which we can now compare to our "desired identity"

$$(1/2)\left[\sqrt{(\rho - A)^2 + z^2} + \sqrt{(\rho + A)^2 + z^2}\right]^2 = (r^2 + A^2) + \sqrt{|r^2 - A^2|^2 + 4A^2z^2}$$

and the are the same.

Conclusion: We have shown that

$\tan^{-1}(1/\xi)$ with

$\xi^2 = (1/2A^2)\{(r^2 - A^2) + \sqrt{(A^2-r^2)^2 + 4A^2z^2}\}$ $\qquad r^2 = x^2+ y^2+ z^2$

is same as

$\sin^{-1}(A/\xi_1)$ with $\qquad$ Jackson

$\xi_1 = (1/2)[\sqrt{(\rho-A)^2+ z^2} + \sqrt{(\rho+A)^2+ z^2})]$ $\qquad \rho^2 = x^2+ y^2$

and therefore Jackson's result 3.178 on page 92 for the charged metal disk,

$\Phi = q \sin^{-1}(A/\xi_1)$

matches our oblate coordinates result for the same problem

$\psi = (2/\pi)V_0 \tan^{-1}(1/\xi)$

4. What about the capacitance of things?

Suppose we have some result which says

$\psi = K_1 V_0 f(\text{coord})$

where we have things normalized so that f(coord=metal) = 1. In all our cases, $\psi(\text{coord} \Rightarrow \infty) = 0$ for going far away, which tells us that $\Delta\psi = \psi$ where $\Delta\psi = \psi(\text{metal}) - \psi(\infty)$. When we go far away, and we look back at our tiny isolated charged metal object with charge Q on it, we say that $\psi = Q/r$. Therefore it must be true that

$\lim_{\text{faraway}} f(\text{coord}) = K_2 / r$

and then we have

$\lim_{\text{faraway}} \psi = K_1V_0K_2/r$

and we then make this interpretation

$Q = K_1V_0K_2$

But capacitance is defined as

$Q = CV$ $\qquad C \equiv Q/\Delta\psi = Q/\psi_{\text{metal}} = K_1V_0K_2 / [K_1 V_0 f(\text{coord} = \text{metal})]$

$C = K_2 / f(\text{coord} = \text{metal}) = K_2$

So the capacitance is simply the constant K_2 in the above scenario.

Restate: just take the large r limit of ψ, and what is there ignoring V_0 and 1/r is C

Example 1: the charged metal general ellipsoid.

$\psi(\xi_1) = V_0 F[\sin^{-1}(a/\xi_1), b/a] / F[\sin^{-1}(a/A), b/a]$ // restatement of MF above
$\psi(\xi_1) = V_0 F[\sin^{-1}(b/\xi_1), a/b] / F[\sin^{-1}(b/A), a/b]$ // this line is "the Kelvin solution"

semis: $\xi_1 = A \geq B \geq C$ focals: $a^2 \equiv A^2 - C^2$ and $b^2 \equiv A^2 - B^2$ $a > b$

The ratio of F functions is our normalized f, and we have coord = a/ξ_1. We claim that

$\lim_{\text{faraway}} f(\text{coord}) = K_2 / r$

How do we find the large distance behavior of ξ_1 (if we did not already know it). Let's try our inversion program

$h = x^2+y^2+z^2 +(a^2+b^2)$ L^2
$f = b^2x^2+ a^2y^2+ (a^2+b^2)z^2 +a^2 b^2$ L^4
$k = a^2 b^2z^2$ L^6
$R = (-9hf +27k + 2h^3)/54$
$Q = (3f - h^2)/9$
$D = Q^3 + R^2$
$S = [R + \sqrt{D}]^{1/3}$
$T = [R - \sqrt{D}]^{1/3}$
$\xi_1^2 = h/3 + (S+T)$ $\Rightarrow \xi_1(x,y,z) = \sqrt{h/3 + (S+T)}$

$h = r^2$ $f,k \sim r^2$ $h^2 >> f$ and k

$R = h^3/27$
$Q = -h^2/9$
$D = (-h^2/9)^3 + (h^3/27)^2 = - h^6/9^3 + h^6/9^3 = 0$
$S = T = R^{1/3} = h/3$
$\xi_1^2 = h/3 + 2h/3 = h = r^2$ $\Rightarrow$ $\xi_1 = r$

Now let's compute

$\lim_{\text{faraway}} f(\text{coord}) = F[\sin^{-1}(a/r), b/a] / F[\sin^{-1}(a/A), b/a]$

We need the small argument behavior of F(x,k) which is F(x,k) ≈ x for small x (as I have now added to my elliptic functions doc and to AS page 594), so we have

$$\lim_{\mathtt{faraway}} f(\text{coord}) = (a/r) \,/\, F[\sin^{-1}(a/A), b/a] = (a/F[\sin^{-1}(a/A), b/a])/r = C/r$$

$$C = a/F[\sin^{-1}(a/A), b/a] = b/F[\sin^{-1}(b/A), a/b] \qquad \text{// from our second form of } \psi$$

so capacitance depends on the two focal distances a > b of our system of confocal ellipsoids, and on the semi major axis of the metal ellipsoid.

If we set a = b to get a **football ellipsoid**, the result is

$$sn^{-1}(x,k) = G(x,k) = F(\sin^{-1}x,k) = F(\sin^{-1}x \mid m) = F(\sin^{-1}x \setminus \alpha) \qquad k = \sin\alpha \qquad m = k^2$$

$$C = a/F[\sin^{-1}(a/A),1] = a/F[\sin^{-1}(a/A) \setminus \pi/2] = a/\ln[\sec\theta + \tan\theta] \qquad \theta = \sin^{-1}(a/A)$$

so

$$C = a/\ln\{(A+a)/\sqrt{A^2-a^2}\}$$

I should then be able to take a→ 0 to get a **sphere**, and it works, we get C = A (as long as we take the limit right, as done earlier in this doc).

Example 2: the round disk in ellipsoidals

$$\psi(\xi_1) = V_o \sin^{-1}(A/\xi_1)(2/\pi) = V_0 f(\text{coord})$$

$$f(\text{coord}) \rightarrow A(2/\pi)\,1/r \qquad C = 2A/\pi$$

Example 3: the round disk in oblate spheroidals

$$\psi(\xi) = V_0 Q_0(i\xi)/Q_0(i0) = V_0 \tan^{-1}(1/\xi)/[\pi/2] = (2/\pi)V_0 \tan^{-1}(1/\xi)$$

where

$$\xi^2 = 1/(2A^2)\left[(r^2 - A^2) + \text{sign}(r^2-A^2)\sqrt{(r^2-A^2)^2 + 4A^2z^2}\right] \qquad a = \text{focus}$$

For large r we get that $\xi = r/A$ so then

$$\psi \rightarrow (2/\pi)V_0(A/r) \qquad \Rightarrow \qquad C = 2A/\pi$$

the same as the previous example.

Part III. Summary of Results

The surface charge distribution on any of these objects is given by the simple Kelvin formula.

$$\sigma = \text{constant}/\sqrt{x^2/A^4 + y^2/B^4 + z^2/C^4}$$

A = largest semi-major axis, or radius of sphere
a > b = focal distances of ellipsoid

general ellipsoid

$Cap = a/ F[\sin^{-1}(a/A), b/a] = b/ F[\sin^{-1}(b/A), a/b]$

$\psi(\xi_1) = V_0 F[\sin^{-1}(a/\xi_1), b/a] / F[\sin^{-1}(a/A), b/a]$

$\psi(\xi_1) = V_0 F[\sin^{-1}(b/\xi_1), a/b] / F[\sin^{-1}(b/A), a/b]$

$x^2/(\xi_1^2-a^2) + y^2/(\xi_1^2- b^2)+ z^2/(\xi_1^2) = 1 \qquad 0 < b < a < \xi_1$

$x^2/C^2 + y^2/B^2 + z^2/A^2 = 1$

semis: $\xi_1 = A \geq B \geq C$

focals: $a^2 \equiv A^2 - C^2$ and $b^2 \equiv A^2 - B^2 \qquad a > b$

large $\xi_1 \rightarrow r \qquad \xi_1 = A$ on metal

$x = (1/a)\sqrt{(\xi_1^2 - a^2)(a^2 - \xi_2^2)(a^2 - \xi_3^2)/(a^2- b^2)}$

$y = (1/b)\sqrt{(\xi_1^2 - b^2)(\xi_2^2-b^2)(b^2 - \xi_3^2)/(a^2- b^2)}$

$z = \xi_1\xi_2\xi_3 / ab$ // inverse formulas are a mess

prolate ellipsoid (as limit of above)

$Cap = a / \ln [(A+a)/\sqrt{A^2-a^2}] \qquad a = b$

$\psi(\xi_1) = V_0 \ln [(\xi_1 + a) / \sqrt{\xi_1^2-a^2}] / \ln [(A + a) / \sqrt{A^2-a^2}]$

$\rho^2 = x^2 + y^2 = (1/a)^2(\xi_1^2 - a^2)(a^2 - \xi_3^2)$

$z = \xi_1\xi_3 / a \qquad \xi_2 = a = b$

$\xi_1^2 = (1/2) [(a^2 + r^2) + \sqrt{(a^2 + r^2)^2 - 4a^2z^2}]$

$\xi_3^2 = (1/2) [(a^2 + r^2) - \sqrt{(a^2 + r^2)^2 - 4a^2z^2}]$

$\xi_2^2 = a^2$

elliptical disk

$Cap = A/K(b/A) \qquad a = A \quad \text{so } C = 0$

$\psi(\xi_1) = V_0 F[\sin^{-1}(A/\xi_1), b/A] /K(b/A)$

ellipsoidal round disk

$Cap = (2A/\pi) \qquad b = 0$

$\psi(\xi_1) = (2/\pi)V_o \sin^{-1}(A/\xi_1)$

needle

$Cap = A/K(b/A) \qquad b/A = .99$ say

$\psi(\xi_1) = V_0 F[\sin^{-1}(A/\xi_1), b/A] / K(b/A)$

$\xi_1 = \xi_2 = A \quad \xi_3 = 0$

sphere

$Cap = A$

$\psi(\xi_1) = V_0 (A/r)$

oblate spheroid

$Cap = a / \tan^{-1}(1/\xi_0)$

$\psi(\xi) = V_0 Q_0(i\xi)/Q_0(i\xi_0) = V_0 \tan^{-1}(1/\xi) / \tan^{-1}(1/\xi_0)$

$x = a\sqrt{\xi^2+1}\sqrt{1-\eta^2}\cos\varphi = a\sqrt{\xi^2+1}\sin\nu\cos\varphi$ $\qquad \nu$ = "cone" angle

$y = a\sqrt{\xi^2+1}\sqrt{1-\eta^2}\sin\varphi = a\sqrt{\xi^2+1}\sin\nu\sin\varphi$

$z = a\,\xi\,\eta = a\,\xi\cos\nu$

$\xi^2 = 1/(2a^2)\ \{ (r^2 - a^2) + \text{sign}(r^2-a^2) \sqrt{(r^2 - a^2)^2 + 4a^2z^2}\ \}$ a = focus

$-\eta^2 = 1/(2a^2)\ \{ (r^2 - a^2) - \text{sign}(r^2-a^2) \sqrt{(r^2 - a^2)^2 + 4a^2z^2}$ $r^2 = x^2 + y^2 + z^2$

$\varphi = \tan^{-1}(y/x)$ $\xi = r/a$ as $\xi \rightarrow \infty$

oblate round disk Cap = $(2A/\pi)$ a = A

$\psi(\xi) = V_0\, Q_0(i\xi)/Q_0(i0) = (2/\pi)V_0 \tan^{-1}(1/\xi)$ $\xi_0 = 0$

$\xi^2 = 1/(2A^2)\ \{ (r^2 - A^2) + \text{sign}(r^2-A^2) \sqrt{(r^2 - A^2)^2 + 4A^2z^2}\ \}$

$-\eta^2 = 1/(2A^2)\ \{ (r^2 - A^2) - \text{sign}(r^2-A^2) \sqrt{(r^2 - A^2)^2 + 4A^2z^2}$