

Charge density on a thin straight wire: The first visit

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The problem of the linear charge density on a long straight conducting wire was considered and solved using a variational technique over 123 years ago. We describe the who and where and summarize his results. We also eat humble pie. © 2002 American Association of Physics Teachers. [DOI: 10.1119/1.1432973]

The question of the linear charge density on a long straight conducting wire and its limiting form as the ratio of radius to length vanishes has been treated in at least six papers both before and after the publication of Griffiths and Li¹ in 1996. My own paper² in 2000 aspired to be the last. Griffiths and Li, in their second paragraph, say, “The question sounds simple enough, and it must surely have been considered long ago by Sommerfeld, Smythe, Stratton, or *maybe even Maxwell himself* {emphasis added}. However, we have found no reference to it in the literature.” They may perhaps be forgiven for overlooking the Russian publications in the 1960s (Refs. 5–7 in Ref. 2) because the Russians were concerned mostly with the total charge and capacitance of a long cylinder. But we are all guilty of a most egregious failure in searching the prior literature, as I now describe.

Recently, in looking for something else, I was astounded and mortified to find, near the end of the second volume of *The Scientific Papers of James Clerk Maxwell*,³ a paper entitled “On the electrical capacity of a long narrow cylinder, and of a disk of sensible thickness.”⁴ The relevant volume of the Proceedings of the London Mathematical Society reveals that the paper was communicated on behalf of Maxwell by a Mr. Tucker at the meeting of the society of March 14, 1878, with Lord Rayleigh, President, in the chair. Maxwell died 20 months later on November 5, 1879. This paper appears to be his last published research in electricity and magnetism and almost his last on any subject. It is a concise, masterful treatment of the topic.

Let me make clear that, in my admiration of Maxwell’s work and my chagrin at not being aware of it, I do not wish to imply that the recent papers are mere rehashes. They approach the problem differently,¹ or in more diversity and depth,² and add to our understanding of the physics involved. Nevertheless, we can only guess how knowledge of Maxwell’s paper would have changed the number and content of our recent publications—surely considerably.

Maxwell’s opening words are, “The distribution of electricity in equilibrium on a straight line without breadth is a uniform one. We may expect, therefore, that the distribution on a cylinder will approximate uniformity as the radius of the cylinder diminishes.” He proceeds to discuss Cavendish’s upper and lower limits for the capacity before launching into a variational calculation that is the equivalent of Problem 1.18 of my textbook.⁵ He chooses to expand the linear charge density $\lambda(z)$ in a finite sum of even order Legendre polynomials, assumes that the length is very long compared to the radius, and ignores the contribution to the potential from the end surfaces. His approach is closely similar to the second method in my paper.² He considers first a two-term approximation ($L=0,2$) and then three ($L=0,2,4$). He gives lower limits for the capacity⁶ and then gives the charge den-

sity $\lambda(z)$ from his three-term approximation in the limit of very large ratio of length to radius. In the notation of my paper, Maxwell’s expression is

$$\lambda(\zeta) = \frac{Q}{2c} \left[1 + \frac{1}{\Lambda} G_M(\zeta) \right], \quad (1)$$

where Q is the total charge, $2c$ is the length of the cylinder, a is its radius, $\zeta = z/c$ with range $(-1, 1)$, and $\Lambda = 2 \ln(2c/a)$. The function $G_M(\zeta)$ of Maxwell is

$$G_M(\zeta) = \frac{7}{16}(9\zeta^4 - 2\zeta^2 - 17/15) = \frac{5}{3}P_2(\zeta) + \frac{9}{10}P_4(\zeta). \quad (2)$$

For comparison, the recent first-order (in $1/\Lambda$) expression has^{2,7}

$$G(\zeta) = 2 \ln 2 - 2 - \ln(1 - \zeta^2). \quad (3)$$

We plot the two functions $G(\zeta)$ in Fig. 1 for comparison. The heavier, slightly oscillatory curve is Maxwell’s approximation. More terms in the expansion in Legendre polynomials would smooth out the oscillations, but the three-term approximation is remarkably good. In fact, if we expand $G(\zeta)$ [Eq. (3)] in Legendre polynomials, we find the first two terms in the series are just Maxwell’s $G_M(\zeta)$ [Eq. (2)].

Below his equivalent of Eqs. (1) and (2), Maxwell continues, “which shows that, as the ratio of the length to the diameter increases, the density becomes more nearly uniform, and the deviation from uniformity becomes more confined to the parts near the ends of the cylinder.” Could the conclusion of the investigation be said more succinctly?

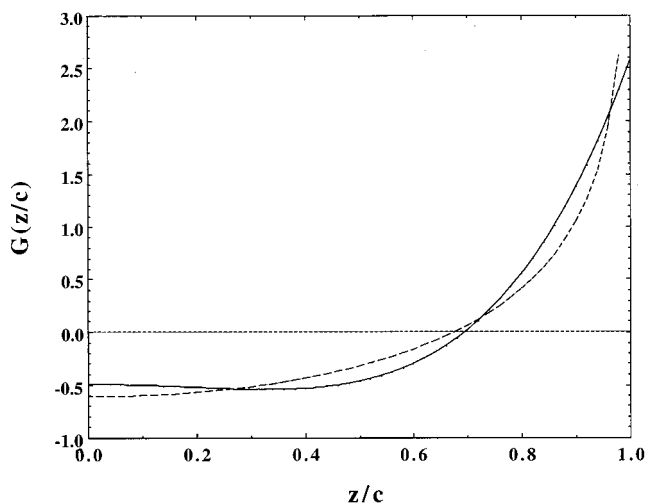


Fig. 1. Comparison of the correction functions $G_M(\zeta)$ and $G(\zeta)$. Maxwell’s is the solid curve.

Of course, as indicated earlier, we as workers in this same vineyard need not be ashamed of having written our papers. Among other things, the linear charge density is now known to higher order in $1/\Lambda$; the results have been generalized to shapes other than the right circular cylinder; the puzzle of how the charges can be in equilibrium on the cylinder with a constant linear density as $a/c \rightarrow 0$ has been answered. Still, I draw the following lessons from this story:

- (a) There is *almost* nothing new under the sun, especially in a mature subject such as electrostatics.⁸
- (b) To avoid embarrassment and/or an editor's wrath, authors must search the previous literature assiduously and search again.
- (c) The giants in any field are far in advance of the mere mortals (in this case, over 123 years!).

Mea culpa atque culpa ceteri

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¹D. J. Griffiths and Y. Li, "Charge density on a conducting needle," *Am. J. Phys.* **64**, 706–714 (1996).

²J. D. Jackson, "Charge density on a thin straight wire, revisited," *Am. J.*

Phys. **68**, 789–799 (2000). This paper contains references to other relevant papers not cited here.

³J. C. Maxwell, *The Scientific Papers of James Clerk Maxwell*, edited by W. D. Nivens, 2 vols. bound as one (Dover reprint, New York), Vol. 2, pp. 672–680.

⁴J. C. Maxwell, "On the electrical capacity of a long narrow cylinder, and of a disk of sensible thickness," *Proc. London Math. Soc.* **IX**, 94–101 (1878).

⁵J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998), 3rd ed., p. 53.

⁶Maxwell reproduces his paper (Ref. 4), *mutatis mutandis*, in *The Electrical Researchers of Henry Cavendish*, edited by J. C. Maxwell (Cambridge U.P., Cambridge, 1879), Note 12, pp. 393–400, and compares his theoretical capacitance to Cavendish's measurements of 1771.

⁷The form of $G(\zeta)$ (to compare with Maxwell's) results from taking the first-order terms in $1/\Lambda$ from my Eqs. (26) and (29) of Ref. 2, and expressing $\lambda(\zeta)$ in terms of the total charge Q rather than the linear charge density parameter λ_0 .

⁸A notable exception is the remarkable theorem discovered in 1956 by Thompson and Lampard [A. M. Thompson, and D. G. Lampard, "A new theorem in electrostatics and its application to calculable standards of capacitance," *Nature (London)* **177**, 888 (1956)]. The practical realization and relevance for electrical standards are discussed briefly by Zimmerman [N. M. Zimmerman, "A primer on electrical units in the *Système International*," *Am. J. Phys.* **66**, 324–331 (1998)]. I have given a pedagogical proof of the theorem with examples [J. D. Jackson, "A curious and useful theorem in two-dimensional electrostatics," *Am. J. Phys.* **67**, 107–115 (1999)].