

תודות

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The pre-potential of the EM field

David Hai Gootvilig

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Abstract

This report is continuation of previous work [8]-[12] on the prepotential of the electromagnetic field.

We define the Hodge dual operator on the standard representation of the Lie algebra of the Lorentz group on Minkowski space, denoted by π in this report. This operator defines a decomposition of the complexification of π into eigen-subspaces $\mathfrak{o}_{\pm}$, corresponding to the eigenvalues $\pm i$. The elements of $\mathfrak{o}_{\pm}$ are called σ -dual, where σ take a value $\pm i$. We define a representation of the Lorentz group $\tilde{\pi}_{\pm}$ on each of the subspaces $\mathfrak{o}_{\pm}$. For any EM field tensor F we associate a σ -dual tensor $\mathcal{F}^{\pm}$. We show that outside of the sources of the field a σ -dual 4-potential can be defined.

For any observer we find a constant $\zeta_{\pm}$, depending only on the null-displacement between the observer and the source of the field which is invariant under $\tilde{\pi}_{\pm}$. The invariant $\zeta_{\pm}$ is used to define the prepotential $S_{\pm}$ of the EM field. We have shown that the prepotential of a point charged particle in *arbitrary* motion produces the correct field. It is shown that outside of the sources the prepotential is always σ -dual.

Maxwell's equations for the prepotential are derived. We find conditions on the current density for the existence of a prepotential. The gauge freedom of the prepotential is calculated. Examples for the relation between the prepotential, the 4-current and the field are presented for: infinite charged rod, infinite charged plate and current through infinite solenoid.

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1 Introduction

1.1 Outline

In section 2 we introduce briefly the language of tensors, and in section 3 we apply it to the classical Electromagnetic theory. In section 4 we introduce the matrix Lie groups and their Lie algebras and the most necessary ideas of representation theory, we show that the canonical Lorentz representation π is a product of two commuting spin-half representations $\tilde{\pi}_+$ and $\tilde{\pi}_-$.

We investigate the σ -dual tensor field arising for the half-spin representations and find that outside of the sources a σ -dual complex 4-potential can be defined, and show that if the prepotential satisfies the wave equation if and only if it is σ -dual. Considering only one of the half-spin representation alone on a complexified Minkowski space we give rise to non-trivial scalar invariant made out of a null displacement vector.

In section 5 we define the prepotential of a charged point particle using the invariant scalar derived in section 4. The relation between the prepotential and the complex 4-potential is defined, and the complex 4-potential induced by this prepotential is calculated. Its real part is the Liénard–Wiechert potentials. The fields induced by this prepotential are calculated. And we show that the prepotential is σ -dual everywhere outside of the source.

The prepotential of a charge distribution is defined through super-position principle. The field tensor is formulated as a function of the prepotential and the gauge freedom of the prepotential is calculated. Maxwell's equations in the prepotential formalism are presented.

1.2 Motivation

- Fewer degrees of freedom in the description of the theory, the Electro-Magnetic field is represented by the Faraday tensor $F_{\mu\nu} \sim (\mathbf{E}, \mathbf{B})$ which has 6 degrees of freedom. Using the Electro-Magnetic 4-potential A_μ we get down to 4 degrees of freedom, when using a gauge choice we reduce them to 3 degrees of freedom. The prepotential S is a complex scalar and has 2 real degrees of freedom.
- Locality. The field strength Tensor $F_{\mu\nu}$ depends on the position, velocity and acceleration of the sources, the 4-potential A_μ depends on the position and velocity of the source, while the prepotential S depends *only* on the position of the source.
- Aharonov-Bohm effect indicates that there is a multiple-valued pre-potential of the 4-potential of an EM field.
- The pre-potential is *Conformally* invariant.

2 A very short introduction to tensors

2.1 An intuitive approach to tensors

In this section we will try to explain what are tensors in an intuitive way, as a naturally required generalization and specialization of scalars and vectors.

2.1.1 Numbers, Scalars, Vectors

Number or 'bare' number is not a physical quantity but is a satisfactory answer when the information we seek is a magnitude. For example: "How many books are on your shelf?", "Ten"

might be a possible answer.

Scalar is a physical quantity, it can be viewed as a number with a name - which gives us the *unit* we use to express the quantity. For example: "How much time does it takes for the world's champion to solve a Rubik's cube?", "Ten seconds" is a possible answer. Examples for scalars: the mass, charge, time, speed, temperature or electric potential at a point inside a medium.

Vectors are abstract objects, which not only posses magnitude but also direction. The magnitude or "length" of a physical vector must have physical units, for example, distance and displacement are measured in meters, velocity is measured in meters/second, force is measured in Newtons, electric field intensity is measured in Volts/meter, and so on. The direction is measured relatively to some possibly arbitrarily chosen reference direction.

As stated vectors are abstract but their components can be manipulated when expressing them in some basis. For example in Newtonian Dynamics, the position or displacement of a point mass is expressed as a vector $\mathbf{r}(t)$ from the origin to the object, which depends on time. Using a coordinate system with a basis e_i , we can write this displacement as

$$\mathbf{r}(t) = \sum_{i=1}^3 r^i(t) e_i = \sum_i r^i(t) e_i = r^i(t) e_i = r^j(t) e_j$$

where in the last two hands Einstein's summation convention is used as defined in 2.1 below.

Definition 2.1. Einstein's summation convention is a notational convention that implies summation over a set of indexed terms in a formula, thus achieving notational brevity. We will define it with the following rules:

1. An index variable can appear up to twice in a single term, one as upper index and one as lower index.
2. When an index variable appears as both upper and lower index in a single term, it implies summation of the term over all the values of the index.
3. An index that is summed over is a *summation index*, and also called a *dummy index* since any symbol can replace it without changing the meaning of the expression, provided that it does not collide with another index symbols on that term.
4. An index that is not summed over is a *free index* and must be found in each term of the expression or equation.
5. Unless stated otherwise:
 - Small Latin alphabet (i, j, k , etc.) indexes will have the set $\{1, 2, 3\}$
 - Greek alphabet (μ, ν , etc.) indexes will have the set $\{0, 1, 2, 3\}$
 - Small bold font Latin alphabet ($\mathbf{i}, \mathbf{j}, \mathbf{k}$, etc.) will be used as index arrays.

in most cases the limits of implied summation is understood from context, an explicit summation is used where there is a possible confusion.

Remark. We will use this convention extensively throughout this report.

2.1.2 Inner product

Definition 2.2. The inner product of two vectors $\mathbf{x} = x^i e_i$ and $\mathbf{y} = y^j e_j$ is a scalar, expressed as

$$\mathbf{x} \circ \mathbf{y} = (e_i \circ e_j) x^i y^j = \delta_{ij} x^i y^j$$

where δ_{ij} is Kronecker's delta. The argument $(e_i \circ e_j)$ here equals δ_{ij} since we assumed an Euclidean space with orthonormal basis, but later we will see this doesn't have to hold in general.

Note. The definition of inner product given in 2.2 will not be used in the rest of the report, and a better definition which generalize 2.2 will be introduced in 2.9.

Definition 2.3. A co-vector v_j is an object that can be combined with a vector u^j to create a scalar $s = v_j u^j$.

Even though a co-vector has similarities to a vector, they transform in a different way under a basis change, co-vectors change in the same way that the basis does and are called co-variant vectors or co-vectors for short, while vectors or contra-variant vectors transform in opposite manner so scalar combined from a co-vector and a vector stays *invariant* under a basis change.

Example 2.4. The differential work dW is a scalar combined from the co-vector force F_j and the differential displacement dr^j as $dW = F_j dr^j$.

Example 2.5. The gradient operator $\partial_j = \frac{\partial}{\partial x^j}$ is a co-variant operator, when applied to scalar it creates a co-vector, the potential energy in classical mechanics U is a scalar, and the force F_j is related to it by $F_j = -\partial_j U = -\frac{\partial U}{\partial x^j}$.

2.1.3 Beyond vectors

There are abstract geometric objects which are more sophisticated than vectors and require more information to be expressed, intuitively these objects have a magnitude and more than one direction. With these objects we can express relations which cannot be formulated when using only scalars and vectors.

For example the magnetic flux density $\mathbf{B}$ in volt*second/m², and the magnetization $\mathbf{H}$ in Amp/m, are related by the magnetic permeability μ in Henry/m by the expression

$$\mathbf{B} = \mu \cdot \mathbf{H} \text{ or } B_i = \mu_{ij} H^j \quad (2.1.3.1)$$

thus $\mathbf{B}$ and $\mathbf{H}$ can differ in magnitude and direction as happens in some materials.

The classical example of the use of tensors in physics is related to stress in material object. Stress σ has units of N/m² so it is clear that (stress) $\times$ (area) should equal (force), which will be the forces that creates the stress on this area. A differential area can be represented by a vector $d\mathbf{S}$ which is normal to the area pointing outward from the convex side, and has magnitude of the of the surface area dS .

Since there are two types of stresses: the tensile stress (normal force) and sheer stress (tangential force) the stress cannot be a scalar, so the force $d\mathbf{F}$ due to the stress σ acting on the differential surface element $d\mathbf{S}$ is given by

$$d\mathbf{F} = \sigma \cdot d\mathbf{S} \text{ or } dF_i = \sigma_{ij} dS^j \quad (2.1.3.2)$$

The stress tensor σ was the first tensor to be described and used by scientists and engineers. The word tensor derives from the Latin *tensus* meaning stress or tension.

2.1.4 Tensors

The defining properties of tensors is that the abstract object they represent remains unchanged under a basis change, and are expressed by a single basis.

It follows that not all scalars and vectors and other entities are tensors, only those that depend on single basis and transform in the correct way. For example the basis transformation matrix is not a tensor since it depends on two bases.

Assume in an N dimensional space we have a basis $\{e_1, e_2, \dots, e_N\}$, therefor for any vector v is represented as $v = v^\mu e_\mu$. Now consider an invertible linear transformation that transforms the basis e_μ into a basis $e_{\mu'}$ as

$$e_\mu = L^{\mu'}_{\mu} e_{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} e_{\mu'} \quad (2.1.4.1)$$

where $\frac{\partial x^{\mu'}}{\partial x^\mu}$ is the inverse of the Jacobian matrix representing the first partial derivatives of the coordinate transformation $x^{\mu'}(x^\mu)$.

A scalar S is a tensor which is invariant under basis transform, thus $S' = S$. Vectors are also tensors, which represent the same object in any reference frame thus $x = x'$. From this requirement we get:

$$x = x^\mu e_\mu = x^\mu L^{\mu'}_{\mu} e_{\mu'} = x^{\mu'} e_{\mu'} = x' \Rightarrow x^{\mu'} = L^{\mu'}_{\mu} x^\mu \Rightarrow x^\mu = (L^{-1})^\mu_{\mu'} x^{\mu'}. \quad (2.1.4.2)$$

We can see from (2.1.4.2) that vectors components are transformed inversely to the basis transform. This is why they are called contra-variant, in contrast to vectors - co-vectors are transformed in the same way as the basis does. Also we see that the transformation matrices L are *not* tensors since they depend on two bases.

Note. Now we can understand why in (2.1.4.1) we substituted $\frac{\partial x^{\mu'}}{\partial x^\mu}$ for $L^{\mu'}_{\mu}$. By differentiating the middle equation in (2.1.4.2) by x^μ assuming L is constant we get:

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = L^{\mu'}_{\mu}.$$

The notion of the partial derivatives will be used in the rest of the report. Moreover it is generalizable to the more general settings of curvilinear coordinates and tangent spaces to differentiable manifolds which will not be discussed in this report.

2.2 Mathematical approach to tensors

Tensors are geometric objects which encapsulate the meaning of the same thing, with respect to an arbitrary basis transformation, in physics the different bases are called a different reference frames.

2.2.1 Definition of tensors by their transformation law

The following definition of tensors that we will use in this report defines tensors by their transformation laws with respect to a basis transformation. This is not the most general and abstract way but will suffice well for our needs.

Definition 2.6. A tensor T (or more precisely tensor field) of type- (n, m) on a finite vector space V with dimension N over the field F (for our cases F is either $\mathbb{R}$ or $\mathbb{C}$), is a multi-dimensional array with indexes $\mathbf{i} = (i_1, i_2, \dots, i_n); \mathbf{j} = (j_1, j_2, \dots, j_m)$ which transforms by the following rule with respect to a basis $e_\mu \rightarrow e_{\mu'}$ and coordinates $x^\mu \rightarrow x^{\mu'}$ transformation as:

$$T_{\mathbf{j}}^{\mathbf{i}'} = \frac{\partial x^{\mathbf{i}'}}{\partial x^{\mathbf{i}}} \frac{\partial x^{\mathbf{j}}}{\partial x^{\mathbf{j}'}} T_{\mathbf{j}}^{\mathbf{i}}$$

where T on the right hand is a tensor field with index arrays $\mathbf{i}' = (i'_1, i'_2, \dots, i'_n); \mathbf{j}' = (j'_1, j'_2, \dots, j'_m)$, and $\frac{\partial x^{\mathbf{i}'}}{\partial x^{\mathbf{i}}} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \dots \frac{\partial x^{i'_n}}{\partial x^{i_n}}$, where $\frac{\partial x^{\mathbf{i}'}}{\partial x^{\mathbf{i}}}$ is given by (2.1.4.1).

Definition 2.7. A tensor *rank* (also *degree* or *order*) is the dimensionality of the array needed to represent it, or the sum of it's type indexes thus a tensor of type- (n, m) is of rank $n + m$.

2.2.2 Metric tensor, inner product and Index Lowering and Raising

Definition 2.8. The inner product in general is defined by the metric tensor $g_{\mu\nu}$ associated with the space, which is a type-(0, 2) tensor which obeys $g_{\mu\nu} = g_{\nu\mu}$ (more precisely it is non-degenerate symmetric form).

Definition 2.9. The inner product $x \circ y$ of vectors x and y is

$$x \circ y = g_{\mu\nu} x^\mu y^\nu = g_{\mu\nu} x^\nu y^\mu$$

Remark. We can now see that $e_\mu \circ e_\nu$ from 2.2 in the general case actually equals to $g_{\mu\nu}$.

Definition 2.10. The inverse metric tensor $g^{\mu\nu}$ is related to the metric by

$$\delta_\nu^\mu = g^{\mu\lambda} g_{\lambda\nu}$$

Definition 2.11. In metric space the metric and the inverse metric are used in order to lower and raise indexes. A vector v^j is related to a co-vector v_i by $v_i = g_{ij} v^j$ or equivalently $v^j = g^{ij} v_i$, this can be done to any index on any tensor.

Example 2.12. The mixed Faraday tensor F_ν^μ which is used in formulation of the Lorentz force $\frac{dp^\mu}{d\tau} = q F_\nu^\mu u^\nu$, is related by a raised index operation to the co-variant form $F_{\alpha\beta}$ as $F_\nu^\mu = g^{\mu\alpha} F_{\alpha\nu}$.

2.2.3 Tensor addition and subtraction

Definition 2.13. Tensors of the same rank can be added or subtracted, this is a component-wise operation, in detail the tensor addition (subtraction) of two tensors A and B of type (n, m) , is a tensor C of type (n, m) where the following holds component-wise:

$$C_j^i = A_j^i \pm B_j^i$$

where a plus sign is used for addition and minus sign for subtraction, i and j are index arrays with n and m indexes respectively.

2.2.4 Tensor product, contraction and transpose

Definition 2.14. The tensor product of two tensors A and B of types (n_1, m_1) and (n_2, m_2) respectively, is a tensor C of type $(n_1 + n_2, m_1 + m_2)$ where the following holds component-wise:

$$C_{j_1 j_2}^{i_1 i_2} = A_{j_1}^{i_1} B_{j_2}^{i_2}$$

where in the left hand side i_1, i_2, j_1 and j_2 are index arrays of n_1, n_2, m_1 and m_2 indexes respectively, and in the right hand side the expressions $i_1 i_2$ and $j_1 j_2$ are index arrays, consisting of the indexes in the first index array followed by the indexes from the second index array.

Example 2.15. The angular momentum L_{ij} in classical mechanics is a subtraction of tensor products of the lowered index displacement x_i and the momentum p_i , as

$$L_{ij} = x_i p_j - x_j p_i = \begin{pmatrix} x_1 p_1 & x_1 p_2 & x_1 p_3 \\ x_2 p_1 & x_2 p_2 & x_2 p_3 \\ x_3 p_1 & x_3 p_2 & x_3 p_3 \end{pmatrix} - \begin{pmatrix} x_1 p_1 & x_2 p_1 & x_3 p_1 \\ x_1 p_2 & x_2 p_2 & x_3 p_2 \\ x_1 p_3 & x_2 p_3 & x_3 p_3 \end{pmatrix}$$

Example 2.16. A multiplication by a scalar λ of a type (n, m) tensor T is defined through the tensor product 2.14, since the scalar is a type $(0, 0)$ tensor the result is a type (n, m) tensor and each of it's components is the respective component of T multiplied by λ thus recapturing the natural meaning of a scalar multiple.

Definition 2.17. Tensor contraction is an operation that reduces a tensor A of type (n, m) with index arrays $\mathbf{i} = (i, \mathbf{i}')$ and $\mathbf{j} = (j, \mathbf{j}')$, into a tensor B of type $(n - 1, m - 1)$ with index arrays $\mathbf{i}'$ and $\mathbf{j}'$ with $n - 1$ and $m - 1$ indexes respectively,

$$B_{\mathbf{j}'}^{\mathbf{i}'} = A_{k\mathbf{j}'}^{k\mathbf{i}'}$$

where N is the dimension of the underlying vector space, and the contraction was chosen over the first contra-variant and co-variant indexes, but any pair of contra-variant and co-variant indexes can be contracted.

Example 2.18. The inner product $\mathbf{x} \circ \mathbf{y}$ defined in 2.9 is a contraction of the tensor $g_{kl}x^i y^j$ (more precisely two contractions i with k and j with l), which is a tensor product of two vectors (x^i, y^j) and the metric tensor g_{kl} defined in 2.8, as

$$\mathbf{x} \circ \mathbf{y} = g_{ij}x^i y^j$$

Definition 2.19. The transpose of a rank 2 tensor A_{ab} , A^{ab} or A^a_b as

$$\begin{aligned} (A_{ab})^t &= A_{ba} \\ (A^{ab})^t &= A^{ba} \\ (A^a_b)^t &= A_b^a = g^{ac} A_c^d g_{bd} \end{aligned}$$

this definition correlates with the known transpose of a matrix.

Example 2.20. The transpose of a matrix $A^i_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in an Euclidean space is

$$(A^i_j)^t = g^{ii'} A^{j'}_{i'} g_{j'j} = \delta^{ii'} A^{j'}_{i'} \delta_{j'j} = A_j^i = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

2.2.5 Symmetric and antisymmetric tensors

Definition 2.21. Symmetry and anti-symmetry in tensors is defined on pairs of indexes. An exchange of symmetric pair of indexes, is equal to the original tensor component-wise, while an exchange of anti-symmetric pair of indexes, is equal to minus the original tensor component-wise. When the index pair is not of the same type (contra-variant or co-variant), the symmetry or anti-symmetry is observed after lowering the contra-variant or raising the co-variant index by the use of the metric (or the inverse-metric).

Example 2.22. The angular momentum L_{ij} is anti-symmetric $L_{ij} = -L_{ji}$, and the stress tensor σ_{ij} is symmetric $\sigma_{ij} = \sigma_{ji}$.

Definition 2.23. Anti-symmetrization notation

$$T_{[a_1 a_2 \dots a_n]} = \frac{1}{n!} \sum_{\sigma \in P} \text{Sign}(\sigma) T_{a_{\sigma_1} a_{\sigma_2} \dots a_{\sigma_n}}$$

where T is a tensor, $!$ is the factorial, P is the set of all permutations of the set $\{1, 2, \dots, n\}$, the sign of the permutation $\text{Sign}(\sigma)$ is 1 for even-permutations and -1 for odd-permutations, and the indexes σ_j are the permutation indexes.

Example 2.24. For $n = 2, 3$ we have:

$$M_{[ab]} = \frac{1}{2!} (M_{ab} - M_{ba})$$

$$T_{[abc]} = \frac{1}{3!} (T_{abc} - T_{acb} + T_{bca} - T_{bac} + T_{cab} - T_{cba})$$

Definition 2.25. The generalized Kronecker delta is related to the anti-symmetrization notation as

$$\delta_{\mathbf{j}}^{\mathbf{i}} T_{\mathbf{i}} = (n!) T_{[\mathbf{j}]}$$

where $\mathbf{i}, \mathbf{j}$ are index arrays of size n , and $!$ is the factorial.

Example 2.26. For $n = 2, 3$ we have:

$$\delta_{ab}^{cd} M_{cd} = (M_{ab} - M_{ba})$$

$$\delta_{abc}^{def} T_{def} = (T_{abc} - T_{acb} + T_{bca} - T_{bac} + T_{cab} - T_{cba})$$

Definition 2.27. A *completely* anti-symmetric tensor is a tensor which is anti-symmetric in each pair of it's indexes.

Theorem 2.28. In an N dimensional space any completely anti-symmetric tensor 2.27 of rank greater then N is trivial, meaning any of it's components in any basis is identically zero.

Proof. Let $T_{\mathbf{i}}$ be a completely anti-symmetric tensor of rank $m > N$, where the index array is $\mathbf{i} = (i_1, \dots, i_m)$. By the pigeonhole principle for any component of the tensor there must be two indexes with the same value, since a change of this two indexes will not change the index array $\mathbf{i}' = \mathbf{i}$ but by the anti-symmetry an exchange flips the sign as $T_{\mathbf{i}} = -T_{\mathbf{i}'}$, we find $T_{\mathbf{i}} = -T_{\mathbf{i}'} = -T_{\mathbf{i}}$ which leads to $T_{\mathbf{i}} \equiv 0$, as required. $\square$

2.2.6 Levi-Civita symbol and the Determinant

Definition 2.29. The Levi-Civita symbol ε is not a tensor, it is a *completely* anti-symmetric object. And in an N dimensional space in *any reference frame* it's components are given by

$$\varepsilon_{\mathbf{i}} = \text{Sign}(\sigma) \varepsilon_{\mathbf{i}_0}$$

where $\mathbf{i}_0 = (1, 2, \dots, N)$ and $\mathbf{i} = \sigma \mathbf{i}_0$ are index arrays of size N , σ is a permutation of index arrays of size N , and $\varepsilon_{\mathbf{i}_0} = 1$. In the cases where $\mathbf{i}$ is not a permutation of $\mathbf{i}_0$ then $\varepsilon_{\mathbf{i}} = 0$.

Example 2.30. The Levi-Civita symbol in two dimensions. It's components are

$$\varepsilon_{11} = 0 \quad \varepsilon_{12} = 1 \quad \varepsilon_{21} = -1 \quad \varepsilon_{22} = 0$$

or in matrix form

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Definition 2.31. The relation between the covariant and the contra-variant Levi-Civita symbol is defined *numerically* to be:

$$\varepsilon^{\mathbf{i}} \equiv \varepsilon_{\mathbf{i}}$$

this is the *one and only* exception to the otherwise universal rule that when we use the same symbol on an object with upstairs indexes and an object with downstairs indexes, the former is related to the latter by raising the indexes with the inverse metric.

Lemma 2.32. *The contraction of the Levi-Civita symbol with itself satisfies $\varepsilon_i \varepsilon^i = N!$.*

Proof. By 2.31 ε_i and ε^i are numerically equivalent, and since by 2.29 in any reference frame the non-zero components are ± 1 thus their square is always 1. Summing on all possible index arrangements without duplicate indexes (which yield the 0 components) which in N dimensional space there are $N!$ arrangements, the result is indeed $N!$ as required. $\square$

Definition 2.33. The determinant of a matrix or type-(1, 1) tensor A^i_j is defined as

$$\det(A^i_j) = \frac{1}{N!} \varepsilon_i \varepsilon^j A^i_j$$

where ε_i and ε^j are the covariant and contra-variant Levi-Civita symbol defined in 2.29 and 2.31, i and j are index arrays of size N , and $A^i_j = A^{i_1}_{j_1} \dots A^{i_N}_{j_N}$.

Example 2.34. The determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using 2.30 and 2.33 we get:

$$\begin{aligned} \det A &= \frac{1}{2!} \varepsilon_{i_1 i_2} \varepsilon^{j_1 j_2} A^{i_1}_{j_1} A^{i_2}_{j_2} = \frac{1}{2} (\varepsilon_{12} \varepsilon_{12} ad + \varepsilon_{21} \varepsilon_{12} bc + \varepsilon_{12} \varepsilon_{21} cb + \varepsilon_{21} \varepsilon_{21} da) = \\ &= \frac{1}{2} (2ad - 2bc) = ad - bc \end{aligned}$$

which is the expected result.

Definition 2.35. In a similar way to 2.33 we define the determinant of a covariant and contra-variant tensors of rank 2 or rather type-(2, 0) tensor A^{ij} and type-(0, 2) tensor A_{ij} as

$$\begin{aligned} \det(A_{ij}) &= \frac{1}{N!} \varepsilon^i \varepsilon^j A_{ij} \\ \det(A^{ij}) &= \frac{1}{N!} \varepsilon_i \varepsilon_j A^{ij} \end{aligned}$$

where ε_i and ε^j are the covariant and contra-variant the Levi-Civita symbol defined in 2.29 and 2.31, i and j are index arrays of size N , and $A_{ij} = A_{i_1 j_1} \dots A_{i_N j_N}$.

Lemma 2.36. *For any rank 2 tensor A^{ij} , A_{ij} or A^i_j , the co-variant ε_i and the contra-variant ε^j Levi-Civita symbols are related by:*

$$\begin{aligned} \varepsilon_i A^{ij} &= \det(A^{\alpha\beta}) \varepsilon^j \\ \varepsilon_i A^i_j &= \det(A^{\alpha}_{\beta}) \varepsilon_j \\ \varepsilon^j A^i_j &= \det(A^{\alpha}_{\beta}) \varepsilon^i \\ \varepsilon^i A_{ij} &= \det(A_{\alpha\beta}) \varepsilon_j \end{aligned} \tag{2.2.6.1}$$

where i and j are index arrays of size N , and $A^{ij} = A^{i_1 j_1} \dots A^{i_N j_N}$.

Proof. Using 2.35 to find the determinant of $A^{\alpha\beta}$ we get

$$\det(A^{\alpha\beta}) = \frac{1}{N!} \varepsilon_i \varepsilon_j A^{ij}.$$

multiplying by $N!$, and using 2.32 we get

$$\varepsilon_j [\varepsilon^j \det(A^{\alpha\beta}) - \varepsilon_i A^{ij}] = 0$$

since ε_j is not zero everywhere we can omit it and find

$$\varepsilon_i A^{ij} = \det(A^{\alpha\beta}) \varepsilon^j$$

as required. All other three relations are proved similarly using 2.35 or 2.33 and 2.32. $\square$

Note. Now it is clear why we said in the definition of the contra-variant Levi-Civita symbol 2.31 that the covariant and the contra-variant Levi-Civita symbols are not related by index raising and lowering, since using 2.36 where the rank 2 tensor is the metric $g_{\mu\nu}$ we get

$$\varepsilon^i g_{ij} = g \varepsilon_j \quad (2.2.6.2)$$

where $g = \det(g_{\alpha\beta})$. This is contrary to the result $\varepsilon^i g_{ij} = \varepsilon_j$ we should have got if the rules of tensor index lowering and raising was applicable to the Levi-Civita symbols.

2.2.7 Tensor Density and the Levi-Civita tensor

Definition 2.37. A tensor density $\mathfrak{T}_j^i$ of weight W and type- (n, m) is a quantity that under a basis transformation, transforms into tensor density $\mathfrak{T}_{j'}^{i'}$ of the same weight and type as

$$\mathfrak{T}_{j'}^{i'} = \left(\det \left[\frac{\partial x^{k'}}{\partial x^k} \right] \right)^W \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \mathfrak{T}_j^i$$

where i, i', j, j' are pairs of index arrays of sizes n, m and $\det \left[\frac{\partial x^{k'}}{\partial x^k} \right]$ is the Jacobian determinant.

A tensor is just a tensor density of weight 0. The tensors operations: addition, subtraction, contraction, index raising and lowering, and the notions of symmetry and anti-symmetry are extended natively to tensor densities.

Theorem 2.38. The Levi-Civita symbol is a co-variant ε_i tensor density of weight $+1$ or a contra-variant ε^i tensor density of weight -1 .

Proof. Using 2.36 with the inverse Jacobian matrix $\frac{\partial x^k}{\partial x^{k'}}$ we get:

$$\varepsilon_{i'} \det \left(\frac{\partial x^k}{\partial x^{k'}} \right) = \frac{\partial x^i}{\partial x^{i'}} \varepsilon_i$$

multiplying by the Jacobian determinant $\det \left(\frac{\partial x^{k'}}{\partial x^k} \right)$ we find

$$\varepsilon_{i'} = \det \left(\frac{\partial x^{k'}}{\partial x^k} \right) \frac{\partial x^i}{\partial x^{i'}} \varepsilon_i$$

thus the transformation of the co-variant Levi-Civita symbol is indeed governed by the transformation of tensor density of weight $+1$ as defined in 2.37, the proof for the contra-variant Levi-Civita symbol follows in a similar way. $\square$

Remark. The Levi-Civita symbol has been defined in 2.29 so it will be an *invariant* tensor density and hence will have the same value in any reference frame.

Definition 2.39. The tensor density product of a tensor density A of type (n_1, m_1) and weight W_1 and a tensor density B of type (n_2, m_2) and weight W_2 is a tensor density C of type $(n_1 + n_2, m_1 + m_2)$ where the component-wise product holds as in 2.14, and the weight of C is $W_1 + W_2$,

Lemma 2.40. The determinant of a rank 2 tensor $L^{\alpha\beta}$, $L_{\alpha\beta}$ and $L^\alpha{}_\beta$ is a scalar density of weight $+2$, -2 and 0 respectively.

Proof. The lemma 2.40 follows immediately from 2.39, 2.38, 2.33 and 2.35. $\square$

Following 2.39 we can produce a tensor out of the Levi-Civita symbol which is a tensor density of weight ± 1 (plus for co-variant, minus for contra-variant), in order to do that we need a *scalar density* of weight ∓ 1 , preferably one which is "natural" and have a geometric meaning.

Lemma 2.41. *Given a tensor density T of weight W then $|g|^{\frac{W}{2}}T$ is a pure tensor, where g is the determinant of the metric tensor.*

Proof. Since following 2.40 g is scalar density of weight -2 , therefor $\sqrt{|g|}$ is a scalar density of weight -1 , hence the total weight in 2.41 is 0 as required. $\square$

Note. The absolute value in 2.41 is necessary when dealing with a space with non-positive definite inner-product, such as the Minkowski space in Special and General relativity.

Definition 2.42. In any metric space the Levi-Civita *tensor* ϵ is defined as a completely anti-symmetric co-variant or contra-variant tensor, which is the tensor density product of the Levi-Civita symbol ε 2.29 and square root of the determinant of the metric $\sqrt{|g|}$ 2.41, thus it is given by

$$\begin{aligned}\epsilon_i &= \sqrt{|g|}\varepsilon_i \\ \epsilon^i &= \frac{(-1)^s}{\sqrt{|g|}}\varepsilon^i\end{aligned}$$

where i is a vector index of size N the dimension of the underlying space, and s is the signature of the metric.

Remark. In 2.42 the definition of both the co-variant and the contra-variant Levi-Civita tensor is redundant since they are related by index lowering and raising.

Lemma 2.43. *The contraction of Levi-Civita tensor with itself obeys the following relation:*

$$\epsilon^{ij}\epsilon_{i'j} = (N - k)!(-1)^s\delta_{i'}^i,$$

where i, i', j are vector indexes of sizes k, k and $N - k$ respectively, $\delta_{i'}^i$ is the generalized Kronecker delta defined in 2.25 and s is the signature of the metric.

Proof. Using 2.42, 2.31, and 2.25 we find

$$\epsilon^{ij}\epsilon_{i'j} = (-1)^s\varepsilon^{ij}\varepsilon_{i'j} = (N - k)!(-1)^s\delta_{i'}^i,$$

The term $(N - k)!$ arise since the contraction of completely anti-symmetric indexes occurs as all the possible arrangements of different indexes hence the factorial. $\square$

2.2.8 Hodge dual

Since we established in 2.28 that the completely anti-symmetric tensors are "bounded" in rank by the space dimension N , we check how much freedom there is in any rank $0 \leq k \leq N$, since a permutation of indexes gives the same object up to a sign flip, thus for tensor rank k there is $\binom{N}{k} = \frac{N!}{k!(N-k)!}$ distinct indexes choices.

Since $\binom{N}{k} = \binom{N}{N-k}$ it is suggestive to define an isomorphism between the k and $N - k$ completely anti-symmetric tensors. While there are many possible isomorphisms the Hodge duality defines a unique one using the inner-product and orientation of the space.

Definition 2.44. The *Hodge dual operator* maps completely anti-symmetric tensors of rank k into completely anti-symmetric tensors of rank $N - k$. The Hodge dual of an arbitrary completely antisymmetric tensor η of rank k is a completely antisymmetric tensor $\star\eta$ of rank $N - k$ as

$$(\star\eta)_{\mathbf{i}} = \frac{1}{k!} \epsilon_{\mathbf{i}}^{\mathbf{j}} \eta_{\mathbf{j}}$$

where $\mathbf{i}$ and $\mathbf{j}$ are index arrays of size $N - k$ and k respectively and ϵ is the Levi-Civita tensor.

Theorem 2.45. The square of the Hodge dual is the identity operator, up to sign. The Hodge dual square of a completely anti-symmetric tensor η of rank k in an N dimensional space is given by

$$(\star\star\eta)_{\mathbf{i}} = (-1)^{k(N-k)+s} \eta_{\mathbf{i}}$$

where $\mathbf{i}$ is an index array of size k and s is the signature of the metric.

Proof. Using 2.43 we find:

$$\begin{aligned} (\star\star\eta)_{\mathbf{i}} &= \frac{1}{k!} \epsilon_{\mathbf{i}}^{\mathbf{j}} (\star\eta)_{\mathbf{j}} = \frac{1}{k!} \epsilon_{\mathbf{i}}^{\mathbf{j}} \frac{1}{(N-k)!} \epsilon_{\mathbf{j}}^{\mathbf{i}'} \eta_{\mathbf{i}'} \\ &= \frac{1}{k!(N-k)!} \epsilon_{\mathbf{ij}} \epsilon^{\mathbf{ji}'} \eta_{\mathbf{i}'} = \frac{(-1)^{k(N-k)}}{k!(N-k)!} \epsilon_{\mathbf{ij}} \epsilon^{\mathbf{i}j} \eta_{\mathbf{i}'} \\ &= \frac{(-1)^{k(N-k)}}{k!(N-k)!} (N-k)! (-1)^s \delta_{\mathbf{i}}^{\mathbf{i}'} \eta_{\mathbf{i}'} = \frac{(-1)^{k(N-k)+s}}{k!} k! \eta_{\mathbf{i}} \\ &= (-1)^{k(N-k)+s} \eta_{\mathbf{i}} \end{aligned}$$

the term $(-1)^{k(N-k)}$ appears when rearranging the indexes $\mathbf{i}'$ and $\mathbf{j}$ on a completely anti-symmetric tensor we get $k(N-k)$ index pair exchanges. $\square$

Example 2.46. In 4-dimensional space with metric signature $s = 1$, the Hodge dual of a completely anti-symmetric tensor of rank 2 $F_{\alpha\beta}$, is $(\star\star F)_{\alpha\beta} = -F_{\alpha\beta}$. On the other hand in an Euclidean space ($s = 0$) we would have $(\star\star F)_{\alpha\beta} = F_{\alpha\beta}$.

3 Tensor formulation of classical EM

The classical four Maxwell's equations (in flat vacuum) in differential formulation, can be divided into two source independent equations

$$\nabla \cdot \mathbf{B} = 0 \tag{3.0.8.1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3.0.8.2}$$

called respectively Gauss's law for magnetism, and the Maxwell-Faraday equation (also called Faraday's law of induction), and two source dependent equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \tag{3.0.8.3}$$

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \tag{3.0.8.4}$$

called respectively Gauss's law and Ampere's circuital law (with Maxwell's addition).

Where ϵ_0 and μ_0 are the electric permittivity and the magnetic permeability of the vacuum, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field components, they do not form vectors or co-vectors, their true nature is as components of unified electromagnetic rank-2 tensor to be introduced below in (3.1.2.2). In this section we will introduce the naive 3D approach.

Using tensorial formulation the source independent equations are

$$B^i{}_{,i} = 0 \quad (3.0.8.5)$$

$$\varepsilon^{ijk} E_{i,j} + \dot{B}^k = 0 \quad (3.0.8.6)$$

where the dot over means differentiation by time $\dot{B}^k = \frac{\partial}{\partial t} B^k$, and for the source dependent equations are

$$E^i{}_{,i} = \frac{1}{\epsilon_0} \rho \quad (3.0.8.7)$$

$$\varepsilon^{ijk} B_{i,j} - \mu_0 \epsilon_0 \dot{E}^k = \mu_0 J^k \quad (3.0.8.8)$$

Since the magnetic field is divergence free $\nabla \cdot \mathbf{B} = 0$ we can define it as a curl of a potential co-vector $\mathbf{A}$

$$B^i = \varepsilon^{ijk} A_{j,k} \quad \text{or} \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (3.0.8.9)$$

substituting (3.0.8.9) into (3.0.8.2) or (3.0.8.6) we find that the curl of the electric field $\mathbf{E}$ equals to the curl of the time derivative of the vector potential $-\frac{\partial \mathbf{A}}{\partial t}$, hence they differ by an arbitrary gradient term $\nabla \phi$ in the following way

$$E_i = -\phi_{,i} - \dot{A}_i \quad \text{or} \quad \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \quad (3.0.8.10)$$

we can verify that ϕ is the known electrostatic potential by substituting (3.0.8.10) into (3.0.8.3) or (3.0.8.7) and setting $\mathbf{A} = 0$ to find the Poisson's equation relating the electric potential to a charge distribution

$$\phi_{,i}{}^{,i} = -\frac{\rho}{\epsilon_0} \quad \text{or} \quad \nabla^2 \phi = -\frac{\rho}{\epsilon_0} \quad (3.0.8.11)$$

in the general case the two source dependent equations takes the form

$$\phi_{,i}{}^{,i} + \dot{A}_i{}^{,i} = -\frac{\rho}{\epsilon_0} \quad (3.0.8.12)$$

$$\mu_0 \epsilon_0 \ddot{A}^i - A^{i,j}{}_{,j} + \partial^i (\mu_0 \epsilon_0 \dot{\phi} + A^j{}_{,j}) = \mu_0 J^i \quad (3.0.8.13)$$

or

$$\nabla^2 \phi + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = -\frac{\rho}{\epsilon_0} \quad (3.0.8.14)$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} + \nabla (\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J} \quad (3.0.8.15)$$

There is a gauge freedom in the potential formulation, if we transform the potentials by the following transformation

$$\begin{aligned} \phi' &= \phi + \frac{\partial \xi}{\partial t} & \text{or} & & \phi' &= \phi + \dot{\xi} \\ \mathbf{A}' &= \mathbf{A} - \nabla \xi & \text{or} & & A'^i &= A^i - \xi^{,i} \end{aligned} \quad (3.0.8.16)$$

where ξ is arbitrary scalar field, the fields they represented by this potential doesn't change. There are several important gauge choices, but we will focus only on the Lorenz gauge since it is the covariant gauge choice. The Lorenz gauge condition is

$$\mu_0 \varepsilon_0 \dot{\phi} + A^j_{,j} = 0 \quad \text{or} \quad \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \quad (3.0.8.17)$$

using this gauge Maxwell's equations in the potential formalism is a wave equation

$$\mu_0 \varepsilon_0 \ddot{\phi} - \phi^{,i}_{,i} = \frac{\rho}{\varepsilon_0} \quad \text{or} \quad (\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} - \nabla^2) \phi = \frac{\rho}{\varepsilon_0} \quad (3.0.8.18)$$

$$\mu_0 \varepsilon_0 \ddot{A}^i - A^{i,j}_{,j} = \mu_0 J^i \quad \text{or} \quad (\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} - \nabla^2) \mathbf{A} = \mu_0 \mathbf{J} \quad (3.0.8.19)$$

3.1 Formulation of EM in the 4D Minkowski space

3.1.1 The Minkowski space M

Definition 3.1. In special-relativity time is one of the space-time dimensions, and the space is called the Minkowski spacetime, a displacement of an event is $x^\mu = (ct, x, y, z)$. The metric of the (flat) Minkowski space is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

this metric express the idea of relative motion in it's formulation which is in the heart of special relativity theory.

The metric $\eta_{\alpha\beta}$ in 3.1 give rise to the following classification of 4-vectors in Minkowski space, let x be a 4-vector then:

$$\begin{cases} x \text{ like time is} & \iff x \cdot x > 0 \\ x \text{ like light is} & \iff x \cdot x = 0 \\ x \text{ like space is} & \iff x \cdot x < 0 \end{cases} \quad (3.1.1.1)$$

3.1.2 EM in Minkowski space

Define the 4-current J^μ is a tensor density of weight -1 which is written in 1 + 3D decomposition as

$$J^\mu = (c\rho, \mathbf{J}) \quad (3.1.2.1)$$

and define a covariant rank-2 anti-symmetric Faraday tensor field $F_{\mu\nu}$ as

$$\begin{aligned} F_{0j} &= \frac{1}{c} E_j \\ F_{ij} &= \varepsilon_{ijk} B^k \end{aligned} \quad (3.1.2.2)$$

or in a matrix form as

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c} E_1 & \frac{1}{c} E_2 & \frac{1}{c} E_3 \\ -\frac{1}{c} E_1 & 0 & B_3 & -B_2 \\ -\frac{1}{c} E_2 & -B_3 & 0 & B_1 \\ -\frac{1}{c} E_3 & B_2 & -B_1 & 0 \end{pmatrix}; \quad F^\mu{}_\nu = \begin{pmatrix} 0 & \frac{1}{c} E_1 & \frac{1}{c} E_2 & \frac{1}{c} E_3 \\ \frac{1}{c} E_1 & 0 & -B_3 & B_2 \\ \frac{1}{c} E_2 & B_3 & 0 & -B_1 \\ \frac{1}{c} E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (3.1.2.3)$$

Using (3.1.2.1), (3.1.2.2) we can express Maxwell's equations in flat vacuum using two equations:

$$\begin{aligned} \partial_\beta F^{\alpha\beta} &= \mu_0 \sqrt{-g} J^\alpha \\ F_{[\alpha\beta,\lambda]} &= 0 \quad \text{or} \quad \partial_\beta (\star F)^{\alpha\beta} = 0 \end{aligned} \quad (3.1.2.4)$$

where μ_0 is the magnetic permeability of the vacuum, and $F^{\alpha\beta}$ is the Faraday tensor with raised indexes. In order to reduce the number of the equations we will define a potential which will always satisfy the second equation. We define the 4-potential A_μ , so that

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = 2\partial_{[\alpha} A_{\beta]} = 2A_{[\beta,\alpha]} \quad (3.1.2.5)$$

which satisfies the second equation because the partial derivatives commute, and under an anti-symmetrization they cancel out. In 1 + 3D decomposition the 4-potential reads

$$A_\mu = \left(\frac{\phi}{c}; -\mathbf{A} \right) \quad (3.1.2.6)$$

the gauge freedom takes a simple form as

$$A'_\mu = A_\mu + \chi_{,\mu} \quad (3.1.2.7)$$

Substituting (3.1.2.5) into the source depended equation we get

$$\square A^\beta - A^{\alpha,\beta}_{,\alpha} = -\mu_0 J^\beta \quad (3.1.2.8)$$

by a gauge choice we can remove one degree of freedom out of the 4-potential. For example the covariant gauge is the Lorenz gauge which states that the divergence of the 4-potential vanishes

$$A^\alpha_{,\alpha} = A_\alpha{}^{,\alpha} = 0 \quad (3.1.2.9)$$

so Maxwell's equation for the 4-potential in the Lorenz gauge is

$$\square A = -\mu_0 J \quad (3.1.2.10)$$

which is the inhomogeneous wave equation.

4 A scalar invariant under the Lorentz Group

4.1 Short introduction to Matrix Lie groups

In this section we will introduce essential aspects of group theory. After that we will define the concept of Lie group and explore matrix Lie groups through construction of the general matrix Lie group $GL_n(F)$. Finally we will develop the matrix Lie algebra as a tool to deal with matrix Lie groups in a linear manner.

4.1.1 Group, subgroup, group action and group homomorphism

Definition 4.1. A group is a set, G , equipped with an operation $\bullet$ (called the *group operation* of G), that combines any two elements a and b to form another element, denoted $a \bullet b$ or as we prefer by the juxtaposition ab . Any group $(G, \bullet)$ or simply G , has to satisfy four requirements called the *group axioms*:

Closure For all $a, b \in G$, the result of the operation ab , is also in G .

Associativity For all $a, b, c \in G$, the following relation holds $(ab)c = a(bc)$.

Identity element There exists an element $e \in G$, so for any $a \in G$, the relation $ea = ae = a$ holds. Such an element is unique, hence it's called *the* identity element.

Inverse element For each $a \in G$, there exists $b \in G$ such that $ab = ba = e$, where e is the identity element.

Example 4.2. The group that consists of the set of integers $\mathbb{Z}$ with the integer addition "+" as group operation, is a group. We get closure since the addition of any two integers is an integer, associativity since integer addition is associative, the identity element is 0 since for any integer z the sum $z + 0 = 0 + z$ equals z , inverse element is exists for any z there exists $-z$ in $\mathbb{Z}$ such that $(z) + (-z) = 0$. This group is also *commutative* since for any two integers z_1, z_2 the sum is commutative $z_1 + z_2 = z_2 + z_1$.

Definition 4.3. A subgroup S of a group G , is a group consisting of a set which is subset of the set of G , and having the same group operation.

Example 4.4. The group of all even integers with integer addition as group operation is a subgroup of the group defined in 4.2.

Remark. Every group G has two trivial subgroups. The subgroup that it's associated set contains only the group's identity element e , and the subgroup which is the group G itself.

The idea of group by itself might have useful outcomes but for our purpose as we want to model symmetries and transformations of space, we will define next the concept of group action which explain how a group "acts" on a set which could be a model of physical space of our interest.

Definition 4.5. A group action ϕ of a group G on a set X is a function

$$\phi : G \times X \rightarrow X$$

that satisfies the following two axioms

Compatibility For all $g, h \in G$ and all $x \in X$, $\phi(gh, x) = \phi(g, \phi(h, x))$, where gh is the result of the group operation on g and h .

Identity For all $x \in X$, $\phi(e, x) = x$, here e is the identity element of G .

Example 4.6. The circle group $U(1)$ consists of all complex numbers $z \in \mathbb{C}$ having unit modulus $\|z\| = 1$ where the group operation is multiplication of complex numbers. We define an action ϕ of $u \in U(1)$ on a point in the Euclidean plane $r \in \mathbb{R}^2$ as

$$\phi(u, r) = (x\Re u - y\Im u, x\Im u + y\Re u)$$

where $r = (x, y)$ while $\Re u$ and $\Im u$ are the real part and the imaginary part of u respectively. Since for some real θ we can express u as $u = e^{i\theta} = \cos \theta + i \sin \theta$ we get

$$\phi(u, r) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$$

which is the known formula for rotating $\mathbb{R}^2$ by angle θ counterclockwise around the origin.

Definition 4.7. Let two groups $(G, \star)$, $(H, \cdot)$, a group homomorphism from $(G, \star)$ to $(H, \cdot)$, is a function $h : G \rightarrow H$, such that for all $g_1, g_2 \in G$

$$h(g_1 \star g_2) = h(g_1) \cdot h(g_2)$$

which is the way of saying that h is "compatible" with the group structure.

Example 4.8. Homomorphism from $U(1)$ defined in 4.6 to $SO(2)$ defined in 4.25 which is the matrix group of rotations of the Euclidean plane $\mathbb{R}^2$ around the origin. Let h be the homomorphism (which is actually an isomorphism) between $U(1)$ to $SO(2)$ defined by:

$$h(u_\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = R_\theta$$

where $u_\theta = e^{i\theta}$. This is expected indeed since by rearranging the result of 4.6 we get:

$$r' = \phi(u_\theta, r) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = h(u_\theta) r = R_\theta r$$

4.1.2 The general matrix group $GL_n(F)$

Definition 4.9. We will note the associative algebra consisting of square matrices of size $n \times n$ over the field F (where F is $\mathbb{R}$ or $\mathbb{C}$) by $M_n(F)$.

Example 4.10. $M_2(\mathbb{R})$ the algebra of 2×2 real matrices. It is the set

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Definition 4.11. The trace of a matrix $A \in M_n(F)$ is:

$$\text{tr } A = A^i_i$$

which could be also seen as a contraction of the indexes of A^i_j .

Example 4.12.

$$\text{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

Definition 4.13. The exponent of a matrix $A \in M_n(F)$ is defined by the series:

$$e^A = \exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

where it can be shown that series converges for any matrix in $M_n(F)$. The always converge property will not be proved on this report.

Example 4.14. The exponent of θA where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that $A^2 = I$ we have

$$\begin{aligned} \exp \theta A &= \sum_{k=0}^{\infty} \frac{(\theta A)^k}{k!} = \sum_{k=0}^{\infty} \left(I \frac{\theta^{2k}}{(2k)!} + A \frac{\theta^{2k+1}}{(2k+1)!} \right) \\ &= I \cosh \theta + A \sinh \theta = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \end{aligned}$$

Lemma 4.15. The matrix exponent satisfies the following properties:

- $e^0 = I$
- $e^{aX} e^{bX} = e^{(a+b)X}$

- $e^X e^{-X} = I$
- If $XY = YX$ (or $[X, Y] = 0$) then $e^{X+Y} = e^X e^Y = e^Y e^X$
- $(e^X)^t = e^{X^t}$.

where $X, Y \in M_n(F)$, X^t is the transpose of X defined in 2.19, I and 0 are the identity matrix and the zero matrix of $M_n(F)$ and $a, b \in F$.

Proof. Item by item proofs:

- $e^0 = I + \sum_{k=1}^{\infty} \frac{0^k}{k!} = I$
- $e^{aX} e^{bX} = \sum_{k=0}^{\infty} \frac{(aX)^k}{k!} \sum_{j=0}^{\infty} \frac{(bX)^j}{j!} = \sum_{k=0}^{\infty} \sum_{j=0}^k \frac{a^{k-j} b^j X^k}{(k-j)! j!} = \sum_{k=0}^{\infty} \frac{(a+b)^k X^k}{k!} = e^{(a+b)X}$
- Using first two properties together we get $e^X e^{-X} = e^{(1-1)X} = e^0 = I$.
- $e^{X+Y} = \sum_{k=0}^{\infty} \frac{(X+Y)^k}{k!} = \sum_{k=0}^{\infty} \sum_{j=0}^k \frac{X^{k-j} Y^j}{(k-j)! j!} = \sum_{k=0}^{\infty} \frac{X^k}{k!} \sum_{j=0}^{\infty} \frac{Y^j}{j!} = \sum_{k=0}^{\infty} \frac{Y^k}{k!} \sum_{j=0}^{\infty} \frac{X^j}{j!}$
- $(e^X)^t = \left(\sum_{k=0}^{\infty} \frac{X^k}{k!} \right)^t = \sum_{k=0}^{\infty} \frac{(X^k)^t}{k!} = \sum_{k=0}^{\infty} \frac{(X^t)^k}{k!} = e^{X^t}$

□

Definition 4.16. The adjugate matrix of any matrix $A \in M_n(F)$ is defined by:

$$\text{adj}(A)^i_j = \frac{\partial \det A}{\partial A^j_i}$$

where the determinant defined in 2.33 is seen as a differentiable function of n^2 independent entries of A .

Example 4.17. The adjugate matrix of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} \text{adj}(A)^i_j &= \frac{\partial \det A}{\partial A^j_i} = \frac{\partial (A^1_1 A^2_2 - A^1_2 A^2_1)}{\partial A^j_i} \\ &= \frac{\partial (ad - bc)}{\partial A^j_i} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \end{aligned}$$

Theorem 4.18. The product of any matrix $A \in M_n(F)$ with it's adjugate matrix $\text{adj } A$ is $I \det A$.

Proof. Using 2.33 we find $\text{adj}(A)^i_k$ explicitly:

$$\frac{\partial \det A}{\partial A^k_i} = \frac{\partial}{\partial A^k_i} \left(\frac{1}{n!} \varepsilon^{\mathbf{i}} \varepsilon_{\mathbf{k}} A^{\mathbf{k}}_{\mathbf{i}} \right) = \frac{1}{(n-1)!} \varepsilon^{i i'} \varepsilon_{k \mathbf{k}'} A^{\mathbf{k}'}_{i'}$$

where $\mathbf{i}$, $\mathbf{k}$, $\mathbf{i}'$ and $\mathbf{k}'$ are index arrays of sizes $n, n, n-1, n-1$ respectively, since there are n symbols of A and each has to be differentiated which gives some $\delta_{\tilde{i}}^i \delta_{\tilde{k}}^k$ where $\tilde{i}$ and $\tilde{k}$ are the indexes of the specific A symbol and since we are moving two indexes on two different

Levi-Civita symbols by the same "distance" we get no sign flip, summing up we get a factor of n .

Now substituting the result into 4.18 and using 2.36, 2.42 and 2.32 we get:

$$\begin{aligned} A^k_j \operatorname{adj}(A)^i_k &= \frac{1}{(n-1)!} \varepsilon^{ii'} \varepsilon_{kk'} A^k_j A^{k'}_{i'} \\ &= \frac{1}{(n-1)!} \varepsilon^{ii'} \varepsilon_{ji'} \det A = \delta^i_j \det A \end{aligned}$$

□

Lemma 4.19. Any matrix $A \in M_n(F)$ having $\det A \neq 0$ is invertible and given by $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

Proof. Since by 4.18 we get $A \operatorname{adj} A = I \det A$ if $\det A \neq 0$ we divide by $\det A$ and see that

$$A \left(\frac{1}{\det A} \operatorname{adj} A \right) = I$$

as required. □

Lemma 4.20. For any matrix $A \in M_n(F)$ the following holds:

$$\det e^A = e^{\operatorname{tr} A}$$

Proof. First we will find $\frac{d \det A(t)}{dt}$ where $A(t)$ is a differentiable function from a real interval into $M_n(F)$, by the chain rule:

$$\frac{d \det A(t)}{dt} = \frac{\partial \det A}{\partial A^j_i} \frac{dA^j_i}{dt} = \operatorname{adj}(A)^i_j \frac{dA^j_i}{dt} = \operatorname{tr} \left(\operatorname{adj} A \frac{dA}{dt} \right)$$

if $A(t)$ is invertible for all t in the interval then by 4.19 we have

$$\frac{d \det A(t)}{dt} = \det A \operatorname{tr} \left(A^{-1} \frac{dA}{dt} \right)$$

now taking $A(t) = e^{tB}$ which from 4.15 is invertible for any $B \in M_n(F)$ by e^{-tB} . Now $\frac{dA}{dt} = AB = BA$ and we get:

$$\frac{d \det e^{tB}}{dt} = \det e^{tB} \operatorname{tr}(B)$$

which it's solution is $\det e^{tB} = e^{t \operatorname{tr} B}$. □

Corollary 4.21. From 4.20 for any $X \in M_n(\mathbb{R})$ we get $\det e^X > 0$, and for any $Y \in M_n(\mathbb{C})$ we get $\det e^Y \neq 0$.

Definition 4.22. The general matrix group $GL_n(F)$ is the most unrestricted subset of $M_n(F)$ defined in (4.9) which could be a group where the group operation is matrix product.

Lemma 4.23. The general matrix group $GL_n(F)$ defined in 4.22 is the subset of all invertible matrices in $M_n(F)$ or more concisely $GL_n(F) = \{A \in M_n(F) : \det A \neq 0\}$.

Proof. Any non-invertible (singular matrix) cannot be included in $GL_n(F)$ since any element in a group must have inverse element.

Closure - $GL_n(F)$ is closed since the product of two invertible matrices in $M_n(F)$ is another invertible matrix in $M_n(F)$.

Associativity - the matrix product is associative.

Identity element - the identity matrix I exists in $GL_n(F)$ since it is an invertible matrix and obey $\forall X \in M_n(F) : IX = XI = X$.

Inverse element - all invertible matrices have an inverse which is invertible and hence in $GL_n(F)$. $\square$

4.1.3 Group representation

Definition 4.24. A representation Π of a group G on a vector space V over a field F ($\mathbb{R}$ or $\mathbb{C}$) is a group homomorphism from G to $GL(V)$, which is the group of linear invertible maps on V . When V is of finite dimension n it is possible to choose a basis for V , using this basis the elements of $GL(V)$ can be written as elements of $GL_n(F)$ defined in 4.22.

We will always work in the setting of V being finite and will identify $GL(V)$ with $GL_n(F)$, as a result we can see the representation Π as a mapping from group elements to matrices, which for any two group elements g_1, g_2 in G has

$$\Pi(g_1 g_2) = \Pi(g_1) \Pi(g_2)$$

which means that the matrix multiplication under the representation is "compatible" with the group operation. This type of representation is called *matrix representation*. We will deal exclusively with matrix representations.

Example 4.25. The group $SO(2)$. This is the group of rotations around the origin in the Euclidean plane $\mathbb{R}^2$ with the group operation of compositing the rotations. To each rotation R in $SO(2)$ an angle θ in $\mathbb{R}$ can be assigned, which specifies the angle of rotation counterclockwise.

This is a *commutative* group, since for any two rotations R_θ and R_ϕ the composition $R_\theta R_\phi = R_\phi R_\theta = R_{\theta+\phi}$ is also a rotation. The identity element is the identity transformation which can be viewed as no rotation at all, and the inverse of R_θ can be represented by $R_{-\theta}$. A representation Π for this group can be defined by:

$$\Pi(R_\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

4.1.4 $GL_n(F)$ and manifold

The group $GL_n(F)$ is not a linear space since for example $A, B \in M_2(\mathbb{R})$ where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ but $A - B, A + B \notin M_2(\mathbb{R})$. This means that $GL_n(F)$ does not have a global sense of displacement, and it is very hard to work with non-linear objects.

We will now develop an idea of subsets of linear space which are *locally* linear. This notion will be applicable to $GL_n(F)$. By locality we mean that when we "zoom in" the space is similar to some linear space, which is related to the infinitesimal "directions" we can "go" from a point to it's "nearby" points.

Definition 4.26. A function (or mapping) that has derivatives of all orders is called smooth.

Example 4.27. The real exponent function $x \in \mathbb{R} \rightarrow e^x$. Any finite polynomial of maximal degree k , $x \in \mathbb{R} \rightarrow P_k(x)$. The bump function

$$f(x) = \begin{cases} x \in [-1, 1] & e^{\frac{1}{x^2-1}} \\ \text{otherwise} & 0 \end{cases}$$

Definition 4.28. A smooth curve γ is a smooth function $\gamma : I \rightarrow S$ from an interval $I \subset \mathbb{F}$ (where F is $\mathbb{R}$ or $\mathbb{C}$) into a subset $S \subset V$ of a linear space V over $\mathbb{F}$. The tangent to the curve $\gamma(t)$ at some point $x \in S$ is $\frac{d\gamma(t)}{dt}\big|_x = \frac{d\gamma(t)}{dt}\big|_{t=t_0}$ where $\gamma(t_0) = x$.

Example 4.29. The path $\gamma : \theta \in (-\pi, \pi) \rightarrow S^1$, defined as $\gamma(\theta) = (\cos \theta, \sin \theta)$, where S^1 is the unit circle of $\mathbb{R}^2$. The tangent at $x = (1, 0)$ is $\frac{d\gamma}{d\theta}\big|_0 = (0, 1)$.

Definition 4.30. A manifold S of dimension n in a vector space V of finite dimension m ($n \leq m$), is a subset of V such that for every point x in S the set of all tangent vectors 4.28 at x is a linear subspace of V of dimension n . The collection of all tangent vectors at x is called the tangent space at x and is noted as $T_x S$.

Note. An alternative definition for $T_x S$ is the space of all directional derivatives at x for function defined on S .

Example 4.31. The $\mathbb{R}^2$ unit circle S^1 . It is the set of all points $(x, y) \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$. It is clearly not a linear space $A = (1, 0), B = (0, 1) \in S^1$ while $A + B, A - B \notin S^1$. But now observe that any point of S^1 has a tangent, which is a line in $\mathbb{R}^2$ and thus a 1-dimensional manifold.

Theorem 4.32. $GL_n(F)$ is a manifold of $M_n(F)$ having at each point a tangent space isomorphic to $M_n(F)$.

Proof. For any $g \in GL_n(F)$ and any $X \in M_n(F)$ by 4.21 $\gamma(t) = ge^{tX}$ is in $GL_n(F)$, hence the tangent at g is $\frac{d\gamma}{dt}\big|_0 = gX$. Since g is invertible then for any $Y \in M_n(F)$ there exists $g^{-1}Y \in M_n(F)$, thus for any $Y \in M_n(F)$ we get that Y is also in $T_g GL_n(F)$. $\square$

4.1.5 Matrix Lie groups and Matrix Lie algebras

We will define *matrix* Lie groups in a way which will suffice to our needs. For proper introduction to Lie groups and Lie algebras see [5].

Definition 4.33. A matrix Lie group G is a subgroup of $GL_n(F)$ (including $GL_n(F)$) which is also a manifold 4.30 of $M_n(F)$.

Definition 4.34. The matrix Lie algebra $\mathfrak{g}$ associated with the matrix Lie group G is identified as the tangent space at the identity element $T_e G$. It is closed under the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which is defined for any $X, Y \in \mathfrak{g}$ as $[X, Y] = XY - YX$, where juxtaposition is matrix multiplication. The elements of a Lie algebra are also called generators.

Remark. The Lie algebra captures the local structure of the Lie group. This is due to the fact that the matrix exponent is depended on the commutation relations of the matrices in it's argument. We will deal from now on mostly with Lie algebras of Lie groups and less with the Lie group by themselves.

Example 4.35. $\mathfrak{so}(2)$ the Lie algebra of $SO(2)$ defined in 4.25. We know that this is one parameter group given by $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, so it has one generator J using 4.34, 4.30 and 4.28 we find:

$$J = \frac{dR_\theta}{d\theta}\bigg|_{\theta=0} = \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

4.2 The Lorentz group, it's Lie algebra and their representations

4.2.1 The Orthogonal group $O(p, q)$ and it's Lie algebra $\mathfrak{o}(p, q)$

The group defined in 4.25 is the rotation group of rotations in the Euclidean space $\mathbb{R}^2$, we called it $SO(2)$ which is the Special Orthogonal group of the Euclidean plane, which is a length preserving transformation. In this section we will introduce the Orthogonal group in a generalized manner, which preserve the generalized "length" defined through the metric.

Definition 4.36. Let $\mathbb{R}^{p,q}$ be N dimensional metric space equipped with a metric $g_{\alpha\beta}$ having signature (p, q) , meaning that in any point the metric is similar to a diagonal matrix $diag(+1, \dots, -1)$ with p entries of $+1$ and q entries of -1 , the signature s of the metric equals here to q .

Definition 4.37. The orthogonal group $O(p, q)$ is a matrix Lie group 4.33 associated with the space $\mathbb{R}^{p,q}$ 4.36, it is the group of all linear transformations that preserves the inner-product of $\mathbb{R}^{p,q}$ induced by the metric. It's Lie algebra is denoted by $\mathfrak{o}(p, q)$.

Theorem 4.38. The elements of the orthogonal Lie algebra $\mathfrak{o}(p, q)$ 4.37 are anti-symmetric matrices, and it is of dimension $\binom{p+q}{2}$. It is closed under the matrix commutator.

Proof. Let any x^a, y^b in $\mathbb{R}^{p,q}$ and a linear transformation $L^{a'}_a$ then by 4.37 we must have:

$$g_{ab}x^ay^b = g_{a'b'}x^{a'}y^{b'} = g_{a'b'}L^{a'}_aL^{b'}_bx^ay^b$$

since x^a and y^b are arbitrary we can omit them and get

$$g_{ab} = g_{a'b'}L^{a'}_aL^{b'}_b$$

using $g_{a'b'}$ to lower $L^{b'}_b$ and contracting with the inverse metric g^{cb} we arrive at

$$\delta^c_a = g^{cb}g_{ab} = g^{cb}L^{a'}_aL_{a'b} = L^{a'}_aL_{a'}^b$$

in a matrix form this reads

$$I = LL^t$$

where L^t is the transpose of L defined in 2.19.

Now take a smooth curve $L(\psi) = e^{\psi m}$ where $m \in \mathfrak{o}(p, q)$ substituted into $LL^t = I$, and using $(e^X)^t = e^{X^t}$ from 4.15 we get at the identity

$$\left. \frac{d}{d\psi} I \right|_0 = 0 = \left. \frac{d}{d\psi} L^t L \right|_0 = \left. \frac{dL^t}{d\psi} L \right|_0 + L^t \left. \frac{dL}{d\psi} \right|_0 = m^t + m$$

rearranging we find

$$m^t = -m$$

as required.

Since the elements of $\mathfrak{o}(p, q)$ are anti-symmetric matrices we have $p + q$ indexes, so the dimension is indeed $\binom{p+q}{2}$ the number of possible choices of two different indexes.

For any $A, B \in \mathfrak{o}(p, q)$ check that also $[A, B] \in \mathfrak{o}(p, q)$ we get:

$$[A, B]^t = (AB - BA)^t = B^t A^t - A^t B^t = BA - AB = -[A, B]$$

as required. □

Following 4.38 An anti-symmetric tensor of rank 2 in an N dimensional space has $\binom{N}{2}$ independent components, this is equivalent to choosing the two directions which define the rotation plane, so we can define a generic notion to rotation generators and find their commutation relations.

Definition 4.39. In N dimensional metric space a generator of rotation in the plane designated by the ordered indexes a, b is

$$R_{ab} = R_{ab}^\mu = \delta_{ab}^{\mu\lambda} g_{\nu\lambda} = \delta_a^\mu g_{\nu b} - \delta_b^\mu g_{\nu a}$$

where a, b are labels and not indexes or can be viewed as indexes of the matrix of all rotation generators.

The result of 4.35 is easily recovered from the general definition 4.39 if we identify J as R_{21} , and recall that in Euclidean space $g_{ab} = \delta_{ab}$, we get

$$J = R_{21} = \delta_{21}^{\mu\alpha} g_{\nu\alpha} = \delta_2^\mu g_{\nu 1} - \delta_1^\mu g_{\nu 2} = \delta_2^\mu \delta_{\nu 1} - \delta_1^\mu \delta_{\nu 2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as was expected. If we had instead identified J with R_{12} we would have gotten a rotation in the reversed direction.

Since the commutation relations of a Lie algebra determinate it fully we now will find the commutation relations of $\mathfrak{o}(p, q)$ from the definition 4.39:

$$\begin{aligned} [R_{ab}, R_{cd}] &= R_{ab}R_{cd} - R_{cd}R_{ab} \\ &= \delta_{ab}^{\mu\alpha} g_{\nu\alpha} \delta_{cd}^{\nu\beta} g_{\lambda\beta} - \delta_{cd}^{\mu\alpha} g_{\nu\alpha} \delta_{ab}^{\nu\beta} g_{\lambda\beta} \\ &= (\delta_a^\mu g_{\nu b} - \delta_b^\mu g_{\nu a}) (\delta_c^\nu g_{\lambda d} - \delta_d^\nu g_{\lambda c}) - (\delta_c^\mu g_{\nu d} - \delta_d^\mu g_{\nu c}) (\delta_a^\nu g_{\lambda b} - \delta_b^\nu g_{\lambda a}) \\ &= g_{ad} (\delta_b^\mu g_{\lambda c} - \delta_c^\mu g_{\lambda b}) + g_{bd} (\delta_c^\mu g_{\lambda a} - \delta_a^\mu g_{\lambda c}) \\ &\quad + g_{ac} (\delta_d^\mu g_{\lambda b} - \delta_b^\mu g_{\lambda d}) + g_{bc} (\delta_a^\mu g_{\lambda d} - \delta_d^\mu g_{\lambda a}) \\ &= g_{ad} \delta_{bc}^{\mu\alpha} g_{\lambda\alpha} + g_{bd} \delta_{ca}^{\mu\alpha} g_{\lambda\alpha} + g_{ac} \delta_{db}^{\mu\alpha} g_{\lambda\alpha} + g_{bc} \delta_{ad}^{\mu\alpha} g_{\lambda\alpha} \end{aligned}$$

and finally using 4.39 we arrive at

$$[R_{ab}, R_{cd}] = g_{ad}R_{bc} + g_{db}R_{ca} + g_{ca}R_{db} + g_{bc}R_{ad} \quad (4.2.1.1)$$

4.2.2 One dimensional Lorentz transform

Consider the Orthogonal group $O(1, 1)$ from the signature we get $g_{ab} = \text{diag}(1, -1)$ using 4.39 we find that the only rotation generator has a matrix form:

$$R_{10} = B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

note that $B^2 = I$, the action of B is

$$e^{\psi B} = I \cosh \psi + B \sinh \psi = \begin{pmatrix} \cosh \psi & \sinh \psi \\ \sinh \psi & \cosh \psi \end{pmatrix}$$

identifying $\gamma = \cosh \psi$ and $\beta = \tanh \psi$ we get

$$e^{\psi B} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$$

putting it as a transformation from $\begin{pmatrix} ct \\ x \end{pmatrix}$ to $\begin{pmatrix} ct' \\ x' \end{pmatrix}$ we get

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \gamma \begin{pmatrix} ct - \beta x \\ x - \beta ct \end{pmatrix}$$

or just

$$\begin{cases} t' = \gamma \left(t + \frac{vx}{c^2} \right) \\ x' = \gamma (x + vt) \end{cases}$$

which is the known formula for Lorentz transform in one dimension, where the relative velocity between frames is v in the x -direction, where the direction of the x axis in each frame is the reverse of the direction in the other frame. A generators like B are called boosts generators.

4.2.3 Representation π

Note. We will use the same symbol for the representation of the Lorentz group and for its Lie algebra representation, except that the Lie algebra symbol will have additional underline.

In special relativity we consider a space which has 3 directions, having a rotation around each direction and having a boost in each direction. Since we can get a boost only with a mixture of indexes with opposite signs, we must have a metric with mixed signature (p, q) . We can summarize this requirement on the signature (p, q) as:

$$\binom{q}{2} = 3 \quad \text{and} \quad pq = 3$$

the result is $q = 3, p = 1$, thus the Lorentz group of special relativity is $O(1, 3)$, and the space is called the Minkowski space having metric $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$, this is in agreement with what was defined in 3.1 above.

Note. The property of having the same number of rotations and boosts generators is unique to Minkowski space in the sense that this is the only Lorentzian signature having this property.

Definition 4.40. The Lorentz group is represented by $O(1, 3)$ it is a six parameter group, its Lie algebra $\mathfrak{o}(1, 3)$ has generators: 3 boosts $\mathfrak{B}_i = R_{i0}$ and 3 rotations $\mathfrak{R}_i = \frac{1}{2}\epsilon_{0i}^{jk} R_{jk}$, where R_{ab} is the rotation generators defined in 4.39.

Lemma 4.41. The rotation generator R_{ij} define in 4.39 is related to the rotation generators $\mathfrak{R}_k$ defined in 4.40 as

$$R_{ij} = -\epsilon_{0ij}^k \mathfrak{R}_k$$

Proof. Starting with the definition of $\mathfrak{R}_k$ in 4.40 and multiplying both sides with $-\epsilon_{0ij}^k$, and using 2.43 we get:

$$\begin{aligned} -\epsilon_{0ij}^k \mathfrak{R}_k &= -\frac{1}{2}\epsilon_{0ij}^k \epsilon_{0k}^{ab} R_{ab} = -\frac{1}{2}\epsilon_{0ij}^k \epsilon_k^{0ab} R_{ab} \\ &= \frac{1}{2}\delta_{0ij}^{0ab} R_{ab} = \frac{1}{2}(R_{ij} - R_{ji}) = R_{ij} \end{aligned}$$

as required. □

Using 4.39 to find a component-wise formulation of the generators from 4.40 $\mathfrak{B}_j$ and $\mathfrak{R}_j$:

$$\begin{aligned} \mathfrak{B}_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathfrak{B}_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \mathfrak{B}_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ \mathfrak{R}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \mathfrak{R}_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} & \mathfrak{R}_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (4.2.3.1)$$

The Faraday tensor of mixed rank $(1, 1)$ can be written as

$$F^\mu{}_\nu = \frac{1}{c} E^j (\mathfrak{B}_j)^\mu{}_\nu + B^j (\mathfrak{R}_j)^\mu{}_\nu \quad (4.2.3.2)$$

in component-wise formulation we get the same result as in (3.1.2.3) above.

Theorem 4.42. *The commutation relations of the generators of the rotations $\mathfrak{R}_j$ and the boosts $\mathfrak{B}_j$ are:*

$$[\mathfrak{R}_i, \mathfrak{R}_j] = -\epsilon_{0ij}{}^k \mathfrak{R}_k \quad (4.2.3.3)$$

$$[\mathfrak{B}_i, \mathfrak{R}_j] = -\epsilon_{0ij}{}^k \mathfrak{B}_k \quad (4.2.3.4)$$

$$[\mathfrak{B}_i, \mathfrak{B}_j] = \epsilon_{0ij}{}^k \mathfrak{R}_k \quad (4.2.3.5)$$

Proof. Using (4.2.1.1), 4.39, 4.40, 2.43 and 4.41 for (4.2.3.3) we find

$$\begin{aligned} [\mathfrak{R}_i, \mathfrak{R}_j] &= \frac{1}{4} \epsilon_{0i}{}^{ab} \epsilon_{0j}{}^{cd} [R_{ab}, R_{cd}] = \frac{1}{4} \epsilon_{0i}{}^{ab} \epsilon_{0j}{}^{cd} (g_{ad} R_{bc} + g_{db} R_{ca} + g_{ca} R_{db} + g_{bc} R_{ad}) \\ &= -\frac{1}{4} g_{lj} g^{ea} g^{fb} \delta_{0ief}^{0lcd} (-g_{ad} \epsilon_{0bc}{}^k - g_{db} \epsilon_{0ca}{}^k - g_{ca} \epsilon_{0db}{}^k - g_{bc} \epsilon_{0ad}{}^k) \mathfrak{R}_k \\ &= \frac{1}{4} g_{lj} (-g^{fb} \delta_{0ief}^{0lcc} \epsilon_{0bc}{}^k + g^{ea} \delta_{0ie}^{0lcc} \epsilon_{0ca}{}^k + g^{fb} \delta_{0if}^{0lcc} \epsilon_{0db}{}^k - g^{ea} \delta_{0ie}^{0lcc} \epsilon_{0ad}{}^k) \mathfrak{R}_k \\ &= -g_{lj} g^{fb} \delta_{0ief}^{0lcc} \epsilon_{0bc}{}^k \mathfrak{R}_k = -(-g_{lj} g^{lb} \epsilon_{0bi}{}^k + g_{ij} g^{cb} \epsilon_{0bc}{}^k) \mathfrak{R}_k = -\epsilon_{0ij}{}^k \mathfrak{R}_k \end{aligned}$$

and for (4.2.3.4) we find

$$\begin{aligned} [\mathfrak{B}_i, \mathfrak{R}_j] &= \frac{1}{2} \epsilon_{0j}{}^{cd} [R_{i0}, R_{cd}] = \frac{1}{2} \epsilon_{0j}{}^{cd} (g_{id} R_{0c} + g_{ci} R_{d0}) \\ &= -\frac{1}{2} \epsilon_{0j}{}^{cd} (g_{ci} \mathfrak{B}_d - g_{id} \mathfrak{B}_c) = -\epsilon_{0ij}{}^k \mathfrak{B}_k \end{aligned}$$

and finally for (4.2.3.5) we find

$$[\mathfrak{B}_i, \mathfrak{B}_j] = [R_{i0}, R_{j0}] = R_{ji} = \epsilon_{0ij}{}^k \mathfrak{R}_k$$

□

Remark. We can see from (4.2.3.3) that the rotations are closed under the commutator and hence generate a sub-group which is isomorphic to the rotation group $SO(3)$.

Another special property of the Lorentz group $O(1, 3)$ is that since it's total dimension is 4 the Hodge dual maps the space of rank 2 anti-symmetric tensors into itself. For example using 2.44, 4.39, 2.45 and (4.2.3.6) we can calculate the Hodge dual of the boost and rotation generators defined in 4.40 as

$$\begin{aligned}\star \mathfrak{B}_i &= \star(R_{i0}) = \star(\delta_{i0}^{\mu\rho} g_{\nu\rho}) = \frac{1}{2} \epsilon_{\lambda\mu}^{\eta}{}^{\nu} \delta_{i0}^{\mu\rho} g_{\nu\rho} \\ &= \frac{1}{2} \left(\epsilon_{\lambda i 0}^{\eta} - \epsilon_{\lambda 0 i}^{\eta} \right) = -\epsilon_{\lambda 0 i}^{\eta} = -\epsilon_{0 i}^{\eta}{}_{\lambda}\end{aligned}\tag{4.2.3.6}$$

$$\begin{aligned}&= -\frac{1}{2} \epsilon_{0 i}^{\eta}{}^{jk} \delta_{jk}^{\eta\rho} g_{\lambda\rho} = -\frac{1}{2} \epsilon_{0 i}^{\eta}{}^{jk} R_{jk} = -\mathfrak{R}_i \\ \star \mathfrak{R}_i &= \star \star \mathfrak{B}_i = \mathfrak{B}_i\end{aligned}\tag{4.2.3.7}$$

summarizing (4.2.3.6) and (4.2.3.7) we get

$$\begin{aligned}\star \mathfrak{B}_j &= -\mathfrak{R}_j \\ \star \mathfrak{R}_j &= \mathfrak{B}_j\end{aligned}\tag{4.2.3.8}$$

We are trying to find an invariant scalar under the Lorentz group which will be made out of the null-displacement connecting an event and the observer. We know that the only thing that the Lorentz group preserves is the inner-product, but since the displacement is null, it's inner product yields 0 which is a useless trivial invariant.

Since the Hodge dual induces an automorphism for the Lorentz generators (rank 2 anti-symmetric tensors), using the Hodge duality we might get a finer structure of the Lorentz group which will preserve non-trivial invariants.

In order to do so we investigate the Hodge dual as linear operator on a vector space, where the vector space is the space of the rank 2 anti-symmetric tensors and try to find it's eigenvalues and eigenvectors. Let F be a rank 2 anti-symmetric tensor which is an eigenvector of the Hodge dual operator having eigenvalue $\lambda \neq 0$ thus

$$\star F = \lambda F$$

applying the Hodge dual we find

$$\star \star F = \star \lambda F = \lambda \star F = \lambda^2 F\tag{4.2.3.9}$$

on the other hand from 2.45 the square of the Hodge dual on rank 2 tensors in Minkowski space is the same tensor multiplied by -1 . Substituting into (4.2.3.9) and omitting the tensor since it is arbitrary we get a condition on the eigenvalues λ as

$$\lambda^2 = -1\tag{4.2.3.10}$$

clearly this equation does not have a solution under the reals so it cannot have a solution in the real Minkowski space. A remediation to this problem will be presented in the next section, which will introduce the complexified Minkowski space which will allow us to solve (4.2.3.10).

4.2.4 Complexified Minkowski space and the σ -dual subspaces $\mathfrak{o}_{\pm}$ of $\mathfrak{o}(1, 3)$

Definition 4.43. The complexified Minkowski space M_c is a natural complex extension of the real Minkowski space 3.1, thus it is a space of 4 complex dimensions equipped with a metric $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$ that acts in the same manner on the real and the imaginary parts of the vectors. The inner-product of the complex 4-vectors x and y is defined through $x \circ y = \eta_{\alpha\beta} x^{\alpha} y^{\beta} = \eta_{\alpha\beta} x^{\beta} y^{\alpha}$.

Now in the context of the complexified Minkowski space 4.43 we can find a solution to (4.2.3.10), since the complexified space M_c is a natural extension of the real Minkowski space M all of the derivations done above for M will hold also in M_c . Now the eigenvalues under M_c can be found as

$$\lambda = \pm \iota \quad (4.2.4.1)$$

The space of rank 2 anti-symmetric tensors which coincides with $\mathfrak{o}(1,3)$ (or $\mathfrak{o}$ for short) the $\underline{\pi}$ Lie algebra representation of the real Lorentz Lie algebra, has by (4.2.4.1) two different eigenvalues for the Hodge dual operator. This means that the matrix space $\mathfrak{o}$ has two sub-spaces each associated with one of the two eigenvalues, we define these two sub-spaces of $\mathfrak{o}$ as $\mathfrak{o}_+$ and $\mathfrak{o}_-$.

Definition 4.44. The sub-space $\mathfrak{o}_\pm$ of $\mathfrak{o}$ is the subspace consisting of eigenvectors of the Hodge dual operator having the eigenvalue $\pm \iota$. It is called a σ -dual subspace and it's elements are called σ -dual, where σ takes the value $\pm \iota$.

Lemma 4.45. The $\mathfrak{o}_\pm$ sub-spaces 4.44 are related through the complex conjugation as $\mathfrak{o}_\pm^* = \mathfrak{o}_\mp$. In detail for any F in $\mathfrak{o}_\pm$ then F^* is in $\mathfrak{o}_\mp$.

Proof. For any F in $\mathfrak{o}_\pm$ by definition we have:

$$\star F = \pm \iota F$$

taking the complex conjugation of this equation we get

$$\star F^* = \mp \iota F^*$$

as required for any F in $\mathfrak{o}_\pm$, F^* is in $\mathfrak{o}_\mp$. □

Theorem 4.46. The projection operators from $\mathfrak{o}$ into $\mathfrak{o}_\pm$ denoted as $P_\pm$ are given by

$$P_\pm = \frac{1}{2} (1 \mp \iota \star)$$

Proof. Validating that the projection operators 4.46 indeed project into the correct sub-space

$$\star P_\pm = P_\pm \star = \star \frac{1}{2} (1 \mp \iota \star) = \frac{1}{2} (\pm \iota + \star) = \pm \iota \frac{1}{2} (1 \mp \iota \star) = \pm \iota P_\pm \quad (4.2.4.2)$$

which agrees with the definition of $\mathfrak{o}_\pm$ 4.44. Now validate that the projection operators 4.46 are indeed projections

$$(P_\pm)^2 = \frac{1}{4} (1 \mp \iota \star)^2 = \frac{1}{4} (1 \mp 2\iota \star - \star^2) = \frac{1}{4} (1 \mp \iota \star) = P_\pm$$

which is the defining property of a projection operator. □

Note. We can show that the sub-spaces $\mathfrak{o}_\pm$ are disjoint since

$$P_\pm P_\mp = \frac{1}{4} (1 \mp \iota \star) (1 \pm \iota \star) = \frac{1}{4} (1 + \star^2) \equiv 0$$

Lemma 4.47. Any element in $\mathfrak{o}$ can be decomposed as a unique sum of element in $\mathfrak{o}_+$ and an element in $\mathfrak{o}_-$.

Proof. Any F in $\mathfrak{o}$ has unique components $P_{\pm}F$ in $\mathfrak{o}_{\pm}$, since from 4.46 we get:

$$P_+ + P_- = \frac{1}{2}(1 - \iota\star) + \frac{1}{2}(1 + \iota\star) = 1 \quad (4.2.4.3)$$

thus we have a unique composition for F as:

$$F = (P_+ + P_-)F = P_+F + P_-F$$

□

4.2.5 Representations $\tilde{\pi}_+$ and $\tilde{\pi}_-$

Using (4.2.3.8) the projections of the generators $\mathfrak{B}_i$ and $\mathfrak{R}_i$ define in 4.40 can be calculate as:

$$\begin{aligned} \tilde{\mathfrak{B}}_i^{\pm} &= P_{\pm}\mathfrak{B}_i = \frac{1}{2}(\mathfrak{B}_i \pm \iota\mathfrak{R}_i) \\ \tilde{\mathfrak{R}}_i^{\pm} &= P_{\pm}\mathfrak{R}_i = \frac{1}{2}(\mathfrak{R}_i \mp \iota\mathfrak{B}_i) = \mp \iota\tilde{\mathfrak{B}}_i^{\pm} \end{aligned} \quad (4.2.5.1)$$

Theorem 4.48. *The projection of the generators calculated in (4.2.5.1) $\tilde{\mathfrak{B}}_i^{\pm}$ and $\tilde{\mathfrak{R}}_i^{\pm}$, each of the two sets induced by the different projection operators $P_{\pm}$ obey the same commutation relation 4.42 as the generators defined in 4.40, and the two sets completely commutes.*

Proof. Using 4.42 and (4.2.5.1) the first commutation relation is

$$\begin{aligned} [\tilde{\mathfrak{R}}_i^{\pm}, \tilde{\mathfrak{R}}_j^{\pm}] &= \frac{1}{4}[\mathfrak{R}_i \mp \iota\mathfrak{B}_i, \mathfrak{R}_j \mp \iota\mathfrak{B}_j] \\ &= \frac{1}{4}([\mathfrak{R}_i, \mathfrak{R}_j] - [\mathfrak{B}_i, \mathfrak{B}_j] \mp 2\iota[\mathfrak{B}_i, \mathfrak{R}_j]) \\ &= -\epsilon_{0ij} \frac{1}{2}(\mathfrak{R}_k \mp \iota\mathfrak{B}_k) = -\epsilon_{0ij} \frac{1}{2}\tilde{\mathfrak{R}}_k^{\pm} \end{aligned}$$

as requiried, since by (4.2.5.1) we have $\tilde{\mathfrak{R}}_i^{\pm} = \mp \iota\tilde{\mathfrak{B}}_i^{\pm}$, the other two relations follows trivially. As for the commutation again using 4.42 and (4.2.5.1) we get

$$\begin{aligned} [\tilde{\mathfrak{R}}_i^{\pm}, \tilde{\mathfrak{R}}_j^{\mp}] &= \frac{1}{4}[\mathfrak{R}_i \mp \iota\mathfrak{B}_i, \mathfrak{R}_j \pm \iota\mathfrak{B}_j] \\ &= \frac{1}{4}([\mathfrak{R}_i, \mathfrak{R}_j] + [\mathfrak{B}_i, \mathfrak{B}_j]) = 0 \end{aligned}$$

and again as before as consequence of $\tilde{\mathfrak{R}}_i^{\pm} = \mp \iota\tilde{\mathfrak{B}}_i^{\pm}$, all the other combinations follows trivially. □

Next we introduce a very useful property of the Lorentz generators in $\tilde{\pi}^{\pm}$ which ease drastically the calculation of the exponent map, thus finding the resulting transformation.

Lemma 4.49. *The projections (4.2.5.1) of the π generators satisfy the relation*

$$\{\tilde{\mathfrak{R}}_i^{\pm}, \tilde{\mathfrak{R}}_j^{\pm}\} = \frac{1}{2}\eta_{ij}I$$

where η is the Minkowski metric, $\{, \}$ stands for the anti-commutator $\{A, B\} = AB + BA$, and I stands for the 2-rank delta δ^{μ}_{ν} .

Proof. First we find $\tilde{\mathfrak{R}}_i^{\pm\mu} \tilde{\mathfrak{R}}_j^{\pm\nu}$ using 4.39, 4.40 and 2.43

$$\begin{aligned}
\tilde{\mathfrak{R}}_i^{\pm\mu} \tilde{\mathfrak{R}}_j^{\pm\nu} &= \frac{1}{4} \left(\epsilon_{0i}^{\mu} \mp \imath \delta_{i0}^{\mu\alpha} \eta_{\nu\alpha} \right) \left(\epsilon_{0j}^{\nu} \mp \imath \delta_{j0}^{\nu\beta} \eta_{\lambda\beta} \right) \\
&= \frac{1}{4} \left(-\epsilon_{0\nu i}^{\mu} \epsilon_{0j\lambda}^{\nu} - \delta_{i0}^{\mu\alpha} \eta_{\nu\alpha} \delta_{j0}^{\nu\beta} \eta_{\lambda\beta} \mp \imath \left(\delta_{i0}^{\mu\alpha} \eta_{\nu\alpha} \epsilon_{0j\lambda}^{\nu} + \epsilon_{0i}^{\mu} \delta_{j0}^{\nu\beta} \eta_{\lambda\beta} \right) \right) \\
&= \frac{1}{4} \left(\delta_{0j\lambda}^{\rho\pi\mu} \eta_{\pi i} \eta_{\rho 0} + \delta_0^{\mu} \eta_{ji} \eta_{\lambda 0} + \delta_i^{\mu} \eta_{\lambda j} \eta_{00} \mp \imath \left(-\delta_0^{\mu} \epsilon_{0ji\lambda} + \epsilon_{0i}^{\mu} \eta_{j\lambda 0} \right) \right) \\
&= \frac{1}{4} \left(\delta_{\lambda}^{\mu} \eta_{ji} - \delta_0^{\mu} \eta_{\lambda 0} \eta_{ji} - \delta_j^{\mu} \eta_{\lambda i} + \delta_i^{\mu} \eta_{\lambda j} + \delta_0^{\mu} \eta_{ji} \eta_{\lambda 0} \mp \imath \left(-\delta_0^{\mu} \epsilon_{0ji\lambda} + \epsilon_{0i}^{\mu} \eta_{j\lambda 0} \right) \right) \\
&= \frac{1}{4} \left(\delta_{\lambda}^{\mu} \eta_{ji} + \delta_i^{\mu} \eta_{\lambda j} - \delta_j^{\mu} \eta_{\lambda i} \mp \imath \left(-\delta_0^{\mu} \epsilon_{0ji\lambda} + \epsilon_{0i}^{\mu} \eta_{j\lambda 0} \right) \right)
\end{aligned}$$

all terms except $\delta_{\lambda}^{\mu} \eta_{ji}$ are anti-symmetric in i, j and thus under the anti-commutator vanishes, the term $\delta_{\lambda}^{\mu} \eta_{ji}$ is symmetric in i, j and hence get multiplied by 2 under the commutator, and since $\delta_{\nu}^{\mu} \equiv I_{\nu}^{\mu}$, we get

$$\{\tilde{\mathfrak{R}}_i^{\pm}, \tilde{\mathfrak{R}}_j^{\pm}\} = \frac{1}{4} 2 \delta_{\lambda}^{\mu} \eta_{ji} = \frac{1}{2} \eta_{ij} I$$

as required. $\square$

Lemma 4.50. *The $\tilde{\mathfrak{B}}^{\pm}$ generators projections (4.2.5.1) obey:*

$$\begin{aligned}
\tilde{\mathfrak{B}}_i^{\pm\alpha\beta} \tilde{\mathfrak{B}}_{j\alpha\beta}^{\pm} &= \eta_{ij} \\
\tilde{\mathfrak{B}}_i^{\pm\alpha\beta} \tilde{\mathfrak{B}}_{j\alpha\beta}^{\mp} &= 0
\end{aligned}$$

Proof. Using 4.49 we get:

$$-\frac{1}{2} \eta_{ij} \delta_{\lambda}^{\mu} = \tilde{\mathfrak{B}}_i^{\pm\mu} \tilde{\mathfrak{B}}_j^{\pm\nu} \lambda + \tilde{\mathfrak{B}}_j^{\pm\mu} \tilde{\mathfrak{B}}_i^{\pm\nu} \lambda$$

contracting on μ and λ we get:

$$-2\eta_{ij} = \tilde{\mathfrak{B}}_i^{\pm\lambda} \tilde{\mathfrak{B}}_j^{\pm\nu} \lambda + \tilde{\mathfrak{B}}_j^{\pm\lambda} \tilde{\mathfrak{B}}_i^{\pm\nu} \lambda = 2\tilde{\mathfrak{B}}_i^{\pm\alpha\beta} \tilde{\mathfrak{B}}_{j\beta\alpha}^{\pm} = -2\tilde{\mathfrak{B}}_i^{\pm\alpha\beta} \tilde{\mathfrak{B}}_{j\alpha\beta}^{\pm}$$

as required. Now using (4.2.5.1) and 2.43 we get:

$$\begin{aligned}
\tilde{\mathfrak{B}}_i^{\pm\alpha\beta} \eta_{\alpha\alpha'} \eta_{\beta\beta'} \tilde{\mathfrak{B}}_j^{\mp\alpha'\beta'} &= \frac{1}{4} \left(\delta_{i0}^{\alpha\beta} \pm \imath \epsilon_{i0}^{\alpha\beta} \right) \eta_{\alpha\alpha'} \eta_{\beta\beta'} \left(\delta_{j0}^{\alpha'\beta'} \mp \imath \epsilon_{j0}^{\alpha'\beta'} \right) \\
&= \frac{1}{4} \left(2\eta_{ij} \eta_{00} - 2\eta_{0i} \eta_{j0} + \epsilon_{i0}^{\alpha\beta} \epsilon_{j0\alpha\beta} + 2\imath [\epsilon_{i0j0} - \epsilon_{j0i0}] \right) \\
&= \frac{1}{4} (2\eta_{ij} \eta_{00} - 2\eta_{ij} \eta_{00} + 2\eta_{i0} \eta_{0j}) = 0
\end{aligned}$$

as required. $\square$

Corollary 4.51. *From 4.50 we see that we can write the projection operators 4.46 in coordinates using as a basis the projected generators (4.2.5.1) as:*

$$P_{\pm\mu\nu}^{\alpha\beta} = \eta^{ij} \tilde{\mathfrak{B}}_i^{\pm\alpha\beta} \tilde{\mathfrak{B}}_{j\mu\nu}^{\pm}$$

Definition 4.52. We define a representations of the Lorentz group $\tilde{\pi}_{\pm}$ induced by the Lie algebra $\tilde{\pi}_{\pm}$ which coincides with $\mathfrak{o}_{\pm}$ the eigen-subspace of $\mathfrak{o}(1, 3)$ under the Hodge operator. These are indeed a representations since by 4.48 each subspace $\mathfrak{o}_{\pm}$ obey the same commutation relations as $\mathfrak{o}(1, 3)$ and has the same dimension, more over also by 4.48, $\tilde{\pi}_{\pm}$ and $\tilde{\pi}_{\mp}$ commutes.

4.2.6 The projections of the Faraday tensor

In this section we explore the separation of π into $\tilde{\pi}_{\pm}$, this in order to find the necessary details of suitable definition of a Faraday tensor for each $\tilde{\pi}_{\pm}$ representation. Using (4.2.3.2), (4.2.5.1) to calculate the projections of the Faraday tensor (3.1.2.2) we get

$$\mathcal{F}^{\pm} = P_{\pm}F = P_{\pm} \left(\frac{1}{c} E^j \mathfrak{B}_j + B^j \mathfrak{R}_j \right) = \left(\frac{1}{c} E^j \mp \imath B^j \right) \tilde{\mathfrak{B}}_j^{\pm} \quad (4.2.6.1)$$

Following (4.2.6.1) we introduce the Riemann–Silberstein vector

$$F_{\pm}^j = \frac{1}{c} E^j \mp \imath B^j \text{ or } \mathbf{F}_{\pm} = \frac{1}{c} \mathbf{E} \mp \imath \mathbf{B} \quad (4.2.6.2)$$

this is indeed a vector since B^j is a pseudo-vector and $\imath$ is a pseudo-scalar their product is a proper vector and can be added to E^j . Now using this we can write

$$\mathcal{F}^{\pm} = F_{\pm}^j \tilde{\mathfrak{B}}_j^{\pm} \quad (4.2.6.3)$$

Since the projection defines a unique decomposition 4.47 we have:

$$F = \mathcal{F}^+ + \mathcal{F}^- \quad (4.2.6.4)$$

Now after we got the separation into $\tilde{\pi}_{\pm}$ we first have to prove that the separation does make sense. Meaning that each $\tilde{\pi}_{\pm}$ representation is "on it's own" and is indeed compatible with π , this is in the sense that we can define a relation between π and $\tilde{\pi}_{\pm}$ which will be useful and preserve covariance in a compatible way.

4.2.7 Lorentz covariance relation of π and $\tilde{\pi}_{\pm}$

Consider the exponent map $\Lambda = e^F$ of some generator $F \in \pi$, using the decomposition (4.2.6.4) and the commutativity 4.48 and the commutation property of the matrix exponent map that if $[A, B] = 0$ then $e^{A+B} = e^A e^B = e^B e^A$ 4.15 to get:

$$\Lambda = e^F = e^{\mathcal{F}^+ + \mathcal{F}^-} = e^{\mathcal{F}^+} e^{\mathcal{F}^-} = e^{\mathcal{F}^-} e^{\mathcal{F}^+} = \Upsilon^+ \Upsilon^- \quad (4.2.7.1)$$

where $\mathcal{F}^{\pm} \in \tilde{\pi}_{\pm}$ are defined in (4.2.6.4) and $\Upsilon^{\pm} = e^{\mathcal{F}^{\pm}}$.

In order to ease calculations we use $\mathcal{F}^{\pm} = F_{\pm}^j \tilde{\mathfrak{B}}_j^{\pm}$ from (4.2.6.3) to find the square of the Faraday tensor projections, using also the anti-commutation relations 4.49 and 4.50, we get

$$(\mathcal{F}^{\pm})^2 = (F_{\pm}^j \tilde{\mathfrak{B}}_j^{\pm})^2 = \frac{\mathbf{F}_{\pm}^2}{4} I \quad (4.2.7.2)$$

where I is the identity matrix and $\mathbf{F}_{\pm}^2 = -\eta_{ij} F_{\pm}^i F_{\pm}^j$ is the norm of the Riemann–Silberstein vector.

This result is a wonderful, because, we see that the square of the projection of the Faraday tensor is an identity matrix multiplied by scalar. This will ease the calculation of the exponent tremendously as the series of the exponent will now have just two "matrix" values. For odd terms that will be the identity matrix. For even terms that will be the projection of the Faraday tensor.

To ease calculation further we can find a complex scalar $\eta_{\pm}$ where $(\eta_+)^* = \eta_-$ such that $\eta_{\pm}^2 = (\mathbf{F}_{\pm})^2$, multiplying both sides of (4.2.7.2) by $\left(\frac{2}{\eta_{\pm}}\right)^2$ we get

$$\left(\frac{2\mathcal{F}^{\pm}}{\eta_{\pm}}\right)^2 = I \quad (4.2.7.3)$$

we just defined from $\mathcal{F}^{\pm}$ a tensor that will square to the unit matrix, now we are ready for some calculations.

Using (4.2.7.1) and (4.2.7.3) to calculate $\Upsilon^{\pm}$, we get

$$\Upsilon^{\pm} = e^{\mathcal{F}^{\pm}} = e^{\frac{\eta_{\pm}}{2} \frac{2\mathcal{F}^{\pm}}{\eta_{\pm}}} = I \cosh\left(\frac{\eta_{\pm}}{2}\right) + \frac{2\mathcal{F}^{\pm}}{\eta_{\pm}} \sinh\left(\frac{\eta_{\pm}}{2}\right) \quad (4.2.7.4)$$

Putting the result for $\Upsilon^{\pm}$ (4.2.7.4) back into $\Lambda = e^F$ (4.2.7.1) we find

$$\begin{aligned} \Lambda &= \Upsilon^+ \Upsilon^- \\ &= \left(I \cosh\left(\frac{\eta_+}{2}\right) + \frac{2\mathcal{F}^+}{\eta_+} \sinh\left(\frac{\eta_+}{2}\right) \right) \left(I \cosh\left(\frac{\eta_-}{2}\right) + \frac{2\mathcal{F}^-}{\eta_-} \sinh\left(\frac{\eta_-}{2}\right) \right) \end{aligned} \quad (4.2.7.5)$$

which is indeed real, and we see clearly that the computation in the half-representation eased the calculation of the whole π representation.

Theorem 4.53. *The covariance of the Faraday tensor F under π is equivalent to the covariance of the projections of the Faraday tensor $\mathcal{F}^{\pm}$ under $\tilde{\pi}_{\pm}$.*

Proof. Take any $\Lambda \in \pi$ which is an exponent map of generator in π by (4.2.7.1) there exists $\Upsilon^{\pm} \in \tilde{\pi}_{\pm}$ which is an exponent map of a generator in $\tilde{\pi}_{\pm}$ such that $\Lambda = \Upsilon^+ \Upsilon^-$ and $(\Upsilon^+)^* = \Upsilon^-$, now for any $F \in \pi$, the transformation Λ sends it to F' as $F' = \Lambda F \Lambda^{-1}$, using the decomposition (4.2.6.4) and the commutation 4.49 we find out:

$$\begin{aligned} F' &= \Lambda F \Lambda^{-1} = \Upsilon^+ \Upsilon^- (\mathcal{F}^+ + \mathcal{F}^-) (\Upsilon^+ \Upsilon^-)^{-1} \\ &= \Upsilon^+ \mathcal{F}^+ (\Upsilon^+)^{-1} + \Upsilon^- \mathcal{F}^- (\Upsilon^-)^{-1} = \mathcal{F}'^+ + \mathcal{F}'^- \end{aligned}$$

as required. $\square$

4.2.8 Complexification of the 4-potential

The projections of the real Faraday tensor into $\tilde{\pi}_{\pm}$ (by $P_{\pm}$) as defined in (4.2.6.1) are

$$\mathcal{F}_{\alpha\beta}^{\pm} = P_{\pm} F_{\alpha\beta} = P_{\pm} 2A_{[\beta,\alpha]}$$

where A_{β} is the 4-potential of the EM field $F_{\alpha\beta}$ as in (3.1.2.5). Consider a complex co-vector field $\mathcal{A}_{\mu}^{\pm}$ such that $\mathcal{F}_{\alpha\beta}^{\pm} = P_{\pm} 2\mathcal{A}_{[\beta,\alpha]}^{\pm}$. Using (4.2.4.2) we get:

$$\mathcal{F}_{\alpha\beta}^{\pm} = P_{\pm} 2\mathcal{A}_{[\beta,\alpha]}^{\pm} = P_{\pm} \left(2\Re \mathcal{A}_{[\beta,\alpha]}^{\pm} + 2i\Im \mathcal{A}_{[\beta,\alpha]}^{\pm} \right) = P_{\pm} \left(2\Re \mathcal{A}_{[\beta,\alpha]}^{\pm} \pm 2\star \Im \mathcal{A}_{[\beta,\alpha]}^{\pm} \right), \quad (4.2.8.1)$$

taking the real part of (4.2.8.1) we get

$$F_{\alpha\beta} = 2\Re \mathcal{A}_{[\beta,\alpha]}^{\pm} \pm 2\star \Im \mathcal{A}_{[\beta,\alpha]}^{\pm}. \quad (4.2.8.2)$$

This relation is the Cabibbo-Ferrari-Shanmugadhasan relation [2] used when considering magnetic charges (magnetic monopoles).

Question arises, when does $\mathcal{F}_{\alpha\beta}^{\pm} = 2\mathcal{A}_{[\beta,\alpha]}^{\pm}$? more precisely when $2\mathcal{A}_{[\beta,\alpha]}^{\pm} \in \mathfrak{o}_{\pm}$? This is equivalent to stating that:

$$P_{\mp} 2\mathcal{A}_{[\beta,\alpha]}^{\pm} = 0. \quad (4.2.8.3)$$

In such cases we will call $\mathcal{A}^{\pm}$ a σ -dual complex 4-potential. If $\mathcal{A}^{\pm}$ is a σ -dual then $2\Re\mathcal{A}_{[\beta,\alpha]}^{\pm} = \pm 2 \star \Im\mathcal{A}_{[\beta,\alpha]}^{\pm}$ and hence $F_{\alpha\beta} = \pm 4 \star \Im\mathcal{A}_{[\beta,\alpha]}^{\pm}$. Now substituting into the Maxwell's equation (3.1.2.4), using Hodge dual definition 2.44 and the anti-symmetry of the Levi-Civita tensor we get

$$\mu_0 \sqrt{-g} J^{\alpha} = \partial_{\beta} F^{\alpha\beta} = \pm 4 \partial_{\beta} \star \Im\mathcal{A}^{\pm[\beta,\alpha]} = \pm 2 \partial_{\beta} \epsilon^{\alpha\beta\mu\nu} \partial_{\mu} \Im\mathcal{A}_{\nu}^{\pm} = 0.$$

This implies that:

Proposition 4.54. *Outside of the sources and only there it is possible to have a σ -dual complex 4-potential. In this case $4\Re\mathcal{A}_{[\beta,\alpha]}^{\pm} = F_{\alpha\beta}$ and $\Re\mathcal{A}_{[\beta,\alpha]}^{\pm} = \pm \star \Im\mathcal{A}_{[\beta,\alpha]}^{\pm}$.*

Note. In [1] a source free potentials inducing Hodge dual fields are used.

If our point of interest x is inside the sources $J(x) \neq 0$, substituting (4.2.8.2) the Maxwell's equation (3.1.2.4) we get:

$$\partial_{\beta} F^{\alpha\beta} = \Re\mathcal{A}_{,\beta}^{\pm\beta,\alpha} - \square \Re\mathcal{A}^{\pm\alpha} = \mu_0 \sqrt{-g} J^{\alpha} \quad (4.2.8.4)$$

$$\mp \partial_{\beta} (\star F)^{\alpha\beta} = \Im\mathcal{A}_{,\beta}^{\pm\beta,\alpha} - \square \Im\mathcal{A}^{\pm\alpha} = 0 \quad (4.2.8.5)$$

where g is the determinant of the metric and μ_0 is the magnetic permeability of the vacuum. This implies that:

Proposition 4.55. *Inside the sources the real part of any complex 4-potential coincides with the real 4-potential A up to some source free potential A_{sf} as $\Re\mathcal{A}^{\pm} = A + A_{sf}$. And the imaginary part $\Im\mathcal{A}^{\pm}$ is a source free potential.*

4.2.9 Prepotential

We will define for the complex 4-potential $\mathcal{A}^{\pm}$ a prepotential, denoted as $S_{\pm}$. We cannot simply relate $\mathcal{A}^{\pm}$ to the prepotential as $\mathcal{A}_{\mu}^{\pm} = S_{\pm,\mu}$ which is a gauge potential, since the partial derivatives commute it would imply that $\mathcal{F}^{\pm} \equiv 0$ which is totally useless. To remedy this we will use a conjugation $\mathcal{C}^{\pm} = 2c^j \tilde{\mathfrak{B}}_j^{\pm}$, for some given constants c^j satisfying $c^2 = -\eta_{ij} c^i c^j = 1$ implying $(\mathcal{C}^{\pm})^2 = I$. The conjugation is an anti-symmetric tensor in $\mathfrak{o}_{\pm}$. The relation between the complex 4-potential and the prepotential is given by:

$$\mathcal{A}_{\mu}^{\pm} = \mathcal{C}^{\mp\lambda}_{\mu} S_{\pm,\lambda}. \quad (4.2.9.1)$$

Lets find the condition on the prepotential inducing a σ -dual 4-potential. Using (4.2.8.3), 4.51, 4.49 and the commutativity of the partial derivatives we get

$$\begin{aligned} P_{\mp\mu\nu}^{\alpha\beta} 2\mathcal{A}_{[\beta,\alpha]}^{\pm} &= P_{\mp\mu\nu}^{\alpha\beta} 2\mathcal{A}_{\beta,\alpha}^{\pm} = 4\eta^{ij} \tilde{\mathfrak{B}}_{j\mu\nu}^{\mp} \tilde{\mathfrak{B}}_i^{\mp\alpha\beta} c^k \tilde{\mathfrak{B}}_k^{\mp\lambda}_{\beta} S_{\pm,\lambda\alpha} \\ &= 2\eta^{ij} \tilde{\mathfrak{B}}_{j\mu\nu}^{\mp} c^k \left(\tilde{\mathfrak{B}}_i^{\mp\alpha\beta} \tilde{\mathfrak{B}}_k^{\mp\lambda}_{\beta} + \tilde{\mathfrak{B}}_k^{\mp\alpha\beta} \tilde{\mathfrak{B}}_i^{\mp\lambda}_{\beta} \right) S_{\pm,\lambda\alpha} \\ &= -\eta^{ij} \tilde{\mathfrak{B}}_{j\mu\nu}^{\mp} c^k \eta_{ik} \eta^{\alpha\lambda} S_{\pm,\lambda\alpha} = - (c^j \square S_{\pm}) \tilde{\mathfrak{B}}_{j\mu\nu}^{\mp} \\ &= -\frac{1}{2} \mathcal{C}_{\mu\nu}^{\mp} \square S_{\pm}, \end{aligned} \quad (4.2.9.2)$$

which implies that $P_{\mp\mu\nu}^{\alpha\beta} 2\mathcal{A}_{[\beta,\alpha]}^{\pm} = 0$ if and only if $\square S_{\pm} = 0$. Thus, we have shown

Proposition 4.56. *A prepotential $S_{\pm}$ has a σ -dual 4-potential if and only if it satisfies the wave equation $\square S_{\pm} = 0$.*

4.3 The $\tilde{\pi}_{\pm}$ invariant $\zeta_{\pm}$ -factor

4.3.1 The eigenvectors of $\tilde{\mathfrak{B}}_i^{\pm}$

By lemma 4.49 the matrices $K_i^{\pm} = 2\tilde{\mathfrak{B}}_i^{\pm}$ obeys

$$K_i^{\pm\mu}{}_{\alpha} K_i^{\pm\alpha}{}_{\nu} = \delta_{\nu}^{\mu} \quad (4.3.1.1)$$

by having their square equals to the identity the $K_i^{\pm}$ does not have zero eigenvalues, moreover any eigenvector v^{μ} of $K_i^{\pm}$ with eigenvalue λ satisfies

$$K_i^{\pm\mu}{}_{\nu} v^{\nu} = \lambda v^{\mu} \quad (4.3.1.2)$$

using (4.3.1.2) and (4.3.1.1) we get:

$$v^{\mu} = \delta_{\nu}^{\mu} v^{\nu} = K_i^{\pm\mu}{}_{\alpha} K_i^{\pm\alpha}{}_{\nu} v^{\nu} = \lambda K_i^{\pm\mu}{}_{\alpha} v^{\alpha} = \lambda^2 v^{\mu}$$

hence any eigenvalue λ of $K_i^{\pm}$ we have:

$$\lambda^2 = 1 \quad (4.3.1.3)$$

which means λ is $+1$ or -1 . Using (4.2.5.1) and 4.39 we rewrite (4.3.1.2) as

$$\left(\delta_{i0}^{\mu\alpha} g_{\nu\alpha} \pm \epsilon_{0i}{}^{\mu}{}_{\nu} \right) v^{\nu} = \lambda v^{\mu}$$

using a cyclic index set i, j, k we can reformulate as system of equations

$$\begin{aligned} \lambda v^0 &= v^i \\ \lambda v^i &= v^0 \\ \lambda v^j &= \mp v^k \\ \lambda v^k &= \pm v^j \end{aligned} \quad (4.3.1.4)$$

Using (4.3.1.4) we define a basis of the complexified Minkowski space M_c out of the eigenvectors of $K_i^{\pm}$ as

$$n_{\pm}^i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm \mathbf{i} \end{pmatrix}; m_{\pm}^i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathbf{j} \pm i\mathbf{k} \end{pmatrix} \quad (4.3.1.5)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is set of ordered orthonormal vectors associated with the cyclic index set i, j, k , the only non-zero inner-products are

$$n_+^i \cdot n_-^i = -m_+^i \cdot m_-^i = 1 \quad (4.3.1.6)$$

and under complex conjugation we have

$$\begin{aligned} (n_{\pm}^i)^* &= n_{\pm}^i \\ (m_{\pm}^i)^* &= m_{\mp}^i \end{aligned} \quad (4.3.1.7)$$

using (4.3.1.4) and (4.3.1.5) we find:

$$\begin{aligned} K_k^+ n_{\pm}^k &= \pm n_{\pm}^k & K_k^- n_{\pm}^k &= \pm n_{\pm}^k \\ K_k^+ m_{\pm}^k &= \pm m_{\pm}^k & K_k^- m_{\pm}^k &= \mp m_{\pm}^k \end{aligned} \quad (4.3.1.8)$$

Theorem 4.57. The $K_i^\pm$ and $K_j^\pm$ matrices defined in (4.3.1.1) act on $n_\pm^k$ and $m_\pm^k$ defined in (4.3.1.5) as:

$$\begin{aligned} K_i^\pm n_+^k &= m_\mp^k & K_i^\pm m_+^k &= n_\mp^k & K_j^\pm n_+^k &= \pm i m_\mp^k & K_j^\pm m_+^k &= +i n_\mp^k \\ K_i^\pm n_-^k &= m_\pm^k & K_i^\pm m_-^k &= n_\pm^k & K_j^\pm n_-^k &= \mp i m_\pm^k & K_j^\pm m_-^k &= -i n_\pm^k \end{aligned}$$

Proof. Using (4.3.1.5) and (4.3.1.8) we prove the first relation:

$$K_i^\pm n_+^k = K_i^\pm \frac{1}{2} (n_+^i + n_-^i - i m_+^i + i m_-^i) = \frac{1}{2} (n_+^i - n_-^i \mp i m_+^i \mp i m_-^i) = m_\mp^k$$

as required. The rest of the relations in 4.57 follows in the same manner. $\square$

4.3.2 The null basis

In a similar way to the treatment of spinors in 3D, we choose an arbitrary direction $\mathbf{k}$, out of a positively oriented orthonormal 3D basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

Using this basis we can construct a null basis of the complexified Minkowski space M_c , written in 1 + 3D decomposition as:

$$\begin{aligned} n &= n_+^k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathbf{k} \end{pmatrix}; \tilde{n} = n_-^k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathbf{k} \end{pmatrix} \\ m &= m_+^k = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathbf{i} + i\mathbf{j} \end{pmatrix}; m^* = m_-^k = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathbf{i} - i\mathbf{j} \end{pmatrix} \end{aligned} \quad (4.3.2.1)$$

where $n_\pm^k, m_\pm^k$ was defined in (4.3.1.5) above. This basis is also called a Bondi null-tetrad [3], or Newman-Penrose [4] basis.

The only non-zero inner-product relations of the null basis (4.3.2.1) elements $n, \tilde{n}, m$ and m^* as required by (4.3.1.6) are:

$$n \cdot \tilde{n} = -m \cdot m^* = 1. \quad (4.3.2.2)$$

Any four-vector $x = x^\mu e_\mu$ can be decomposed in the null basis (4.3.2.1) as:

$$x = An + D\tilde{n} + Bm + Cm^*. \quad (4.3.2.3)$$

Using (4.3.2.2) and (4.3.2.3) we find that the relation between the parameters A, B, C, D in (4.3.2.3) and the inner products of the null basis with vector x is

$$\begin{aligned} A &= x \cdot \tilde{n} & D &= x \cdot n \\ B &= -x \cdot m^* & C &= -x \cdot m \end{aligned} \quad (4.3.2.4)$$

Now substituting (4.3.2.4) into (4.3.2.3) any four-vector x is represented in the null basis (4.3.2.1) as

$$x = (x \cdot \tilde{n})n + (x \cdot n)\tilde{n} - (x \cdot m^*)m - (x \cdot m)m^*. \quad (4.3.2.5)$$

4.3.3 The matrix associated with 4-vector

For any 4-vector x using (4.3.2.3) and (4.3.2.2) we find the inner-product with itself as:

$$\begin{aligned} x \cdot x &= (An + D\tilde{n} + Bm + Cm^*) \cdot (An + D\tilde{n} + Bm + Cm^*) \\ &= 2(AD - BC) = 2 \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 2 \det X \end{aligned} \quad (4.3.3.1)$$

Definition 4.58. We define Ψ as the invertible linear mapping from any 4-vector x to a matrix X given in (4.3.3.1). And also sends the inner-product as $\Psi(x \cdot y) = \det(\Psi(x + y)) - \det \Psi(x) - \det \Psi(y)$.

We easily see that under this definition 4.58 the following identification relations hold:

$$\Psi(n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \Psi(m) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \Psi(m^*) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ; \Psi(\tilde{n}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.3.3.2)$$

Lemma 4.59. The complex conjugate operation on any 4-vector x in the complexified Minkowski space $M_{\mathbb{C}}$ is equivalent to the conjugate transpose operation on its associated matrix $X = \Psi(x)$ by 4.58 as:

$$\Psi(x^*) = X^\dagger$$

where the † is the conjugate transpose operation on the matrices (without notion of metric).

Proof. Using (4.3.2.3) and 4.58

$$\Psi(x^*) = \Psi(A^*n + D^*\tilde{n} + B^*m^* + C^*m) = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^\dagger = X^\dagger$$

□

Remark. Following 4.59 any real 4-vector $x = x^*$ the matrix $X = \Psi(x)$ is hermitian $X = X^\dagger$.

Now we want to describe the action of Lorentz representations $\tilde{\pi}_\pm$ by the matrix notation as multiplication by matrices from $M_2(\mathbb{C})$, the space of 2×2 complex matrices. We begin by introducing the Pauli matrices.

Definition 4.60. The Pauli matrices are the following hermitian $M_2(\mathbb{C})$ matrices:

$$\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \sigma_j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.3.3.3)$$

where the cyclic index set i, j, k are associated with the ordered orthonormal vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Lemma 4.61. Applying K_l^+ (K_l^-) defined in (4.3.1.1) on any 4-vector x is equivalent to the application of a respective Pauli matrix σ_l 4.60 on the left (right) of $X = \Psi(x)$ defined in 4.58, or simply:

$$\begin{aligned} \Psi(K^+x) &= \sigma_l X \\ \Psi(K^-x) &= X \sigma_l \end{aligned}$$

Proof. The lemma 4.61 is direct outcome of 4.58, (4.3.3.2), (4.3.1.8) and 4.57. □

Lemma 4.62. The Pauli matrices σ_l defined in 4.60 satisfies the commutation and anti-commutation relations:

$$\begin{aligned} [\sigma_a, \sigma_b] &= -2i\epsilon_{0ab}{}^c \sigma_c \\ \{\sigma_a, \sigma_b\} &= -2\eta_{ab} I \end{aligned}$$

Moreover they are also hermitian $\sigma_l^\dagger = \sigma_l$.

Proof. Using 4.61, 4.42, 4.48 and 4.49 we get

$$\begin{aligned} [\sigma_a, \sigma_b] X &= \Psi ([K_a^+, K_b^+] x) = \Psi (4 [\tilde{\mathfrak{B}}_a^+, \tilde{\mathfrak{B}}_b^+] x) \\ &= \Psi (4 \epsilon_{0ab} {}^c \tilde{\mathfrak{K}}_c^+) = \Psi (-2 \iota \epsilon_{0ab} {}^c K_c^+ x) = -2 \iota \epsilon_{0ab} {}^c \sigma_c X \end{aligned}$$

since X is arbitrary we can omit it and get the desired result. Now as for the anti-commutation relation:

$$\{\sigma_a, \sigma_b\} X = \Psi (\{K_a^+, K_b^+\} x) = \Psi (-4 \{\tilde{\mathfrak{K}}_a^+, \tilde{\mathfrak{K}}_b^+\} x) = \Psi (-2 \eta_{ab} x) = -2 \eta_{ab} X$$

since X is arbitrary we can omit it and get the desired result. As for the hermitian property:

$$\Psi \left((K_l^+ x)^* \right) = \Psi (K_l^- x^*) = (\sigma_l X)^\dagger = X^\dagger \sigma_l^\dagger = X^\dagger \sigma_l$$

since $X^\dagger$ is arbitrary we can omit it and get the desired result. $\square$

Lemma 4.63. *The action of Υ^+ (Υ^-) defined in (4.2.7.4) on any 4-vector x is equivalent to the action of U ($U^\dagger$) on X as*

$$\begin{aligned} \Psi (\Upsilon^+ x) &= UX \\ \Psi (\Upsilon^- x) &= XU^\dagger \end{aligned}$$

where $U = e^{\frac{1}{2} F_+^l \sigma_l}$ and the F_+^l is the Riemann–Silberstein vector defined in (4.2.6.2).

Proof. Using (4.2.7.4), (4.2.6.3) and $K_l^\pm = 2 \tilde{\mathfrak{B}}_l^\pm$ we find

$$\Upsilon^\pm = (\Upsilon^\mp)^* = e^{F_\pm^l \tilde{\mathfrak{B}}_l^\pm} = I \cosh \left(\frac{\eta_\pm}{2} \right) + \frac{F_\pm^l K_l^\pm}{\eta_\pm} \sinh \left(\frac{\eta_\pm}{2} \right)$$

by 4.61, 4.58 and 4.62 we get

$$\begin{aligned} \Psi (\Upsilon^+ x) &= \left(I \cosh \left(\frac{\eta_+}{2} \right) + \frac{F_+^l \sigma_l}{\eta_+} \sinh \left(\frac{\eta_+}{2} \right) \right) X = e^{\frac{1}{2} F_+^l \sigma_l} X = UX \\ \Psi (\Upsilon^- x) &= X \left(I \cosh \left(\frac{\eta_-}{2} \right) + \frac{F_-^l \sigma_l}{\eta_-} \sinh \left(\frac{\eta_-}{2} \right) \right) = X e^{\frac{1}{2} F_-^l \sigma_l} = XU^\dagger \end{aligned}$$

$\square$

Lemma 4.64. *The matrices $U, U^\dagger$ from 4.63 have unit determinant.*

$$\det U = \det U^\dagger = 1$$

Proof. Since $\Upsilon^\pm$ are part of the orthogonal group they preserve the inner-product of any two 4-vectors, while specifically using 4.58:

$$x' \cdot x' = x \cdot x \leftrightarrow \det \Psi(x') = \det \Psi(x)$$

and following separately for U and $U^\dagger$ we get

$$\begin{aligned} \det X' &= \det (UX) = \det U \det X = \det X \\ \det X' &= \det (XU^\dagger) = \det X \det U^\dagger = \det X \end{aligned}$$

which leads directly to $\det U = \det U^\dagger = 1$ as expected. $\square$

Theorem 4.65. *The action of any Λ in π defined in (4.2.7.5) on a four vector x is equivalent to the action of U and $U^\dagger$ on X by*

$$\Psi(\Lambda x) = UXU^\dagger$$

where $\Lambda \in \pi$ is defined as an exponent map of a generator $F \in \underline{\pi}$ as in (4.2.7.1), and $U, U^\dagger$ are defined as in 4.63

Proof. Since by (4.2.7.1) and 4.48 the action of Λ can be split into action of $\Upsilon^\pm$ and afterwards $\Upsilon^\mp$, using for each action 4.63 we get the desired result. $\square$

Remark. From 4.65 and 4.59 we can see that the full group π preserves the real and imaginary parts of the complexified Minkowski space M_c , since

$$X'^\dagger = (UXU^\dagger)^\dagger = UX^\dagger U^\dagger$$

thus the (anti)-hermitian property of X' depends solely on X .

4.3.4 Decomposition of null-vectors in the null-basis

Using (4.3.2.5) and (4.3.2.2) for any null-vector $r = (\|\mathbf{r}\|, \mathbf{r})$, we get

$$r \cdot r = 2(r \cdot \tilde{n})(r \cdot n) - 2(r \cdot m^*)(r \cdot m) = 0 \quad (4.3.4.1)$$

rearranging we find out

$$2(r \cdot \tilde{n})(r \cdot n) = 2(r \cdot m^*)(r \cdot m)$$

we define a scalar $\dot{\rho}$ as

$$\dot{\rho} = \sqrt{(r \cdot \tilde{n})(r \cdot n)} = \sqrt{(r \cdot m^*)(r \cdot m)} \quad (4.3.4.2)$$

we can now satisfy (4.3.4.1) generally using a decomposition for a *real* r as

$$r = \dot{\rho} (e^\theta n + e^{-\theta} \tilde{n} + e^{-i\varphi} m + e^{i\varphi} m^*) \quad (4.3.4.3)$$

using (4.3.2.2) we can extract from (4.3.4.3) θ and φ :

$$\theta = \frac{1}{2} \ln \frac{r \cdot \tilde{n}}{r \cdot n} = \ln \frac{r \cdot \tilde{n}}{\dot{\rho}} = -\ln \frac{r \cdot n}{\dot{\rho}} \quad (4.3.4.4)$$

$$\varphi = -\frac{i}{2} \ln \frac{r \cdot m}{r \cdot m^*} = -i \ln \frac{-r \cdot m}{\dot{\rho}} = i \ln \frac{-r \cdot m^*}{\dot{\rho}} \quad (4.3.4.5)$$

4.3.5 Local orthonormal basis, and it's null-vector decomposition

Following (4.3.4.3) we define on the light-cone a *real* local orthonormal basis $(\Theta, \Phi, \tilde{\Phi}, \tilde{\Theta})$ as

$$\begin{aligned} \Theta &= \frac{1}{\sqrt{2}} (e^\theta n + e^{-\theta} \tilde{n}) = (\cosh \theta; \mathbf{k} \sinh \theta) \\ \Phi &= \frac{1}{\sqrt{2}} (e^{i\varphi} m^* + e^{-i\varphi} m) = (0; \mathbf{i} \cos \varphi + \mathbf{j} \sin \varphi) \\ \tilde{\Phi} &= \frac{i}{\sqrt{2}} (e^{i\varphi} m^* - e^{-i\varphi} m) = (0; -\mathbf{i} \sin \varphi + \mathbf{j} \cos \varphi) \\ \tilde{\Theta} &= \frac{1}{\sqrt{2}} (e^\theta n - e^{-\theta} \tilde{n}) = (\sinh \theta; \mathbf{k} \cosh \theta) \end{aligned} \quad (4.3.5.1)$$

It is useful to note that the basis vectors (4.3.5.1) are depended on θ or φ and related by differentiation by θ or φ as:

$$\begin{aligned} \tilde{\Theta} &= \Theta_{,\theta} & \Theta &= \tilde{\Theta}_{,\theta} \\ \tilde{\Phi} &= \Phi_{,\varphi} & \Phi &= -\tilde{\Phi}_{,\varphi} \end{aligned} \quad (4.3.5.2)$$

We can view the null basis (4.3.2.1) as depended on the orthonormal basis (4.3.5.1) as:

$$\begin{aligned} n &= \frac{e^{-\theta}}{\sqrt{2}} (\Theta + \tilde{\Theta}) \\ m &= \frac{e^{i\varphi}}{\sqrt{2}} (\Phi + i\tilde{\Phi}) \\ m^* &= \frac{e^{-i\varphi}}{\sqrt{2}} (\Phi - i\tilde{\Phi}) \\ \tilde{n} &= \frac{e^{\theta}}{\sqrt{2}} (\Theta - \tilde{\Theta}) \end{aligned} \quad (4.3.5.3)$$

Now substituting (4.3.5.3) into (4.3.4.3) we can decompose any null vector r also as:

$$r = \varrho (\Theta + \Phi) = \varrho \hat{r}, \quad (4.3.5.4)$$

where $\varrho = \sqrt{2}\hat{\varrho}$, and (4.3.5.4) can be viewed as a "length" and "direction" of r in a bi-polar decomposition.

4.3.6 The invariant $\zeta_{\pm}$ -factor

Definition 4.66. Let $\zeta : \mathcal{N} \rightarrow \mathbb{C}$, be called the ζ -factor as a function from the null-cone to the complex plane, for any chosen null-basis (4.3.2.1), and a null-vector $r \in \mathcal{N}$, by using θ (4.3.4.4) and φ (4.3.4.5) derived from (4.3.4.3) as

$$\zeta_{\pm} = \zeta_{\pm}(r) = e^{\theta \mp i\varphi} = -\frac{\tilde{n} \cdot r}{m_{\mp} \cdot r}.$$

Theorem 4.67. The $\zeta_{\pm}$ -factor 4.66 is invariant under $\tilde{\pi}_{\pm}$ respectively.

Proof. Since r is null then its related matrix $R = \Psi(r)$ as by 4.58 must have zero determinant $\det R \equiv 0$, since zero determinant means linearly dependent lines/rows then R can be decomposed as an outer product of a two complex component column and row

$$R = \begin{bmatrix} w \\ v \end{bmatrix} \otimes \begin{bmatrix} u^* & z^* \end{bmatrix} = \begin{pmatrix} wu^* & wz^* \\ vu^* & vz^* \end{pmatrix}.$$

By employing only one of $\tilde{\pi}_{\pm}$ by 4.63 we get that under $\tilde{\pi}_{+}$ only the column is transformed, and under $\tilde{\pi}_{-}$ only the row is transformed (more accurately they are transformed by the identity transformation) as

$$\begin{aligned} UR &= U \begin{bmatrix} w \\ v \end{bmatrix} \otimes \begin{bmatrix} u^* & z^* \end{bmatrix} = \begin{bmatrix} w' \\ v' \end{bmatrix} \otimes \begin{bmatrix} u^* & z^* \end{bmatrix} \\ RU^{\dagger} &= \begin{bmatrix} w \\ v \end{bmatrix} \otimes \begin{bmatrix} u^* & z^* \end{bmatrix} U^{\dagger} = \begin{bmatrix} w \\ v \end{bmatrix} \otimes \begin{bmatrix} u'^* & z'^* \end{bmatrix} \end{aligned}$$

Hence under $\tilde{\pi}_{+}$ the fraction $\frac{u^*}{z^*}$ is left invariant and under $\tilde{\pi}_{-}$ the fraction $\frac{w}{v}$ is left invariant. Since by 4.58 and (4.3.2.5) we have

$$R = \begin{pmatrix} wu^* & wz^* \\ vu^* & vz^* \end{pmatrix} = \Psi(r) = \begin{pmatrix} r \cdot \tilde{n} & -r \cdot m^* \\ -r \cdot m & r \cdot n \end{pmatrix}.$$

Using 4.66 we find out:

$$\begin{aligned} \frac{u^*}{z^*} &= \frac{wu^*}{wz^*} = -\frac{\tilde{n} \cdot r}{m^* \cdot r} = \zeta_{+} \\ \frac{w}{v} &= \frac{wu^*}{vu^*} = -\frac{\tilde{n} \cdot r}{m \cdot r} = \zeta_{-} \end{aligned}$$

as required. □

Note. The $\zeta_{\pm}$ -factor is also *conformally* invariant since it depends on the ratio of two inner-products which are evaluated in the same point in space-time.

Note. In the proof of theorem 4.67 we used complex 2-component row and column vectors, as an outer product decomposition of the matrix related to null vector, and each of them was acted upon deferent half-spin representation $\tilde{\pi}_{\pm}$ while the other remained invariant. These 2-component vectors are actually objects by their own right called 2-spinors, for an introduction about 2-spinors see [6].

5 The prepotential of the EM field

5.1 The prepotential of a point charge

Note. We will work in this section under $\tilde{\pi}_{-}$, here ζ is always ζ_{-} and $\mathcal{F}$ is always $\mathcal{F}^{-}$.

Definition 5.1. Let x be an observer location in Minkowski space M , let $y(\tau)$ be the world-line of a point charge and τ it's proper time. Define the past pointing null-sight/light-like 4-vector from the observer to the point charge as:

$$r(x) = x - y(\tau(x)) \text{ where } r \cdot r \equiv 0$$

Definition 5.2. The prepotential at $x \in M$ of point particle with charge q is defined as the logarithm of the ζ_{-} -factor 4.66 of the null-vector r define in 5.1

$$S(x) = \frac{q}{8\pi\epsilon_0 c} \ln \zeta_{-}(r(x)) = \frac{q}{8\pi\epsilon_0 c} (\theta(r(x)) + \iota\varphi(r(x)))$$

where ϵ_0 is the electric permittivity from the maxwell equation (3.0.8.3), and c is the speed of light in vacuum. The units of the prepotential are $\frac{J \cdot sec}{C}$ so when differentiating twice we get $\frac{N}{C \cdot \frac{m}{sec}}$ which are the units of the Faraday tensor.

Note. The term $\frac{q}{4\pi\epsilon_0 c}$ in 5.2 will **not** be present in the derivations in the rest of the report for simplicity, and we will use instead $S = \frac{1}{2} \ln \zeta_{-}(r) = \frac{1}{2} (\theta(r) + \iota\varphi(r))$.

5.2 The gradient of the prepotential

By (4.2.9.1) the 4-potential has to be a conjugation of the gradient of the prepotential (5.2.2.1). To calculate the gradient of the prepotential, which depends on r the null-vector 5.1, which in turn depends on τ the parameter of the world line of the charge, each of which we have to find how to differentiate.

We will start by finding the gradient of τ . Using 5.1 we find the gradient of τ

$$\begin{aligned} 0 &= \partial_{\mu} (r \cdot r) = 2r^{\nu} r_{\nu,\mu} \\ &= 2r^{\nu} (x_{\nu,\mu} - y_{\nu}(\tau(x))_{,\mu}) \\ &= 2r^{\nu} (g_{\nu\mu} - y_{\nu,\tau} \tau_{,\mu}). \end{aligned}$$

Defining $u \equiv y_{,\tau}$, rearranging and dividing by $r \cdot u$, we finally arrive at

$$\tau_{,\mu} = \frac{r_{\mu}}{r \cdot u}. \quad (5.2.0.1)$$

Now using (5.2.0.1) we calculate the gradient of r defined in 5.1 we get:

$$r^{\mu}_{,\nu} = \delta^{\mu}_{\nu} - u^{\mu} r_{\nu} (r \cdot u)^{-1}. \quad (5.2.0.2)$$

Since the prepotential 5.2 is depended on the same θ and φ parameters which the local orthonormal basis (4.3.5.1) was defined with. We will develop derivatives of θ and φ using this basis. Later it will be a very useful calculation tool.

We use the decomposition of null vector (4.3.5.4) by the local orthonormal basis (4.3.5.1) so in this basis (5.2.0.2) the null vector r is written as

$$r_{,\nu}^\mu = \varrho_{,\nu} (\Theta^\mu + \Phi^\mu) + \varrho \theta_{,\nu} \tilde{\Theta}^\mu + \varrho \varphi_{,\nu} \tilde{\Phi}^\mu \quad (5.2.0.3)$$

we will use this which is equal to (5.2.0.2) to find the gradients of the null parameters θ , φ and ϱ .

Lemma 5.3. *The gradients of θ , φ and $\ln \varrho$ are given by:*

$$\theta_{,\nu} = 2 \frac{u^\alpha}{r \cdot u} \tilde{\Theta}_{[\alpha} \hat{r}_{\nu]} \quad (5.2.0.4)$$

$$\varphi_{,\nu} = 2 \frac{u^\alpha}{r \cdot u} \tilde{\Phi}_{[\alpha} \hat{r}_{\nu]} \quad (5.2.0.5)$$

$$(\ln \varrho)_{,\nu} = 2 \frac{u^\alpha}{r \cdot u} \Theta_{[\nu} \Phi_{\alpha]} \quad (5.2.0.6)$$

Proof. Using both (5.2.0.2) and (5.2.0.3) and calculating the inner product of each equation with a different basis vector of (4.3.5.1), we can separate each parameter and find it's gradient.

Contracting with Θ_μ we get:

$$\begin{aligned} \varrho_{,\nu} &= \Theta_\mu (\delta^\mu_{\nu} - u^\mu r_\nu (r \cdot u)^{-1}) = \Theta_\nu - r_\nu \frac{\Theta \cdot u}{r \cdot u} \\ &= \frac{u^\alpha}{r \cdot u} (\Theta_\nu r_\alpha - r_\nu \Theta_\alpha) = \varrho \frac{u^\alpha}{r \cdot u} (\Theta_\nu \Phi_\alpha - \Phi_\nu \Theta_\alpha) \\ &= 2\varrho \frac{u^\alpha}{r \cdot u} \Theta_{[\nu} \Phi_{\alpha]}. \end{aligned}$$

Which is equivalent to the formula for $\partial_\nu \ln \varrho$ we expected.

Contracting with $\tilde{\Theta}_\mu$ we get;

$$\begin{aligned} -\varrho \theta_{,\nu} &= \tilde{\Theta}_\mu (\delta^\mu_{\nu} - u^\mu r_\nu (r \cdot u)^{-1}) = \tilde{\Theta}_\nu - r_\nu \frac{\tilde{\Theta} \cdot u}{r \cdot u} \\ &= \frac{u^\alpha}{r \cdot u} (\tilde{\Theta}_\nu r_\alpha - r_\nu \tilde{\Theta}_\alpha) = \varrho \frac{u^\alpha}{r \cdot u} (\tilde{\Theta}_\nu \hat{r}_\alpha - \hat{r}_\nu \tilde{\Theta}_\alpha) \\ &= 2\varrho \frac{u^\alpha}{r \cdot u} \tilde{\Theta}_{[\nu} \hat{r}_{\alpha]}. \end{aligned}$$

After cancellation of ϱ and slight rearrangement we get the desired result.

Contracting with $\tilde{\Phi}_\mu$ we get:

$$\begin{aligned} -\varrho \varphi_{,\nu} &= \tilde{\Phi}_\mu (\delta^\mu_{\nu} - u^\mu r_\nu (r \cdot u)^{-1}) = \tilde{\Phi}_\nu - r_\nu \frac{\tilde{\Phi} \cdot u}{r \cdot u} \\ &= \frac{u^\alpha}{r \cdot u} (\tilde{\Phi}_\nu r_\alpha - r_\nu \tilde{\Phi}_\alpha) = \varrho \frac{u^\alpha}{r \cdot u} (\tilde{\Phi}_\nu \hat{r}_\alpha - \hat{r}_\nu \tilde{\Phi}_\alpha) \\ &= 2\varrho \frac{u^\alpha}{r \cdot u} \tilde{\Phi}_{[\nu} \hat{r}_{\alpha]}. \end{aligned}$$

after cancellation of ϱ and slight rearrangement we get the desired result. $\square$

Now using (5.2.0.4) and (5.2.0.5) we find directly that the gradient of the prepotential S is

$$\partial_\nu S = \frac{1}{2} (\theta_{,\nu} + \varphi_{,\nu}) = \frac{u^\alpha}{r \cdot u} (\tilde{\Theta}_{[\alpha} \hat{r}_{\nu]} + \tilde{\Phi}_{[\alpha} \hat{r}_{\nu]}) \quad (5.2.0.7)$$

5.2.1 Conjugation

By (4.2.9.1) we are required to choose a conjugation so we can define the relation between the prepotential and the complex 4-potential. The chosen conjugation is related to the chosen null-basis as

$$\mathcal{C} = K_3^+ \quad (5.2.1.1)$$

the following relations arise for the local basis vectors (4.3.5.1) and the conjugation (5.2.1.1)

$$\begin{aligned} \mathcal{C}^\nu_\mu \Theta_\nu &= -\tilde{\Theta}_\mu & \mathcal{C}^\nu_\mu \Phi_\nu &= \imath \tilde{\Phi}_\mu \\ \mathcal{C}^\nu_\mu \tilde{\Phi}_\nu &= -\imath \Phi_\mu & \mathcal{C}^\nu_\mu \tilde{\Theta}_\nu &= -\Theta_\mu \end{aligned} \quad (5.2.1.2)$$

this can be viewed as operator $-\partial_\theta + \imath \partial_\phi$.

5.2.2 Complex 4-potential of a point charge

The complex 4-potential is defined as required by (4.2.9.1) as

$$\mathcal{A}_\mu = \mathcal{C}^\nu_\mu \partial_\nu S. \quad (5.2.2.1)$$

Substituting the gradient of the prepotential (5.2.0.7) into (5.2.2.1) and using the relations (5.2.1.2) we get

$$\begin{aligned} \mathcal{A}_\mu &= \mathcal{C}^\nu_\mu \frac{u^\alpha}{r \cdot u} (\tilde{\Theta}_{[\alpha} \hat{r}_{\nu]} + \imath \tilde{\Phi}_{[\alpha} \hat{r}_{\nu]}) \\ &= \frac{u^\alpha}{2r \cdot u} \left[\tilde{\Theta}_\alpha (-\tilde{\Theta} + \imath \tilde{\Phi})_\mu - (-\Theta_\mu) (\Theta + \Phi)_\alpha \right. \\ &\quad \left. + \imath \tilde{\Phi}_\alpha (-\tilde{\Theta} + \imath \tilde{\Phi})_\mu - \imath (-\imath \Phi_\mu) (\Theta + \Phi)_\alpha \right] \\ &= \frac{u^\alpha}{2r \cdot u} (\Theta_\mu \Theta_\alpha - \Phi_\mu \Phi_\alpha - \tilde{\Phi}_\mu \tilde{\Phi}_\alpha - \tilde{\Theta}_\mu \tilde{\Theta}_\alpha + 2\Theta_{[\mu} \Phi_{\alpha]} + 2\imath \tilde{\Phi}_{[\mu} \tilde{\Theta}_{\alpha]}). \end{aligned}$$

Since the (4.3.5.1) basis is orthonormal so $\eta_{\mu\alpha} = \Theta_\mu \Theta_\alpha - \Phi_\mu \Phi_\alpha - \tilde{\Phi}_\mu \tilde{\Phi}_\alpha - \tilde{\Theta}_\mu \tilde{\Theta}_\alpha$, and using (5.2.0.6) the complex 4-potential (5.2.2.1) can be written as

$$\mathcal{A}_\mu = \frac{1}{2} \frac{u_\mu}{r \cdot u} + \frac{1}{2} \partial_\mu \ln \varrho + \imath \frac{u^\alpha}{r \cdot u} \tilde{\Phi}_{[\mu} \tilde{\Theta}_{\alpha]}. \quad (5.2.2.2)$$

We can see that the real part is half the Liénard–Wiechert potentials with a scalar gauge (5.2.0.6), and we have additional complex term which cannot be described fully as a gradient.

5.2.3 EM Field of a point charge

Before we find the projection of the Faraday EM tensor field $\mathcal{F}_{\mu\nu}$, we find some useful relations. Since the term $\frac{u_\alpha}{r \cdot u}$ appears in (5.2.2.2) twice we need to find it's gradient. We first will calculate the gradient of $r \cdot u$, we get

$$\begin{aligned} \partial_\mu (r \cdot u) &= \left(\delta_\mu^\alpha - u^\alpha r_\mu (r \cdot u)^{-1} \right) u_\alpha + r_\mu \frac{r \cdot u, \tau}{r \cdot u} \\ &= u_\mu - r_\mu (r \cdot u)^{-1} (u \cdot u - r \cdot a) \end{aligned} \quad (5.2.3.1)$$

where $a \equiv u_{,\tau}$. Now using (5.2.3.1) for $\frac{u_\alpha}{r \cdot u}$ we find

$$\begin{aligned} \partial_\mu \frac{u_\alpha}{r \cdot u} &= \frac{a_\alpha r_\mu}{(r \cdot u)^2} - \frac{u_\alpha}{(r \cdot u)^2} \left(u_\mu - r_\mu (r \cdot u)^{-1} (u \cdot u - r \cdot a) \right) \\ &= \frac{r_\mu u_\alpha}{(r \cdot u)^3} - \frac{u_\mu u_\alpha}{(r \cdot u)^2} + \frac{r_\mu a_\alpha (r \cdot u) - r_\mu u_\alpha (r \cdot a)}{(r \cdot u)^3} \end{aligned} \quad (5.2.3.2)$$

Lemma 5.4. *Outside for the sources the prepotential defined in 5.2 obeys $\square S = 0$.*

Proof. Using (5.2.0.2) and (5.2.3.2) we get:

$$\begin{aligned} \square S &= \partial^\mu \frac{\partial S}{\partial r^\nu} r_{,\mu}^\nu = \frac{\partial^2 S}{\partial r^\lambda \partial r^\nu} r_{,\mu}^\nu r^{\lambda,\mu} + \frac{\partial S}{\partial r^\nu} \square r^\nu \\ &= \frac{\partial^2 S}{\partial r^\lambda \partial r^\nu} (\delta_\mu^\nu - u^\nu r_\mu (r \cdot u)^{-1}) (\eta^{\lambda\mu} - u^\lambda r^\mu (r \cdot u)^{-1}) \\ &\quad + \frac{\partial S}{\partial r^\nu} \left(-3u^\nu (r \cdot u)^{-1} - r^\mu \left[\frac{a^\nu r_\mu}{(r \cdot u)^2} - \frac{u^\nu}{(r \cdot u)^2} (u_\mu - r_\mu (r \cdot u)^{-1} (u \cdot u - r \cdot a)) \right] \right) \\ &= \frac{\partial^2 S}{\partial r^\mu \partial r_\mu} - 2 \frac{u^\nu}{r \cdot u} \left(\frac{\partial^2 S}{\partial r^\mu \partial r^\nu} r^\mu + \frac{\partial S}{\partial r^\mu} \frac{\partial r^\mu}{\partial r^\nu} \right) \\ &= \frac{\partial^2 S}{\partial r^\mu \partial r_\mu} - 2 \frac{u^\nu}{r \cdot u} \frac{\partial}{\partial r^\nu} \left(\frac{\partial S}{\partial r^\mu} r^\mu \right). \end{aligned}$$

All what is left to prove is that $\frac{\partial^2 S}{\partial r^\mu \partial r_\mu} = 0$ and $\frac{\partial S}{\partial r^\mu} r^\mu = 0$.

Using 5.2 and 4.66 to differentiate S by r^μ we get:

$$\frac{\partial S}{\partial r^\mu} = \frac{\tilde{n}_\mu}{2\tilde{n} \cdot r} - \frac{m_\mu^*}{2m^* \cdot r}.$$

This immediately gives $\frac{\partial S}{\partial r^\mu} r^\mu = 0$ as required. Now differentiating again by r_μ we get

$$\frac{\partial^2 S}{\partial r_\mu \partial r^\mu} = -\frac{\tilde{n}_\mu \tilde{n}^\mu}{2(\tilde{n} \cdot r)^2} + \frac{m_\mu^* m^{*\mu}}{2(m^* \cdot r)^2} = 0.$$

as required. $\square$

Now since by 5.4 the prepotential satisfies the homogeneous wave equation $\square S = 0$, then by 4.56 we know that the complex 4-potential $\mathcal{A}^\mu$ (5.2.2.2) is a σ -dual potential 4.54, hence it is related to the real Faraday tensor as $F_{\alpha\beta} = 4\Re \mathcal{A}_{[\beta,\alpha]}$. Leading to

$$F_{\mu\nu} = 4\Re \partial_{[\mu} \mathcal{A}_{\nu]} = \delta_{\mu\nu}^{\alpha\beta} \partial_\alpha \frac{u_\beta}{(r \cdot u)} + 2\partial_{[\mu} \partial_{\nu]} \ln \varrho \quad (5.2.3.3)$$

the second term of in the last hand vanishes since the partial derivatives commute, thus using (5.2.3.2) the real Faraday tensor is

$$F_{\mu\nu} = 2 \frac{r_{[\mu} u_{\nu]}}{(r \cdot u)^3} + 2 \frac{r_{[\mu} a_{\nu]} (r \cdot u) - r_{[\mu} u_{\nu]} (r \cdot a)}{(r \cdot u)^3} \quad (5.2.3.4)$$

and the projections of F into $\mathfrak{o}_\pm$ as $\mathcal{F}^\pm = P_\pm F$ is given by

$$\mathcal{F}_{\mu\nu}^\pm = \frac{1}{2} (F_{\mu\nu} \mp \imath \star F_{\alpha\beta}) \quad (5.2.3.5)$$

where $F_{\mu\nu}$ is given by (5.2.3.4).

5.3 Prepotential of charge distribution

Definition 5.5. The pre-potential of a charge distribution is defined through the super-position principle based on the result for charged point particle. We define the scalar distribution $\rho(x) = \sqrt{j^\mu(x)j_\mu(x)}$ and use it to define

$$S(x) = \int_{\mathbb{R}^3} [\ln \zeta(r)] \rho(x-r) d^3r \quad (5.3.0.6)$$

or

$$= \int_{K^-(x)} [\ln \zeta(x-y(x))] \rho(y(x)) d^3y \quad (5.3.0.7)$$

where $r(\mathbf{r}) = (\|\mathbf{r}\|, \mathbf{r})$ while $\mathbf{r} \in \mathbb{R}^3$, $K^-(x)$ is the hyper-surface representing the past light-cone in the position $x = (x^0, \mathbf{x})$ and $y(x) = (x^0 - \|\mathbf{x} - \mathbf{y}\|, \mathbf{y})$ while $\mathbf{y} \in \mathbb{R}^3$.

5.4 Riemann–Silberstein vector F_- by the prepotential

In this subsection we will find the direct relation of the Fields from some prepotential S , which by superposition principle is well defined.

By (4.2.6.3) the Faraday projection $\mathcal{F} = \mathcal{F}^-$ has a decomposition by the generators $\tilde{\mathfrak{B}}_j^-$ as $\mathcal{F}^- = F_-^j \tilde{\mathfrak{B}}_j^-$. We will extract a direct formula for F_-^j , using 4.50 we can multiply (4.2.6.3) by $\tilde{\mathfrak{B}}_i^{-\alpha\beta}$ which gives:

$$\tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{F}_{\alpha\beta}^- = \tilde{\mathfrak{B}}_i^{-\alpha\beta} F_-^j \tilde{\mathfrak{B}}_{j\alpha\beta}^- = \eta_{ij} F_-^j$$

lowering the j index we finally get:

$$F_{-i} = \tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{F}_{\alpha\beta}^- \quad (5.4.0.8)$$

Since anywhere we can consider a complex potential as in (4.2.8.1), using (5.4.0.8) we get:

$$\begin{aligned} F_{-i} &= \tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{F}_{\alpha\beta}^- = \tilde{\mathfrak{B}}_i^{-\alpha\beta} \frac{1}{2} \left(\mathcal{A}_{[\beta,\alpha]} + \frac{1}{2} \epsilon_{\alpha\beta}^{\mu\nu} \mathcal{A}_{[\nu,\mu]} \right) \\ &= \frac{1}{2} \left(\tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{A}_{[\beta,\alpha]} + \frac{1}{2} \epsilon_{\alpha\beta}^{\mu\nu} \tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{A}_{[\nu,\mu]} \right) \\ &= \frac{1}{2} \left(\tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{A}_{[\beta,\alpha]} + \imath (-\imath) \tilde{\mathfrak{B}}_i^{-\mu\nu} \mathcal{A}_{[\nu,\mu]} \right) \\ &= \tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{A}_{[\beta,\alpha]} = \tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{A}_{\beta,\alpha} \end{aligned}$$

summarizing

$$F_{-i} = \tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{A}_{\beta,\alpha} \quad (5.4.0.9)$$

substituting the complex 4-potential induced by the prepotential (5.2.2.1) into the relation (5.4.0.9) we get

$$F_{-i} = \tilde{\mathfrak{B}}_i^{-\alpha\beta} \mathcal{C}_{\beta,\alpha}^\lambda S_{,\alpha\lambda} = -\mathcal{C}_{\beta}^\lambda \tilde{\mathfrak{B}}_i^{-\beta}{}_\mu \eta^{\mu\alpha} S_{,\alpha\lambda} = \frac{1}{2} \alpha_i^{\mu\nu} S_{,\mu\nu} \quad (5.4.0.10)$$

where the α_i matrices coincides in the null-basis with the Dirac alpha matrices in the Weyl or chiral basis [13].

Using (5.4.0.10), (4.2.5.1) and (4.2.3.1) we get a system of equations relating the Riemann–Silberstein vector (4.2.6.2) to the prepotential 5.2 as:

$$\begin{aligned}
F^1 &= -F_{-1} = \mathcal{C}^\alpha_\nu \tilde{\mathfrak{B}}_1^{-\nu}{}_\mu \eta^{\mu\beta} S_{,\alpha\beta} \\
&= \frac{1}{2} \left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\imath & 0 \\ 0 & \imath & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \imath \\ 0 & 0 & -\imath & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right]^{\alpha\beta} S_{,\alpha\beta} \\
&= \frac{1}{2} \begin{pmatrix} 0 & 0 & \imath & 0 \\ 0 & 0 & 0 & -1 \\ \imath & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^{\alpha\beta} S_{,\alpha\beta} = -S_{,13} + \imath S_{,02}
\end{aligned}$$

$$\begin{aligned}
F^2 &= -F_{-2} = \mathcal{C}^\alpha_\nu \tilde{\mathfrak{B}}_2^{-\nu}{}_\mu \eta^{\mu\beta} S_{,\alpha\beta} \\
&= \frac{1}{2} \left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\imath & 0 \\ 0 & \imath & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\imath \\ 1 & 0 & 0 & 0 \\ 0 & \imath & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right]^{\alpha\beta} S_{,\alpha\beta} \\
&= \frac{1}{2} \begin{pmatrix} 0 & -\imath & 0 & 0 \\ -\imath & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}^{\alpha\beta} S_{,\alpha\beta} = -S_{,23} - \imath S_{,01}
\end{aligned}$$

$$\begin{aligned}
F^3 &= -F_{-3} = \mathcal{C}^\alpha_\nu \tilde{\mathfrak{B}}_3^{-\nu}{}_\mu \eta^{\mu\beta} S_{,\alpha\beta} \\
&= \frac{1}{2} \left[\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\imath & 0 \\ 0 & \imath & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \imath & 0 \\ 0 & -\imath & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right]^{\alpha\beta} S_{,\alpha\beta} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{\alpha\beta} S_{,\alpha\beta} = \frac{1}{2} (S_{,00} + S_{,11} + S_{,22} - S_{,33})
\end{aligned}$$

or summarizing:

$$\begin{aligned}
F^1 &= -S_{,13} + \imath S_{,02} \\
F^2 &= -S_{,23} - \imath S_{,01} \\
F^3 &= \frac{1}{2} (S_{,00} + S_{,11} + S_{,22} - S_{,33})
\end{aligned} \tag{5.4.0.11}$$

5.5 Maxwell's equation, 4-current condition and gauge conditions

5.5.1 Maxwell's Equation

Now we are about to find Maxwell's equations in the prepotential formalism. Using (4.2.6.1), 4.46 and (3.1.2.4) we get:

$$\partial_\beta \mathcal{F}^{\pm\alpha\beta} = \partial_\beta P_\pm F^{\alpha\beta} = \partial_\beta \frac{1}{2} (F^{\alpha\beta} \mp \iota(\star F)^{\alpha\beta}) = \frac{1}{2} \mu_0 j^\alpha,$$

now substituting (4.2.9.2), (4.2.9.1), and using (4.2.4.3) and the anti-symmetry of the conjugation we get:

$$\begin{aligned} \frac{1}{2} \mu_0 j^\alpha &= \partial_\beta \mathcal{F}^{\pm\alpha\beta} = 2\partial_\beta P_\pm \mathcal{A}^{\pm[\beta,\alpha]} = 2\partial_\beta \mathcal{A}^{\pm[\beta,\alpha]} - 2\partial_\beta P_\mp \mathcal{A}^{\pm[\beta,\alpha]} \\ &= 2\partial_\beta \mathcal{C}^{\pm\lambda[\beta} S_{\pm,\lambda}^{\alpha]} - 2\partial_\beta P_\mp \mathcal{C}^{\pm\lambda[\beta} S_{\pm,\lambda}^{\alpha]} \\ &= -\mathcal{C}^{\pm\lambda\alpha} \partial_\lambda \square S_\pm + \frac{1}{2} \partial_\beta \mathcal{C}^{\pm\alpha\beta} \square S_\pm \\ &= -\frac{1}{2} \mathcal{C}^{\pm\lambda\alpha} \partial_\lambda \square S_\pm. \end{aligned}$$

This implies that:

Proposition 5.6. *Maxwell's equation for the prepotential is $\mathcal{C}^{\lambda\alpha} \partial_\lambda \square S = -\mu_0 j^\alpha$.*

5.5.2 Current condition

We can rewrite the prepotential Maxwell equation 5.6 as pair of equations

$$\begin{aligned} \square S &= \vartheta \\ \mathcal{C}^\lambda_\mu \vartheta_{,\lambda} &= -\mu_0 j_\mu \end{aligned} \tag{5.5.2.1}$$

where ϑ is a complex scalar field. Taking the second equation and multiplying both sides by $\mathcal{C}^\mu_\alpha$ we get $-\mu_0 \mathcal{C}^\mu_\alpha j_\mu = \mathcal{C}^\mu_\alpha \mathcal{C}^\lambda_\mu \vartheta_{,\lambda} = \delta^\lambda_\alpha \vartheta_{,\lambda} = \vartheta_{,\alpha}$, differentiating by x^β we get $-\mu_0 \mathcal{C}^\mu_\alpha j_{\mu,\beta} = \vartheta_{,\alpha\beta}$, multiplying both sides by $\delta^{\alpha\beta}_{\pi\rho}$ we get $-\mu_0 \delta^{\alpha\beta}_{\pi\rho} \mathcal{C}^\mu_\alpha j_{\mu,\beta} = \delta^{\alpha\beta}_{\pi\rho} \Phi_{,\alpha\beta} = 0$, relabeling $\pi, \rho \rightarrow \alpha, \beta$ we get $\mathcal{C}^\mu_\alpha j_{\mu,\beta} = \mathcal{C}^\mu_\beta j_{\mu,\alpha}$, multiplying again by $\mathcal{C}^\alpha_\nu$ and relabeling $\alpha, \nu \rightarrow \nu, \alpha$ we finally get

$$j_{\alpha,\beta} = \mathcal{C}^\nu_\alpha \mathcal{C}^\mu_\beta j_{\mu,\nu}. \tag{5.5.2.2}$$

This is a condition on the 4-current in order to have a prepotential, which is equivalent to the non-trivial conditions

$$\begin{aligned} j_{0,0} - j_{3,3} &= 0 & j_{1,1} + j_{2,2} &= 0 \\ j_{0,1} - \iota j_{2,3} &= 0 & j_{1,0} - \iota j_{3,2} &= 0 \\ j_{0,2} + \iota j_{1,3} &= 0 & j_{2,0} + \iota j_{3,1} &= 0 \end{aligned} .$$

This raise a question whatever any real 4-current can have complex extension that satisfies (5.5.2.2)? If the answer is no then not every real 4-current has a prepotential.

5.5.3 Prepotential Gauge

The gauge equations for the prepotential by (5.4.0.11) is:

$$\begin{aligned} 0 &= -S_{,13} + \iota S_{,02} \\ 0 &= -S_{,23} - \iota S_{,01} \\ 0 &= \frac{1}{2} (S_{,00} + S_{,11} + S_{,22} - S_{,33}) \end{aligned} .$$

Since S is a function of the position x^μ as $S = f(x^0, x^1, x^2, x^3)$, we want to find the conditions on the functions that could satisfy the equations. Factoring the first equation into linear operators we get:

$$\begin{aligned}
0 &= -S_{,13} + \imath S_{,02} \\
&= -\frac{1}{2} ([\partial_1 + \partial_0] [\partial_3 - \imath \partial_2] + [\partial_1 - \partial_0] [\partial_3 + \imath \partial_2]) S \\
&= -\frac{1}{2} ([\partial_1 + \partial_2] [\partial_3 - \imath \partial_0] + [\partial_1 - \partial_2] [\partial_3 + \imath \partial_0]) S \\
&= -\frac{1}{2} ([\partial_3 + \partial_0] [\partial_1 - \imath \partial_2] + [\partial_3 - \partial_0] [\partial_1 + \imath \partial_2]) S \\
&= -\frac{1}{2} ([\partial_3 + \partial_2] [\partial_1 - \imath \partial_0] + [\partial_3 - \partial_2] [\partial_1 + \imath \partial_0]) S.
\end{aligned}$$

This are four separate options to factor the first equation. For each option respectively we have a linear solution as:

$$\begin{aligned}
S &= g_1(x^1 - x^0, x^1 + x^0) + g_2(x^1 - x^0, x^3 + \imath x^2) + g_3(x^1 + x^0, x^3 - \imath x^2) + g_4(x^3 + \imath x^2, x^3 - \imath x^2) \\
S &= g_5(x^1 - x^2, x^1 + x^2) + g_6(x^1 - x^2, x^3 + \imath x^0) + g_7(x^1 + x^2, x^3 - \imath x^0) + g_8(x^3 + \imath x^0, x^3 - \imath x^0) \\
S &= g_9(x^3 - x^0, x^3 + x^0) + g_{10}(x^3 - x^0, x^1 + \imath x^2) + g_{11}(x^3 + x^0, x^1 - \imath x^2) + g_{12}(x^1 + \imath x^2, x^1 - \imath x^2) , \\
S &= g_{13}(x^3 - x^2, x^3 + x^2) + g_{14}(x^3 - x^2, x^1 + \imath x^0) + g_{15}(x^3 + x^2, x^1 - \imath x^0) + g_{16}(x^1 + \imath x^0, x^1 - \imath x^0)
\end{aligned}$$

where any $g_\#$ is an arbitrary function in two variables. Under similar argument the linear factorization of the second equation is:

$$\begin{aligned}
0 &= -S_{,23} - \imath S_{,01} \\
&= -\frac{1}{2} ([\partial_2 + \partial_0] [\partial_3 + \imath \partial_1] + [\partial_2 - \partial_0] [\partial_3 - \imath \partial_1]) S \\
&= -\frac{1}{2} ([\partial_2 + \partial_1] [\partial_3 + \imath \partial_0] + [\partial_2 - \partial_1] [\partial_3 - \imath \partial_0]) S \\
&= -\frac{1}{2} ([\partial_3 + \partial_0] [\partial_2 + \imath \partial_1] + [\partial_3 - \partial_0] [\partial_2 - \imath \partial_1]) S \\
&= -\frac{1}{2} ([\partial_3 + \partial_1] [\partial_2 + \imath \partial_0] + [\partial_3 - \partial_1] [\partial_2 - \imath \partial_0]) S.
\end{aligned}$$

And the solutions to the four options are respectively:

$$\begin{aligned}
S &= h_1(x^2 - x^0, x^2 + x^0) + h_2(x^2 - x^0, x^3 + \imath x^1) + h_3(x^2 + x^0, x^3 - \imath x^1) + h_4(x^3 + \imath x^1, x^3 - \imath x^1) \\
S &= h_5(x^1 - x^2, x^1 + x^2) + h_6(x^1 + x^2, x^3 + \imath x^0) + h_7(x^1 - x^2, x^3 - \imath x^0) + h_8(x^3 + \imath x^0, x^3 - \imath x^0) \\
S &= h_9(x^3 - x^0, x^3 + x^0) + h_{10}(x^3 + x^0, x^2 + \imath x^1) + h_{11}(x^3 - x^0, x^2 - \imath x^1) + h_{12}(x^2 + \imath x^1, x^2 - \imath x^1) , \\
S &= h_{13}(x^3 - x^1, x^3 + x^1) + h_{14}(x^3 + x^1, x^2 + \imath x^0) + h_{15}(x^3 - x^1, x^2 - \imath x^0) + h_{16}(x^2 + \imath x^0, x^2 - \imath x^0)
\end{aligned}$$

where any $h_\#$ is an arbitrary function in two variables.

The only solution that satisfies the first equation and the second equation simultaneously is of the form

$$S = f_1(x^0 + x^3, x^0 - x^3) + f_2(x^0 + x^3, x^1 - \imath x^2) + f_3(x^0 - x^3, x^1 + \imath x^2) + f_4(x^1 + \imath x^2, x^1 - \imath x^2),$$

which coincides with the third solution for both the first and second equations given above. Moreover this solution also satisfy the third equation, hence, this implies that:

Proposition 5.7. *The prepotential gauge is of the form*

$$S = f_1(x^0 + x^3, x^0 - x^3) + f_2(x^0 + x^3, x^1 - \imath x^2) + f_3(x^0 - x^3, x^1 + \imath x^2) + f_4(x^1 + \imath x^2, x^1 - \imath x^2).$$

5.6 Examples of calculations involving the prepotential

5.6.1 Infinitely long charged rod

In [12, p. 11] Friedman found that the pre-potential of an infinitely long charged rod placed on the x^3 -axis, as

$$S(x) = -\lambda x^3 \ln \varrho \quad (5.6.1.1)$$

where λ is the charge density, and $\varrho = \sqrt{(x^1)^2 + (x^2)^2}$ is the distance from the x^3 -axis in the x^1 - x^2 plane. Using (5.4.0.11) we find the Riemann–Silberstein vector that corresponds to this prepotential:

$$\begin{aligned} F^1 &= -S_{,13} + \imath S_{,02} = \lambda \frac{x^1}{\varrho^2} \\ F^2 &= -S_{,23} - \imath S_{,01} = \lambda \frac{x^2}{\varrho^2} \\ F^3 &= \frac{1}{2} (S_{,00} + S_{,11} + S_{,22} - S_{,33}) = -2\pi\lambda x^3 \delta(x^1) \delta(x^2) \end{aligned} \quad (5.6.1.2)$$

the 4-current is

$$\begin{aligned} j_\mu &= -C_\mu^\alpha \partial_\alpha \square S = C_\mu^\alpha \partial_\alpha 2\pi\lambda x^3 \delta(x^1) \delta(x^2) \\ &= 2\pi\lambda (\delta(x^1) \delta(x^2); \imath x^3 \delta(x^1) \delta'(x^2); -\imath x^3 \delta'(x^1) \delta(x^2); 0) \end{aligned} \quad (5.6.1.3)$$

where δ' is the distributional derivative of the dirac delta δ . Notice that imaginary part of the current is inside the sources and is a complex extension of the real current so it satisfies (5.5.2.2). A factor of 2π was introduced which is because $\ln \varrho$ is a green function of $\Delta_{2D} = \partial_1 \partial_1 + \partial_2 \partial_2$, this shows that there are still fine subjects to differentiation and integration in relation to the prepotential which has to be understood.

5.6.2 Infinite charged plate

The current of an infinite charged plate positioned in the x^1 - x^2 plane, thus the current density is $j_\mu = (\sigma \delta(x^3); 0; 0; 0)$, which satisfies the current condition (5.5.2.2). We will use (5.5.2.1) in order to find the prepotential. The conjugation of the current density is $C^\lambda_\mu j_\lambda = (0; 0; 0; -\sigma \delta(x^3))$, thus it is a gradient of $\vartheta = \sigma (h(x^3) - \frac{1}{2})$ where h is the Heaviside function and the prepotential is

$$S = \frac{1}{4} \sigma x^3 |x^3|. \quad (5.6.2.1)$$

the force is

$$\begin{aligned} F_1 &= F_2 = 0 \\ F_3 &= \frac{1}{2} \sigma \text{Sign}(x^3) = \sigma (h(x^3) - \frac{1}{2}) \end{aligned}$$

which is the known result for an infinite charged plate.

5.6.3 Infinitely long solenoid

In [7, p. 14-5] the current density of a long solenoid is considered as circumferential surface current, where the pitch is neglected, so the current of solenoid of radius R is

$$j_\mu = J_0 \delta(R - \varrho) \hat{\varphi}_\mu \quad (5.6.3.1)$$

where $\varrho = \sqrt{(x^1)^2 + (x^2)^2}$ and $\hat{\varphi}_\mu = (0; \frac{-x^2}{\varrho}; \frac{x^1}{\varrho}; 0) = \partial_\mu \varphi = iC_\mu^\nu \partial_\nu (\ln \varrho)$. This distribution satisfies (5.5.2.2) so there is no need to add imaginary currents to satisfy this condition. The 4-potential is

$$A_\mu = \frac{\mu_0 J_0}{4} \hat{\varphi}_\mu \left(h(\varrho - R) \frac{R^2}{\varrho} + h(R - \varrho) \varrho \right).$$

Since $\partial_\nu S = C_\nu^\mu A_\mu$ we get

$$\begin{aligned} \partial_\nu S &= C_\nu^\mu A_\mu = \frac{\mu_0 J_0}{4} \left(h(\varrho - R) \frac{R^2}{\varrho} + h(R - \varrho) \varrho \right) \partial_\mu \ln \varrho \\ &= \frac{\mu_0 J_0}{4} \left(h(\varrho - R) \frac{R^2}{\varrho^2} + h(R - \varrho) \right) \partial_\mu \varrho. \end{aligned}$$

Which leads to

$$S = \frac{\mu_0 J_0}{4} \left(h(\varrho - R) \left[2R - \frac{R^2}{\varrho} \right] + h(R - \varrho) \varrho \right). \quad (5.6.3.2)$$

This prepotential does not indicate that a closed loop will change the phase by 2π . If instead we try to find a "direct" prepotential ϕ as $\partial_\mu \phi = A_\mu$, we see this cannot be done or else it would imply that there is no force or charge in the configuration, on the other hand we can see that

$$A_\mu dx^\mu = \frac{\mu_0 J_0}{4} \left(h(\varrho - R) \frac{R^2}{\varrho} + h(R - \varrho) \varrho \right) \hat{\varphi}_\mu dx^\mu = f(\varrho) \frac{\partial \varphi}{\partial x^\mu} dx^\mu = f(\varrho) d\varphi \quad (5.6.3.3)$$

which shows why we can't find a "direct" prepotential, and also why this 4-potential gets a winding-number when line-integrated around the z -axis.

6 Discussion

We have proposed a prepotential model for the classical EM theory which provides less degrees of freedom and still have conformal invariance. Some progress and advances has been made over previous works, but there are still open problems especially in the understanding of the physical meaning of our results.

6.1 Open problems

1. Finding the physical meaning of the complexified Minkowski space M_c .
2. Finding the physical meaning of $\tilde{\pi}_\pm$ representations.
3. Finding the physical meaning of the current density condition (5.5.2.2).
4. Does any real 4-current density can be complexly extended to density that satisfies (5.5.2.2)?

6.2 Advances with respect to previous works on the subject

In [9] and [8] the prepotential and the complexified Minkowski space was firstly introduced, and the field of non-accelerating point charged particle was calculated. In [10] it was argued that the prepotential might account for the Aharonov-Bohm effect.

In [11] a gauge of the prepotential was found and it was proved that the prepotential of stationary point particle satisfies the wave equation, also the prepotential was calculated for an infinite charged rod and for current of an infinite rod.

The advances in this report are:

- We re-defined the Lie algebra representations $\tilde{\pi}_{\pm}$ 4.52 as the σ -dual subspaces of Lie algebra representation π , which corresponds to the two eigenvalues (4.2.4.1) of the Hodge dual operator. We also introduce the projection operators $P_{\pm}$ 4.46.
- We have shown in 4.54 that outside of the sources a σ -dual complex 4-potential can be defined. As for the prepotential we showed in 4.56 that $\square S_{\pm} = 0$ if and only if it has a σ -dual complex 4-potential.
- We generalized the definition of the null-basis (4.3.2.1) and the scalar invariant 4.66, and shown how they are related to the 2-spinor formalism. This simplified greatly the proof that the $\zeta_{\pm}$ -factor is invariant under $\tilde{\pi}_{\pm}$.
- We introduced the θ , φ and ϱ parameters of the decomposition of a null vector (4.3.4.3), using them we introduced the local orthonormal basis (4.3.5.1), (4.3.5.4).
- We used the local orthonormal basis to simplify greatly the calculation of the gradient of the prepotential (5.2.0.7) and the complex 4-potential (5.2.2.2).
- We showed that 5.4 outside of the sources the prepotential always satisfies the wave equation. This allowed using 4.56 which made the calculation of the field of an accelerating point charged particle (5.2.3.4) very simple.
- We calculated Maxwell's equations in the prepotential formalism 5.6.
- The condition on the 4-current density (5.5.2.2) in order to have a prepotential was found.
- We found the gauge freedom of the prepotential 5.7.
- We calculated the prepotential of infinite charged plate (5.6.2.1) and of an infinitely long solenoid (5.6.3.2).

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תקציר

דו"ח זה הוא המשך פיתוח נושא הקדם-פוטנציאל של שדה אלמ"ג. הגדרנו פעולת דואליות ה-Hodge על הצגה הסטנדרטית של אלגברת Lie של חבורת לורנץ על מרחב מינקובסקי, המסומנת בדו"ח זה כ- π . הראנו כי פעולה זו יוצרת פירוק של π לתתי-מרחבים עצמיים $\mathfrak{o}_{\pm}$ המתאימים לערכים העצמיים ± 1 . קראנו לאיברים של $\mathfrak{o}_{\pm}$ דואליים כך ש- σ מקבל ערך ± 1 . הגדרנו הצגות של חבורת לורנץ $\tilde{\pi}_{\pm}$ על כל אחד מתתי-המרחבים $\mathfrak{o}_{\pm}$. התאמנו לטנזור שדה האלמ"ג F טנזור σ -דואלי $\mathcal{F}^{\pm}$. הראנו שמחוץ למקורות ניתן להגדיר ארבע-פוטנציאל σ -דואלי. עבור כל צופה מצאנו סקלר $\zeta_{\pm}$, התלוי רק במיקום הצופה והמקור שדה האלמ"ג, אינווריאנטי תחת ההצגה $\tilde{\pi}_{\pm}$. בעזרת האינווריאנט $\zeta_{\pm}$ הגדרנו את הקדם-פוטנציאל $S_{\pm}$ של שדה האלמ"ג. הראנו שקדם-פוטנציאל של מטען נקודתי בתנועה כלשהי מגדיר שדה האלמ"ג הידוע. הראנו שקדם-פוטנציאל מחוץ למקור תמיד σ -דואלי. פיתחנו את משוואות מקסוול עבור הקדם-פוטנציאל. מצאנו תנאים על התפלגות הזרם לקיום קדם-פוטנציאל. חישבנו את חופש הכיל עבור הקדם-פוטנציאל. הצגנו דוגמאות לקשר בין הקדם-פוטנציאל, התפלגות הזרם ושדה אלמ"ג עבור: תיל טעון אינסופי, משטח טעון אינסופי וזרם בסליל אינסופי.