

Classical physics: Spacetime, fields and particles

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July 15, 2011

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1 Spacetime

Einstein's principle of general covariance states that all physical laws do not change their form (are covariant) under continuous coordinate transformations in four-dimensional spacetime.

1.1 Tensors

1.1.1 Vectors

We consider a coordinate transformation from old (unprimed) to new (primed) *coordinates* in a four-dimensional manifold:

$$x^i \rightarrow x'^j(x^i), \quad (1.1.1)$$

where x'^j are differentiable and nondegenerate functions of x^i and the index i can be 0,1,2,3. Thus the matrix $\frac{\partial x'^j}{\partial x^i}$ has the nonzero determinant $|\frac{\partial x'^j}{\partial x^i}| \neq 0$, so x^i are differentiable and nondegenerate functions of x'^j . The matrix $\frac{\partial x^i}{\partial x'^j}$ is the inverse of $\frac{\partial x'^j}{\partial x^i}$:

$$\sum_i \frac{\partial x'^i}{\partial x^j} \frac{\partial x^k}{\partial x'^i} = \delta_j^k, \quad (1.1.2)$$

where

$$\delta_k^i = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}. \quad (1.1.3)$$

The differentials and derivatives transform according to

$$dx'^j = \frac{\partial x'^j}{\partial x^i} dx^i, \quad (1.1.4)$$

$$\frac{\partial}{\partial x'^j} = \frac{\partial x^i}{\partial x'^j} \frac{\partial}{\partial x^i}. \quad (1.1.5)$$

A *scalar* (invariant) is defined as a quantity that does not change:

$$\phi' = \phi. \quad (1.1.6)$$

A *contravariant vector* is defined as a quantity that transform like a differential:

$$A'^j = \frac{\partial x'^j}{\partial x^i} A^i. \quad (1.1.7)$$

A *covariant vector* is defined as a quantity that transforms like a derivative:

$$B'_j = \frac{\partial x^i}{\partial x'^j} B_i. \quad (1.1.8)$$

Therefore a derivative of a scalar is a covariant vector. The coordinates x^i do not form a vector.

1.1.2 Tensors

A product of several vectors transforms such that each coordinate index transforms separately:

$$A'^i B'^j \dots C'_k D'_l \dots = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^k} \frac{\partial x^q}{\partial x'^l} A^m B^n \dots C_p D_q \dots \quad (1.1.9)$$

A *tensor* is defined as a quantity that transforms like a product of vectors:

$$T'^{ij\dots}_{kl\dots} = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^k} \frac{\partial x^q}{\partial x'^l} T'^{mn\dots}_{pq\dots}. \quad (1.1.10)$$

A tensor is of rank (k, l) if it has k contravariant and l covariant indices. A scalar is a tensor of rank $(0, 0)$, a contravariant vector is a tensor of rank $(1, 0)$, and a covariant vector is a tensor of rank $(0, 1)$. A linear combination of two tensors of rank (k, l) is a tensor of rank (k, l) . The product of two tensors of ranks (k_1, l_1) and (k_2, l_2) is a tensor of rank $(k_1 + k_2, l_1 + l_2)$. Tensor indices (all contravariant or all covariant) can be *symmetrized*:

$$T_{(ij\dots k)} = \frac{1}{n!} \sum_{\text{permutations}} T_{\{ij\dots k\}}, \quad (1.1.11)$$

or *antisymmetrized*:

$$T_{[ij\dots k]} = \frac{1}{n!} \sum_{\text{permutations}} T_{\{ij\dots k\}} (-1)^m, \quad (1.1.12)$$

where n is the number of symmetrized or antisymmetrized indices and m is the number of permutations that bring $T_{ij\dots k}$ into $T_{\{ij\dots k\}}$. For example, for two indices: $T_{(ik)} = \frac{1}{2}(T_{ik} + T_{ki})$ and $T_{[ik]} = \frac{1}{2}(T_{ik} - T_{ki})$, and for three indices: $T_{[ijk]} = \frac{1}{3}(T_{ijk} + T_{jki} + T_{kij})$. If $n > 4$ then $T_{[ij\dots k]} = 0$. Symmetrized and antisymmetrized tensors of rank (k, l) are tensors of rank (k, l) . Symmetrization of an antisymmetric tensor or antisymmetrization of a symmetric tensor bring these tensors to zero. Any tensor of rank $(0, 2)$ is the sum of its symmetric and antisymmetric part,

$$T_{(ik)} + T_{[ik]} = T_{ik}. \quad (1.1.13)$$

The number 0 can be regarded as a tensor of arbitrary rank. Therefore all covariant equations of classical physics must be represented in the tensor form: $T^{ij\dots}_{kl\dots} = 0$.

1.1.3 Densities

The element of volume in four-dimensional spacetime transforms according to

$$d^4x' = \left| \frac{\partial x'^i}{\partial x^k} \right| d^4x. \quad (1.1.14)$$

A *scalar density* is defined as a quantity that transforms such that its product with the element of volume is a scalar, $\mathfrak{s}' d^4x' = \mathfrak{s} d^4x$:

$$\mathfrak{s}' = \left| \frac{\partial x^i}{\partial x'^k} \right| \mathfrak{s}. \quad (1.1.15)$$

A *tensor density*, which includes a contravariant and covariant vector density, is defined as a quantity that transforms like a product of a tensor and a scalar density:

$$\mathfrak{T}'^{ij\dots}_{kl\dots} = \left| \frac{\partial x^i}{\partial x'^k} \right| \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^k} \frac{\partial x^q}{\partial x'^l} \mathfrak{T}^{mn\dots}_{pq\dots}. \quad (1.1.16)$$

For example, the square root of the determinant of a tensor of rank $(0, 2)$ is a scalar density of weight 1:

$$\sqrt{|T'_{ik}|} = \sqrt{\left| \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} T_{lm} \right|} = \sqrt{\left| \frac{\partial x^j}{\partial x'^n} \right|^2 |T_{ik}|} = \left| \frac{\partial x^j}{\partial x'^n} \right| \sqrt{|T_{ik}|}. \quad (1.1.17)$$

The above densities are said to be of weight 1. One can generalize this definition of densities by introducing densities of weight w , which transform like normal densities except that $\left| \frac{\partial x^i}{\partial x'^k} \right|$ is replaced by $\left| \frac{\partial x^i}{\partial x'^k} \right|^w$. For example, d^4x is a scalar density of weight -1. A linear combination of two densities of weight w is a density of weight w . The product of two densities of weights w_1 and w_2 is a density of weight $w_1 + w_2$. Symmetrized and antisymmetrized densities of weight w are densities of weight w . Densities of weight 1 are simply referred to as densities. Tensors are densities of weight 0.

1.1.4 Contraction

We adopt Einstein's convention: if the same coordinate index i appears twice (as a contravariant index and covariant index) then we perform the summation $\sum_i$ over a given tensor or density. Such a tensor or density is said to be *contracted* over index i . A contracted tensor of rank (k, l) transforms like a tensor of rank $(k-1, l-1)$:

$$T'^{ij\dots}_{il\dots} = \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^i} \frac{\partial x^q}{\partial x'^l} T'^{mn\dots}_{pq\dots} = \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} \delta^p_m T'^{mn\dots}_{pq\dots} = \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} T'^{mn\dots}_{mq\dots}. \quad (1.1.18)$$

For example, the contraction of a contravariant and covariant vector $A^i B_i$ is a scalar (*scalar product*). A contracted tensor density of rank (k, l) and weight w transforms like a tensor density of rank $(k-1, l-1)$ and weight w :

$$\begin{aligned} \mathfrak{T}'^{ij\dots}_{il\dots} &= \left| \frac{\partial x^i}{\partial x'^k} \right|^w \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x^p}{\partial x'^i} \frac{\partial x^q}{\partial x'^l} \mathfrak{T}'^{mn\dots}_{pq\dots} = \left| \frac{\partial x^i}{\partial x'^k} \right|^w \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} \delta^p_m \mathfrak{T}'^{mn\dots}_{pq\dots} \\ &= \left| \frac{\partial x^i}{\partial x'^k} \right|^w \frac{\partial x'^j}{\partial x^n} \frac{\partial x^q}{\partial x'^l} \mathfrak{T}'^{mn\dots}_{mq\dots}. \end{aligned} \quad (1.1.19)$$

Contraction of a symmetric tensor with an antisymmetric tensor (over indices with respect to which these tensors are symmetric or antisymmetric) gives zero. If contraction of two tensors gives zero, these tensors are said to be *orthogonal*. Two orthogonal vectors (one contravariant and one covariant) are said to be *perpendicular*.

1.1.5 Kronecker and Levi-Civita symbols

The *Kronecker symbol* δ^i_k (1.1.3) is a tensor with constant components:

$$\delta'^i_k = \frac{\partial x'^i}{\partial x^j} \frac{\partial x^l}{\partial x'^k} \delta^j_l = \frac{\partial x'^i}{\partial x^j} \frac{\partial x^j}{\partial x'^k} = \delta^i_k. \quad (1.1.20)$$

A totally antisymmetric tensor of rank $(4, 0)$, $T^{ijkl} = T^{[ijkl]}$ has 1 independent component T : $T^{ijkl} = T \epsilon^{ijkl}$, where ϵ^{ijkl} is the totally antisymmetric, contravariant permutation *Levi-Civita symbol*:

$$\epsilon^{0123} = 1, \quad \epsilon^{ijkl} = (-1)^m, \quad (1.1.21)$$

and m is the number of permutations that bring ϵ^{ijkl} into ϵ^{0123} . The determinant of a matrix S^i_k , $\det(S^i_k) = |S^i_k|$, is defined through the permutation symbol:

$$|S^r_s| \epsilon^{ijkl} = S^i_m S^j_n S^k_p S^l_q \epsilon^{mnpq}. \quad (1.1.22)$$

Taking $S^i_k = \frac{\partial x'^i}{\partial x^k}$ gives

$$\epsilon^{ijkl} = \left| \frac{\partial x^r}{\partial x'^s} \right| \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^n} \frac{\partial x'^k}{\partial x^p} \frac{\partial x'^l}{\partial x^q} \epsilon^{mnpq}. \quad (1.1.23)$$

This equation looks like a transformation law for a tensor density with constant components: $\epsilon'^{ijkl} = \epsilon^{ijkl}$. Accordingly, T is a scalar density of weight -1. We also introduce the covariant Levi-Civita symbol ϵ_{ijkl} through:

$$\epsilon^{ijkl} \epsilon_{mnpq} = - \begin{vmatrix} \delta^i_m & \delta^i_n & \delta^i_p & \delta^i_q \\ \delta^j_m & \delta^j_n & \delta^j_p & \delta^j_q \\ \delta^k_m & \delta^k_n & \delta^k_p & \delta^k_q \\ \delta^l_m & \delta^l_n & \delta^l_p & \delta^l_q \end{vmatrix}. \quad (1.1.24)$$

Thus the covariant Levi-Civita symbol is a tensor density of weight -1 and its product with a scalar density is a tensor. The covariant Levi-Civita symbol is given by

$$\epsilon_{0123} = -1, \quad \epsilon_{ijkl} = (-1)^m, \quad (1.1.25)$$

where m is the number of permutations that bring ϵ_{ijkl} into ϵ_{0123} , and satisfies

$$|S_s^r| \epsilon_{ijkl} = S_i^m S_j^n S_k^p S_l^q \epsilon_{mnpq}. \quad (1.1.26)$$

Contracting (1.1.24) gives the following relations:

$$\begin{aligned} \epsilon^{ijkl} \epsilon_{mnpq} &= - \begin{vmatrix} \delta_m^i & \delta_n^i & \delta_p^i \\ \delta_m^j & \delta_n^j & \delta_p^j \\ \delta_m^k & \delta_n^k & \delta_p^k \end{vmatrix}, \\ \epsilon^{ijkl} \epsilon_{mnkl} &= -2(\delta_m^i \delta_n^j - \delta_n^i \delta_m^j), \\ \epsilon^{ijkl} \epsilon_{mjkl} &= -6\delta_m^i, \\ \epsilon^{ijkl} \epsilon_{ijkl} &= -24. \end{aligned} \quad (1.1.27)$$

1.1.6 Dual densities

A contracted product of a covariant tensor and the contravariant Levi-Civita symbol gives a *dual* contravariant tensor density:

$$\epsilon^{iklm} A_m = \mathfrak{A}^{ikl}, \quad \epsilon^{iklm} B_{lm} = \mathfrak{B}^{ik}, \quad \epsilon^{iklm} C_{klm} = \mathfrak{C}^i. \quad (1.1.28)$$

A contracted product of a contravariant tensor and the covariant Levi-Civita symbol gives a dual covariant tensor density:

$$\epsilon_{iklm} A^m = \mathfrak{A}_{ikl}, \quad \epsilon_{iklm} B^{lm} = \mathfrak{B}_{ik}, \quad \epsilon_{iklm} C^{klm} = \mathfrak{C}_i. \quad (1.1.29)$$

Therefore there exists an algebraic correspondence between covariant tensors and contravariant densities, and between contravariant tensors and covariant densities.

1.1.7 Covariant integrals

A covariant line integral is an integral of a tensor contracted with the line differential dx^i : $\int T_{i\dots}^{j\dots} dx^i$. A covariant surface integral is an integral of a tensor contracted with the surface differential $df^{ik} = dx^i dx'^k - dx^k dx'^i$ (which can be geometrically represented as a parallelogram spanned by the vectors dx^i and dx'^i): $\int T_{ik\dots}^{j\dots} df^{ik}$. A covariant hypersurface (volume) integral is an integral of a tensor

contracted with the volume differential $dS^{ikl} = \begin{vmatrix} dx^i & dx'^i & dx''^i \\ dx^k & dx'^k & dx''^k \\ dx^l & dx'^l & dx''^l \end{vmatrix}$ (which can be geometrically

represented as a parallelepiped spanned by the vectors dx^i , dx'^i) and dx''^i : $\int T_{ikl\dots}^{j\dots} dS^{ikl}$. A covariant four-volume integral is an integral of a tensor contracted with the four-volume differential dS^{ijkl} , defined analogously to dS^{ikl} . The dual density corresponding to the surface element is given by

$$df_{ik}^* = \frac{1}{2} \epsilon_{lmik} df^{lm}. \quad (1.1.30)$$

The dual density corresponding to the hypersurface element is given by

$$dS_i = \frac{1}{6} \epsilon_{klmi} dS^{klm}. \quad (1.1.31)$$

The dual density corresponding to the four-volume element is given by

$$d\Omega = \frac{1}{24} \epsilon_{iklm} dS^{iklm} = dx^0 dx^1 dx^2 dx^3. \quad (1.1.32)$$

Covariant integrands that include the above dual densities of weight -1 must be multiplied by a scalar density, for example, by the square root of the determinant of a tensor of rank (0, 2). According to Gauß' and Stokes' theorems, there exists relations between integrals over different elements:

$$dx^i \leftrightarrow df^{ik} \frac{\partial}{\partial x^k}, \quad (1.1.33)$$

$$df_{ik}^* \leftrightarrow dS_i \frac{\partial}{\partial x^k} - dS_k \frac{\partial}{\partial x^i}, \quad (1.1.34)$$

$$dS_i \leftrightarrow d\Omega \frac{\partial}{\partial x^i}. \quad (1.1.35)$$

1.1.8 Derivatives

A derivative of a covariant vector does not transform like a tensor:

$$\frac{\partial A'_k}{\partial x'^i} = \frac{\partial x^l}{\partial x'^i} \frac{\partial}{\partial x^l} \left(\frac{\partial x^m}{\partial x'^k} A_m \right) = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} \frac{\partial A_m}{\partial x^l} + \frac{\partial^2 x^m}{\partial x'^i \partial x'^k} A_m, \quad (1.1.36)$$

because of the second term which is linear and homogeneous in A_i , unless x^i are linear functions of x'^j . This term is symmetric in the indices i, k so the antisymmetric part of $\frac{\partial A_k}{\partial x^i}$ with respect to these indices is a tensor:

$$\partial_{[i} A'_{k]} = \frac{\partial x^l}{\partial x'^{[i}} \frac{\partial x^m}{\partial x'^{k]}} \partial_l A_m = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} \partial_{[l} A_{m]}, \quad (1.1.37)$$

where we denote $\partial_i = \frac{\partial}{\partial x^i}$. The *curl* of a covariant vector A_i is defined as twice the antisymmetric part of $\partial_i A_k$: $\partial_i A_k - \partial_k A_i$, and is a tensor. We will also use $_{,i} = \frac{\partial}{\partial x^i}$ to denote a partial derivative with respect to x^i . Similarly, totally antisymmetrized derivatives of tensors of rank (0, 2) and (0, 3), $\partial_{[i} B_{kl]}$ and $\partial_{[i} C_{klm]}$, are tensors. If $B_{kl} = A_{[k,l]}$ then $\partial_{[i} B_{kl]} = 0$, or conversely, if $\partial_{[i} B_{kl]} = 0$ then there exists a vector A_i such that $B_{kl} = A_{[k,l]}$. The *divergence* of a tensor (or density) is a contracted derivative of this tensor (density): $\partial_i T^{...i}_{jk...}$. Because of the correspondence between tensors and dual densities, divergences of (totally antisymmetric if more than 1 index) contravariant densities are densities, dual to totally antisymmetrized derivatives of tensors:

$$\partial_i \mathfrak{C}^i = \epsilon^{iklm} \partial_{[i} C_{klm]}, \quad \partial_k \mathfrak{B}^{ik} = \epsilon^{iklm} \partial_{[k} B_{lm]}, \quad \partial_l \mathfrak{A}^{ikl} = \epsilon^{iklm} \partial_{[l} A_{m]}. \quad (1.1.38)$$

For example, the equations $\mathfrak{F}^{ik}_{,i} = j^k$ and $F_{[ik,l]} = 0$, that describe Maxwell's electrodynamics, are tensorial.

References: [1, 2].

1.2 Affine connection

1.2.1 Covariant differentiation of tensors

An ordinary derivative of a covariant vector A_i is not a tensor, because its coordinate transformation law contains an additional noncovariant term, linear and homogeneous in A_i . We consider the expression

$$A_{i;k} = A_{i,k} - \Gamma_{ik}^l A_l, \quad (1.2.1)$$

where the quantity Γ_{ik}^l (in the second term which is linear and homogeneous in A_i) transforms such that $A_{i;k}$ is a tensor:

$$A'_{i;k} = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} A_{l;m} = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} (A_{l,m} - \Gamma_{lm}^n A_n). \quad (1.2.2)$$

On the other hand (1.1.36) gives

$$A'_{i;k} = A'_{i,k} - \Gamma'^l_{ik} A'_l = \frac{\partial x^m}{\partial x'^k} \frac{\partial x^l}{\partial x'^i} A_{l,m} + \frac{\partial^2 x^n}{\partial x'^k \partial x'^i} A_n - \frac{\partial x^n}{\partial x'^l} \Gamma'^l_{ik} A_n, \quad (1.2.3)$$

so we obtain

$$\frac{\partial x^n}{\partial x'^l} \Gamma'_{ik} = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} \Gamma_{lm}^n + \frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \quad (1.2.4)$$

Multiplying this equation by $\frac{\partial x'^j}{\partial x^n}$ gives the transformation law for Γ_{ik}^l :

$$\Gamma_{ik}^j = \frac{\partial x'^j}{\partial x^n} \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} \Gamma_{lm}^n + \frac{\partial x'^j}{\partial x^n} \frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \quad (1.2.5)$$

The algebraic object Γ_{ik}^l , which equips spacetime in order to covariantize a derivative of a vector, is referred to as the *affine connection*, affinity or simply connection. The connection has generally 64 independent components. The tensor $A_{i;k}$ is the *covariant derivative* of a vector A_i with respect to x^i . We will also use $\nabla_i =_{;i}$ to denote a covariant derivative. The contracted affine connection transforms according to

$$\Gamma'^i_{ik} = \frac{\partial x^m}{\partial x'^k} \Gamma_{lm}^l + \frac{\partial x'^i}{\partial x^n} \frac{\partial^2 x^n}{\partial x'^k \partial x'^i}. \quad (1.2.6)$$

The affine connection is not a tensor because of the second term on the right-hand side of (1.2.5).

A derivative of a scalar is a covariant vector. Therefore a covariant derivative of a scalar is equal to an ordinary derivative:

$$\phi_{;i} = \phi_{,i}. \quad (1.2.7)$$

If we also assume that a covariant derivative of the product of two tensors obeys the same chain rule as an ordinary derivative:

$$(TU)_{;i} = T_{;i}U + TU_{;i}, \quad (1.2.8)$$

then

$$(A_k B^k)_{;i} = (A_k B^k)_{,i} = A_{k;i} B^k + A_k B^k_{;i} = A_{k,i} B^k - \Gamma_{li}^k A_k B^l + A_k B^k_{;i}. \quad (1.2.9)$$

Therefore we obtain a covariant derivative of a contravariant vector:

$$B^k_{;i} = B^k_{,i} + \Gamma_{li}^k B^l. \quad (1.2.10)$$

The chain rule (1.2.8) also implies that a covariant derivative of a tensor is equal to the sum of the corresponding ordinary derivative of this tensor and terms with the affine connection that covariantize each index:

$$T^{ij\dots}_{kl\dots;m} = T^{ij\dots}_{kl\dots,m} + \Gamma_{nm}^i T^{nj\dots}_{kl\dots} + \Gamma_{nm}^j T^{in\dots}_{kl\dots} + \dots - \Gamma_{km}^n T^{ij\dots}_{nl\dots} - \Gamma_{lm}^n T^{ij\dots}_{kn\dots} - \dots \quad (1.2.11)$$

A covariant derivative of the Kronecker symbol vanishes:

$$\delta^k_{l;i} = \Gamma_{ji}^k \delta_l^j - \Gamma_{li}^j \delta_j^k = 0. \quad (1.2.12)$$

The second term on the right-hand side of (1.2.5) does not depend on the affine connection, but only on the coordinate transformation. Therefore the difference between two different connections transforms like a tensor of rank (1,2). Consequently, the variation $\delta \Gamma_{ik}^j$, which is an infinitesimal difference between two connections, is a tensor of rank (1,2).

1.2.2 Parallel transport

We consider two infinitesimally separated points in spacetime, $P(x^i)$ and $Q(x^i + dx^i)$, and a vector field A which takes the value A^k at P and $A^k + dA^k$ at Q . Because $dA^k = A^k_{,i} dx^i$ and $A^k_{,i}$ is not a tensor, the difference dA^k is not a vector, which is related to subtracting of two vectors at two points with different coordinate transformation laws. In order to calculate the covariant difference between two vectors at two different points, we must bring these vectors to the same point. Instead of subtracting from the vector $A^k + dA^k$ at Q the vector A^k at P , we must subtract a vector $A^k + \delta A^k$ at Q that corresponds to A^k at P , so the resulting difference (covariant differential) $DA^k = dA^k - \delta A^k$ is a vector. The vector $A^k + \delta A^k$ is the *parallel-transported* or parallel-translated A^k from P to Q . A parallel-transported linear combination of vectors must be equal to the same linear combination

of parallel-transported vectors. Therefore δA^k is a linear and homogeneous function of A^k . It is also on the order of a differential, thus a linear and homogeneous function of dx^i . The most general form of δA^k is

$$\delta A^k = -\Gamma_{li}^k A^l dx^i, \quad (1.2.13)$$

so

$$DA^k = dA^k + \Gamma_{li}^k A^l dx^i = A^k_{;i} dx^i. \quad (1.2.14)$$

Because δA^k is not a vector, Γ_{li}^k is not a tensor. Because DA^k is a vector, $A^k_{;i}$ is a tensor. The expressions for covariant derivatives of a covariant vector and tensors result from

$$\delta\phi = 0, \quad \delta(TU) = \delta TU + T\delta U. \quad (1.2.15)$$

1.2.3 Torsion tensor

The second term on the right-hand side of (1.2.5) is symmetric in the indices i, k so the antisymmetric part of the connection with respect to these indices,

$$S^j_{ik} = \Gamma_{[ik]}^j, \quad (1.2.16)$$

is a tensor:

$$S'^j_{ik} = \frac{\partial x'^j}{\partial x^n} \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^k} S^n_{lm}. \quad (1.2.17)$$

This tensor is called the Cartan *torsion tensor*. The torsion tensor has generally 24 independent components. The contracted torsion tensor,

$$S^k_{ik} = S_i, \quad (1.2.18)$$

is called the torsion trace or *torsion vector*.

1.2.4 Covariant differentiation of densities

A derivative of a scalar density of weight w , $\mathfrak{s}$, does not transform like a covariant vector density:

$$\begin{aligned} \partial_i \mathfrak{s}' &= \frac{\partial x^l}{\partial x'^i} \partial_l \left(\left| \frac{\partial x^j}{\partial x'^k} \right|^w \mathfrak{s} \right) = \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^w \partial_l \mathfrak{s} + w \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^{w-1} \partial_l \left| \frac{\partial x^r}{\partial x'^s} \right| \mathfrak{s} \\ &= \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^w \partial_l \mathfrak{s} + w \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^{w-1} \left| \frac{\partial x^r}{\partial x'^s} \right| \frac{\partial x'^n}{\partial x^m} \frac{\partial}{\partial x'^n} \frac{\partial x^m}{\partial x'^i} \mathfrak{s} \\ &= \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^w \partial_l \mathfrak{s} + w \left| \frac{\partial x^j}{\partial x'^k} \right|^w \frac{\partial x'^n}{\partial x^m} \frac{\partial^2 x^m}{\partial x'^n \partial x'^i} \mathfrak{s}. \end{aligned} \quad (1.2.19)$$

We consider the expression

$$\mathfrak{s}_{;i} = \mathfrak{s}_{,i} - w \Gamma_i \mathfrak{s}, \quad (1.2.20)$$

where the quantity Γ_i transforms such that $\mathfrak{s}_{;i}$ is a vector density of weight w :

$$\mathfrak{s}'_{;i} = \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^w \mathfrak{s}_{;l} = \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^w (\mathfrak{s}_{,l} - w \Gamma_l \mathfrak{s}). \quad (1.2.21)$$

On the other hand (1.2.19) gives

$$\mathfrak{s}'_{;i} = \mathfrak{s}'_{,i} - w \Gamma'_i \mathfrak{s}' = \frac{\partial x^l}{\partial x'^i} \left| \frac{\partial x^j}{\partial x'^k} \right|^w \partial_l \mathfrak{s} + w \left| \frac{\partial x^j}{\partial x'^k} \right|^w \frac{\partial x'^n}{\partial x^m} \frac{\partial^2 x^m}{\partial x'^n \partial x'^i} \mathfrak{s} - w \left| \frac{\partial x^j}{\partial x'^k} \right|^w \Gamma'_i \mathfrak{s}, \quad (1.2.22)$$

so we obtain the transformation law for Γ_i :

$$\Gamma'_i = \frac{\partial x^l}{\partial x'^i} \Gamma_l + \frac{\partial x'^n}{\partial x^m} \frac{\partial^2 x^m}{\partial x'^n \partial x'^i}, \quad (1.2.23)$$

which is the same as the transformation law for Γ_{ki}^k (1.2.6). Therefore the difference $\Gamma_i - \Gamma_{ki}^k$ is some covariant vector V_i .

If we assume that parallel transport of the product of a scalar density of any weight and a tensor obeys the chain rule:

$$\delta(\mathfrak{s}T) = \delta\mathfrak{s}T + \mathfrak{s}\delta T, \quad (1.2.24)$$

so a covariant derivative of such product behaves like an ordinary derivative:

$$(\mathfrak{s}T)_{;i} = \mathfrak{s}_{;i}T + \mathfrak{s}T_{;i}, \quad (1.2.25)$$

then a covariant derivative of a tensor density of weight w is equal to the sum of the corresponding ordinary derivative of this tensor, terms with the affine connection that covariantize each index, and the term with Γ_i :

$$\begin{aligned} \mathfrak{T}_{kl\dots m}^{ij\dots} &= \mathfrak{T}_{kl\dots m}^{ij\dots} + \Gamma_{nm}^i \mathfrak{T}_{kl\dots}^{nj\dots} + \Gamma_{nm}^j \mathfrak{T}_{kl\dots}^{in\dots} + \dots \\ &- \Gamma_{km}^n \mathfrak{T}_{nl\dots}^{ij\dots} - \Gamma_{lm}^n \mathfrak{T}_{kn\dots}^{ij\dots} - \dots - w\Gamma_m \mathfrak{T}_{kl\dots}^{ij\dots}. \end{aligned} \quad (1.2.26)$$

A covariant derivative of the contravariant Levi-Civita density is

$$\epsilon^{ijkl}_{;m} = \Gamma_{nm}^i \epsilon^{njkl} + \Gamma_{nm}^j \epsilon^{in kl} + \Gamma_{nm}^k \epsilon^{ij nl} + \Gamma_{nm}^l \epsilon^{ijk n} - \Gamma_m \epsilon^{ijkl}. \quad (1.2.27)$$

In the summations over n only one term does not vanish for each term on the right-hand side of (1.2.27), so

$$\begin{aligned} \epsilon^{ijkl}_{;m} &= \Gamma_{n=i|m}^i \epsilon^{n=jkl} + \Gamma_{n=j|m}^j \epsilon^{i=nkl} + \Gamma_{n=k|m}^k \epsilon^{ij=n l} + \Gamma_{n=l|m}^l \epsilon^{ijk=n} - \Gamma_m \epsilon^{ijkl} \\ &= (\Gamma_{nm}^n - \Gamma_m) \epsilon^{ijkl} = -V_m \epsilon^{ijkl}. \end{aligned} \quad (1.2.28)$$

The Levi-Civita symbol is a tensor density with constant components, so it does not change under a parallel transport, $\delta\epsilon = 0$. Therefore $\epsilon^{ijkl}_{;m} = 0$, so $V_i = 0$ and

$$\Gamma_i = \Gamma_{ki}^k. \quad (1.2.29)$$

1.2.5 Covariant derivatives

Totally antisymmetrized ordinary derivatives of covariant tensors, $A_{[i;k]}$, $B_{[ik;l]}$ and $C_{[ikl;m]}$, are tensors because of antisymmetrization. Totally antisymmetrized covariant derivatives of tensors are clearly tensors because ∇_i is a covariant operation, and are given by direct calculation using the definition of a covariant derivative:

$$A_{[i;k]} = A_{[i,k]} - S_{ik}^l A_l, \quad B_{[ik;l]} = B_{[ik,l]} - 2S_{[ik}^m B_{l]m}. \quad (1.2.30)$$

Divergences of (totally antisymmetric if more than 1 index) contravariant densities, $\mathfrak{C}^i_{;i}$, $\mathfrak{B}^{ik}_{;i}$ and $\mathfrak{A}^{ikl}_{;i}$, are densities because of the correspondence between tensors and dual densities. Covariant divergences of contravariant densities are clearly densities, and are given by direct calculation:

$$\mathfrak{C}^i_{;i} = \mathfrak{C}^i_{;i} + 2S_i \mathfrak{C}^i, \quad \mathfrak{B}^{ik}_{;i} = \mathfrak{B}^{ik}_{;i} - S_{ik}^l \mathfrak{B}^{il} + 2S_i \mathfrak{B}^{ik}. \quad (1.2.31)$$

1.2.6 Partial integration

If the product of two quantities (tensors or densities) TU is a contravariant density $\mathfrak{C}^k$ then

$$\int TU_{;k} d\Omega = \int (TU)_{;k} d\Omega - \int T_{;k} U d\Omega = \int (TU)_{;k} d\Omega + 2 \int S_k TU d\Omega - \int T_{;k} U d\Omega. \quad (1.2.32)$$

The first term on the right-hand side can be transformed into a hypersurface integral $\int TU dS_k$. If the region of integration extends to infinity and $\mathfrak{C}^k$ corresponds to some physical quantity then the boundary integral $\int TU dS_k$ vanishes, giving

$$\int TU_{;k} d\Omega = 2 \int S_k TU d\Omega - \int T_{;k} U d\Omega. \quad (1.2.33)$$

If $T = \delta_i^k$ then $U = \mathfrak{C}^i$ and

$$\int \mathfrak{C}^i_{;i} d\Omega = 2 \int S_i \mathfrak{C}^i d\Omega. \quad (1.2.34)$$

1.2.7 Geodesic frame of reference

We consider a coordinate transformation

$$x^k = x'^k + \frac{1}{2} a^k_{lm} x'^l x'^m, \quad (1.2.35)$$

where a^k_{lm} is symmetric in the indices l, m . Substituting this transformation to (1.2.5) and calculating it at $x^k = x'^k = 0$ gives

$$\frac{\partial x^i}{\partial x'^k} = \delta_k^i \quad (1.2.36)$$

and

$$\Gamma'^j_{ik} = \Gamma^j_{ik} + a^j_{ik}. \quad (1.2.37)$$

Putting

$$a^j_{ik} = -\Gamma^j_{(ik)}|_{x'=0} \quad (1.2.38)$$

gives

$$\Gamma'^j_{(ik)} = 0. \quad (1.2.39)$$

Therefore there always exists a coordinate frame of reference in which the symmetric part of the connection vanishes locally (at one point). If the affine connection is symmetric in the covariant indices, $\Gamma^j_{ik} = \Gamma^j_{ki}$ (the torsion tensor vanishes) then (1.2.39) gives

$$\Gamma'^j_{ik} = 0. \quad (1.2.40)$$

The coordinate frame of reference in which the connection vanishes (locally) is referred to as *geodesic*.

1.2.8 Affine geodesics and four-velocity

We consider a point in spacetime $P(x^k)$ and a vector dx^k at this point. Construct a point $P'(x^k + dx^k)$ and find the vector $d'x^k$ which is the parallel-transported dx^k from P to P' . Then construct a point $P''(x^k + dx^k + d'x^k)$ and find the vector $d''x^k$ which is the parallel-transported $d'x^k$ from P' to P'' . The next point is $P'''(x^k + dx^k + d'x^k + d''x^k)$ etc. Repeating this step constructs a polygonal line which in the limit $dx^k \rightarrow 0$ becomes a curve such that the vector $\frac{dx^k}{d\lambda}$ (where λ is a parameter along the curve) tangent to it at any point, when parallelly translated to another point on this curve, coincides with the tangent vector there. Such curve is referred to as an autoparallel curve or *affine geodesic*. Affine geodesics can be attributed with the concept of length, which, for the polygonal curve, is proportional to the number of parallel-transport steps described above.

The condition that parallel transport of a tangent vector be a tangent vector is

$$\frac{dx^i}{d\lambda} + \delta \left(\frac{dx^i}{d\lambda} \right) = \frac{dx^i}{d\lambda} - \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = M \left(\frac{dx^i}{d\lambda} + \frac{d^2 x^i}{d\lambda^2} d\lambda \right), \quad (1.2.41)$$

where the proportionality factor M is some function of λ , or

$$M \frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = \frac{1 - M}{d\lambda} \frac{dx^i}{d\lambda}, \quad (1.2.42)$$

from which it follows that M must differ from 1 by the order of $d\lambda$. In the first term on the left-hand side of (1.2.42) we can therefore put $M = 1$, and we denote $1 - M$ by $\phi(\lambda)d\lambda$, so

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = \phi(\lambda) \frac{dx^i}{d\lambda}. \quad (1.2.43)$$

If we replace λ by a new variable $s(\lambda)$ then (1.2.43) becomes

$$\frac{d^2 x^i}{ds^2} + \Gamma_{kl}^i \frac{dx^k}{ds} \frac{dx^l}{ds} = \frac{\phi s' - s''}{s'^2} \frac{dx^i}{ds}, \quad (1.2.44)$$

where the prime denotes differentiation with respect to λ . Requiring $\phi s' - s'' = 0$, which has a general solution $s = \int^\lambda d\lambda \exp[-\int^\lambda \phi(x) dx]$, brings (1.2.44) into

$$\frac{d^2 x^i}{ds^2} + \Gamma_{kl}^i \frac{dx^k}{ds} \frac{dx^l}{ds} = 0, \quad (1.2.45)$$

where the scalar variable s is the affine parameter. The autoparallel equation (1.2.45) is invariant under linear transformations $s \rightarrow as + b$ since the two lower limits of integration in the expression for $s(\lambda)$ are arbitrary. Defining the *four-velocity* vector

$$u^i = \frac{dx^i}{ds} \quad (1.2.46)$$

brings (1.2.44) into

$$\frac{DA^k}{ds} = A^k_{;i} u^i, \quad \frac{dA^k}{ds} = A^k_{;i} u^i, \quad (1.2.47)$$

so

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma_{kl}^i u^k u^l = u^i_{;j} u^j = 0. \quad (1.2.48)$$

The relations (1.2.47) can be generalized to any tensor density T :

$$\frac{DT}{ds} = T_{;i} u^i, \quad \frac{dT}{ds} = T_{;i} u^i, \quad (1.2.49)$$

The vector $\frac{dx^i}{ds}|_Q$ is a parallel translation of $\frac{dx^i}{ds}|_P$. Because ds is a scalar, it is invariant under parallel transport, $ds|_Q = ds|_P$. Therefore the vector $dx^i|_Q$ is a parallel translation of $dx^i|_P$, so ds measures the length of an infinitesimal section of an affine geodesic.

Only the symmetric part $\Gamma_{(kl)}^i$ of the connection enters the autoparallel equation (1.2.45) because of the symmetry of $\frac{dx^k}{ds} \frac{dx^l}{ds}$ with respect to the indices k, l ; affine geodesics do not depend on torsion. At any point, a coordinate transformation to the geodesic frame (1.2.35) brings all the components $\Gamma_{(kl)}^i$ to zero, so the autoparallel equation becomes $\frac{du^i}{ds} = 0$. The autoparallel equation is also invariant under a *projective* transformation

$$\Gamma_{kl}^i \rightarrow \Gamma_{kl}^i + \delta_k^i A_l, \quad (1.2.50)$$

where A_i is an arbitrary vector. Substituting this transformation to (1.2.48) gives

$$\frac{du^i}{ds} + \Gamma_{kl}^i u^k u^l = -u^i u^k A_k. \quad (1.2.51)$$

If we replace s by a new variable $\tilde{s}(s)$ then (1.2.51) becomes

$$\frac{dU^i}{d\tilde{s}} + \Gamma_{kl}^i U^k U^l = -\frac{u^k A_k \tilde{s}' + \tilde{s}''}{\tilde{s}'^2} \frac{dx^i}{d\tilde{s}}, \quad (1.2.52)$$

where

$$U^i = \frac{dx^i}{d\tilde{s}} \quad (1.2.53)$$

and the prime denotes differentiation with respect to s . Requiring $u^k A_k \tilde{s}' + \tilde{s}'' = 0$, which has a general solution $\tilde{s} = -\int^s ds \exp[\int^s A_k u^k(x) dx]$, brings (1.2.52) into

$$\frac{dU^i}{d\tilde{s}} + \Gamma_{kl}^i U^k U^l = 0. \quad (1.2.54)$$

1.2.9 Infinitesimal coordinate transformations

We consider a coordinate transformation

$$x'^i = x^i + \xi^i, \quad (1.2.55)$$

where $\xi^i = \delta x^i$ is an infinitesimal vector (variation of x^i). For a tensor or density T define

$$\delta T = T'(x'^i) - T(x^i), \quad (1.2.56)$$

$$\bar{\delta} T = T'(x^i) - T(x^i) = \delta T - \xi^k T_{,k}. \quad (1.2.57)$$

For a scalar we find

$$\delta \phi = 0, \quad \bar{\delta} \phi = -\xi^k \phi_{,k}. \quad (1.2.58)$$

For a covariant vector

$$\delta A_i = \frac{\partial x^k}{\partial x'^i} A_k - A_i \approx -\xi^k_{,i} A_k, \quad (1.2.59)$$

$$\bar{\delta} A_i \approx -\xi^k_{,i} A_k - \xi^k A_{i,k}. \quad (1.2.60)$$

The variation (1.2.59) is not a tensor, but (1.2.60) is:

$$\bar{\delta} A_i = -\xi^k_{,i} A_k - \xi^k A_{i,k} - 2S^j_{ik} \xi^k A_j. \quad (1.2.61)$$

We refer to $-\bar{\delta} T$ as a *Lie derivative* of T , $\mathcal{L}_\xi T$. For a contravariant vector

$$\delta B^i = \frac{\partial x'^i}{\partial x^k} B^k - B^i \approx \xi^i_{,k} B^k, \quad (1.2.62)$$

$$\bar{\delta} B^i \approx \xi^i_{,k} B^k - \xi^k B^i_{,k} = \xi^i_{,k} B^k - \xi^k B^i_{,k} + 2S^i_{jk} \xi^k B^j. \quad (1.2.63)$$

For a scalar density

$$\delta \mathfrak{s} = \left(\left| \frac{\partial x^i}{\partial x'^i} \right| - 1 \right) \mathfrak{s} \approx -\xi^i_{,i} \mathfrak{s}, \quad (1.2.64)$$

$$\bar{\delta} \mathfrak{s} \approx -\xi^i_{,i} \mathfrak{s} - \xi^k \mathfrak{s}_{,k} = -\xi^i_{,i} \mathfrak{s} - \xi^k \mathfrak{s}_{,k} + 2S_i \xi^i \mathfrak{s}. \quad (1.2.65)$$

The chain rule for δ implies that for a tensor density of weight w (which includes tensors as densities of weight 0)

$$\begin{aligned} \delta \mathfrak{T}^{ij\dots}_{kl\dots} &\approx \xi^i_{,m} \mathfrak{T}^{mj\dots}_{kl\dots} + \xi^j_{,m} \mathfrak{T}^{im\dots}_{kl\dots} + \dots - \xi^m_{,k} \mathfrak{T}^{ij\dots}_{ml\dots} - \xi^m_{,l} \mathfrak{T}^{ij\dots}_{km\dots} - \dots \\ &- w \xi^m_{,m} \mathfrak{T}^{ij\dots}_{kl\dots}, \end{aligned} \quad (1.2.66)$$

$$\begin{aligned} \bar{\delta} \mathfrak{T}^{ij\dots}_{kl\dots} &\approx \xi^i_{,m} \mathfrak{T}^{mj\dots}_{kl\dots} + \xi^j_{,m} \mathfrak{T}^{im\dots}_{kl\dots} + \dots - \xi^m_{,k} \mathfrak{T}^{ij\dots}_{ml\dots} - \xi^m_{,l} \mathfrak{T}^{ij\dots}_{km\dots} - \dots \\ &- w \xi^m_{,m} \mathfrak{T}^{ij\dots}_{kl\dots} - \xi^m \mathfrak{T}^{ij\dots}_{kl\dots,m} + 2S^i_{nm} \xi^m \mathfrak{T}^{nj\dots}_{kl\dots} + 2S^j_{nm} \xi^m \mathfrak{T}^{in\dots}_{kl\dots} + \dots \\ &- 2S^n_{km} \xi^m \mathfrak{T}^{ij\dots}_{nl\dots} - 2S^n_{lm} \xi^m \mathfrak{T}^{ij\dots}_{kn\dots} - \dots + 2w S_m \xi^m \mathfrak{T}^{ij\dots}_{kl\dots}. \end{aligned} \quad (1.2.67)$$

A Lie derivative of a tensor density of rank (k, l) and weight w is a tensor density of rank (k, l) and weight w .

The formula for a covariant derivative of T can be written as

$$T_{;k} = T_{,k} + \Gamma^j_{ik} \hat{C}^i_j T, \quad (1.2.68)$$

where $\hat{C}$ is an operator acting on tensor densities:

$$\hat{C}^i_j \phi = 0, \quad \hat{C}^i_j A_k = -\delta^i_k A_j, \quad \hat{C}^i_j B^k = \delta^k_j B^i, \quad \hat{C}^i_j \mathfrak{s} = -\delta^i_j \mathfrak{s}, \quad (1.2.69)$$

or generally

$$\hat{C}^m_n \mathfrak{T}^{ij\dots}_{kl\dots} = \delta^i_n \mathfrak{T}^{mj\dots}_{kl\dots} + \delta^j_n \mathfrak{T}^{im\dots}_{kl\dots} + \dots - \delta^m_k \mathfrak{T}^{ij\dots}_{nl\dots} - \delta^m_l \mathfrak{T}^{ij\dots}_{kn\dots} - \dots - w \delta^m_n \mathfrak{T}^{ij\dots}_{kl\dots}. \quad (1.2.70)$$

Such defined operator also enters the formula for δT :

$$\delta T = \hat{C}^k_i T \xi^i_{,k}. \quad (1.2.71)$$

1.2.10 Killing vectors

A vector ζ_i that satisfies

$$\zeta_{(i;k)} = 0 \quad (1.2.72)$$

is referred to as a *Killing vector*. Along an affine geodesic

$$\frac{D}{ds}(u^i \zeta_i) = u^k (u^i \zeta_i)_{;k} = u^i u^k \zeta_{i;k} + \zeta_i u^k u^i_{;k} = 0. \quad (1.2.73)$$

The first term in the sum in (1.2.73) vanishes because of the definition of ζ_i and the second term vanishes because of the affine geodesic equation. Therefore, to each Killing vector ζ_i there corresponds a quantity $u^i \zeta_i$ which is constant along an affine geodesic.

References: [1, 2, 3].

1.3 Curvature

1.3.1 Curvature tensor

The commutator of covariant derivatives of a contravariant vector is a tensor:

$$\begin{aligned} [\nabla_j, \nabla_k] B^i &= 2\nabla_{[j} \nabla_{k]} B^i = 2\partial_{[j} \nabla_{k]} B^i - 2\Gamma_{[k j]}^l \nabla_l B^i + 2\Gamma_{l[j}^i \nabla_{k]} B^l \\ &= 2\partial_{[j} (\Gamma_{|m|k]}^i B^m) + 2S_{jk}^l \nabla_l B^i + 2\Gamma_{l[j}^i \partial_{k]} B^l + 2\Gamma_{l[j}^i \Gamma_{|m|k]}^l B^m \\ &= 2(\partial_{[j} \Gamma_{|m|k]}^i + \Gamma_{l[j}^i \Gamma_{|m|k]}^l) B^m + 2S_{jk}^l \nabla_l B^i = R^i_{mjk} B^m + 2S_{jk}^l \nabla_l B^i, \end{aligned} \quad (1.3.1)$$

where $||$ an index which is excluded from symmetrization or antisymmetrization. Therefore R^i_{mjk} , defined as

$$R^i_{mjk} = \partial_j \Gamma_{mk}^i - \partial_k \Gamma_{mj}^i + \Gamma_{lj}^i \Gamma_{mk}^l - \Gamma_{lk}^i \Gamma_{mj}^l, \quad (1.3.2)$$

is a tensor, referred to as the *curvature tensor*. The curvature tensor R^i_{mjk} is antisymmetric in the indices j, k and has generally 96 independent components. The commutator of covariant derivatives of a covariant vector is

$$[\nabla_j, \nabla_k] A_i = -R^m_{ijk} A_m + 2S_{jk}^l \nabla_l A_i, \quad (1.3.3)$$

and the commutator of covariant derivatives of a tensor is

$$\begin{aligned} [\nabla_j, \nabla_k] T^{in...}_{lp...} &= R^i_{mjk} T^{mn...}_{lp...} + R^n_{mjk} T^{im...}_{lp...} + \dots - R^m_{ljk} T^{in...}_{mp...} - R^m_{pj k} T^{in...}_{lm...} \\ &- \dots + 2S_{jk}^l \nabla_l T^{in...}_{lp...}. \end{aligned} \quad (1.3.4)$$

A change in the connection

$$\Gamma_{jk}^i \rightarrow \Gamma_{jk}^i + T_{jk}^i, \quad (1.3.5)$$

where T_{jk}^i is a tensor, results in the following change of the curvature tensor:

$$R^i_{klm} \rightarrow R^i_{klm} + T^i_{km;l} - T^i_{kl;m} + T^j_{km} T^i_{jl} - T^j_{kl} T^i_{jm}. \quad (1.3.6)$$

For a projective transformation (1.2.50), $T_{jk}^i = \delta_j^i A_k$, so

$$R^i_{klm} \rightarrow R^i_{klm} + \delta_k^i (A_{m;l} - A_{l;m}). \quad (1.3.7)$$

The variation of the curvature tensor is

$$\begin{aligned} \delta R^i_{klm} &= (\delta \Gamma_{km}^i)_{;l} - (\delta \Gamma_{kl}^i)_{;m} + \delta \Gamma_{jl}^i \Gamma_{km}^j + \Gamma_{jl}^i \delta \Gamma_{km}^j - \delta \Gamma_{jm}^i \Gamma_{kl}^j - \Gamma_{jm}^i \delta \Gamma_{kl}^j \\ &= (\delta \Gamma_{km}^i)_{;l} - \Gamma_{jl}^i \delta \Gamma_{km}^j + \Gamma_{kl}^j \delta \Gamma_{jm}^i + \Gamma_{ml}^j \delta \Gamma_{kj}^i - (\delta \Gamma_{kl}^i)_{;m} + \Gamma_{jm}^i \delta \Gamma_{kl}^j - \Gamma_{km}^j \delta \Gamma_{jl}^i \\ &- \Gamma_{lm}^j \delta \Gamma_{kj}^i + \delta \Gamma_{jl}^i \Gamma_{km}^j + \Gamma_{jl}^i \delta \Gamma_{km}^j - \delta \Gamma_{jm}^i \Gamma_{kl}^j - \Gamma_{jm}^i \delta \Gamma_{kl}^j \\ &= (\delta \Gamma_{km}^i)_{;l} - (\delta \Gamma_{kl}^i)_{;m} - 2S_{lm}^n \delta \Gamma_{kn}^i. \end{aligned} \quad (1.3.8)$$

1.3.2 Integrability of connection

The affine connection is *integrable* if parallel transport of a vector from point P to point Q is independent of a path along which this vector is parallelly translated, or equivalently, parallel transport of a vector around a closed curve does not change this vector. For an integrable connection, we can uniquely translate parallelly a given vector h^i at point P to all points in spacetime:

$$\delta h^i = dh^i, \quad (1.3.9)$$

or

$$h^i_{,k} = -\Gamma_{jk}^i h^j. \quad (1.3.10)$$

Therefore

$$(\Gamma_{jk}^i h^j)_{,l} - (\Gamma_{jl}^i h^j)_{,k} = \Gamma_{jk,l}^i h^j - \Gamma_{jk}^i \Gamma_{ml}^j h^m - \Gamma_{jl,k}^i h^j + \Gamma_{jl}^i \Gamma_{mk}^j h^m = R_{jlk}^i h^j = 0, \quad (1.3.11)$$

so, because h^i is arbitrary,

$$R_{klm}^i = 0. \quad (1.3.12)$$

Spacetime with a vanishing curvature tensor $R_{klm}^i = 0$ is *flat*. We consider 4 linearly independent vectors h_a^i , where a is 1,2,3,4, and vectors inverse to h_a^i :

$$\sum_a h_a^i h_{ka}^i = \delta_k^i. \quad (1.3.13)$$

If the affine connection is integrable then (1.3.10) becomes

$$h_{a,k}^i = -\Gamma_{lk}^i h_a^l. \quad (1.3.14)$$

Multiplying (1.3.14) by h_{ja}^i gives

$$\Gamma_{jk}^i = -h_{ja}^i h_{a,k}^i = h_{ja,k}^i h_a^i. \quad (1.3.15)$$

An integrable connection has thus 16 independent components. If the connection is also symmetric, $S_{jk}^i = 0$, then

$$h_{ja,k}^i - h_{ka,j}^i = 0, \quad (1.3.16)$$

which is the condition for the independence of the coordinates

$$y_a = \int_P^Q h_{ia} dx^i \quad (1.3.17)$$

of the path of integration PQ . Adopting y_a as the new coordinates (with point $P = (0,0,0,0)$ in the center) gives

$$\frac{\partial y_a}{\partial x^i} = h_{ia}, \quad \frac{\partial x^i}{\partial y_a} = h_a^i, \quad (1.3.18)$$

so (1.3.15) becomes

$$\Gamma_{jk}^i(x^i) = \frac{\partial x^i}{\partial y_a} \frac{\partial^2 y_a}{\partial x^k \partial x^j}. \quad (1.3.19)$$

The transformation law for the connection (1.2.5) gives (with y_a corresponding to x'^j)

$$\Gamma_{jk}^i(y_a) = 0. \quad (1.3.20)$$

A torsionless integrable connection can be thus transformed to zero; one can always find a system of coordinates which is geodesic everywhere. If a connection is symmetric but nonintegrable then a geodesic frame of reference can be constructed only at a given point (or along a given world line).

1.3.3 Parallel transport along closed curve

We consider parallel transport of a covariant vector around an infinitesimal closed curve. Such transport changes this vector by

$$\begin{aligned}\Delta A_k &= \oint \delta A_k = \oint \Gamma_{kl}^i A_i dx^l = \frac{1}{2} \int \left(\frac{\partial(\Gamma_{km}^i A_i)}{\partial x^l} - \frac{\partial(\Gamma_{kl}^i A_i)}{\partial x^m} \right) df^{lm} \\ &\approx \frac{1}{2} \int \left[\left(\frac{\partial \Gamma_{km}^i}{\partial x^l} - \frac{\partial \Gamma_{kl}^i}{\partial x^m} \right) A_i + (\Gamma_{km}^i \Gamma_{il}^n - \Gamma_{kl}^i \Gamma_{im}^n) A_n \right] df^{lm} \\ &= \frac{1}{2} R_{klm}^i A_i \Delta f^{lm},\end{aligned}\tag{1.3.21}$$

where we use Stokes' theorem (1.1.33) and $A_{k,l} = \Gamma_{kl}^i A_i$ which is valid along the curve and thus is approximately valid (to terms of first order in Δf^{lm}) inside this curve. The change of a contravariant vector due to parallel transport around an infinitesimal closed curve results from $\Delta(A_k B^k) = 0$:

$$\delta B^k = -\frac{1}{2} R_{ilm}^k B^i \Delta f^{lm},\tag{1.3.22}$$

and the corresponding change of a tensor results from the chain rule for parallel transport:

$$\delta T_{np\dots}^{ik\dots} = -\frac{1}{2} (R_{jlm}^i T_{np\dots}^{jk\dots} + R_{jlm}^k T_{np\dots}^{ij\dots} + \dots - R_{nlm}^j T_{jp\dots}^{ik\dots} - R_{plm}^j T_{nj\dots}^{ik\dots} - \dots) \Delta f^{lm}.\tag{1.3.23}$$

1.3.4 Bianchi identities

We consider

$$\nabla_j \nabla_{[k} \nabla_{l]} B^i = \frac{1}{2} \nabla_j (R_{mkl}^i B^m) + \nabla_j (S_{kl}^m \nabla_m B^i)\tag{1.3.24}$$

and

$$\begin{aligned}\nabla_{[j} \nabla_k \nabla_{l]} B^i &= -\frac{1}{2} R_{ljk}^m \nabla_m B^i + \frac{1}{2} R_{mjk}^i \nabla_l B^m + S_{jk}^m \nabla_m \nabla_l B^i = -\frac{1}{2} R_{ljk}^m \nabla_m B^i \\ &+ \frac{1}{2} R_{mjk}^i \nabla_l B^m + S_{jk}^m \nabla_l \nabla_m B^i + S_{jk}^m R_{nml}^i B^n + 2S_{jk}^m S_{ml}^n \nabla_n B^i.\end{aligned}\tag{1.3.25}$$

Total antisymmetrization of the indices j, k, l in (1.3.24) and (1.3.25) gives

$$\nabla_{[j} \nabla_k \nabla_{l]} B^i = \frac{1}{2} \nabla_{[j} R_{m|kl]}^i B^m + \frac{1}{2} R_{m[kl]}^i \nabla_{j]} B^m + \nabla_{[j} S_{kl]}^m \nabla_m B^i + S_{kl}^m \nabla_{j]} \nabla_m B^i\tag{1.3.26}$$

and

$$\begin{aligned}\nabla_{[j} \nabla_k \nabla_{l]} B^i &= -\frac{1}{2} R_{[ljk]}^m \nabla_m B^i + \frac{1}{2} R_{m[jk]}^i \nabla_{l]} B^m + S_{[jk]}^m \nabla_{l]} \nabla_m B^i \\ &+ S_{[jk]}^m R_{|nm|l]}^i B^n + 2S_{[jk]}^m S_{|m|l]}^n \nabla_n B^i,\end{aligned}\tag{1.3.27}$$

so

$$\begin{aligned}\frac{1}{2} \nabla_{[j} R_{m|kl]}^i B^m + \nabla_{[j} S_{kl]}^m \nabla_m B^i &= -\frac{1}{2} R_{[ljk]}^m \nabla_m B^i + S_{[jk]}^m R_{|nm|l]}^i B^n \\ &+ 2S_{[jk]}^m S_{|m|l]}^n \nabla_n B^i.\end{aligned}\tag{1.3.28}$$

Comparing terms in (1.3.28) with B^i gives the first Bianchi identity or simply *Bianchi identity*:

$$R_{n[jk;l]}^i = 2R_{nm[j}^i S_{kl]}^m,\tag{1.3.29}$$

while comparing terms with $\nabla_k B^i$ gives the second Bianchi identity or *cyclic identity*:

$$R_{[jkl]}^m = -2S_{[jk;l]}^m + 4S_{n[j}^m S_{kl]}^n.\tag{1.3.30}$$

For a symmetric connection, $S_{jk}^i = 0$, these identities reduce to

$$R^i_{n[jk;l]} = 0, \quad (1.3.31)$$

$$R^m_{[jkl]} = 0. \quad (1.3.32)$$

The cyclic identity (1.3.32) imposes 16 constraints on the curvature tensor, so the curvature tensor with a vanishing torsion has 80 independent components.

1.3.5 Ricci tensor

Contraction of the curvature tensor with respect to the contravariant index and the second covariant index gives the *Ricci tensor*:

$$R_{ik} = R^j_{ijk} = \Gamma^j_{ik,j} - \Gamma^j_{ij,k} + \Gamma^l_{ik}\Gamma^j_{lj} - \Gamma^l_{ij}\Gamma^j_{lk}. \quad (1.3.33)$$

Contraction of the curvature tensor with respect to the contravariant index and the third covariant index gives the Ricci tensor with the opposite sign due to the antisymmetry of the curvature tensor with respect to its last indices. Contraction of the curvature tensor with respect to the contravariant index and the first covariant index gives the homothetic or *segmental curvature* tensor:

$$Q_{ik} = R^j_{jik} = \Gamma^j_{jk,i} - \Gamma^j_{ji,k}, \quad (1.3.34)$$

which is a curl. A change in the connection (1.3.5) results in the following changes of the Ricci tensor and segmental curvature tensor:

$$R_{ik} \rightarrow R_{ik} + T^l_{ik;l} - T^l_{il;k} + T^j_{ik}T^l_{jl} - T^j_{il}T^l_{jk}, \quad (1.3.35)$$

$$Q_{ik} \rightarrow Q_{ik} + T^j_{jk,i} - T^j_{ji,k}. \quad (1.3.36)$$

For a projective transformation (1.2.50)

$$R_{ik} \rightarrow R_{ik} + A_{k;i} - A_{i;k}, \quad (1.3.37)$$

$$Q_{ik} \rightarrow Q_{ik} + 4(A_{k,i} - A_{i,k}). \quad (1.3.38)$$

Therefore the symmetric part of the Ricci tensor is invariant under projective transformations. The variation of the Ricci tensor is

$$\delta R_{ik} = (\delta \Gamma^l_{ik})_{;l} - (\delta \Gamma^l_{il})_{;k} - 2S^j_{lk}\delta \Gamma^l_{ij}, \quad (1.3.39)$$

while the variation of the segmental curvature tensor is

$$\delta Q_{ik} = (\delta \Gamma^j_{jk})_{;i} - (\delta \Gamma^j_{ji})_{;k}. \quad (1.3.40)$$

1.3.6 Geodesic deviation

We consider a family of affine geodesics characterized by the affine parameter s and distinguished by a scalar parameter t . We define the separation vector

$$v^i = \frac{dx^i}{dt}, \quad (1.3.41)$$

so

$$v^i_{;k}u^k - u^i_{;k}v^k = v^i_{;k}u^k - u^i_{;k}v^k - 2S^i_{kl}u^k v^l = \frac{du^i}{dt} - \frac{dv^i}{ds} - 2S^i_{kl}u^k v^l = -2S^i_{kl}u^k v^l. \quad (1.3.42)$$

Therefore

$$\begin{aligned}
\frac{D^2 v^i}{ds^2} &= (v^i_{;j} u^j)_{;k} u^k = (u^i_{;j} v^j)_{;k} u^k - 2(S^i_{kl} u^k v^l)_{;j} u^j \\
&= u^i_{;jk} v^j u^k + u^i_{;j} v^j_{;k} u^k - 2(S^i_{kl} u^k v^l)_{;j} u^j \\
&= u^i_{;kj} v^j u^k - R^i_{ljk} u^l v^j u^k - 2S^l_{jk} u^i_{;l} v^j u^k + u^i_{;j} v^j_{;k} u^k - 2(S^i_{kl} u^k v^l)_{;j} u^j \\
&= u^i_{;kj} v^j u^k - R^i_{ljk} u^l v^j u^k - 2S^l_{jk} u^i_{;l} v^j u^k + u^i_{;j} (u^j_{;k} v^k - 2S^j_{kl} u^k v^l) \\
&\quad - 2(S^i_{kl} u^k v^l)_{;j} u^j = (u^i_{;k} u^k)_{;j} v^j + R^i_{jkl} u^j u^k v^l - 2(S^i_{kl} u^k v^l)_{;j} u^j \\
&= R^i_{jkl} u^j u^k v^l - 2 \frac{D}{ds} (S^i_{kl} u^k v^l)
\end{aligned} \tag{1.3.43}$$

or

$$\frac{D}{ds} \left(\frac{Dv^i}{ds} + 2S^i_{kl} u^k v^l \right) = R^i_{jkl} u^j u^k v^l. \tag{1.3.44}$$

This is the equation of *geodesic deviation*. If we replace affine geodesics by arbitrary curves then $u^i_{;k} u^k \neq 0$ and (1.3.44) becomes

$$\frac{D}{ds} \left(\frac{Dv^i}{ds} + 2S^i_{kl} u^k v^l \right) = R^i_{jkl} u^j u^k v^l + (u^i_{;k} u^k)_{;j} v^j. \tag{1.3.45}$$

References: [1, 2, 3, 4].

1.4 Metric

1.4.1 Metric tensor

An affine parameter s is a measure of the length only along an affine geodesic. In order to extend the concept of length to all points in spacetime, we equip spacetime with an algebraic object g_{ik} , referred to as the covariant *metric tensor* and defined as

$$ds^2 = g_{ik} dx^i dx^k. \tag{1.4.1}$$

The metric tensor is a symmetric tensor of rank (0,2):

$$g_{ik} = g_{ki}. \tag{1.4.2}$$

The affine parameter s , whose differential is given by (1.4.1), is referred to as the *interval*. Because ds does not change under parallel transport along an affine geodesic from point $P(x^i)$ to point $Q(x^i + dx^i)$, $ds|_Q = ds|_P$, and $dx^i|_Q$ is a parallel translation of $dx^i|_P$, $g_{ik}|_Q = g_{ik}|_P + g_{ik,j} dx^j$ is a parallel translation of $g_{ik}|_P$:

$$g_{ik}|_Q = g_{ik}|_P + \delta g_{ik}, \tag{1.4.3}$$

so

$$Dg_{ik} = g_{ik;j} dx^j = dg_{ik} - \delta g_{ik} = g_{ik,j} dx^j - \delta g_{ik} = 0. \tag{1.4.4}$$

Therefore a covariant derivative of the covariant metric tensor vanishes:

$$N_{jik} = -g_{ik;j} = 0 \tag{1.4.5}$$

or

$$g_{ik,j} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{kj} g_{il} = 0, \tag{1.4.6}$$

where N_{ijk} is the *nonmetricity tensor*. The symmetric contravariant metric tensor $g^{ik} = g^{ki}$ is defined as the inverse of g_{ik} :

$$g_{ij} g^{jk} = \delta_j^k. \tag{1.4.7}$$

A covariant derivative of the contravariant metric tensor also vanishes:

$$g^{ik}{}_{;j} = 0. \quad (1.4.8)$$

The metric tensor allows to associate covariant and contravariant vectors:

$$A^i = g^{ik} A_k, \quad (1.4.9)$$

$$B_i = g_{ik} B^k, \quad (1.4.10)$$

because such association works for the covariant differentials of these vectors which are vectors:

$$DA^i = D(g^{ik} A_k) = g^{ik} DA_k, \quad DB_i = D(g_{ik} B^k) = g_{ik} DB^k \quad (1.4.11)$$

(raising and lowering of coordinate indices commutes with covariant differentiation with respect to $\Gamma_{\mu\nu}^\rho$). For covariant and contravariant indices of tensors and densities this association is

$$g_{im} \mathfrak{T}^{ij\dots}_{kl\dots} = \mathfrak{T}_m{}^{j\dots}_{kl\dots}, \quad (1.4.12)$$

$$g^{km} \mathfrak{T}^{ij\dots}_{kl\dots} = \mathfrak{T}^{ijm\dots}_{l\dots}. \quad (1.4.13)$$

The contravariant and covariant components of a two-dimensional vector are shown in Fig. 1. The square root of the absolute value of the determinant

$$\mathfrak{g} = |g_{ik}| \quad (1.4.14)$$

of the metric tensor is a scalar density, which we can use to multiply covariant integrands that contain dual densities of weight -1, since

$$e_{iklm} = \sqrt{|\mathfrak{g}|} \epsilon_{iklm}, \quad e^{iklm} = \frac{1}{\sqrt{|\mathfrak{g}|}} \epsilon^{iklm} \quad (1.4.15)$$

are tensors. Thus the relations (1.1.27) are also valid if we replace ϵ by e . The variation of the determinant of the metric tensor is

$$\delta \mathfrak{g} = \mathfrak{g} g^{ik} \delta g_{ik} = -\mathfrak{g} g_{ik} \delta g^{ik}. \quad (1.4.16)$$

A covariant derivative of the determinant of the metric tensor vanishes:

$$\mathfrak{g}_{;j} = 0. \quad (1.4.17)$$

A Lie derivative of the metric tensor is

$$\mathcal{L}_\xi g^{ik} = -2\xi^{(i;k)} - 4S^{(ik)}{}_l \xi^l, \quad (1.4.18)$$

where ${}^{;i} = {}_{;k} g^{ik}$. The four-velocity vector (1.2.46) is normalized due to (1.4.1):

$$u^i u_i = 1, \quad (1.4.19)$$

thus having 3 independent components.

The commutator of covariant derivatives (1.3.4) of the metric tensor gives

$$R^{(ij)}{}_{kl} = -N_{[k}{}^{ij}{}_{;l]} - S^m{}_{kl} N_m{}^{ij} = -N_{[k}{}^{ij}{}_{;l]}, \quad (1.4.20)$$

so the segmental curvature tensor (1.3.34) is

$$Q_{kl} = -N_{[k}{}^{ij}{}_{;l]} g_{ij}. \quad (1.4.21)$$

Because the nonmetricity tensor (1.4.5) vanishes, the curvature tensor is antisymmetric in its first two indices:

$$R_{ijkl} = -R_{jikl}. \quad (1.4.22)$$

Thus the segmental curvature tensor also vanishes, and

$$R_{ijkl} g^{jl} = R_{ik}, \quad (1.4.23)$$

so there is only one independent way to contract the curvature tensor, which gives the Ricci tensor up to the sign.

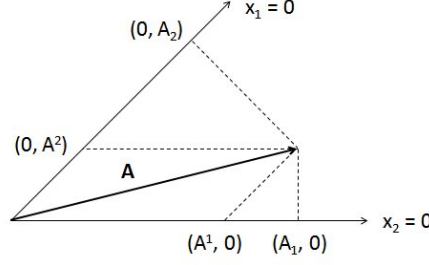


Figure 1: Contravariant and covariant components of a vector.

1.4.2 Christoffel symbols

The condition (1.4.5) is referred to as metricity or *metric compatibility* of the affine connection, and imposes 40 constraints on the connection:

$$g_{ik;j} + g_{kj;i} - g_{ji;k} = g_{ik,j} - \Gamma_{ij}^l g_{lk} - \Gamma_{kj}^l g_{il} + g_{kj,i} - \Gamma_{ki}^l g_{lj} - \Gamma_{ji}^l g_{kl} - g_{ji,k} + \Gamma_{jk}^l g_{li} + \Gamma_{ik}^l g_{jl} = g_{ik,j} + g_{kj,i} - g_{ji,k} - 2\Gamma_{(ij)}^l g_{kl} - 2S_{kj}^l g_{il} - 2S_{ki}^l g_{jl} = 0. \quad (1.4.24)$$

Multiplying (1.4.24) by g^{km} gives

$$\Gamma_{(ij)}^m = \{i_j^m\} + 2S_{(ij)}^m, \quad (1.4.25)$$

where

$$\{i_j^m\} = \frac{1}{2}g^{mk}(g_{ki,j} + g_{kj,i} - g_{ij,k}) \quad (1.4.26)$$

are the *Christoffel symbols*, symmetric in their covariant indices:

$$\{i_j^k\} = \{j_i^k\}. \quad (1.4.27)$$

Because $\Gamma_{ij}^k = \Gamma_{(ij)}^k + S_{ij}^k$, the metric-compatible affine connection equals

$$\Gamma_{ij}^k = \{i_j^k\} + C_{ij}^k, \quad (1.4.28)$$

where

$$C_{ij}^k = 2S_{(ij)}^k + S_{ij}^k \quad (1.4.29)$$

is the *contortion tensor*, antisymmetric in its first two indices:

$$C_{ijk} = -C_{jik}. \quad (1.4.30)$$

The inverse relation between the torsion and contortion tensor is

$$S_{jk}^i = C_{[jk]}^i. \quad (1.4.31)$$

The difference between two affine connections is a tensor, so the sum of a connection and a tensor of rank (1,2) is a connection. Therefore the Christoffel symbols form a connection, referred to as the *Levi-Civita connection*. We define the covariant derivative with respect to the Levi-Civita connection analogously to (1.2.11), with Γ_{ij}^k replaced by $\{i_j^k\}$, and denote it $\cdot_i$ instead of $\cdot_{;i}$, or $\nabla_i^{\{ \}$ instead of ∇_i . A covariant derivative with respect to the Levi-Civita connection of the metric tensor vanishes due to the definition of the Christoffel symbols:

$$g_{ik;j} = g_{ik,j} - \{i_j^l\}g_{lk} - \{k_j^l\}g_{il} = 0, \quad (1.4.32)$$

which gives the inverse relation between ordinary derivatives of the metric tensor and the Christoffel symbols. The variation of the Levi-Civita connection is a tensor:

$$\delta\{i_j^k\} = \frac{1}{2}g^{kl}((\delta g_{li})_{;j} + (\delta g_{lj})_{;i} - (\delta g_{ij})_{;l}). \quad (1.4.33)$$

The covariant derivative over s of a tensor density with respect to the Levi-Civita connection is, analogously to (1.2.49),

$$\frac{D^{\{\}} T}{ds} = T_{;i} u^i. \quad (1.4.34)$$

One can show that the following formulae hold:

$$\{^k_i\} = (\ln \sqrt{|\mathbf{g}|})_{,i}, \quad (1.4.35)$$

$$\{^k_j\} g^{ij} = -\frac{1}{\sqrt{|\mathbf{g}|}} (\sqrt{|\mathbf{g}|} g^{ik})_{,i}, \quad (1.4.36)$$

$$B^i_{;i} = \frac{1}{\sqrt{|\mathbf{g}|}} (\sqrt{|\mathbf{g}|} B^i)_{,i}, \quad (1.4.37)$$

$$F^{ik}_{;i} = \frac{1}{\sqrt{|\mathbf{g}|}} (\sqrt{|\mathbf{g}|} F^{ik})_{,i}, \quad (1.4.38)$$

$$A_{i;k} - A_{k;i} = A_{i,k} - A_{k,i}, \quad (1.4.39)$$

$$\oint B^i \sqrt{|\mathbf{g}|} dS_i = \int B^i_{;i} \sqrt{|\mathbf{g}|} d\Omega, \quad (1.4.40)$$

where $F^{ik} = -F^{ki}$. The Christoffel symbols satisfy all formulae that are satisfied by Γ^k_{ij} in which $S^i_{jk} = 0$. Because the Levi-Civita connection is a symmetric connection, it can be brought to zero by transforming the coordinates to a geodesic frame. In a geodesic frame, the covariant derivative with respect to the Levi-Civita connection, $\nabla^{\{\}}_i$, coincides with the ordinary derivative ∂_i . A Lie derivative of the metric tensor (1.4.18) can be written as

$$\mathcal{L}_\xi g^{ik} = -2\xi^{(i;k)}, \quad \mathcal{L}_\xi g_{ik} = 2\xi_{(i;k)}, \quad (1.4.41)$$

where $^{i;k} = g^{ik}_{;k}$. A Killing vector (1.2.72) for the Levi-Civita connection satisfies

$$\zeta_{(i;k)} = 0, \quad (1.4.42)$$

thus becomes a generator of *isometries*, transformations that do not change the metric tensor.

If the nonmetricity tensor does not vanish, the general formula for the affine connection (1.4.28) is

$$\Gamma^k_{ij} = \{^k_{ij}\} + C^k_{ij} - N^k_{ij} + \frac{1}{2} N^k_{(i j)}. \quad (1.4.43)$$

1.4.3 Riemann curvature tensor

The curvature tensor constructed from the Levi-Civita connection is referred to as the *Riemann tensor*:

$$P^i_{mjk} = \partial_j \{^i_{mk}\} - \partial_k \{^i_{mj}\} + \{^i_lj\} \{^l_{mk}\} - \{^i_lk\} \{^l_{mj}\}. \quad (1.4.44)$$

The commutator of covariant derivatives of the metric tensor vanishes:

$$[\nabla^{\{\}}_j, \nabla^{\{\}}_k] g_{lp} = -P^m_{ljk} g_{mp} - P^m_{pjk} g_{lm} = 0, \quad (1.4.45)$$

so the covariant Riemann tensor P_{imjk} is also antisymmetric in the indices i, m . Substituting (1.4.26) in (1.4.44) gives

$$P_{iklm} = \frac{1}{2} (g_{im,kl} + g_{kl,im} - g_{il,km} - g_{km,il}) + g_{jn} (\{^j_{im}\} \{^n_{kl}\} - \{^j_{il}\} \{^n_{km}\}), \quad (1.4.46)$$

which explicitly shows the following symmetry and antisymmetry properties:

$$P_{iklm} = -P_{ikml}, \quad (1.4.47)$$

$$P_{iklm} = -P_{kilm}, \quad (1.4.48)$$

$$P_{iklm} = P_{lmik}. \quad (1.4.49)$$

Accordingly, the Riemannian Ricci tensor is symmetric:

$$P_{ik} = P^j_{ijk} = P_{ki}. \quad (1.4.50)$$

Substituting (1.4.28) in (1.3.5) and (1.3.6) gives the relation between the curvature and Riemann tensors:

$$R^i_{klm} = P^i_{klm} + C^i_{km;l} - C^i_{kl;m} + C^j_{km} C^i_{jl} - C^j_{kl} C^i_{jm}. \quad (1.4.51)$$

Contracting (1.4.51) with respect to in the indices i, l gives

$$R_{km} = P_{km} + C^i_{km;i} - C^i_{ki;m} + C^j_{km} C^i_{ji} - C^j_{ki} C^i_{jm}. \quad (1.4.52)$$

Consequently, the Ricci or *curvature scalar*,

$$R = R_{ik} g^{ik}, \quad (1.4.53)$$

is given by

$$R = P - g^{ik} (2C^l_{il;k} + C^j_{ij} C^l_{kl} - C^l_{im} C^m_{kl}), \quad (1.4.54)$$

where P is the Riemannian curvature scalar,

$$P = P_{ik} g^{ik}. \quad (1.4.55)$$

The variation of the Riemann tensor is, analogously to (1.3.8),

$$\delta P^i_{klm} = (\delta\{^i_{km}\})_{;l} - (\delta\{^i_{kl}\})_{;m}, \quad (1.4.56)$$

and the variation of the Riemannian Ricci tensor is

$$\delta P_{ik} = (\delta\{^l_{ik}\})_{;l} - (\delta\{^l_{il}\})_{;k}. \quad (1.4.57)$$

The Bianchi identities (1.3.29) and (1.3.30) contracted with respect to one contravariant and one covariant index give

$$R^i_{n[ik;l]} = 2R^i_{nm[i} S^m_{kl]}, \quad (1.4.58)$$

$$R^k_{[jkl]} = -2S^k_{[jk;l]} + 4S^k_{n[j} S^n_{kl]}. \quad (1.4.59)$$

Contracting these equations with the metric tensor gives

$$R_{nk;l} - R_{nl;k} + R^i_{nkl;i} = -2R_{nm} S^m_{kl} - 2R^i_{nmk} S^m_{il} + 2R^i_{nml} S^m_{ik} \quad (1.4.60)$$

and the *contracted cyclic identity*:

$$R_{jl} - R_{lj} = -2S_{j;l} + 2S_{l;j} - 2S^k_{lj;k} + 4S_n S^n_{lj}. \quad (1.4.61)$$

Further contraction of (1.4.60) with the metric tensor gives the *contracted Bianchi identity*:

$$R^i_{l;i} - \frac{1}{2} R_{;l} = 2R_{km} S^m_{kl} - R^{ik}_{ml} S^m_{ik}. \quad (1.4.62)$$

The Bianchi identities (1.3.31) and (1.3.32) for the Riemann tensor are

$$P^i_{n[jk;l]} = 0, \quad (1.4.63)$$

$$P^m_{[jkl]} = 0. \quad (1.4.64)$$

Contracting these equations with the metric tensor gives

$$P_{nk;l} + P^i_{nkl;i} - P_{nl;k} = 0, \quad (1.4.65)$$

$$P_{jl} - P_{lj} = 0, \quad (1.4.66)$$

in agreement with (1.4.50). Further contraction of (1.4.65) with the metric tensor gives the covariant conservation,

$$G^i_{k;i} = 0, \quad (1.4.67)$$

of the symmetric *Einstein tensor*,

$$G_{ik} = P_{ik} - \frac{1}{2} P g_{ik}. \quad (1.4.68)$$

1.4.4 Properties of Riemann tensor

In two dimensions there is only 1 independent component of the Riemann tensor, P_{1212} . The Riemann scalar is

$$P = \frac{2P_{1212}}{\mathfrak{s}}, \quad (1.4.69)$$

where $\mathfrak{s}$ is the determinant of the two-dimensional metric tensor γ_{ik} :

$$\mathfrak{s} = |\gamma_{ik}| = \gamma_{11}\gamma_{22} - \gamma_{12}^2. \quad (1.4.70)$$

A surface near point $x = 0, y = 0$ is given by

$$z = \frac{x^2}{2\rho_1} + \frac{y^2}{2\rho_2}, \quad (1.4.71)$$

where ρ_1 and ρ_2 are the radii of curvature. Substituting (1.4.71) to

$$dl^2 = dx^2 + dy^2 + dz^2 = \gamma_{ik}dx^i dx^k \quad (1.4.72)$$

gives $\gamma_{ik}(x, y)$, which then gives

$$\frac{P}{2} \Big|_{x=y=0} = K = \frac{1}{\rho_1\rho_2}, \quad (1.4.73)$$

where K is the Gaußcurvature.

In three dimensions there are 3 independent pairs, 12, 23, and 31, so the Riemann tensor has 6 independent components: 3 with identical pairs and $\frac{3 \cdot 2}{2} = 3$ with different pairs (the cyclic identity does not reduce the number of independent components). The Ricci tensor has also 6 components, which are related to the components of the Riemann tensor by

$$P_{\alpha\beta\gamma\delta} = P_{\alpha\gamma\beta\delta} - P_{\alpha\delta\beta\gamma} + P_{\beta\delta\alpha\gamma} - P_{\beta\gamma\alpha\delta} + \frac{P}{2}(\gamma_{\alpha\delta}\gamma_{\beta\gamma} - \gamma_{\alpha\gamma}\gamma_{\beta\delta}). \quad (1.4.74)$$

Choosing the *Cartesian coordinates* at a given point, defined by the condition

$$g_{\alpha\beta} = \text{diag}(1, 1, 1), \quad (1.4.75)$$

and diagonalizing $P_{\alpha\beta}$, which is equivalent to 3 rotations, brings $P_{\alpha\beta}$ to the canonical form with $6 - 3 = 3$ independent components. Consequently, the Riemann tensor in three dimensions has 3 physically independent components. The Gaußcurvature of a surface perpendicular to the x^3 axis is given by

$$K = \frac{P_{1212}}{\gamma_{11}\gamma_{22} - \gamma_{12}^2}. \quad (1.4.76)$$

In four dimensions there are 6 independent pairs, 01, 02, 03, 12, 23, and 31, so there are 6 components with identical pairs and $\frac{6 \cdot 5}{2} = 15$ with different pairs. The cyclic identity reduces the number of independent components by 1, so the Riemann tensor in four dimensions has generally 20 independent components. Choosing the Cartesian coordinates at a given point and applying 6 rotations brings P_{ijkl} to the canonical form with $20 - 6 = 14$ physically independent components.

1.4.5 Weyl tensor

In four dimensions the *Weyl tensor* is defined as

$$W_{iklm} = P_{iklm} - \frac{1}{2}(P_{il}g_{km} + P_{km}g_{il} - P_{im}g_{kl} - P_{kl}g_{im}) + \frac{1}{6}P(g_{il}g_{km} - g_{im}g_{kl}). \quad (1.4.77)$$

This tensor has all the symmetry and antisymmetry properties of the Riemann tensor, and is also traceless (any contraction of the Weyl tensor vanishes).

1.4.6 Metric geodesics

We consider two points in spacetime, P and Q . Among curves that connect these points, one curve has the minimal value of the interval $s = \int ds$, and is referred to as a *metric geodesic*. The equation of a metric geodesic is given by the condition that $\int ds$ be an extremum with the endpoints of the curve fixed:

$$\begin{aligned} \delta \int ds &= \delta \int (g_{ik} dx^i dx^k)^{1/2} = \int \frac{\delta dx^i g_{ij} dx^j}{ds} + \frac{1}{2} \int \frac{\delta g_{ij} dx^i dx^j}{ds} = \int g_{ij} u^j \delta dx^i \\ &+ \frac{1}{2} \int g_{ij,k} \delta x^k u^i u^j ds = \int d(u_i \delta x^i) - \int du_i \delta x^i + \frac{1}{2} \int g_{ij,k} \delta x^k u^i u^j ds \\ &= - \int \frac{du_i}{ds} \delta x^i ds + \frac{1}{2} \int g_{jk,i} \delta x^i u^j u^k ds = 0, \end{aligned} \quad (1.4.78)$$

where we omit the total differential term $\int d(u_i \delta x^i)$ because $\delta x^i = 0$ at the endpoints. Since δx^i is arbitrary, we obtain

$$\begin{aligned} \frac{d}{ds}(g_{ij} u^j) - \frac{1}{2} \int g_{jk,i} u^j u^k ds &= g_{ij} \frac{du^j}{ds} + u^k g_{ij,k} u^j - \frac{1}{2} \int g_{jk,i} u^j u^k ds \\ &= g_{ij} \frac{du^j}{ds} + \{g_{ij}^m\} g_{im} u^j u^k = 0 \end{aligned} \quad (1.4.79)$$

or, after multiplying (1.4.79) by g^{il} :

$$\frac{D\{u^l\}}{ds} = \frac{du^l}{ds} + \{g^l_k\} u^k = u^i u^l_{;i} = 0. \quad (1.4.80)$$

The metric geodesic equation (1.4.80) can be written as

$$\frac{d^2 x^i}{ds^2} + \{g^i_{kl}\} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0. \quad (1.4.81)$$

Using (1.4.28) and (1.4.29), the affine geodesic equation (1.2.45) can be written as

$$\frac{d^2 x^i}{ds^2} + \{g^i_{kl}\} \frac{dx^k}{ds} \frac{dx^l}{ds} + 2S_{kl}^i \frac{dx^k}{ds} \frac{dx^l}{ds} = 0. \quad (1.4.82)$$

If the torsion tensor is completely antisymmetric then the last term in (1.4.82) vanishes and the affine geodesic equation coincides with the metric geodesic equation. The equation of geodesic deviation with respect to the Levi-Civita connection is, analogously to (1.3.44),

$$\frac{D\{v^i\}}{ds^2} = P^i_{jkl} u^j u^k v^l. \quad (1.4.83)$$

1.4.7 Galilean frame of reference and Minkowski tensor

At a given point, the nondegenerate ($g \neq 0$) metric tensor can be brought to a diagonal (canonical) form $g_{ik} = \text{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$. *Physical* systems are described by the metric tensor with $g < 0$. Without loss of generality, we assume that the canonical form of the metric tensor is

$$g_{ik} = \eta_{ik} = \text{diag}(1, -1, -1, -1), \quad g^{ik} = \eta^{ik} = \text{diag}(1, -1, -1, -1). \quad (1.4.84)$$

A frame of reference in which g_{ik} has the canonical form is referred to as *Galilean*. The spatial coordinates in the Galilean frame of reference are Cartesian (1.4.75). The transformation (1.2.35) with (1.2.38) brings a symmetric affine connection, thus the Christoffel symbols, to zero at a given point without changing the components of the metric tensor because of (1.2.36). Therefore a frame of reference can be both geodesic and Galilean. In such *locally inertial* frame first derivatives of the

metric tensor vanish because of (1.4.32). The corresponding metric tensor (1.4.84) is referred to as the *Minkowski tensor*. The interval for this metric is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \quad (1.4.85)$$

In a locally inertial frame the coordinates x^i , not only the differentials dx^i , are components of a contravariant vector.

In the absence of torsion, spacetime with a vanishing Riemann tensor $P^i_{klm} = 0$ is flat. In the new coordinates y_a (1.3.17), (1.3.18) gives

$$g^{ab}(y) = g_{ik}(x) \frac{\partial x^i}{\partial y_a} \frac{\partial x^k}{\partial y_b} = g_{ik}(x) h^{ia} h^{kb} = \eta^{ab}. \quad (1.4.86)$$

Therefore in a flat spacetime without torsion one can always find a system of coordinates which is Galilean everywhere.

1.4.8 Intervals, proper time and distances

The form of the Minkowski tensor distinguishes the coordinate x^0 from the rest of the coordinates x^α , where the index α can be 1,2,3. The *temporal* coordinate $x^0 = ct$, where t is referred to as *time* and c is referred to as the *velocity of propagation of interaction*. The coordinates x^α are *spatial* and span *space*. The set of 4 coordinates x^i describe an *event* and span *spacetime*. The curve $x^i(\lambda)$, where λ is a parameter, is referred to as a *world line* of a given point. The quantities

$$v^\alpha = \frac{dx^\alpha}{dt} \quad (1.4.87)$$

are the components of a three-dimensional vector, the *velocity* of this point. An infinitesimal interval ds is *timelike* if $ds^2 > 0$, *spacelike* if $ds^2 < 0$, and *null* if $ds^2 = 0$. In the Galilean frame, the interval between two infinitesimally separated points (events) is

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^\alpha dx^\alpha, \quad (1.4.88)$$

where dx^i are infinitesimal coordinate differences between the two points. The interval between two finitely separated points is

$$\Delta s^2 = \eta_{ik} \Delta x^i \Delta x^k = c^2 \Delta t^2 - \Delta x^\alpha \Delta x^\alpha, \quad (1.4.89)$$

where Δx^i are finite coordinate differences between the two points. If Δs is timelike, one can always find a frame of reference in which the two events occur at the same place, $\Delta x^\alpha = 0$. A frame of reference in which $dx^\alpha = 0$ describes a point at *rest* and is referred to as the *rest frame* or the *comoving frame*. In this frame $t = \tau$,

$$ds^2 = c^2 d\tau^2, \quad (1.4.90)$$

where τ is the *proper time*. If $dx^\alpha \neq 0$ along a world line then the point *moves* or is in *motion*. The proper time for a moving point is equal to the time measured by a clock moving with this point. If Δs is spacelike, one can always find a frame of reference in which the two events occur at the same time (are *synchronous*), $\Delta x^0 = 0$. If $ds = 0$ along a world line, this world line describes the propagation of a signal (*interaction*), with $v = (v^\alpha v^\alpha)^{1/2} = c$. Equations (1.4.88) and (1.4.90) give

$$d\tau^2 = dt^2 - \frac{1}{c^2} dx^\alpha dx^\alpha, \quad (1.4.91)$$

so the proper time τ goes more slowly than the coordinate time t . If Δs is timelike, the two events occur at different times: $t_1 \neq t_2$. If $t_2 > t_1$ then t_2 is in the *future* with respect to t_1 and t_1 is in the *past* with respect to t_2 . The time of a measurement t_0 is called the *present* time.

All events for which $t < t_0$ form the *absolute past* relative to the event O at the present (events in this region occur *before* O in all systems of reference). All events for which $t > t_0$ form the *absolute*

future relative to the event O at the present (events in this region occur *after* O in all systems of reference). Such a division into the absolute past and the absolute future with respect to O is possible only for events for which their intervals with respect to O are timelike, as shown in Fig. 2. For $O = (0, 0, 0, 0)$, these events (ct, x, y, z) lie within a cone $(ct)^2 - x^2 - y^2 - z^2 = 0$ which is called the null cone or *light cone*. All events for which their intervals with respect to O are spacelike are *absolutely remote* relative to O . The principle of *causality* states that any event O can be affected only by events in the absolute past relative to O .

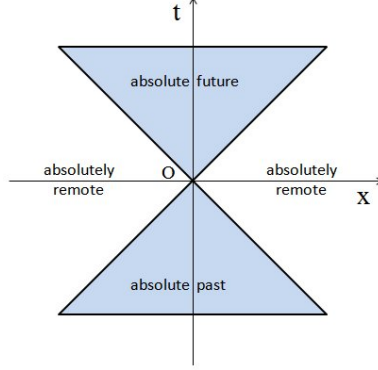


Figure 2: Light cone.

In the rest frame $dx^\alpha = 0$ gives $u^\alpha = 0$. At each point in space, the condition $dx^\alpha = 0$ gives the relation between the proper time and the coordinate time:

$$d\tau = \frac{1}{c} \sqrt{g_{00}} dx^0, \quad (1.4.92)$$

which requires

$$g_{00} \geq 0. \quad (1.4.93)$$

The relation (1.4.19) gives

$$u^0 = (g_{00})^{-1/2}. \quad (1.4.94)$$

The distance between two infinitesimally separated points cannot be obtained by imposing dx^0 because x^0 transforms differently at these points. Instead, consider a signal that leaves point $B(x^\alpha + dx^\alpha)$ at $x^0 + dx^0_-$, reaching point $A(x^\alpha)$ at x^0 and coming back to point B at $x^0 + dx^0_+$, as shown in Fig. 3. Therefore

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta = 0 \quad (1.4.95)$$

gives

$$dx^\alpha_\pm = \frac{1}{g_{00}} (-g_{0\alpha}dx^0 \pm \sqrt{(g_{0\alpha}g_{0\beta} - g_{00}g_{\alpha\beta})dx^\alpha dx^\beta}). \quad (1.4.96)$$

The difference in the time coordinate between emitting and receiving the signal at point B is equal to the difference between dx^0_+ and dx^0_- times $\sqrt{g_{00}}/c$, and the *distance* dl between points A and B is equal to this difference times $c/2$:

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta, \quad (1.4.97)$$

where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}} \quad (1.4.98)$$

is the symmetric *spatial metric tensor* of spacetime, i.e. the metric tensor of space. The event at point A at x^0 is *synchronized* with the event at point B at the arithmetic mean of the time coordinates of emitting and receiving the signal, i.e. at

$$x^0 + \frac{1}{2}(dx^0_- + dx^0_+) = x^0 + g_\alpha dx^\alpha, \quad (1.4.99)$$

where

$$g_\alpha = -\frac{g_{0\alpha}}{g_{00}}. \quad (1.4.100)$$

Therefore

$$\delta x^0 = g_\alpha \delta x^\alpha, \quad (1.4.101)$$

which is equivalent to $\delta x_0 = 0$, is the difference in x^0 between two synchronized infinitesimally separated points.

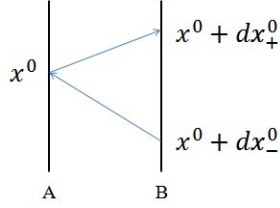


Figure 3: Distance.

1.4.9 Spatial vectors

The spatial components of a contravariant four-vector A^i form a three-dimensional, *spatial vector* $\mathbf{A}$:

$$A^i = (A^0, A^\alpha) = (A^0, \mathbf{A}). \quad (1.4.102)$$

The contravariant four-vector index α is also the contravariant spatial-vector index. The covariant components of a spatial vector are related to the contravariant components by the spatial metric tensor (1.4.98) which raises and lowers indices of spatial vectors analogously to the metric tensor acting on four-vectors:

$$A_\alpha = \gamma_{\alpha\beta} A^\beta, \quad (1.4.103)$$

$$B^\alpha = \gamma^{\alpha\beta} B_\beta, \quad (1.4.104)$$

where $\gamma^{\alpha\beta}$ is the inverse of $\gamma_{\alpha\beta}$:

$$\gamma^{\alpha\delta} \gamma_{\beta\delta} = \delta_\beta^\alpha. \quad (1.4.105)$$

One can show that the following formulae hold:

$$\gamma^{\alpha\beta} = -g^{\alpha\beta}, \quad (1.4.106)$$

$$\mathfrak{g} = -g_{00}\mathfrak{s}, \quad (1.4.107)$$

$$g^\alpha = -g^{0\alpha}, \quad (1.4.108)$$

$$g^{00} = \frac{1}{g_{00}} - g_\alpha g^\alpha, \quad (1.4.109)$$

where

$$\mathfrak{s} = \det \gamma_{\alpha\beta}. \quad (1.4.110)$$

The components g^α form a spatial vector $\mathbf{g}$.

The *scalar product* of two spatial vectors is

$$\mathbf{A} \cdot \mathbf{B} = \gamma_{\alpha\beta} A^\alpha B^\beta. \quad (1.4.111)$$

The *square* of a spatial vector $\mathbf{A}$ is

$$A^2 = \mathbf{A} \cdot \mathbf{A} \quad (1.4.112)$$

and its *norm* is

$$A = \sqrt{A^2}. \quad (1.4.113)$$

The *angle* between two spatial vectors θ is defined through

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta. \quad (1.4.114)$$

In three-dimensional space, the *permutation symbol* is defined as

$$\epsilon_{\alpha\beta\gamma} = -\epsilon_{0\alpha\beta\gamma}. \quad (1.4.115)$$

The three-dimensional equivalent of (1.4.15) is

$$e_{\alpha\beta\gamma} = \sqrt{\mathfrak{s}} \epsilon_{\alpha\beta\gamma}, \quad e^{\alpha\beta\gamma} = \frac{1}{\sqrt{\mathfrak{s}}} \epsilon^{\alpha\beta\gamma}. \quad (1.4.116)$$

The *cross product* of two three-dimensional vectors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is

$$C_\alpha = e_{\alpha\beta\gamma} A^\beta B^\gamma, \quad C^\alpha = e^{\alpha\beta\gamma} A_\beta B_\gamma. \quad (1.4.117)$$

The permutation symbol (1.4.115) satisfies

$$\epsilon_{\alpha\beta\gamma} \epsilon^\alpha_{\delta\zeta} = \delta_{\beta\delta} \delta_{\gamma\zeta} - \delta_{\beta\zeta} \delta_{\gamma\delta}, \quad (1.4.118)$$

$$\epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\beta}_{\delta} = 2\delta_{\gamma\delta}, \quad (1.4.119)$$

$$\epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\beta\gamma} = 6. \quad (1.4.120)$$

The *spatial covariant derivative* ∇_α acts on spatial vectors analogously to the metric covariant derivative acting on four-vectors:

$$\nabla_\alpha A^\beta = \partial_\alpha A^\beta + \{\gamma_\alpha^\beta\}_\gamma A^\gamma, \quad (1.4.121)$$

$$\nabla_\alpha A_\beta = \partial_\alpha A_\beta - \{\gamma_\beta^\alpha\}_\gamma A_\gamma, \quad (1.4.122)$$

where $\{\delta_{\alpha\beta}\}_\gamma$ are the three-dimensional, *spatial Christoffel symbols*:

$$\{\delta_{\alpha\beta}\}_\gamma = \frac{1}{2} \gamma^{\delta\gamma} (\gamma_{\gamma\alpha,\beta} + \gamma_{\gamma\beta,\alpha} - \gamma_{\alpha\beta,\gamma}). \quad (1.4.123)$$

The *gradient* operator is given by

$$(\text{grad})^\alpha = (\nabla)^\alpha = \gamma^{\alpha\beta} \nabla_\beta. \quad (1.4.124)$$

The spatial components of a covariant-vector operator ∂_i acting on a scalar ϕ form the gradient of ϕ :

$$\partial_i \phi = \left(\frac{\partial \phi}{c \partial t}, \frac{\partial \phi}{\partial x^\alpha} \right) = \left(\frac{\partial \phi}{c \partial t}, \nabla \phi \right). \quad (1.4.125)$$

The *divergence* of a spatial vector $\mathbf{A}$ is, analogously to (1.4.37),

$$\text{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{1}{\sqrt{\mathfrak{s}}} \partial_\alpha (\sqrt{\mathfrak{s}} A^\alpha). \quad (1.4.126)$$

The curl of a spatial vector is

$$(\text{curl} \mathbf{A})^\alpha = (\nabla \times \mathbf{A})^\alpha = e^{\alpha\beta\gamma} \partial_\beta A_\gamma. \quad (1.4.127)$$

The Laplace-Beltrami operator or *Laplacian* is the divergence of the gradient,

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{1}{\sqrt{\mathfrak{s}}} \partial_\alpha (\sqrt{\mathfrak{s}} \gamma^{\alpha\beta} \partial_\beta). \quad (1.4.128)$$

The d'Alembert operator or *d'Alembertian* is defined as

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta. \quad (1.4.129)$$

In a locally Galilean frame of reference, the covariant and contravariant three-dimensional components of a vector are identical because

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta}, \quad (1.4.130)$$

where $\delta_{\alpha\beta}$ is the *Cartesian metric tensor*,

$$\delta_{\alpha\beta} = \text{diag}(1, 1, 1), \quad \delta^{\alpha\beta} = \text{diag}(1, 1, 1). \quad (1.4.131)$$

In this frame we refer to the coordinates x^1, x^2, x^3 , which are Cartesian, as x, y, z . One can show that the following formulae hold:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \quad (1.4.132)$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \quad (1.4.133)$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \quad (1.4.134)$$

$$(\mathbf{A} \cdot \mathbf{B})^2 + (\mathbf{A} \times \mathbf{B})^2 = A^2 B^2, \quad (1.4.135)$$

$$\text{curl grad} \phi = 0, \quad (1.4.136)$$

$$\text{div curl} \mathbf{A} = 0, \quad (1.4.137)$$

$$\text{grad}(\phi\psi) = \text{grad} \phi \psi + \phi \text{grad} \psi, \quad (1.4.138)$$

$$\begin{aligned} \text{grad}(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times \text{curl} \mathbf{B} \\ &\quad + \mathbf{B} \times \text{curl} \mathbf{A}, \end{aligned} \quad (1.4.139)$$

$$\text{div}(\phi \mathbf{A}) = \text{grad} \phi \cdot \mathbf{A} + \phi \text{div} \mathbf{A}, \quad (1.4.140)$$

$$\text{curl}(\phi \mathbf{A}) = \text{grad} \phi \times \mathbf{A} + \phi \text{curl} \mathbf{A}, \quad (1.4.141)$$

$$\text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl} \mathbf{A} - \mathbf{A} \cdot \text{curl} \mathbf{B}, \quad (1.4.142)$$

$$\text{curl}(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A} \text{div} \mathbf{B} - \mathbf{B} \text{div} \mathbf{A}, \quad (1.4.143)$$

$$\text{curl curl} \mathbf{A} = \text{grad div} \mathbf{A} - \Delta \mathbf{A}, \quad (1.4.144)$$

where

$$(\mathbf{A} \cdot \nabla) \mathbf{B} = A^\alpha \partial_\alpha \mathbf{B}. \quad (1.4.145)$$

The cross product of two spatial vectors $\mathbf{A}$ and $\mathbf{B}$ satisfies

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \mathbf{n}, \quad (1.4.146)$$

where $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, in the direction given by the right-handed corkscrew rule.

References: [1, 2, 3, 4].

1.5 Tetrad and spin connection

1.5.1 Tetrad

In addition to the coordinate systems, at each spacetime point we set up four linearly independent vectors e_a^i such that

$$e_a^i e_{ib} = \eta_{ab}, \quad (1.5.1)$$

where $a, b = 0, 1, 2, 3$ are *Lorentz indices* and $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is the coordinate-invariant Minkowski metric tensor in a locally geodesic frame of reference at this point. This set of four vectors is referred to as a *tetrad*. The inverse tetrad e^{ai} satisfies

$$e_a^i e_i^b = \delta_a^b, \quad (1.5.2)$$

$$e_a^i e_k^a = \delta_k^i. \quad (1.5.3)$$

The coordinate metric tensor g_{ik} is related to the Minkowski metric tensor through the tetrad:

$$g_{ik} = e_i^a e_k^b \eta_{ab}. \quad (1.5.4)$$

Accordingly, the determinant $\mathbf{g}$ of the metric tensor g_{ik} is related to the determinant of the tetrad $\mathbf{e} = |e_i^a|$ by

$$\sqrt{|\mathbf{g}|} = \mathbf{e}. \quad (1.5.5)$$

Any vector V can be specified by its components V^i with respect to the coordinate system or by the coordinate-invariant projections V^a of the vector onto the tetrad field:

$$V^a = e_i^a V^i, \quad V_a = e_a^i V_i, \quad (1.5.6)$$

$$V^i = e_a^i V^a, \quad V_i = e_i^a V_a, \quad (1.5.7)$$

and similarly for tensors and densities with more indices. We can use η_{ab} and its inverse η^{ab} to lower and raise Lorentz indices, as we use g_{ik} and its inverse g^{ik} to lower and raise coordinate indices.

1.5.2 Lorentz transformation

The relation (1.5.4) imposes 10 constraints on the 16 components of the tetrad, leaving 6 components arbitrary. If we change from one tetrad e_a^i to another, $\tilde{e}_b^i$, then the vectors of the new tetrad are linear combinations of the vectors of the old tetrad:

$$\tilde{e}_a^i = \Lambda_a^b e_b^i. \quad (1.5.8)$$

The relation (1.5.4) applied to the tetrad field $\tilde{e}_b^i$,

$$g_{ik} = \tilde{e}_i^a \tilde{e}_k^b \eta_{ab}, \quad (1.5.9)$$

imposes on the matrix Λ_a^b the orthogonality condition:

$$\Lambda_a^c \Lambda_b^d \eta_{cd} = \eta_{ab}. \quad (1.5.10)$$

We refer to Λ_a^b as a *Lorentz matrix*, and to a transformation of form (1.5.8) as the *Lorentz transformation*.

1.5.3 Tetrad transport

A natural choice for the zeroth component of a tetrad at a given point is

$$e_0^i = u^i. \quad (1.5.11)$$

Along a world line this tetrad should be transported such that the zeroth component always coincides with the four-velocity. The *Fermi-Walker transport* of a tetrad is defined as

$$\frac{\nabla e_a^i}{ds} = -u^i e_a^j \frac{Du_j}{ds} + \frac{Du^i}{ds} e_a^j u_j. \quad (1.5.12)$$

Putting $a = 0$ in (1.5.12) gives

$$\frac{\nabla u^i}{ds} = \frac{Du^i}{ds}, \quad (1.5.13)$$

so the Fermi-Walker transport of the four-velocity is equivalent to its covariant change and thus (1.5.11) is valid at all points. This transport does not change the orthogonality relation for tetrads (1.5.1) since (1.5.12) gives

$$\frac{\nabla}{ds}(e_a^i e_{ib}) = 0. \quad (1.5.14)$$

1.5.4 Spin connection

We define

$$\omega^i_{ak} = e^i_{a;k} = e^i_{a,k} + \Gamma^i_{jk} e^j_a. \quad (1.5.15)$$

The quantities

$$\omega^a_{bi} = e^a_j \omega^j_{bi} \quad (1.5.16)$$

transform like vectors under coordinate transformations. We can extend the notion of covariant differentiation to quantities with Lorentz coordinate-invariant indices by regarding ω^{ab}_i as a connection, referred to as Lorentz or *spin connection*. For a contravariant Lorentz vector

$$V^a_{|i} = V^a_{,i} + \omega^a_{bi} V^b, \quad (1.5.17)$$

where $|_i$ is a covariant derivative of such a quantity with respect to x^i . The covariant derivative of a scalar $V^a W_a$ coincides with its ordinary derivative:

$$(V^a W_a)_{|i} = (V^a W_a)_{,i}, \quad (1.5.18)$$

which gives a covariant derivative of a covariant Lorentz vector:

$$W_{a|i} = W_{a,i} - \omega^b_{ai} W_b. \quad (1.5.19)$$

The chain rule implies that a covariant derivative of a Lorentz tensor is equal to the sum of the corresponding ordinary derivative of this tensor and terms with spin connection corresponding to each Lorentz index:

$$T^{ab...}_{cd...|i} = T^{ab...}_{cd...,i} + \omega^a_{ei} T^{eb...}_{cd...} + \omega^b_{ei} T^{ae...}_{cd...} + \dots - \omega^e_{ci} T^{ab...}_{ed...} - \omega^e_{di} T^{ab...}_{ce...} - \dots \quad (1.5.20)$$

We assume that the covariant derivative $|_i$ is total, that is, also recognizes coordinate indices, acting on them like $,i$. For a tensor with both coordinate and Lorentz indices

$$T^{aj...}_{bk...|i} = T^{aj...}_{bk...,i} + \omega^a_{ei} T^{ej...}_{bk...} + \Gamma^j_{li} T^{al...}_{bk...} + \dots - \omega^e_{bi} T^{aj...}_{ek...} - \Gamma^l_{ki} T^{aj...}_{bl...} - \dots \quad (1.5.21)$$

A total covariant derivative of a tetrad is

$$e^i_{a|k} = e^i_{a,k} + \Gamma^i_{jk} e^j_a - \omega^b_{ak} e^i_b = 0, \quad (1.5.22)$$

due to (1.5.15). Therefore total covariant differentiation commutes with converting between coordinate and Lorentz indices. Equation (1.5.22) also determines the spin connection ω^a_{bi} in terms of the affine connection, tetrad and its ordinary derivatives:

$$\omega^a_{bi} = e^a_k (e^k_{b,i} + \Gamma^k_{ji} e^j_b). \quad (1.5.23)$$

Conversely, the affine connection is determined by the spin connection, tetrad and its derivatives:

$$\Gamma^j_{ik} = \omega^j_{ik} + e^a_{i,k} e^j_a. \quad (1.5.24)$$

The torsion tensor is then

$$S^j_{ik} = \omega^j_{[ik]} + e^a_{[i,k]} e^j_a, \quad (1.5.25)$$

and the torsion vector is

$$S_i = \omega^k_{[ik]} + e^a_{[i,k]} e^k_a. \quad (1.5.26)$$

Metric compatibility of the affine connection leads to

$$g_{ik;j} = g_{ik|j} = e^a_i e^b_k \eta_{ab|j} = -e^a_i e^b_k (\omega^c_{aj} \eta_{cb} + \omega^c_{bj} \eta_{ac}) = -(\omega_{kij} + \omega_{ikj}) = 0, \quad (1.5.27)$$

so the spin connection is antisymmetric in its first two indices:

$$\omega^a_{bi} = -\omega^a_{ib}. \quad (1.5.28)$$

Accordingly, the spin connection has 24 independent components. The contortion tensor is

$$C_{ijk} = \omega_{ijk} + \Delta_{ijk}, \quad (1.5.29)$$

where

$$\Delta_{ijk} = e_{ia}e_{[j,k]}^a - e_{ja}e_{[i,k]}^a - e_{ka}e_{[i,j]}^a \quad (1.5.30)$$

are the *Ricci rotation coefficients*. The first term on the right-hand side in (1.5.29) is expected because both the contortion tensor and spin connection are antisymmetric in their first two indices. The quantities

$$\varpi_{ak}^i = e_{a:k}^i = e_{a,k}^i + \{^i_{jk}\}e_a^j \quad (1.5.31)$$

form the *Levi-Civita spin connection* and are related to the Ricci rotation coefficients by (1.5.29) with $C_{ijk} = 0$,

$$\varpi_{ijk} = -\Delta_{ijk}, \quad (1.5.32)$$

so

$$C_{ijk} = \omega_{ijk} - \varpi_{ijk}. \quad (1.5.33)$$

1.5.5 Tetrad representation of curvature tensor

The commutator of the covariant derivatives of a tetrad with respect to the affine connection is

$$2e_{a:[j}^k e_{i]}^\sigma = R^k_{lij} e_a^\sigma + 2S^l_{ij} e_{a;l}^k. \quad (1.5.34)$$

This commutator can also be expressed in terms of the spin connection:

$$\begin{aligned} e_{a:[j}^k e_{i]}^\sigma &= \omega_{a[j;i]}^k (e_b^k \omega_{a[j]}^\sigma)_{;i]} = \omega_{ba[j} \omega_{i]}^{kb} + \omega_{a[j;i]}^b e_b^k \\ &= \omega_{ba[j} \omega_{i]}^{kb} + \omega_{a[j;i]}^b e_b^k + S^l_{ij} \omega_{al}^k. \end{aligned} \quad (1.5.35)$$

Consequently, the curvature tensor with two Lorentz and two coordinate indices depends only on the spin connection and its ordinary derivatives:

$$R^a_{bij} = \omega^a_{bj,i} - \omega^a_{bi,j} + \omega^a_{ci} \omega^c_{bj} - \omega^a_{cj} \omega^c_{bi}. \quad (1.5.36)$$

Because the spin connection is antisymmetric in its first two indices, the tensor (1.5.36) is antisymmetric in its first two (Lorentz) indices, like the Riemann tensor. The contraction of the curvature tensor (1.5.36) with a tetrad gives the Ricci tensor with one Lorentz and one coordinate index:

$$R_{bj} = R^a_{bij} e_a^i. \quad (1.5.37)$$

The contraction of the tensor R^a_i with a tetrad gives the Ricci scalar,

$$R = R^a_i e_a^i = R^{ab}_{ij} e_a^i e_b^j. \quad (1.5.38)$$

The Riemann tensor with two Lorentz and two coordinate indices depends on the Levi-Civita connection (1.5.31) the same way the curvature tensor depends on the affine connection:

$$P^a_{bij} = \varpi^a_{bj,i} - \varpi^a_{bi,j} + \varpi^a_{ci} \varpi^c_{bj} - \varpi^a_{cj} \varpi^c_{bi}. \quad (1.5.39)$$

The contraction of (1.5.39) with a tetrad gives the Riemannian Ricci tensor with one Lorentz and one coordinate index:

$$P_{bj} = P^a_{bij} e_a^i. \quad (1.5.40)$$

The contraction of the tensor P^a_i with a tetrad gives the Riemann scalar,

$$P = P^a_i e_a^i = P^{ab}_{ij} e_a^i e_b^j. \quad (1.5.41)$$

1.6 Lorentz group

1.6.1 Subgroups of Lorentz group and principle of relativity

A composition of two Lorentz transformations Λ_1 and Λ_2 ,

$$\Lambda^a_b = \Lambda^a_{(1)c} \Lambda^c_{(2)b}, \quad (1.6.1)$$

satisfies (1.5.10), so it is a Lorentz transformation. The Kronecker symbol δ^a_b also satisfies (1.5.10), so it can be regarded as the identity Lorentz transformation. Therefore Lorentz transformations form a group, referred to as the *Lorentz group*. Taking the determinant of the relation (1.5.10) gives

$$|\Lambda^a_b| = \pm 1. \quad (1.6.2)$$

A Lorentz transformation with $|\Lambda^a_b| = 1$ is *proper* and with $|\Lambda^a_b| = -1$ is *improper*. Proper Lorentz transformations form a group because the determinant of the product of two proper Lorentz transformations is 1. Improper Lorentz transformations include the *parity* transformation P

$$\Lambda^a_b(P) = \text{diag}(1, -1, -1, -1), \quad t \rightarrow t, \quad \mathbf{x} \rightarrow -\mathbf{x}, \quad (1.6.3)$$

and the *time reversal* T

$$\Lambda^a_b(T) = \text{diag}(-1, 1, 1, 1), \quad t \rightarrow -t, \quad \mathbf{x} \rightarrow \mathbf{x}. \quad (1.6.4)$$

The relation (1.5.10) gives $\Lambda^0_0 \Lambda^0_0 - \Lambda^0_\alpha \Lambda^0_\alpha = 1$, so

$$|\Lambda^0_0| \geq 1. \quad (1.6.5)$$

Lorentz transformations with $\Lambda^0_0 \geq 1$ are *orthochronous* and form a group. If x^i is a timelike vector, $x^i x_i > 0$, then for an orthochronous transformation $x'^0 = \Lambda^0_0 x^0 + \Lambda^0_\alpha x^\alpha$,

$$|\Lambda^0_\alpha x^\alpha| \leq \sqrt{\Lambda^0_\alpha \Lambda^0_\alpha x^\beta x^\beta} < \sqrt{(\Lambda^0_0)^2 (x^0)^2} = |\Lambda^0_0 x^0|. \quad (1.6.6)$$

Thus the time component of a timelike vector does not change the sign under orthochronous transformations. Einstein's *principle of relativity* states that all physical laws are invariant under transformations within the orthochronous proper subgroup of the Lorentz group.

Under the parity transformation, the spatial components of contravariant and covariant vectors (three-dimensional vectors) change the sign, while the spatial components of dual vectors (such as cross products of vectors) do not change the sign. Similarly, the scalar contraction of the Levi-Civita symbol and a tensor changes the sign, while a scalar does not. Quantities that transform under proper Lorentz transformations like vectors and do not change the sign in their spatial components under parity are referred to as *axial* vectors or *pseudovectors*. Quantities that transform under proper Lorentz transformations like scalars and change the sign under parity are referred to as *pseudoscalars*.

1.6.2 Infinitesimal Lorentz transformations

We consider an infinitesimal Lorentz transformation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \epsilon^\mu_\nu, \quad (1.6.7)$$

where ϵ^μ_ν are infinitesimal quantities. The relation (1.5.10) gives

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \quad (1.6.8)$$

where the indices are raised and lowered using the Minkowski metric tensor. Therefore Lorentz transformations are given by 6 independent antisymmetric parameters $\epsilon_{\mu\nu}$. The corresponding transformation of a contravariant vector A^μ is

$$A'^\mu = A^\mu + \epsilon^\mu_\nu A^\nu = A^\mu + \frac{1}{2} \epsilon^{\rho\sigma} (\delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu}) A^\nu = A^\mu + \frac{1}{2} \epsilon^{\rho\sigma} J^\mu_{\nu\rho\sigma} A^\nu, \quad (1.6.9)$$

where

$$J_{\nu\rho\sigma}^\mu = \delta_\rho^\mu \eta_{\sigma\nu} - \delta_\sigma^\mu \eta_{\rho\nu}. \quad (1.6.10)$$

We define matrices $J_{\rho\sigma}$ such that

$$(J_{\rho\sigma})_\nu^\mu = J_{\nu\rho\sigma}^\mu. \quad (1.6.11)$$

Therefore, in the matrix notation (with A^μ treated as a column),

$$A' = \left(1 + \frac{1}{2}\epsilon^{\rho\sigma} J_{\rho\sigma}\right) A. \quad (1.6.12)$$

The 6 matrices $J_{\rho\sigma}$ are the infinitesimal *generators* of the *vector representation* of the Lorentz group. The explicit form of the generators of the Lorentz group in the vector representation is

$$\begin{aligned} J_{01} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & J_{02} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ J_{03} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & J_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ J_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & J_{31} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (1.6.13)$$

1.6.3 Generators and Lie algebra of Lorentz group

The commutator of the generators of the Lorentz group in the vector representation is, using (1.6.10) and (1.6.11),

$$[J_{\kappa\tau}, J_{\rho\sigma}]_\nu^\mu = (J_{\kappa\tau})_\lambda^\mu (J_{\rho\sigma})_\nu^\lambda - (J_{\rho\sigma})_\lambda^\mu (J_{\kappa\tau})_\nu^\lambda = (-J_{\kappa\rho}\eta_{\tau\sigma} - J_{\tau\sigma}\eta_{\kappa\rho} + J_{\kappa\sigma}\eta_{\tau\rho} + J_{\tau\rho}\eta_{\kappa\sigma})_\nu^\mu, \quad (1.6.14)$$

so

$$[J_{\kappa\tau}, J_{\rho\sigma}] = -J_{\kappa\rho}\eta_{\tau\sigma} - J_{\tau\sigma}\eta_{\kappa\rho} + J_{\kappa\sigma}\eta_{\tau\rho} + J_{\tau\rho}\eta_{\kappa\sigma}. \quad (1.6.15)$$

The relation (1.6.15) constitutes the *Lie algebra* of the *Lorentz group*. If a set of quantities ϕ transforms under a Lorentz transformation Λ with a matrix $D(\Lambda)$

$$\phi \rightarrow D(\Lambda)\phi, \quad (1.6.16)$$

then D is a representation of the Lorentz group if

$$D(I) = I, \quad D(\Lambda_1\Lambda_2) = D(\Lambda_1)D(\Lambda_2), \quad (1.6.17)$$

where I denotes the identity transformation, and Λ_1 and Λ_2 are two Lorentz transformations. Therefore

$$D(\Lambda^{-1}) = D^{-1}(\Lambda), \quad (1.6.18)$$

where Λ^{-1} is the Lorentz transformation to Λ : $\Lambda\Lambda^{-1} = I$. For an infinitesimal Lorentz transformation in any representation,

$$D(\Lambda) = I + \frac{1}{2}\epsilon^{\rho\sigma} J_{\rho\sigma}, \quad (1.6.19)$$

according to (1.6.12). The relation

$$D(\Lambda_1\Lambda_2\Lambda_1^{-1}) = D(\Lambda_1)D(\Lambda_2)D^{-1}(\Lambda_1) \quad (1.6.20)$$

gives (1.6.15), valid for any representation of the Lorentz group.

If Λ_1 and Λ_2 are two group transformations then $\Lambda_3 = \Lambda_1 \Lambda_2 \Lambda_1^{-1}$ is a group transformation. If $\Lambda_2 = I + \epsilon_2 G_2$ is an infinitesimal group transformation with generator G_2 then $\Lambda_3 = I + \epsilon_2 \Lambda_1 G_2 \Lambda_1^{-1}$ is an infinitesimal group transformation with generator $G_3 = \Lambda_1 G_2 \Lambda_1^{-1}$. If $\Lambda_1 = I + \epsilon_1 G_1$ is an infinitesimal group transformation with generator G_1 then, neglecting terms in ϵ_1 of higher order, $G_3 = G_2 + \epsilon_1 [G_1, G_2]$, so $[G_1, G_2]$ is a generator. For a finite number N of linearly independent generators, a general infinitesimal group transformation is $\Lambda = I + \sum_{a=1}^N \epsilon_a G_a$. Because $[G_a, G_b]$ is a generator, it is a linear combination of the N generators: $[G_a, G_b] = \sum_{c=1}^N f_{abc} G_c$, where f_{abc} are *structure constants* of the Lie algebra of the given group. For the Lorentz group, $\epsilon_a G_a = D(\Lambda) - I$, where $D(\Lambda)$ is given by (1.6.19).

1.6.4 Rotations and boosts

Rotations are proper orthochronous Lorentz transformations with

$$\Lambda^0_{\alpha} = \Lambda^{\alpha}_0 = 0, \quad \Lambda^0_0 = 1. \quad (1.6.21)$$

Rotations act only on the spatial coordinates x^{α} and form a group, referred to as the *rotation group*. *Boosts* are proper orthochronous Lorentz transformations with

$$\Lambda^{\alpha}_{\beta} = 0. \quad (1.6.22)$$

We define

$$J_{\alpha} = \frac{1}{2} e_{\alpha\beta\gamma} J^{\beta\gamma}, \quad (1.6.23)$$

$$K_{\alpha} = J_{0\alpha}, \quad (1.6.24)$$

and

$$\vartheta_{\alpha} = \frac{1}{2} e_{\alpha\beta\gamma} \epsilon^{\beta\gamma}, \quad (1.6.25)$$

$$\eta_{\alpha} = \epsilon_{0\alpha} \quad (1.6.26)$$

(for the Lorentz group $\mathfrak{g} = 1$, so the tensors e and densities ϵ are numerically identical). The explicit form of the generators of the rotation group J_{α} in the vector representation is

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1.6.27)$$

For an infinitesimal Lorentz transformation (1.6.19)

$$D = 1 + \boldsymbol{\vartheta} \cdot \mathbf{J} + \boldsymbol{\eta} \cdot \mathbf{K}. \quad (1.6.28)$$

A finite Lorentz transformation can be regarded as a composition of successive identical infinitesimal Lorentz transformations:

$$D = \lim_{n \rightarrow \infty} (1 + \boldsymbol{\theta} \cdot \mathbf{J}/n + \boldsymbol{\eta} \cdot \mathbf{K}/n)^n = e^{\boldsymbol{\theta} \cdot \mathbf{J} + \boldsymbol{\eta} \cdot \mathbf{K}}. \quad (1.6.29)$$

The finite parameters $\boldsymbol{\theta}$, $\boldsymbol{\eta}$ are the *canonical parameters* for a given Lorentz transformations. For a finite Lorentz transformation, (1.6.19) gives

$$D(\Lambda) = e^{\frac{1}{2} \epsilon^{\rho\sigma} J_{\rho\sigma}}, \quad (1.6.30)$$

so

$$J_{\mu\nu} = \left. \frac{\partial D(\Lambda)}{\partial \epsilon^{\mu\nu}} \right|_{\Lambda=I}. \quad (1.6.31)$$

The explicit form of a finite Lorentz transformation in the vector representation is

$$\begin{aligned}
R_1 = e^{\theta J_1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_2 = e^{\theta J_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}, \\
R_3 = e^{\theta J_3} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_1 = e^{\eta K_1} = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
B_2 = e^{\eta K_2} &= \begin{pmatrix} \cosh\eta & 0 & \sinh\eta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\eta & 0 & \cosh\eta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_3 = e^{\eta K_3} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix},
\end{aligned} \tag{1.6.32}$$

where R_α denotes a rotation about the x^α -axis and B_α denotes a boost along this axis. The canonical parameters $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are respectively referred to as the *angle of rotation* and *rapidity*. The parameters $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ in (1.6.25) and (1.6.26) are thus respectively infinitesimal values of the angle of rotation and rapidity. A rotation about any axis, say z , by an angle θ turns the two other axes, x and y , into new axes, x' and y' , such that the angle between x and x' (or y and y') (1.4.114) is θ . The rotation group is *compact*: $\theta_\alpha \in [0, 2\pi]$ and $\theta = 2\pi \Leftrightarrow \theta = 0$. The explicit form of a finite rotation in the three-dimensional vector representation is

$$\begin{aligned}
R_1(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \\
R_3(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{1.6.33}$$

For instance,

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \rightarrow \begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = R_3 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x \cos\theta - V_y \sin\theta \\ V_x \sin\theta + V_y \cos\theta \\ V_z \end{pmatrix}. \tag{1.6.34}$$

The relation (1.6.31) gives

$$J_\alpha = \left. \frac{\partial R_\alpha(\theta)}{\partial \theta} \right|_{\theta=0}. \tag{1.6.35}$$

The orthogonality relation (1.5.10) applied to any of the rotation matrices (1.6.33) shows that a rotation matrix R is orthogonal, that is, its transpose R^T is equal to its inverse R^{-1} :

$$R_\alpha^T = R_\alpha^{-1}, \quad R_\alpha R_\alpha^T = R_\alpha^T R_\alpha = I, \tag{1.6.36}$$

where I is the identity matrix.

The commutation relation (1.6.15) gives

$$[J_\alpha, J_\beta] = e_{\alpha\beta\gamma} J_\gamma, \tag{1.6.37}$$

$$[J_\alpha, K_\beta] = e_{\alpha\beta\gamma} K_\gamma, \tag{1.6.38}$$

$$[K_\alpha, K_\beta] = -e_{\alpha\beta\gamma} J_\gamma. \tag{1.6.39}$$

Therefore rotations do not commute and form a nonabelian group, rotations and boosts do not commute, and boosts do not commute. Changing the order of two nonparallel boosts is equivalent

to applying a rotation, referred to as the *Thomas-Wigner rotation*. The structure constants of the Lie algebra of the rotation group are $f_{abc} = e_{abc}$. Moreover, the square of the generators of rotation,

$$J^2 = J_\alpha J_\alpha, \quad (1.6.40)$$

commutes with J_α :

$$[J^2, J_\beta] = [J_\alpha, J_\beta] J_\alpha + J_\alpha [J_\alpha, J_\beta] = e_{\alpha\beta\gamma} (J_\gamma J_\alpha + J_\alpha J_\gamma) = 0. \quad (1.6.41)$$

Defining

$$\mathbf{L} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \quad (1.6.42)$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}), \quad (1.6.43)$$

gives

$$[L_\alpha, L_\beta] = e_{\alpha\beta\gamma} L_\gamma, \quad (1.6.44)$$

$$[Q_\alpha, Q_\beta] = e_{\alpha\beta\gamma} Q_\gamma, \quad (1.6.45)$$

$$[L_\alpha, Q_\beta] = 0, \quad (1.6.46)$$

so the Lorentz group is isomorphic with the product of two complex rotation groups. Accordingly, the Lorentz group can be regarded as the group of four-dimensional rotations in the Minkowski space, or the group of *tetrad rotations*.

1.6.5 Poincaré group

Under the infinitesimal coordinate transformation (1.2.55) in a locally flat spacetime, (1.4.41) gives

$$\eta_{ik} \rightarrow \eta_{ik} - \xi_{i,k} - \xi_{k,i}. \quad (1.6.47)$$

Thus the tensor η_{ik} is invariant under (1.2.55) (isometric) if ξ^i is a Killing vector,

$$\xi_{(i,k)} = 0, \quad (1.6.48)$$

which has the solution

$$\xi^i = \epsilon^{ik} x_k + \epsilon^i, \quad (1.6.49)$$

where ϵ^{ik} and ϵ^i are constant. The first term on the right-hand side of (1.6.49) corresponds to a Lorentz rotation described by 6 parameters ϵ^{ik} . The second term on the right-hand side of (1.6.49) corresponds to a *translation*. A combination of two translations does not change if their order is reversed, so translations commute:

$$[T_\mu, T_\nu] = 0, \quad (1.6.50)$$

where T_μ is the generator of translation. The relations (1.6.37) and (1.6.38) mean that J^α and K^α are spatial vectors under rotations. Spatial translations are spatial vectors under rotations, while a time translation is a scalar:

$$[J_\alpha, T_\beta] = e_{\alpha\beta\gamma} T_\gamma, \quad (1.6.51)$$

$$[J_\alpha, T_0] = 0. \quad (1.6.52)$$

The last relation indicates that the generators of rotations, like generators of spatial translation, correspond to *conserved* quantities, which are quantities that do not change in time. The covariant generalization of (1.6.51) and (1.6.52) is

$$[J_{\mu\nu}, T_\rho] = T_\mu \eta_{\nu\rho} - T_\nu \eta_{\mu\rho}. \quad (1.6.53)$$

The relations (1.6.15), (1.6.50) and (1.6.53) constitute the Lie algebra of the inhomogeneous Lorentz or *Poincaré group*. In particular,

$$[K_\alpha, T_\beta] = -T_0 \delta_{\alpha\beta}, \quad (1.6.54)$$

$$[K_\alpha, T_0] = -T_\alpha. \quad (1.6.55)$$

The last relation indicates that the generators of boosts do not correspond to conserved quantities.

For an infinitesimal rotation about the z -axis,

$$\begin{aligned} (1 + \theta J_z) f(ct, \mathbf{x}) &= D(R_z(\theta)) f(ct, \mathbf{x}) = f(ct, R_z(\theta) \mathbf{x}) \approx f(ct, x - \theta y, \theta x + y, z) \\ &= f(ct, \mathbf{x}) - \theta y \frac{\partial f}{\partial x} + \theta x \frac{\partial f}{\partial y}, \end{aligned} \quad (1.6.56)$$

or

$$J_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad (1.6.57)$$

which gives the differential representation of rotations:

$$J_\alpha = e_{\alpha\beta\gamma} x_\beta \partial_\gamma. \quad (1.6.58)$$

For an infinitesimal boost along the z -axis,

$$\begin{aligned} (1 + \eta K_z) f(ct, \mathbf{x}) &= D(B_z(\eta)) f(ct, \mathbf{x}) = f(B_z(\eta)(ct, \mathbf{x})) \approx f(ct + \eta z, y, z + \eta ct) \\ &= f(ct, \mathbf{x}) + \eta z \frac{\partial f}{\partial ct} + \eta ct \frac{\partial f}{\partial z}, \end{aligned} \quad (1.6.59)$$

or

$$K_z = z \frac{\partial}{\partial ct} + ct \frac{\partial}{\partial z}, \quad (1.6.60)$$

which gives the differential representation of boosts:

$$K_\alpha = x_\alpha \frac{\partial}{\partial ct} + ct \frac{\partial}{\partial x^\alpha}. \quad (1.6.61)$$

The relation for an infinitesimal translation, analogous to (1.6.19), is

$$D(t) = I + \epsilon^\mu T_\mu, \quad (1.6.62)$$

so a finite translation is given by

$$D(t) = e^{\epsilon^\mu T_\mu}. \quad (1.6.63)$$

Translation in (1.6.49) can also be written as

$$t_\mu(\epsilon) x^\nu = x^\nu + \epsilon \delta_\mu^\nu. \quad (1.6.64)$$

The relation analogous to (1.6.35) is

$$T_\mu = \left. \frac{\partial t_\mu(\epsilon)}{\partial \epsilon} \right|_{\epsilon=0}. \quad (1.6.65)$$

The differential representation of a translation is thus

$$T_\mu = \frac{\partial}{\partial x^\mu}. \quad (1.6.66)$$

1.6.6 Invariants of Lorentz and Poincaré group

Analogously to (1.6.41),

$$[L^2, L_\beta] = 0, \quad (1.6.67)$$

$$[Q^2, Q_\beta] = 0, \quad (1.6.68)$$

so L^2 and Q^2 commute with all 6 generators of the Lorentz group. Consequently, $J^2 + K^2$ and $\mathbf{J} \cdot \mathbf{K}$ commute with all generators of the Lorentz group, that is, are the invariants or *Casimir operators* of the Lorentz group. The Casimir operators of Lorentz group do not commute with the generators of translation T_μ , so they are not the invariants of the Poincaré group. Instead, the *mass operator*

$$m^2 = -T^\mu T_\mu \quad (1.6.69)$$

and

$$W^2 = W^\mu W_\mu, \quad (1.6.70)$$

where W^μ is the *Pauli-Lubański pseudovector*

$$W^\mu = \frac{1}{2} e^{\mu\nu\rho\sigma} J_{\rho\sigma} T_\nu, \quad (1.6.71)$$

commute with all generators of the Poincaré group, so they are the Casimir operators of the Poincaré group. The Pauli-Lubański pseudovector obeys the commutation relations

$$[T_\mu, W_\nu] = 0, \quad (1.6.72)$$

$$[J_{\mu\nu}, W_\rho] = W_\mu \eta_{\nu\rho} - W_\nu \eta_{\mu\rho}, \quad (1.6.73)$$

$$[W^\mu, W^\nu] = e^{\mu\nu\rho\sigma} W_\rho T_\sigma. \quad (1.6.74)$$

The relation (1.6.73) is analogous to (1.6.53) because W^μ behaves like a vector under proper Lorentz transformations.

We define the *four-momentum operator*

$$P_\mu = iT_\mu, \quad (1.6.75)$$

whose time component is the *energy operator* $P_0 = iT_0$ and spatial components form the *momentum operator* $P_\alpha = iT_\alpha$. We define the *angular four-momentum operator*

$$M_{\mu\nu} = iJ_{\mu\nu}, \quad (1.6.76)$$

whose spatial components form the *angular momentum operator*

$$M_\alpha = iJ_\alpha. \quad (1.6.77)$$

Therefore the following relations are satisfied:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\rho}\eta_{\nu\sigma} + M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma}), \quad (1.6.78)$$

$$[P_\mu, P_\nu] = 0, \quad (1.6.79)$$

$$[M_{\mu\nu}, P_\rho] = i(P_\mu\eta_{\nu\rho} - P_\nu\eta_{\mu\rho}), \quad (1.6.80)$$

$$m^2 = P^\mu P_\mu, \quad (1.6.81)$$

$$W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\rho\sigma} P_\nu, \quad (1.6.82)$$

$$[P_\mu, W_\nu] = 0, \quad (1.6.83)$$

$$[M_{\mu\nu}, W_\rho] = i(W_\mu\eta_{\nu\rho} - W_\nu\eta_{\mu\rho}), \quad (1.6.84)$$

$$[W^\mu, W^\nu] = -ie^{\mu\nu\rho\sigma} W_\rho P_\sigma, \quad (1.6.85)$$

$$[M_\alpha, M_\beta] = ie_{\alpha\beta\gamma} M_\gamma. \quad (1.6.86)$$

1.6.7 Relativistic kinematics

We consider a boost in the direction of the z -axis

$$x'^i = e^{-\eta K_3} x^i, \quad (1.6.87)$$

where x^i and x'^i have a form of a column (4×1 matrix), and $e^{\eta K_3}$ is given by (1.6.32). Therefore the coordinates in an inertial K -system (unprimed) are related to the coordinates in an inertial K' -system (primed) by

$$\begin{aligned} ct &= ct' \cosh \eta + z' \sinh \eta, \\ x &= x', \quad y = y', \\ z &= z' \cosh \eta + ct' \sinh \eta. \end{aligned} \quad (1.6.88)$$

We consider the origin of the K' -system, $x' = y' = z' = 0$, in the K -system. Therefore

$$\begin{aligned} ct &= ct' \cosh \eta, \\ z &= ct' \sinh \eta, \end{aligned} \quad (1.6.89)$$

which gives the relation between the rapidity η and velocity $V = \frac{dz}{dt}$ of K' relative to K :

$$\tanh \eta = \beta, \quad (1.6.90)$$

where

$$\beta = \frac{V}{c}. \quad (1.6.91)$$

Accordingly, $\cosh \eta = \gamma$ and $\sinh \eta = \beta \gamma$, where

$$\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}. \quad (1.6.92)$$

The relations (1.6.88) become

$$\begin{aligned} t &= \gamma \left(t' + \frac{V}{c^2} z' \right), \\ x &= x', \quad y = y', \\ z &= \gamma (z' + V t'), \end{aligned} \quad (1.6.93)$$

and are referred to as a *special Lorentz transformation* in the z -direction. The reverse transformation is

$$\begin{aligned} t' &= \gamma \left(t - \frac{V}{c^2} z \right), \\ x' &= x, \quad y' = y, \\ z' &= \gamma (z - V t). \end{aligned} \quad (1.6.94)$$

For a boost along an arbitrary direction, the spatial vector $\mathbf{x} = (x, y, z)$ transforms such that its component parallel to the velocity $\mathbf{V} = c\boldsymbol{\beta}$ of K' relative to K , $\mathbf{x}_{\parallel} = (\mathbf{x} \cdot \mathbf{V})\mathbf{V}/V^2$ (similarly for primed), behaves like z in (1.6.93) and its component perpendicular to $\mathbf{V}$, $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\parallel}$, behaves like x in (1.6.93):

$$\begin{aligned} t &= \gamma \left(t' + \frac{\mathbf{V} \cdot \mathbf{x}'}{c^2} \right), \\ \mathbf{x}_{\perp} &= \mathbf{x}'_{\perp}, \\ \mathbf{x}_{\parallel} &= \gamma (\mathbf{x}'_{\parallel} + \mathbf{V} t'), \end{aligned} \quad (1.6.95)$$

so

$$\mathbf{x} = \gamma (\mathbf{x}'_{\parallel} + \mathbf{V} t') + \mathbf{x}'_{\perp} = \gamma \mathbf{V} t' + \mathbf{x}' + \frac{(\gamma - 1)(\mathbf{V} \cdot \mathbf{x}')\mathbf{V}}{V^2}. \quad (1.6.96)$$

Therefore the transformation law for the coordinates in two inertial frames of reference is

$$\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2} \end{pmatrix} \begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix}, \quad (1.6.97)$$

or equivalently

$$\begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ -\gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2} \end{pmatrix} \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}. \quad (1.6.98)$$

The matrix in (1.6.98) is called a *boost matrix*. In the local Minkowski spacetime, contravariant vectors transform like x^i , according to (1.6.95) and (1.6.97),

$$\begin{pmatrix} W^0 \\ \mathbf{W} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2} \end{pmatrix} \begin{pmatrix} W'^0 \\ \mathbf{W}' \end{pmatrix}, \quad (1.6.99)$$

covariant vectors transform such that they remain related to contravariant vectors by the Minkowski metric tensor, and tensors transform like products of vectors. For example, if $\mathbf{V} = c\beta\hat{z}$ is parallel to the z -axis, a tensor of rank (0,2) transforms according to

$$\begin{aligned} T_{00} &= \gamma(T_{00'} + \beta T_{03'}) = \gamma^2(T_{0'0'} + \beta T_{3'0'} + \beta T_{0'3'} + \beta^2 T_{3'3'}), \\ T_{0\perp} &= \gamma(T_{0'\perp'} + \beta T_{3'\perp'}), \\ T_{03} &= \gamma(T_{03'} + \beta T_{00'}) = \gamma^2(T_{0'3'} + \beta T_{3'3'} + \beta T_{0'0'} + \beta^2 T_{3'0'}), \\ T_{\perp\perp} &= T_{\perp'\perp'}, \\ T_{3\perp} &= \gamma(T_{3'\perp'} + \beta T_{0'\perp'}), \\ T_{33} &= \gamma(T_{33'} + \beta T_{30'}) = \gamma^2(T_{3'3'} + \beta T_{0'3'} + \beta T_{3'0'} + \beta^2 T_{0'0'}), \end{aligned} \quad (1.6.100)$$

where the index $\perp$ denotes either 1 or 2, and the transposed components $T_{ik}^T = T_{ki}$ transform like the transpositions of the right-hand sides in (1.6.100). If T_{ik} is antisymmetric then $T_{03} = T_{0'3'}$.

The relations (1.6.93) can be written as

$$\begin{aligned} dt &= \gamma\left(dt' + \frac{V}{c^2}dz'\right), \\ dx &= dx', \quad dy = dy', \\ dz &= \gamma(dz' + Vdt'), \end{aligned} \quad (1.6.101)$$

which gives

$$\begin{aligned} v_x &= \frac{v'_x}{\gamma(1 + Vv'_z/c^2)}, \\ v_y &= \frac{v'_y}{\gamma(1 + Vv'_z/c^2)}, \\ v_z &= \frac{v'_z + V}{1 + Vv'_z/c^2}, \end{aligned} \quad (1.6.102)$$

where

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \quad \mathbf{v}' = \frac{d\mathbf{x}'}{dt'}. \quad (1.6.103)$$

Two special Lorentz transformations in the same direction commute because of (1.6.39). If a Lorentz transformation from K' to K has parameters β_1 and γ_1 , and a Lorentz transformation from K'' to K' has parameters β_2 and γ_2 , then a Lorentz transformation from K'' to K has parameters β_3 and γ_3 such that

$$\beta_3 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}, \quad \gamma_3 = \gamma_1\gamma_2(1 + \beta_1\beta_2). \quad (1.6.104)$$

For a boost along an arbitrary direction, (1.6.97) gives the Lorentz transformation of velocities:

$$\mathbf{v} = \frac{\mathbf{v}' + \gamma \mathbf{V} + (\gamma - 1)(\mathbf{v}' \cdot \mathbf{V})\mathbf{V}/V^2}{\gamma(1 + \mathbf{v}' \cdot \mathbf{V}/c^2)}. \quad (1.6.105)$$

If $v' = |\mathbf{V}'| = c$ then $v = |\mathbf{V}| = c$, in agreement with the constancy of the velocity of propagation of interaction.

We consider two points at rest in the inertial frame of reference K with positions z_1 and z_2 , so the distance between them is $\Delta z = z_2 - z_1$. In the inertial frame K' , moving relative to K in the z -direction with velocity V , $z_1 = \gamma(z'_1 + Vt'_1)$ and $z_2 = \gamma(z'_2 + Vt'_2)$, so if $t'_1 = t'_2$ is the time at which we measure (simultaneously) the positions of the two points then $\Delta z = \gamma(z'_2 - z'_1) = \gamma\Delta z'$. Therefore the length of an object in K' , whose length in the rest frame K is l (*proper length*), is

$$l' = \frac{l}{\gamma} < l, \quad (1.6.106)$$

which is referred to as the *Lorentz-FitzGerald contraction*. The volume of an object in K' , whose volume in the rest frame K is V (*proper volume*), is

$$V' = \frac{V}{\gamma}. \quad (1.6.107)$$

Suppose that there are two rods of equal lengths, moving parallel relative to each other. From the point of view of an observer moving with the first rod, the second one is shorter, and from the point of view of an observer moving with the second rod, the first one is shorter. There is no contradiction in this statement because the positions of both ends of a rod must be measured simultaneously and the simultaneity is not invariant: from the transformation law (1.6.93) it follows that if $\delta t = 0$ then $\delta t' \neq 0$ and if $\delta t' = 0$ then $\delta t \neq 0$.

We consider a clock (any mechanism with a periodic or evolutionary behavior) at rest in K' with position z' ; the time difference between two events with t'_1 and t'_2 , as measured by this clock, is $\Delta t' = t'_2 - t'_1$. In the frame K , $t_1 = \gamma(t'_1 + Vz'/c^2)$ and $t_2 = \gamma(t'_2 + Vz'/c^2)$, so

$$\Delta t = t_2 - t_1 = \gamma\Delta t' > \Delta t'. \quad (1.6.108)$$

Thus the rate of time is slower for moving clocks than those at rest (*time dilation*), in agreement with (1.4.91) and (1.4.97), from which $c^2 d\tau^2 = c^2 dt^2 - dl^2$ and

$$d\tau = \frac{1}{\gamma} dt. \quad (1.6.109)$$

Suppose that there are two clocks linked to the inertial frames K and K' , and that when the clock in K passes by the clock in K' the readings of the two clocks coincide. From the point of view of an observer in K clocks in K' go more slowly, and from the point of view of an observer in K' clocks in K go more slowly. There is no contradiction in this statement because to compare the rates of the two clocks in K and K' we must compare the readings of the same moving clock in K' with different clocks in K ; we require several clocks in one frame and one in the other, thus the measurement process is not symmetric with respect to the two frames of reference. The clock that goes more slowly is the one which is being compared with different clocks in the other frame. The time interval measured by a clock is equal to the integral

$$\Delta t = \frac{1}{c} \int ds \quad (1.6.110)$$

along its world line. Since the world line is a straight line for a clock at rest and a curved line for a clock moving such that it returns to the starting point, the integral $\int ds$ taken between two world points has its maximum value if it is taken along the straight line connecting these two points.

For a Lorentz transformation with velocity $V = |\mathbf{V}|$, (1.6.105) gives

$$\tan\theta = \frac{v'\sin\theta'}{\gamma(v'\cos\theta' + V)}, \quad (1.6.111)$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{V}$, and θ' is the angle between $\mathbf{v}'$ and $\mathbf{V}$. If $v = v' = c$ then

$$\cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta'}, \quad (1.6.112)$$

which is referred to as the *aberration* of a signal. Suppose an observer in frame K measures a periodic signal with period T , frequency $\nu = \frac{1}{T}$ and wavelength $\lambda = \frac{c}{\nu}$, propagating in the $-z$ direction; the number of pulses in time dt is $n = \nu dt$. A second observer in frame K' , moving in the z direction with velocity V relative to the first one, travels a distance Vdt and measures $\frac{Vdt}{\lambda}$ more pulses: $n' = \nu(1 + \frac{V}{c})dt$. Because the time interval dt with respect to K' is $dt' = \frac{dt}{\gamma}$, the frequency of the signal in K' is $\nu' = \gamma\nu(1 + \frac{V}{c})$ or

$$\nu' = e^\eta \nu. \quad (1.6.113)$$

This dependence of the frequency of a signal on a frame of reference is referred to as the *Doppler effect*.

When $c \rightarrow \infty$ (at which $\gamma \rightarrow 1$) the above formulae, referring to *relativistic kinematics*, reduce to their *nonrelativistic* limit. The Lorentz transformation (1.6.97) reduces to the *Galileo transformation*,

$$\begin{aligned} t &= t', \\ \mathbf{x} &= \mathbf{x}' + \mathbf{V}t', \end{aligned} \quad (1.6.114)$$

so the time is an absolute (invariant) quantity in nonrelativistic (*Newtonian*) physics. Any two Galileo transformations commute. The transformation law for velocities (1.6.105) reduces to the simple addition of vectors,

$$\mathbf{v} = \mathbf{v}' + \mathbf{V}. \quad (1.6.115)$$

1.6.8 Four-acceleration

In a locally inertial frame of reference, the four-velocity is

$$u^i = \left(\gamma, \gamma \frac{\mathbf{v}}{c} \right), \quad u_i = \left(\gamma, -\gamma \frac{\mathbf{v}}{c} \right), \quad (1.6.116)$$

where $\mathbf{v}$ is the velocity and $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$. We define the *four-acceleration*

$$w^i = \frac{Du^i}{ds} = \frac{D^2x^i}{ds^2} = u^k u^i_{;k}, \quad (1.6.117)$$

which is orthogonal to u^i because of (1.4.19):

$$w^i u_i = 0, \quad (1.6.118)$$

thus having 3 independent components. In a locally inertial frame of reference, the four-acceleration is

$$w^i = \frac{du^i}{ds} = \frac{d^2x^i}{ds^2} = c^{-2} \left(\gamma^4 \frac{\mathbf{v} \cdot \mathbf{a}}{c}, \gamma^2 \mathbf{a} + \gamma^4 \frac{(\mathbf{v} \cdot \mathbf{a})\mathbf{v}}{c^2} \right), \quad (1.6.119)$$

where $\mathbf{a}$ is the three-dimensional *acceleration* vector

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}. \quad (1.6.120)$$

The invariant square of the four-acceleration is thus

$$w^i w_i = -\frac{\gamma^4}{c^4} \left(\mathbf{a}^2 + \frac{\gamma^2}{c^2} (\mathbf{v} \cdot \mathbf{a})^2 \right). \quad (1.6.121)$$

If $\mathbf{v} = 0$ at a given instant of time, the corresponding frame of reference is referred to as the *instantaneous rest frame*. In this frame

$$w^i w_i = -\frac{a^2}{c^4}, \quad (1.6.122)$$

so

$$a_0 = c^2 \sqrt{-w^i w_i} \quad (1.6.123)$$

is the absolute value of the acceleration in the instantaneous rest frame, called the *proper acceleration*.
References: [2, 3].

1.7 Spinors

1.7.1 Spinor representation of Lorentz group

Let γ^a be the coordinate-invariant 4×4 *Dirac matrices* defined as

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} I, \quad (1.7.1)$$

where I is the unit 4×4 matrix (4 is the lowest dimension for which (1.7.1) has solutions). Accordingly, the spacetime-dependent Dirac matrices, $\gamma^i = e_a^i \gamma^a$, satisfy

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ij} I. \quad (1.7.2)$$

Under a tetrad rotation, (1.5.8) gives

$$\tilde{\gamma}^a = \Lambda^a_b \gamma^b. \quad (1.7.3)$$

Let L be a 4×4 matrix such that

$$\gamma^a = \Lambda^a_b L \gamma^b L^{-1} = L \tilde{\gamma}^a L^{-1}, \quad (1.7.4)$$

where L^{-1} is the matrix inverse to L : $LL^{-1} = L^{-1}L = I$. The condition (1.7.4) represents the constancy of the Dirac matrices γ^a under the combined tetrad rotation and transformation $\gamma \rightarrow L\gamma L^{-1}$. We refer to L as the *spinor representation* of the Lorentz group. The relation (1.7.4) gives the matrix L as a function of the Lorentz matrix Λ^a_b . For an infinitesimal Lorentz transformation (1.6.7), the solution for L is

$$L = I + \frac{1}{2} \epsilon_{ab} G^{ab}, \quad L^{-1} = I - \frac{1}{2} \epsilon_{ab} G^{ab}, \quad (1.7.5)$$

where G^{ab} are the *generators* of the spinor representation of the Lorentz group:

$$G^{ab} = \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a). \quad (1.7.6)$$

A *spinor* ψ is defined as a quantity that, under tetrad rotations, transforms according to

$$\tilde{\psi} = L\psi. \quad (1.7.7)$$

An *adjoint spinor* $\bar{\psi}$ is defined as a quantity that transforms according to

$$\tilde{\bar{\psi}} = \bar{\psi} L^{-1}, \quad (1.7.8)$$

so the product $\bar{\psi}\psi$ is a scalar:

$$\tilde{\bar{\psi}}\tilde{\psi} = \bar{\psi}\psi. \quad (1.7.9)$$

The indices of the γ^a and L that are implicit in the 4×4 matrix multiplication in (1.7.1), (1.7.2) and (1.7.4) are spinor indices. The relation (1.7.4) implies that the Dirac matrices γ^a can be regarded as quantities that have, in addition to the invariant index a , one spinor index and one adjoint-spinor index. The product $\bar{\psi}\psi$ transforms like the Dirac matrices:

$$\tilde{\bar{\psi}}\tilde{\psi} = L\bar{\psi}\psi L^{-1}. \quad (1.7.10)$$

The spinors ψ and $\bar{\psi}$ can be used to construct tensors. For example, $\bar{\psi}\gamma^a\psi$ transforms like a contravariant Lorentz vector:

$$\bar{\psi}\gamma^a\psi \rightarrow \bar{\psi}L^{-1}\Lambda^a_b L\gamma^b L^{-1}L\psi = \Lambda^a_b \bar{\psi}\gamma^b\psi. \quad (1.7.11)$$

1.7.2 Spinor connection

The derivative of a spinor does not transform like a spinor:

$$\tilde{\psi}_{;i} = L\psi_{;i} + L_{,i}\psi. \quad (1.7.12)$$

If we introduce the *spinor connection* Γ_i that transforms according to

$$\tilde{\Gamma}_i = L\Gamma_i L^{-1} + L_{,i}L^{-1}, \quad (1.7.13)$$

then a *covariant derivative* of a spinor,

$$\psi_{;i} = \psi_{,i} - \Gamma_i \psi, \quad (1.7.14)$$

is a spinor:

$$\tilde{\psi}_{;i} = \tilde{\psi}_{,i} - \tilde{\Gamma}_i \tilde{\psi} = L\psi_{;i} + L_{,i}\psi - (L\Gamma_i L^{-1} + L_{,i}L^{-1})L\psi = L\psi_{;i}. \quad (1.7.15)$$

Because $\bar{\psi}\psi$ is a scalar,

$$(\bar{\psi}\psi)_{;i} = (\bar{\psi}\psi)_{,i}, \quad (1.7.16)$$

the chain rule for covariant differentiation gives a covariant derivative of an adjoint spinor

$$\bar{\psi}_{;i} = \bar{\psi}_{,i} + \bar{\psi}\Gamma_i. \quad (1.7.17)$$

Also

$$\psi_{|i} = \psi_{;i}, \quad \bar{\psi}_{|i} = \bar{\psi}_{;i}. \quad (1.7.18)$$

The Dirac matrices γ^a transform like $\psi\bar{\psi}$, whose covariant derivative is

$$(\psi\bar{\psi})_{;i} = \psi_{;i}\bar{\psi} + \psi\bar{\psi}_{;i} = (\psi\bar{\psi})_{,i} - \Gamma_i\psi\bar{\psi} + \psi\bar{\psi}\Gamma_i = (\psi\bar{\psi})_{,i} - [\Gamma_i, \psi\bar{\psi}]. \quad (1.7.19)$$

Therefore a covariant derivative of the Dirac matrices is

$$\gamma^a_{;i} = \gamma^a_{,i} - [\Gamma_i, \gamma^a] = -[\Gamma_i, \gamma^a], \quad (1.7.20)$$

so

$$\gamma^j_{;i} = \gamma^j_{,i} = \gamma^j_{,i} + \Gamma_{k,i}\gamma^k - [\Gamma_i, \gamma^j]. \quad (1.7.21)$$

Accordingly

$$\gamma^a_{|i} = \omega^a_{bi}\gamma^b - [\Gamma_i, \gamma^a]. \quad (1.7.22)$$

The quantity $\bar{\psi}\gamma^i\psi_{|i}$ transforms under Lorentz rotations like a scalar:

$$\bar{\psi}\gamma^i\psi_{|i} \rightarrow \bar{\psi}L^{-1}L\gamma^iL^{-1}L\psi_{|i} = \bar{\psi}\gamma^i\psi_{|i}. \quad (1.7.23)$$

The relation $\eta_{ab|i} = 0$ implies that

$$\gamma^a_{|i} = 0, \quad (1.7.24)$$

because the Dirac matrices γ^a only depend on η_{ab} . Multiplying both sides of (1.7.22) by γ_a from the left gives

$$\omega_{abi}\gamma^a\gamma^b - \gamma_a\Gamma_i\gamma^a + 4\Gamma_i = 0. \quad (1.7.25)$$

We seek the solution of (1.7.25) in the form

$$\Gamma_i = -\frac{1}{4}\omega_{abi}\gamma^a\gamma^b - A_i, \quad (1.7.26)$$

where A_i is a spinor-tensor quantity with one vector index. Substituting (1.7.26) to (1.7.25), together with the identity $\gamma_c\gamma^a\gamma^b\gamma^c = 4\eta^{ab}$, gives

$$-\gamma_a A_i \gamma^a + 4A_i = 0, \quad (1.7.27)$$

so A_i is an arbitrary vector multiple of I . Therefore the spinor connection Γ_i is given, up to the addition of an arbitrary vector multiple of I , by the *Fock-Ivanenko coefficients*:

$$\Gamma_i = -\frac{1}{4}\omega_{abi}\gamma^a\gamma^b = -\frac{1}{2}\omega_{abi}G^{ab}. \quad (1.7.28)$$

Using the definition (1.5.15), we can also write (1.7.28) as

$$\Gamma_i = -\frac{1}{8}e^j_{c;i}[\gamma_j, \gamma^c] = \frac{1}{8}[\gamma^j_{;i}, \gamma_j]. \quad (1.7.29)$$

1.7.3 Curvature spinor

The commutator of total covariant derivatives of a spinor is

$$\begin{aligned}\psi_{|ji} - \psi_{|ij} &= (\psi_{|j})_{,i} - \Gamma_i \psi_{|j} - \Gamma_j^k \psi_{|k} - (\psi_{|i})_{,j} + \Gamma_j \psi_{|i} + \Gamma_i^k \psi_{|k} \\ &= -\Gamma_{j,i} \psi + \Gamma_i \Gamma_j \psi + \Gamma_{i,j} \psi - \Gamma_j \Gamma_i \psi + 2S_{ij}^k \psi_{|k} = K_{ij} \psi + 2S_{ij}^k \psi_{|k},\end{aligned}\quad (1.7.30)$$

where $K_{ij} = -K_{ji}$ is defined as

$$K_{ij} = \Gamma_{i,j} - \Gamma_{j,i} + [\Gamma_i, \Gamma_j]. \quad (1.7.31)$$

Substituting (1.7.13) to (1.7.31) gives

$$\tilde{K}_{ij} = \tilde{\Gamma}_{i,j} - \tilde{\Gamma}_{j,i} + [\tilde{\Gamma}_i, \tilde{\Gamma}_j] = L(\Gamma_{i,j} - \Gamma_{j,i} + [\Gamma_i, \Gamma_j])L^{-1} = LK_{ij}L^{-1}, \quad (1.7.32)$$

so K_{ij} transforms under tetrad rotations like the Dirac matrices γ^a , that is, K_{ij} is a spinor with one spinor index and one adjoint-spinor index. We refer to K_{ij} as the *curvature spinor*.

The relation (1.7.24) leads to

$$\gamma^k_{|i} = 0. \quad (1.7.33)$$

Thus the commutator of covariant derivatives of the spacetime-dependent Dirac matrices vanishes:

$$2\gamma^k_{|[j} = R^k_{lij} \gamma^l + 2S^l_{ij} \gamma^k_{|l} + [K_{ij}, \gamma^k] = R^k_{lij} \gamma^l + [K_{ij}, \gamma^k] = 0. \quad (1.7.34)$$

Multiplying both sides of (1.7.34) by γ_k from the left gives

$$R_{klij} \gamma^k \gamma^l + \gamma_k K_{ij} \gamma^k - 4K_{ij} = 0. \quad (1.7.35)$$

We seek the solution of (1.7.35) in the form

$$K_{ij} = \frac{1}{4} R_{klij} \gamma^k \gamma^l + B_{ij}, \quad (1.7.36)$$

where B_{ij} is a spinor-tensor quantity with two vector indices. Substituting (1.7.36) to (1.7.35) gives

$$\gamma_k B_{ij} \gamma^k - 4B_{ij} = 0, \quad (1.7.37)$$

so B_{ij} is an antisymmetric-tensor multiple of I . The tensor B_{ij} is related to the vector A_i in (1.7.26) by

$$B_{ij} = A_{j,i} - A_{i,j} + [A_i, A_j]. \quad (1.7.38)$$

Because ψ has no indices other than spinor indices, A_i is a vector and $[A_i, A_j] = 0$. The invariance of (1.7.35) under the addition of an antisymmetric-tensor multiple B_{ij} of the unit matrix to the curvature spinor is related to the invariance of (1.7.25) under the addition of a vector multiple A_i of the unit matrix to the spinor connection. Setting $A_i = 0$, which corresponds to the Fock-Ivanenko spinor connection, gives $B_{ij} = 0$. Therefore the curvature spinor K_{ij} is given, up to the addition of an arbitrary antisymmetric-tensor multiple of I , by

$$K_{ij} = \frac{1}{4} R_{klij} \gamma^k \gamma^l = \frac{1}{2} R_{klij} G^{kl}. \quad (1.7.39)$$

References: [3, 4].

2 Fields

2.1 Principle of least action

The most general formulation of the law that governs the dynamics of classical systems is Hamilton's *principle of least action*, according to which every classical system is characterized by a definite scalar-density function $\mathfrak{L}$, and the dynamics of the system is such that a certain condition is satisfied. Let $\phi_A(x^i)$ be a set of physical *fields*, being differentiable functions of the coordinates, and let $\mathfrak{L}$ be a Lorentz covariant quantity constructed from the ϕ_A and their derivatives. We consider a scalar quantity

$$S = \frac{1}{c} \int \mathfrak{L} d\Omega, \quad (2.1.1)$$

where the integration is over some region in locally Minkowski spacetime. Let $\delta\phi_A$ be arbitrary small changes in ϕ_A (regarded as a dynamical variable) over the region of integration, which vanish on the boundary. Then the change in S can be written as

$$\delta S = \frac{1}{c} \int F^A \delta\phi_A d\Omega. \quad (2.1.2)$$

The principle of least action states that the dynamics of a physical system is given by the condition the scalar S be a local minimum. Therefore any infinitesimal change in the dynamics of the system does not alter the value of S :

$$\delta S = 0 \quad (2.1.3)$$

(S is a local extremum). If $\mathfrak{L}$ is covariant and ϕ_A transform covariantly under the Lorentz group, the variational condition (2.1.3) gives the Lorentz covariant equations

$$\phi_A = 0. \quad (2.1.4)$$

These equations are also invariant for any other transformations (internal symmetries) for which $\mathfrak{L}$ is invariant. $\mathfrak{L}$ is referred to as the *Lagrangian density*, S is the *action* functional, $\delta S = 0$ is the principle of least action, and $F^A = 0$ are the *field equations*. The field equations of a physical system are the result of the action being a local extremum. The condition that the action be a local minimum imposes additional restrictions on possible choices for S . The number of independent field equations for a given system is referred to as the number of the *degrees of freedom* representing this system.

In most cases $\mathfrak{L}$ contains only ϕ_A and their first derivatives (the Lagrangian density for the gravitational field contains second derivatives). A Lagrangian density containing higher derivatives can always be written in terms of first derivatives by increasing the number of the components of ϕ_A . We consider a physical system in the galilean frame of reference. If $\mathfrak{L}$ depends only on ϕ_A and $\partial_i \phi_A$, $\mathfrak{L} = \mathfrak{L}(\phi, \phi_{,i})$, then

$$\begin{aligned} \delta S &= \frac{1}{c} \int \left(\frac{\partial \mathfrak{L}}{\partial \phi} \delta\phi + \frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \delta(\phi_{,i}) \right) d\Omega = \frac{1}{c} \int \left(\frac{\partial \mathfrak{L}}{\partial \phi} \delta\phi + \frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} (\delta\phi)_{,i} \right) d\Omega \\ &= \frac{1}{c} \int \left(\frac{\partial \mathfrak{L}}{\partial \phi} \delta\phi - \partial_i \left(\frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \right) \delta\phi + \partial_i \left(\frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \delta\phi \right) \right) d\Omega. \end{aligned} \quad (2.1.5)$$

The last term in the second line of (2.1.5) is a divergence, which, after integration, can be transformed into a hypersurface integral over the boundary of integration region, where $\delta\phi = 0$ on the boundary, so this term does not contribute to the action variation:

$$\delta S = \frac{1}{c} \int \left(\frac{\partial \mathfrak{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \right) \right) \delta\phi d\Omega + \int \frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \delta\phi dS_i = \frac{1}{c} \int \left(\frac{\partial \mathfrak{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \right) \right) \delta\phi d\Omega. \quad (2.1.6)$$

If $\delta S = 0$ for arbitrary variations $\delta\phi$ that vanish on the boundary then

$$\frac{\partial \mathfrak{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathfrak{L}}{\partial(\phi_{,i})} \right) = 0, \quad (2.1.7)$$

or

$$\frac{\delta \mathfrak{L}}{\delta \phi} = 0, \quad (2.1.8)$$

where

$$\frac{\delta \mathfrak{L}}{\delta \phi} = \frac{\partial \mathfrak{L}}{\partial \phi} - \partial_i \left(\frac{\partial \mathfrak{L}}{\partial (\phi_{,i})} \right) \quad (2.1.9)$$

is a *variational derivative* of $\mathfrak{L}$ with respect to ϕ . This set of equations, for each component ϕ_A , is referred to as the *Lagrange equations*. Generalizing the Lagrange equations to an arbitrary coordinate frame gives

$$\frac{\partial \mathfrak{L}}{\partial \phi} - \nabla_i^{\{\}} \left(\frac{\partial \mathfrak{L}}{\partial (\phi_{,i})} \right) = 0. \quad (2.1.10)$$

There is some arbitrariness in the choice of $\mathfrak{L}$; adding to it the divergence of an arbitrary vector density or multiplying it by a constant produces the same field equations. If a system consists of two noninteracting parts A and B , with corresponding Lagrangian densities $\mathfrak{L}_A(\phi_A, \partial\phi_A)$ and $\mathfrak{L}_B(\phi_B, \partial\phi_B)$, then the Lagrangian for this system is the sum $\mathfrak{L}_A + \mathfrak{L}_B$. This additivity of the Lagrangian density means that the field equations for either of the two parts do not involve quantities pertaining to the other part. If $\mathfrak{L}_A$ also depends on ϕ_B and/or $\partial\phi_B$, and/or $\mathfrak{L}_B$ depends on ϕ_A and/or $\partial\phi_A$, then the subsystems A and B interact.

References: [1, 2, 3].

2.2 Action for gravitational field

We consider a Lagrangian density that depends on the affine (or spin) connection and its first derivatives. Such Lagrangian density can be decomposed into the covariant part that contains derivatives of the affine/spin connection, which is referred to as the Lagrangian density for the *gravitational field*, and the covariant part that does not contain these derivatives, which is referred to as the Lagrangian density for *matter*. The simplest covariant scalar that can be constructed from the affine/spin connection and its first derivatives is the Ricci scalar R . The corresponding Lagrangian density for the gravitational field is proportional to the product of R and the scalar density $\sqrt{-g}$:

$$\mathfrak{L}_g = -\frac{1}{2\kappa} \sqrt{-g} R, \quad (2.2.1)$$

where κ is *Einstein's gravitational constant*. There exist two variational principles in the theory of the gravitational field. The *metric* variational principle regards the metric tensor or tetrad as a dynamical variable and assumes the affine connection to be the Levi-Civita connection. The *metric-affine* variational principle regards both the metric tensor (or tetrad) and the metric-compatible affine connection (or spin connection) as dynamical variables.

In the metric variational formulation, the Lagrangian density for the gravitational field is proportional to the Riemann scalar P :

$$\mathfrak{L}_g = -\frac{1}{2\kappa} \sqrt{-g} P. \quad (2.2.2)$$

Because P is linear in derivatives of $\{^i_{kl}\}$:

$$\begin{aligned} \sqrt{-g} P &= \sqrt{-g} g^{ik} (\{^l_{ik}\}_{,l} - \{^l_{il}\}_{,k} + \{^m_{ik}\} \{^l_{ml}\} - \{^m_{il}\} \{^l_{mk}\}) \\ &= (\sqrt{-g} g^{ik} \{^l_{ik}\})_{,l} - \{^l_{ik}\} (\sqrt{-g} g^{ik})_{,l} - (\sqrt{-g} g^{ik} \{^l_{il}\})_{,k} + \{^l_{il}\} (\sqrt{-g} g^{ik})_{,k} \\ &\quad + \sqrt{-g} g^{ik} (\{^m_{ik}\} \{^l_{ml}\} - \{^m_{il}\} \{^l_{mk}\}), \end{aligned} \quad (2.2.3)$$

we can subtract from $\sqrt{-g} P$ total derivatives without altering the field equations, replacing P by a

noncovariant quantity G :

$$\begin{aligned}
\sqrt{-g}G &= \{^l_{il}\}(\sqrt{-g}g^{ik})_{,k} - \{^l_{ik}\}(\sqrt{-g}g^{ik})_{,l} + \sqrt{-g}g^{ik}(\{^m_{ik}\}\{^l_{ml}\} - \{^m_{il}\}\{^l_{mk}\}) \\
&= \{^l_{il}\}((\sqrt{-g}g^{ik})_{,k} + \{^j_{jk}\}\sqrt{-g}g^{ik} - \sqrt{-g}\{^i_{jk}\}g^{jk} - \sqrt{-g}\{^k_{jk}\}g^{ij}) \\
&\quad - \{^l_{ik}\}((\sqrt{-g}g^{ik})_{,l} + \{^j_{jl}\}\sqrt{-g}g^{ik} - \sqrt{-g}\{^i_{jl}\}g^{jk} - \sqrt{-g}\{^k_{jl}\}g^{ij}) \\
&\quad + \sqrt{-g}g^{ik}(\{^m_{ik}\}\{^l_{ml}\} - \{^m_{il}\}\{^l_{mk}\}) = \{^l_{il}\}(\{^j_{jk}\}\sqrt{-g}g^{ik} - \sqrt{-g}\{^i_{jk}\}g^{jk} \\
&\quad - \sqrt{-g}\{^k_{jk}\}g^{ij}) - \{^l_{ik}\}(\{^j_{jl}\}\sqrt{-g}g^{ik} - \sqrt{-g}\{^i_{jl}\}g^{jk} - \sqrt{-g}\{^k_{jl}\}g^{ij}) \\
&\quad + \sqrt{-g}g^{ik}(\{^m_{ik}\}\{^l_{ml}\} - \{^m_{il}\}\{^l_{mk}\}) = \sqrt{-g}g^{ik}(\{^m_{il}\}\{^l_{mk}\} - \{^m_{ik}\}\{^l_{ml}\}). \tag{2.2.4}
\end{aligned}$$

Therefore

$$G = g^{ik}(\{^m_{il}\}\{^l_{mk}\} - \{^m_{ik}\}\{^l_{ml}\}), \tag{2.2.5}$$

and the Lagrangian density for the gravitational field is

$$\mathfrak{L}_g = -\frac{1}{2\kappa}\sqrt{-g}G. \tag{2.2.6}$$

Any coordinate transformation results in variations of g^{ik} , so S is not necessarily a minimum with respect to these variations (only an extremum) because not all δg^{ik} correspond to actual variations of the gravitational field. In order to exclude the variations δg^{ik} resulting from changing the coordinates, we must impose on the metric tensor 4 arbitrary constraints. If we choose

$$g_{0\alpha} = 0, \quad |g_{\alpha\beta}| = \text{const}, \tag{2.2.7}$$

then G becomes

$$G = -\frac{1}{4}g^{00}g^{\alpha\beta}g^{\gamma\delta}g_{\alpha\gamma,0}g_{\beta\delta,0}. \tag{2.2.8}$$

In the locally galilean frame $g_{\alpha\beta} = -\delta_{\alpha\beta}$, so

$$G = -\frac{1}{4}g^{00}(g_{\alpha\beta,0})^2. \tag{2.2.9}$$

For physical systems $g^{00} > 0$. Therefore in order for S to have a minimum, κ must be positive, otherwise an arbitrarily rapid change of $g_{\alpha\beta}$ in time would result in an arbitrarily low value of S and there would be no minimum.

References: [2, 3].

2.3 Matter

2.3.1 Metric dynamical energy-momentum density

The variation of the matter action $S_m = \int \mathfrak{L}_m d\Omega$ with respect to the metric tensor,

$$\delta S_m = \frac{1}{2c} \int \mathcal{T}_{ij} \delta g^{ij} d\Omega = -\frac{1}{2c} \int \mathcal{T}^{ij} \delta g_{ij} d\Omega, \tag{2.3.1}$$

defines the *metric dynamical energy-momentum density* $\mathcal{T}_{ij}$, which is symmetric:

$$\mathcal{T}_{ij} = \mathcal{T}_{ji}. \tag{2.3.2}$$

Equivalently

$$\mathcal{T}_{ij} = 2 \frac{\delta \mathfrak{L}_m}{\delta g^{ij}} = 2 \left(\frac{\partial \mathfrak{L}_m}{\partial g^{ij}} - \partial_k \left(\frac{\partial \mathfrak{L}_m}{\partial g^{ikj}} \right) \right). \tag{2.3.3}$$

The *metric dynamical energy-momentum tensor* T_{ij} is defined as

$$T_{ij} = \frac{\mathcal{T}_{ij}}{\sqrt{-g}}. \tag{2.3.4}$$

2.3.2 Tetrad dynamical energy-momentum density

The variation of the matter action S_m with respect to the tetrad,

$$\delta S_m = \frac{1}{c} \int \mathfrak{T}_i^a \delta e_a^i d\Omega, \quad (2.3.5)$$

defines the *tetrad dynamical energy-momentum density* $\mathfrak{T}_i^a$. Equivalently

$$\delta \mathfrak{L}_m = \mathfrak{T}_i^a \delta e_a^i \quad (2.3.6)$$

or

$$\mathfrak{T}_i^a = \frac{\delta \mathfrak{L}_m}{\delta e_a^i}. \quad (2.3.7)$$

If $\mathfrak{L}_m$ depends only on tensor matter fields expressed in terms of the coordinate indices and it depends on neither derivatives of the metric tensor nor derivatives of the tetrad then the tetrad enters $\mathfrak{L}_m$ only through the metric tensor, in a combination $g^{ij} = \eta^{ab} e_a^i e_b^j$. Thus

$$\delta e_a^i = \frac{1}{2} e_{aj} \delta g^{ij}. \quad (2.3.8)$$

Substituting (2.3.8) to (2.3.5) gives

$$\delta S_m = \frac{1}{2c} \int \mathfrak{T}_{ij} \delta g^{ij} d\Omega, \quad (2.3.9)$$

where

$$\mathfrak{T}_{ij} = e_{aj} \mathfrak{T}_i^a. \quad (2.3.10)$$

The tensor $\mathfrak{T}_{ij}$ is generally not symmetric. Comparing (2.3.9) with (2.3.1) gives the relation between the tetrad dynamical energy-momentum density and the metric dynamical energy-momentum density for tensor matter fields:

$$\mathfrak{T}_{(ij)} = \mathcal{T}_{ij}. \quad (2.3.11)$$

2.3.3 Canonical energy-momentum density

If we express the matter Lagrangian density $\mathfrak{L}_m$, depending on matter fields ϕ and their first derivatives $\phi_{,i}$, only in terms of Lorentz and spinor indices, then the tetrad appears in $\mathfrak{L}_m$ only through a derivative of ϕ , in a covariant combination $e_a^i \phi_{|i}$. Since $\mathfrak{L}_m = \mathfrak{e}L$, where L is a scalar, we obtain

$$\begin{aligned} \delta \mathfrak{L}_m &= \mathfrak{e} \delta L - \mathfrak{e} e_i^a L \delta e_a^i = \mathfrak{e} \frac{\partial L}{\partial \phi_{|a}} \phi_{|i} \delta e_a^i - \mathfrak{L}_m e_i^a \delta e_a^i = \left(\frac{\partial \mathfrak{L}_m}{\partial \phi_{|a}} \phi_{|i} - \mathfrak{L}_m e_i^a \right) \delta e_a^i \\ &= \left(\frac{\partial \mathfrak{L}_m}{\partial \phi_{,a}} \phi_{|i} - \mathfrak{L}_m e_i^a \right) \delta e_a^i. \end{aligned} \quad (2.3.12)$$

The last term in (2.3.12),

$$\Theta_i^a = \frac{\partial \mathfrak{L}_m}{\partial \phi_{,a}} \phi_{|i} - e_i^a \mathfrak{L}_m, \quad (2.3.13)$$

is referred to as the *canonical energy-momentum density*. Accordingly

$$\Theta_j^i = \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \phi_{|j} - \delta_j^i \mathfrak{L}_m. \quad (2.3.14)$$

Comparing (2.3.12) with (2.3.6) shows that the canonical energy-momentum density is identical with the dynamical tetrad energy-momentum density:

$$\Theta_i^a = \mathfrak{T}_i^a. \quad (2.3.15)$$

2.3.4 Spin density

The variation of the matter action S_m with respect to the spin connection,

$$\delta S_m = \frac{1}{2c} \int \mathfrak{S}_{ab}{}^i \delta \omega^{ab}{}_i d\Omega, \quad (2.3.16)$$

defines the *dynamical spin density* $\mathfrak{S}_{ab}{}^i$:

$$\mathfrak{S}_{ab}{}^i = 2 \frac{\delta \mathfrak{L}_m}{\delta \omega^{ab}{}_i}, \quad (2.3.17)$$

which is antisymmetric in the Lorentz indices:

$$\mathfrak{S}_{ab}{}^i = -\mathfrak{S}_{ba}{}^i. \quad (2.3.18)$$

In the metric variational formulation of gravity, the variations $\delta \omega^{ab}{}_i = \delta \omega^{\{ab\}i}$ are functions of the variations δe_a^i and their derivatives, so the spin density is a function of the energy-momentum density. In the metric-affine variational formulation of gravity, the variations $\delta \omega^{ab}{}_i$ are independent of δe_a^i and their derivatives. The relation (1.5.29) indicates that the spin density is generated by the contortion tensor:

$$\mathfrak{S}_{ij}{}^k = 2 \frac{\delta \mathfrak{L}_m}{\delta C^{ij}{}_k}. \quad (2.3.19)$$

Accordingly, the variation of $\mathfrak{L}_m$ with respect to the torsion tensor,

$$\tau_i{}^{jk} = 2 \frac{\delta \mathfrak{L}_m}{\delta S^i{}_{jk}}, \quad (2.3.20)$$

is a homogeneous linear function of the spin connection because of (1.4.29):

$$\begin{aligned} \tau_{ijk} &= 2 \frac{\delta \mathfrak{L}_m}{\delta S^{ijk}} = 2 \frac{\delta \mathfrak{L}_m}{\delta C^{lmn}} \frac{\partial C^{lmn}}{\partial S^{ijk}} = \mathfrak{S}_{lmn} (\delta_i^l \delta_{[j}^m \delta_{k]}^n + \delta_i^m \delta_{[j}^n \delta_{k]}^l + \delta_i^n \delta_{[j}^l \delta_{k]}^m) \\ &= \mathfrak{S}_{ijk} - \mathfrak{S}_{jki} + \mathfrak{S}_{kij}, \end{aligned} \quad (2.3.21)$$

$$\mathfrak{S}_{ijk} = \tau_{[ij]k}, \quad (2.3.22)$$

antisymmetric in the last two indices:

$$\tau_{ijk} = -\tau_{ikj}. \quad (2.3.23)$$

The variation of $\mathfrak{L}_m$ with respect to the metric-compatible affine connection in the metric-affine variational formulation of gravity is equivalent to the variation with respect to the torsion (or contortion) tensor. The spin connection $\omega^{ab}{}_i$ enters $\mathfrak{L}_m$ only through derivatives of ϕ , in a combination $-\frac{\partial \mathfrak{L}}{\partial \phi_{,i}} \Gamma_i \phi$, where Γ_i is the covariant derivative acting on ϕ :

$$\Gamma_i = -\frac{1}{2} \omega_{abi} G^{ab}. \quad (2.3.24)$$

Consequently, the dynamical spin density $\mathfrak{S}_{ab}{}^i$ is identical with

$$\Sigma_{ab}{}^i = \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} G_{ab} \phi, \quad (2.3.25)$$

referred to as the *canonical spin density*. *Spin tensor* is defined as

$$s_{ijk} = \frac{\mathfrak{S}_{ijk}}{\sqrt{-g}}. \quad (2.3.26)$$

2.3.5 Belinfante-Rosenfeld relation

The total variation of the matter action with respect to geometrical variables is either

$$\delta S_m = \frac{1}{c} \int d\Omega \mathfrak{T}_i^a \delta e_a^i + \frac{1}{2c} \int d\Omega \mathfrak{S}_{ab}^i \delta \omega^{ab}_i \quad (2.3.27)$$

or

$$\delta S_m = \frac{1}{2c} \int d\Omega \mathcal{T}_{ik} \delta g^{ik} + \frac{1}{2c} \int d\Omega \tau_j^{ik} \delta S^j_{ik}. \quad (2.3.28)$$

Equation (1.5.25) gives

$$\begin{aligned} \frac{1}{2} \int d\Omega \tau_j^{ik} \delta S^j_{ik} &= \frac{1}{2} \int d\Omega \tau_j^{ik} \left(\delta(e_a^j e_{ib} \omega^{ab}_k) + \delta e_{i,k}^a e_a^j + e_{i,k}^a \delta e_a^j \right) \\ &= \frac{1}{2} \int d\Omega \left(\tau_j^{li} \delta(e_a^j e_{lb}) \omega^{ab}_i + \tau_{ab}^i \delta \omega^{ab}_i + (\tau_j^{ik} e_a^j \delta e_i^a)_{,k} - (\tau_j^{ik} e_a^j)_{,k} \delta e_i^a + \tau_j^{ik} e_{i,k}^a \delta e_a^j \right) \\ &= \frac{1}{2} \int d\Omega \left(\tau_j^{lk} \omega^{cb}_k e_{lb} \delta e_c^j + \tau_j^{li} \omega^{ab}_i e_a^j \delta e_{lb} + \tau_{ab}^i \delta \omega^{ab}_i - (\tau_j^{ik} e_a^j)_{,k} \delta e_i^a + \tau_j^{lm} e_{l,m}^b \delta e_b^j \right) \\ &\quad + \frac{1}{2} \int d\Omega \tau_j^{ik} e_a^j \delta e_i^a = \frac{1}{2} \int d\Omega \left(-\tau_j^{lk} \omega^{cb}_k e_{lb} e_c^j \delta e_i^a + \tau_j^{il} \omega_{al}^b e_b^j \delta e_i^a + \tau_{ab}^i \delta \omega^{ab}_i \right. \\ &\quad \left. - (\tau_a^{ik}{}_{|k} - S_{jk}^i \tau_a^{jk} - 2S_j \tau_a^{ij} + \omega_{ak}^b \tau_b^{ik}) \delta e_i^a - \tau_j^{lm} e_{l,m}^b e_b^j \delta e_i^a \right) \\ &= \frac{1}{2} \int d\Omega \left(\tau_{ab}^i \delta \omega^{ab}_i - \tau_j^{ik}{}_{;k} e_a^j \delta e_i^a + 2S_j \tau_a^{ij} \delta e_i^a \right), \end{aligned} \quad (2.3.29)$$

so comparing of (2.3.27) with (2.3.28) leads to

$$\begin{aligned} \int d\Omega \mathfrak{T}_i^a \delta e_a^i + \frac{1}{2} \int d\Omega \mathfrak{S}_{ab}^i \delta \omega^{ab}_i &= \frac{1}{2} \int d\Omega \mathcal{T}_{ik} \delta g^{ik} + \frac{1}{2} \int d\Omega \left(\tau_{ab}^i \delta \omega^{ab}_i - \tau_j^{ik}{}_{;k} e_a^j \delta e_i^a \right. \\ &\quad \left. + 2S_j \tau_a^{ij} \delta e_i^a \right) = \int d\Omega \mathcal{T}_{ik} e^{ka} \delta e_a^i + \frac{1}{2} \int d\Omega \tau_{ab}^i \delta \omega^{ab}_i + \frac{1}{2} \int d\Omega \tau_i^{jk}{}_{;k} e_j^a \delta e_a^i \\ &\quad - \int d\Omega S_j \tau_b^{kj} e_k^a e_i^b \delta e_a^i. \end{aligned} \quad (2.3.30)$$

The terms with $\delta \omega^{ab}_i$ give (2.3.22), while the terms with δe_a^i give

$$\mathfrak{T}_i^a = \mathcal{T}_{ik} e^{ka} + \frac{1}{2} \tau_i^{jk}{}_{;k} e_j^a - S_j \tau_i^{aj} \quad (2.3.31)$$

or

$$\mathcal{T}_{ik} = \mathfrak{T}_{ik} - \frac{1}{2} \nabla_j (\mathfrak{S}_{ik}^j - \mathfrak{S}_k^j{}_i + \mathfrak{S}_{ik}^j) + S_j (\mathfrak{S}_{ik}^j - \mathfrak{S}_k^j{}_i + \mathfrak{S}_{ik}^j). \quad (2.3.32)$$

Equation (2.3.32) is referred to as the *Belinfante-Rosenfeld relation* between the dynamical metric and dynamical tetrad (canonical) energy-momentum densities. In the absence of torsion, (2.3.32) is consistent with (2.3.11). The Belinfante-Rosenfeld relation can be written as

$$T_{ik} = \frac{1}{\sqrt{-g}} \mathfrak{T}_{ik} - \frac{1}{2} \nabla_j^* (s_{ik}^j - s_k^j{}_i + s_{ik}^j), \quad (2.3.33)$$

where

$$\nabla_i^* = \nabla_i - 2S_i \quad (2.3.34)$$

is the *modified covariant derivative*.

References: [2, 3, 4, 6, 8]

2.4 Symmetries and conservation laws

2.4.1 Noether theorem

We consider a physical system in the galilean frame of reference, described by the Lagrangian density $\mathfrak{L}$ that depends on matter fields ϕ_A , their first derivatives $\phi_{,i}$, and the coordinates x^i . The change of the Lagrangian density $\delta\mathfrak{L}$ under an infinitesimal coordinate transformation (1.2.55) is thus

$$\delta\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\delta(\phi_{,i}) + \frac{\bar{\partial}\mathfrak{L}}{\partial x^i}\xi^i, \quad (2.4.1)$$

where the changes $\delta\phi$ and $\delta(\phi_{,i})$ are brought by the transformation (1.2.55) and $\bar{\partial}$ denotes partial differentiation with respect to x^i at constant ϕ and $\phi_{,i}$. The variation $\delta\mathfrak{L}$ under this transformation is also given by (1.2.64):

$$\delta\mathfrak{L} = \xi^i_{,i}\mathfrak{L}. \quad (2.4.2)$$

Using the Lagrange equations (2.1.7) and the identities

$$\mathfrak{L}_{,i} = \frac{\bar{\partial}\mathfrak{L}}{\partial x^i} + \frac{\partial\mathfrak{L}}{\partial\phi}\phi_{,i} + \frac{\partial\mathfrak{L}}{\partial\phi_{,j}}\phi_{,ji}, \quad (2.4.3)$$

$$\delta(\phi_{,i}) = (\delta\phi)_{,i} - \xi^j_{,i}\phi_{,j}, \quad (2.4.4)$$

we bring (2.4.1) to

$$\delta\mathfrak{L} = \xi^i\mathfrak{L}_{,i} + \left(\frac{\partial\mathfrak{L}}{\partial\phi_{,i}}(\delta\phi - \xi^j\phi_{,j})\right)_{,i}. \quad (2.4.5)$$

Combining (2.4.2) and (2.4.5) gives the conservation law,

$$\mathfrak{J}^i_{,i} = 0, \quad (2.4.6)$$

for the current

$$\mathfrak{J}^i = \xi^i\mathfrak{L} + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}(\delta\phi - \xi^j\phi_{,j}) = \xi^i\mathfrak{L} + \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\bar{\delta}\phi = \frac{\partial\mathfrak{L}}{\partial\phi_{,i}}\delta\phi - \Theta_j^i\xi^j. \quad (2.4.7)$$

Equations (2.4.6) and (2.4.7) represent the *Noether theorem*, which states that to each continuous symmetry of a Lagrangian density there corresponds a conservation law. Generalizing (2.4.6) to an arbitrary coordinate frame gives

$$\mathfrak{J}^i_{;i} = 0. \quad (2.4.8)$$

2.4.2 Conservation of spin

The Lorentz group is the group of tetrad rotations. Since a physical matter Lagrangian density $\mathfrak{L}_m(\phi, \phi_{,i})$ is invariant under local, proper Lorentz transformations, it is invariant under tetrad rotations:

$$\delta\mathfrak{L}_m = \frac{\partial\mathfrak{L}_m}{\partial\phi}\delta\phi + \frac{\partial\mathfrak{L}_m}{\partial\phi_{,i}}\delta(\phi_{,i}) + \mathfrak{T}_i^a\delta e_a^i + \frac{1}{2}\mathfrak{S}_{ab}^i\delta\omega^{ab}_i = 0, \quad (2.4.9)$$

where the changes δ correspond to a tetrad rotation. Under integration of (2.4.9) over spacetime, the first two terms vanish because of the Lagrange equations for ϕ (2.1.7):

$$\int \left(\mathfrak{T}_i^a\delta e_a^i + \frac{1}{2}\mathfrak{S}_{ab}^i\delta\omega^{ab}_i \right) d^4x = 0. \quad (2.4.10)$$

For an infinitesimal Lorentz transformation (1.6.7), the tetrad e_i^a changes by

$$\delta e_i^a = \tilde{e}_i^a - e_i^a = \Lambda^a_b{}^i\delta\omega^{ab}_i - e_i^a = \epsilon^a_i, \quad (2.4.11)$$

and the tetrad e_a^i , because of the identity $\delta(e_i^a e_a^i) = 0$, according to

$$\delta e_a^i = -\epsilon^i_a. \quad (2.4.12)$$

The spin connection changes by

$$\delta\omega^{ab}{}_i = \delta(e^a_j \omega^{jb}{}_i) = \epsilon^a{}_j \omega^{jb}{}_i - e^a_j \epsilon^{jb}{}_{;i} = \epsilon^a{}_c \omega^{cb}{}_i - e^a_j \epsilon^{jb}{}_{|i} + \epsilon^a{}_c \omega^{bc}{}_i = -\epsilon^{ab}{}_{|i}. \quad (2.4.13)$$

Substituting (2.4.12) and (2.4.13) to (2.4.10), together with partial integration (1.2.34), gives

$$\begin{aligned} & - \int \left(\mathfrak{T}_i^a \epsilon^i{}_a + \frac{1}{2} \mathfrak{S}_{ab}{}^i \epsilon^{ab}{}_{|i} \right) d^4x = - \int \left(\mathfrak{T}_{ij} \epsilon^{ij} + \frac{1}{2} \mathfrak{S}_{ij}{}^k \epsilon^{ij}{}_{|k} \right) d^4x \\ & = \int \left(-\mathfrak{T}_{[ij]} - S_k \mathfrak{S}_{ij}{}^k + \frac{1}{2} \mathfrak{S}_{ij}{}^k{}_{;k} \right) \epsilon^{ij} d^4x. \end{aligned} \quad (2.4.14)$$

Since the infinitesimal Lorentz rotation ϵ^{ij} is arbitrary, we obtain the covariant *conservation law for the spin density*:

$$\mathfrak{S}_{ij}{}^k{}_{;k} = \mathfrak{T}_{ij} - \mathfrak{T}_{ji} + 2S_k \mathfrak{S}_{ij}{}^k \quad (2.4.15)$$

or

$$\nabla_k^* s_{ij}{}^k = \frac{1}{\sqrt{-g}} (\mathfrak{T}_{ij} - \mathfrak{T}_{ji}). \quad (2.4.16)$$

The conservation law for the spin density (2.4.16) also results from antisymmetrizing the Belinfante-Rosenfeld relation (2.3.33) with respect to the indices i, k . If we use the metric-compatible affine connection Γ_{ij}^k , which is invariant under tetrad rotations, instead of the spin connection $\omega^{ab}{}_i$ as a variable in $\mathfrak{L}_m$ then we must replace the term with $\delta\omega^{ab}{}_i$ in (2.4.9) by a term with $\delta(e^i_{a,j})$.

2.4.3 Conservation of metric energy-momentum

We consider the metric variational formulation of gravity. Under an infinitesimal coordinate transformation (1.2.55), the matter Lagrangian density $\mathfrak{L}_m(\phi, \phi_{,i})$ changes according to

$$\delta\mathfrak{L}_m = \frac{\partial\mathfrak{L}_m}{\partial\phi} \delta\phi + \frac{\partial\mathfrak{L}_m}{\partial\phi_{,i}} \delta(\phi_{,i}) + \frac{\partial\mathfrak{L}_m}{\partial g^{ik}} \delta g^{ik} + \frac{\partial\mathfrak{L}_m}{\partial g^{ik}{}_{,l}} \delta(g^{ik}{}_{,l}). \quad (2.4.17)$$

The matter action $S_m = \frac{1}{c} \int \mathfrak{L}_m(\phi, \phi_{,i}) d\Omega$ is a scalar, so it does not change under this transformation:

$$\delta S_m = \frac{1}{c} \int \left(\frac{\partial\mathfrak{L}_m}{\partial\phi} \delta\phi + \frac{\partial\mathfrak{L}_m}{\partial\phi_{,i}} \delta(\phi_{,i}) + \frac{\partial\mathfrak{L}_m}{\partial g^{ik}} \delta g^{ik} + \frac{\partial\mathfrak{L}_m}{\partial g^{ik}{}_{,l}} \delta(g^{ik}{}_{,l}) \right) d\Omega = 0. \quad (2.4.18)$$

The first two terms in (2.4.18) vanish because of the Lagrange equations for ϕ (2.1.7), so

$$\delta S_m = \frac{1}{c} \int \left(\frac{\partial\mathfrak{L}_m}{\partial g^{ik}} - \partial_l \frac{\partial\mathfrak{L}_m}{\partial g^{ik}{}_{,l}} \right) \delta g^{ik} d\Omega = \frac{1}{c} \int \frac{\delta\mathfrak{L}_m}{\delta g^{ik}} \delta g^{ik} d\Omega = \frac{1}{2c} \int \mathcal{T}_{ij} \delta g^{ij} d\Omega = 0. \quad (2.4.19)$$

If the components of the metric tensor change because of an infinitesimal coordinate transformation (1.2.55) then the corresponding variation of the metric tensor is given by (1.4.41):

$$\delta g_{ij} = \bar{\delta} g_{ij} = -2\xi_{(i;j)}, \quad (2.4.20)$$

so

$$\begin{aligned} \delta S_m &= \bar{\delta} S_m = -\frac{1}{2c} \int \mathcal{T}^{ij} \bar{\delta} g_{ij} d\Omega = -\frac{1}{c} \int \mathcal{T}^{ij} \xi_{(i;j)} d\Omega = -\frac{1}{c} \int \mathcal{T}^{ij} \xi_{i;j} d\Omega \\ &= -\frac{1}{c} \int (\mathcal{T}^{ij} \xi_i)_{;j} d\Omega + \frac{1}{c} \int \mathcal{T}^{ij}{}_{;j} \xi_i d\Omega = -\frac{1}{c} \int (\mathcal{T}^{ij} \xi_i)_{;j} d\Omega + \frac{1}{c} \int \mathcal{T}^{ij}{}_{;j} \xi_i d\Omega \\ &= -\frac{1}{c} \int \mathcal{T}^{ij} \xi_i dS_j + \frac{1}{c} \int \mathcal{T}^{ij}{}_{;j} \xi_i d\Omega = 0. \end{aligned} \quad (2.4.21)$$

If the variation of the coordinates ξ^i vanishes on the boundary of the region of integration then

$$\int \mathcal{T}^{ij}{}_{;j} \xi_i d\Omega = 0, \quad (2.4.22)$$

which, for arbitrary variations ξ^i gives the covariant conservation of the metric energy-momentum density (4 equations):

$$\mathcal{T}^{ij}_{;j} = 0. \quad (2.4.23)$$

Equivalently

$$T^{ij}_{;j} = 0. \quad (2.4.24)$$

Note that vanishing of $\int \mathcal{T}^{ij} \bar{\delta} g_{ij} d\Omega$ in (2.4.21) does not imply $\mathcal{T}^{ij} = 0$, because 10 variations $\bar{\delta} g_{ij}$ are functions of 4 variations ξ^i and thus not independent.

2.4.4 Conservation of tetrad energy-momentum

The matter Lagrangian density $\mathcal{L}_m$ is invariant under infinitesimal translations of the coordinate system (1.2.55). The corresponding changes of the tetrad and spin connection are given by Lie derivatives

$$\bar{\delta} e_a^i = -\mathcal{L}_\xi e_a^i = \xi^j_{;j} e_a^i - \xi^j e_{a,j}^i, \quad (2.4.25)$$

$$\bar{\delta} \omega_i^{ab} = -\mathcal{L}_\xi \omega_i^{ab} = -\xi^j_{;i} \omega_j^{ab} - \xi^j \omega_{i,j}^{ab}. \quad (2.4.26)$$

Equation (2.4.10) becomes now

$$\int \left(\mathfrak{T}_i^a \bar{\delta} e_a^i + \frac{1}{2} \mathfrak{S}_{ab}^i \bar{\delta} \omega_i^{ab} \right) d^4x = 0. \quad (2.4.27)$$

Substituting (2.4.25) and (2.4.26) into (2.4.27) gives

$$\begin{aligned} & \int \left(\mathfrak{T}_i^a \xi^i_{;j} e_a^j - \mathfrak{T}_i^a \xi^j e_{a,j}^i - \frac{1}{2} \mathfrak{S}_{ab}^i \xi^j_{;i} \omega_j^{ab} - \frac{1}{2} \mathfrak{S}_{ab}^i \xi^j \omega_{i,j}^{ab} \right) d^4x \\ &= \int \left(-\mathfrak{T}_i^j_{;j} - \mathfrak{T}_j^a e_{a,i}^j + \frac{1}{2} (\mathfrak{S}_{ab}^j \omega_i^{ab})_{;j} - \frac{1}{2} \mathfrak{S}_{ab}^j \omega_{j,i}^{ab} \right) \xi^i d^4x = 0. \end{aligned} \quad (2.4.28)$$

This equation holds for an arbitrary vector ξ^i , so we obtain

$$\begin{aligned} & \mathfrak{S}_{ab}^j \omega_{i,j}^{ab} + \mathfrak{S}_{ab}^j (\omega_{i,j}^{ab} - \omega_{j,i}^{ab}) - 2\mathfrak{T}_i^j_{;j} - 2\mathfrak{T}_j^a e_{a,i}^j \\ &= (\mathfrak{S}_{ab}^j|_j - 2S_k \mathfrak{S}_{ab}^k + \mathfrak{S}_{cb}^j \omega_{aj}^c + \mathfrak{S}_{ac}^j \omega_{bj}^c) \omega_i^{ab} - 2\mathfrak{T}_i^j_{;j} - 2\mathfrak{T}_j^a e_{a,i}^j \\ &+ \mathfrak{S}_{ab}^j (-R^{ab}_{ij} + \omega_{ci}^a \omega_{cj}^{cb} - \omega_{cj}^a \omega_i^{cb}) = 0, \end{aligned} \quad (2.4.29)$$

which reduces to

$$\begin{aligned} & (\mathfrak{S}_{ab}^j|_j - 2S_k \mathfrak{S}_{ab}^k) \omega_i^{ab} - R^{ab}_{ij} \mathfrak{S}_{ab}^j - 2\mathfrak{T}_i^j_{;j} + 4S_j \mathfrak{T}_i^j - 2\mathfrak{T}_{jk} \omega_i^{jk} + 4S^{jk}_i \mathfrak{T}_{jk} \\ &= (\mathfrak{S}_{jl}^k|_{;k} - 2S_k \mathfrak{S}_{jl}^k) \omega_i^{jl} - R^{kl}_{ij} \mathfrak{S}_{kl}^j - 2\mathfrak{T}_i^j_{;j} + 4S_j \mathfrak{T}_i^j - 2\mathfrak{T}_{jk} \omega_i^{jk} \\ &+ 4S^{jk}_i \mathfrak{T}_{jk} = 0. \end{aligned} \quad (2.4.30)$$

The conservation law for the spin density (2.4.15) brings (2.4.30) to the covariant *conservation law for the energy-momentum density*:

$$\mathfrak{T}_i^j_{;j} = 2S_j \mathfrak{T}_i^j + 2S^j_{ki} \mathfrak{T}_j^k + \frac{1}{2} \mathfrak{S}_{kl}^j R^{kl}_{ji} \quad (2.4.31)$$

or

$$\mathfrak{T}^{ij}_{;j} = C_{jk}^i \mathfrak{T}^{jk} + \frac{1}{2} \mathfrak{S}_{klj} R^{klji}. \quad (2.4.32)$$

2.4.5 Conservation laws for Lorentz group

We consider a matter Lagrangian $\mathfrak{L}_m$ for a physical system in the galilean and geodesic frame of reference, that depends on the coordinates only through a field ϕ and its first derivatives $\phi_{,i}$. Therefore

$$\partial_i \mathfrak{L}_m = \frac{\mathfrak{L}_m}{\partial \phi} \phi_{,i} + \frac{\partial \mathfrak{L}_m}{\partial \phi_{,j}} \phi_{,ji} = \partial_j \left(\frac{\partial \mathfrak{L}_m}{\partial \phi_{,j}} \right) \phi_{,i} + \frac{\partial \mathfrak{L}_m}{\partial \phi_{,j}} \phi_{,ji} = \partial_j \left(\frac{\mathfrak{L}_m}{\partial \phi_{,j}} \phi_{,i} \right), \quad (2.4.33)$$

where we use the Lagrange equations (2.1.7), from which we obtain the conservation law,

$$\theta_i^j{}_{,j} = 0, \quad (2.4.34)$$

for

$$\theta_i^j = \frac{\partial \mathfrak{L}_m}{\partial \phi_{,j}} \phi_{,i} - \delta_i^j \mathfrak{L}_m. \quad (2.4.35)$$

The conservation law (2.4.34) is a special case of (2.4.31) in the absence of torsion and spin, expressed in the Galilean and geodesic frame. The quantity (2.4.35) is a special case of the canonical energy-momentum density (2.3.14) in the absence of torsion and spin, expressed in the galilean and geodesic frame.

If x^i are Cartesian coordinates then for translations, $\xi^i = \epsilon^i = \text{const}$ and $\delta\phi = 0$, the current (2.4.7) is

$$\mathfrak{J}^i = \epsilon^i \mathfrak{L}_m - \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \epsilon^j \phi_{,j}. \quad (2.4.36)$$

The conservation law (2.4.6) gives

$$\epsilon^j \theta_j^i{}_{,i} = 0, \quad (2.4.37)$$

which gives (2.4.34) because ϵ^i are arbitrary. For Lorentz rotations, $\xi^i = \epsilon^i_j x^j$ and $\phi = \frac{1}{2} \epsilon_{ij} G^{ij} \phi$, where G^{ij} are the generators of the Lorentz group, the current (2.4.7) is

$$\mathfrak{J}^i = \epsilon^{ij} x_j \mathfrak{L}_m + \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \left(\frac{1}{2} \epsilon^{kl} G_{kl} \phi - \epsilon^{jk} x_k j \phi_{,j} \right) = \epsilon^{kl} \left(x_k \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \phi_{,l} - x_k \delta_l^i \mathfrak{L}_m + \frac{1}{2} \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} G_{kl} \phi \right). \quad (2.4.38)$$

The conservation law (2.4.6) gives

$$\epsilon^{kl} \left(\frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} \phi_{,[l} x_{k]} - \delta_{[l}^i x_{k]} \mathfrak{L}_m + \frac{1}{2} \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} G_{kl} \phi \right)_{,i}, \quad (2.4.39)$$

which, because ϵ^{kl} are arbitrary, gives

$$\mathfrak{M}_{kl}^i{}_{,i} = 0, \quad (2.4.40)$$

where

$$\mathfrak{M}_{kl}^i = x_k \theta_l^i - x_l \theta_k^i + \frac{\partial \mathfrak{L}_m}{\partial \phi_{,i}} G_{kl} \phi. \quad (2.4.41)$$

The quantity $\mathfrak{M}_{kl}^i$ is referred to as the *angular momentum density*, and is the sum,

$$\mathfrak{M}_{kl}^i = \Lambda_{kl}^i + \Sigma_{kl}^i, \quad (2.4.42)$$

of two densities: the *orbital angular momentum density*,

$$\Lambda_{kl}^i = x_k \theta_l^i - x_l \theta_k^i, \quad (2.4.43)$$

and the canonical spin density (2.3.25).

The conservation law (2.4.40) for the angular momentum density is equivalent to

$$\theta_{kl} - \theta_{lk} - \Sigma_{kl}^i{}_{,i} = 0, \quad (2.4.44)$$

which is a special case of the conservation law for the spin density (2.4.15) in the absence of torsion, expressed in the galilean and geodesic frame. The canonical energy-momentum density θ_{ik} is not symmetric. However, the quantity

$$\tau_{ik} = \theta_{ik} - \frac{1}{2} \partial_j (\Sigma_{ik}^j - \Sigma_k^j{}_i + \Sigma_{ik}^j), \quad (2.4.45)$$

is symmetric, which follows from (2.4.44), and conserved:

$$\tau_{ik} = \tau_{ki}, \quad (2.4.46)$$

$$\tau_k^i{}_{,i} = 0. \quad (2.4.47)$$

The symmetric energy-momentum density τ_{ik} corresponds to the metric dynamical energy-momentum density (2.3.3), expressed in the galilean and geodesic frame. Equation (2.4.45) is a special case of the Belinfante-Rosenfeld relation (2.3.32) in the absence of torsion, expressed in the Galilean and geodesic frame. The second term on the right-hand side of (2.4.45) has the form $\partial_j \psi^{ikj}$, where $\psi^{ikj} = -\psi^{ijk}$. Adding such term to θ^{ik} preserves the conservation law (2.4.34) and brings θ^{ik} to a symmetric form.

2.4.6 Components of energy-momentum tensor

Integrating the conservation law (2.4.34), valid in the galilean and geodesic frame of reference, over a hypersurface enclosing matter represented by τ^{ik} and using the Gauß-Stokes theorem gives

$$\oint \tau^{ik} dS_k = 0, \quad (2.4.48)$$

which gives the conservation of the *four-momentum vector*

$$P^i = \frac{1}{c} \int \tau^{ik} dS_k = \text{const.} \quad (2.4.49)$$

Choosing the volume hypersurface $dV = dS_0$ gives

$$P^i = \frac{1}{c} \int \tau^{i0} dV, \quad (2.4.50)$$

so the components $\frac{1}{c} \tau^{i0}$ form the *four-momentum density*. The component τ^{00} , referred to as the *energy density*,

$$W = \tau^{00} = \dot{\phi} \frac{\partial \mathfrak{L}_m}{\partial \dot{\phi}} - \mathfrak{L}_m, \quad (2.4.51)$$

integrated over the volume gives the time component of the four-momentum, the *energy*

$$E = cP_0, \quad cP^0 = \int \tau^{00} dV = \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - L, \quad (2.4.52)$$

where

$$L = \int \mathfrak{L}_m dV \quad (2.4.53)$$

is the Lagrange function or *Lagrangian*. Hereinafter, a dot above any quantity ϕ denotes the partial derivative of ϕ with respect to time, $\dot{\phi} = \frac{d\phi}{dt}$, and two dots above ϕ denote the second derivative of ϕ with respect to time, $\ddot{\phi} = \frac{d^2\phi}{dt^2}$. Consequently, the action of a physical system is the time integral of the Lagrangian,

$$S = \int L dt. \quad (2.4.54)$$

The components $\frac{1}{c} \tau^{\alpha 0}$, referred to as the *momentum density*, integrated over the volume give the spatial components of the four-momentum, the *momentum vector*

$$\mathbf{P} : P^\alpha = \frac{1}{c} \int \tau^{\alpha 0} dV. \quad (2.4.55)$$

Adding a total divergence $\partial_j \psi^{ikj}$ to τ^{ik} does not alter the definition of the four-momentum vector (2.4.49).

The symmetry of τ^{ik} can be written as

$$\partial_l (x^i \tau^{kl} - x^k \tau^{il}) = 0, \quad (2.4.56)$$

which upon the integration over a hypersurface enclosing matter represented by τ^{ik} and using the Gauß-Stokes theorem gives

$$\oint (x^i \tau^{kl} - x^k \tau^{il}) dS_l = 0, \quad (2.4.57)$$

which gives the conservation of the *angular momentum tensor*

$$M^{ik} = \int (x^i dP^k - x^k dP^i) = \frac{1}{c} \int (x^i \tau^{kl} - x^k \tau^{il}) dS_l = \text{const.} \quad (2.4.58)$$

Choosing the volume hypersurface $dV = dS_0$ gives

$$M^{ik} = \frac{1}{c} \int (x^i \tau^{k0} - x^k \tau^{i0}) dV. \quad (2.4.59)$$

The conservation of $M^{0\alpha}$,

$$M^{\alpha 0} = \frac{1}{c} \left(\int x^\alpha \tau^{00} dV - x^0 \int \tau^{\alpha 0} dV \right) = \frac{1}{c} \int x^\alpha \tau^{00} dV - ct P^\alpha = \text{const}, \quad (2.4.60)$$

divided by the conservation of P^0 (2.4.50), $P^0 = \text{const}$, gives

$$X^\alpha = V^\alpha t + \text{const}, \quad (2.4.61)$$

where

$$V^\alpha = \frac{c P^\alpha}{P^0} \quad (2.4.62)$$

and

$$X^\alpha = \frac{\int x^\alpha \tau^{00} dV}{\int \tau^{00} dV}. \quad (2.4.63)$$

The relation (2.4.61) describes a uniform motion of the *center of inertia*, whose coordinates are X^α , with velocity V^α . The coordinates of the center of inertia (2.4.63) are not the spatial components of a four-dimensional vector.

The conservation law (2.4.34) can be written as

$$\frac{1}{c} \frac{\partial \tau^{00}}{\partial t} + \frac{\partial \tau^{0\alpha}}{\partial x^\alpha} = 0, \quad (2.4.64)$$

$$\frac{1}{c} \frac{\partial \tau^{\alpha 0}}{\partial t} + \frac{\partial \tau^{\alpha\beta}}{\partial x^\beta} = 0. \quad (2.4.65)$$

Integrating these equations over the volume hypersurface and using the Gauß-Stokes theorem gives

$$\frac{\partial}{\partial t} \int \tau^{00} dV = -c \oint \tau^{0\alpha} df_\alpha, \quad (2.4.66)$$

$$\frac{\partial}{\partial t} \int \frac{1}{c} \tau^{\alpha 0} dV = - \oint \tau^{\alpha\beta} df_\beta, \quad (2.4.67)$$

where

$$df_\alpha = df_{0\alpha}^* \quad (2.4.68)$$

is the spatial surface element (1.1.30). The integral of a three-dimensional vector V^α over the two-dimensional surface element df_α , $\oint V^\alpha df_\alpha$, is referred to as the *flux* of this vector. Therefore the components

$$S^\alpha = c \tau^{0\alpha} \quad (2.4.69)$$

of the *energy current* $\mathbf{S}$ form, upon integrating over df_α , the *energy flux*. The components $\tau^{\alpha\beta}$ represent the *momentum current* and give, upon integrating over df_α , the *momentum flux*. The *stress tensor* is defined as

$$\sigma_{\alpha\beta} = -\tau_{\alpha\beta}. \quad (2.4.70)$$

The components of the energy-momentum tensor form the matrix

$$\tau^{ik} = \begin{pmatrix} W & \frac{\mathbf{S}}{c} \\ \frac{\mathbf{S}}{c} & -\sigma_{\alpha\beta} \end{pmatrix}. \quad (2.4.71)$$

We define the spatial *surface force* vector,

$$F^\alpha = \oint \sigma^{\alpha\beta} df_\beta. \quad (2.4.72)$$

The relations (2.4.55), (2.4.67), (2.4.70) and (2.4.72) equal the time derivative of the momentum P^α to the surface force F^α ,

$$\dot{P}^\alpha = F^\alpha. \quad (2.4.73)$$

In an arbitrary frame of reference, the metric dynamical energy-momentum density $\mathcal{T}_{ik}$ describing *isotropic* spinless matter (without a preferred direction in its rest frame) can be decomposed into the part proportional to $u_i u_k$, as in a system of particles (2.4.194), the part proportional to the *projection tensor*,

$$h_{ik} = g_{ik} - u_i u_k, \quad (2.4.74)$$

which is orthogonal to u^i ,

$$h_{ik} u^k = 0, \quad (2.4.75)$$

and parts containing covariant derivatives of u^i . The projection tensor satisfies

$$h_i^j h_j^k = h_i^k. \quad (2.4.76)$$

We assume that $\mathcal{T}_{ik}$ does not depend on derivatives of u^i . Therefore

$$T_{ik} = \epsilon u_i u_k - p h_{ik} = (\epsilon + p) u_i u_k - p g_{ik}, \quad (2.4.77)$$

where a scalar ϵ is equal to the energy density W in the locally Galilean rest frame and a scalar p is the *pressure*. In this frame $T^{ik} = \text{diag}(\epsilon, p, p, p)$ and the stress tensor $\sigma_{\alpha\beta} = -p \delta_{\alpha\beta}$, so (2.4.72) gives

$$F^\alpha = -p \oint df^\alpha = -p \oint n^\alpha df, \quad (2.4.78)$$

which states that the force per unit surface df acting on a surface is parallel, with the opposite sign, to the outward normal vector of this surface n^α , $\frac{dF^\alpha}{df} = -p n^\alpha$, and which is referred to as *Pascal's law*. The scalars ϵ and p can also be written as

$$\epsilon = T_{ik} u^i u^k, \quad (2.4.79)$$

$$p = -\frac{1}{3} T_{ik} h^{ik}. \quad (2.4.80)$$

Matter described by the tensor (2.4.77) represents an *ideal fluid*. The relation (2.4.77) can be written as

$$T_{ik} = (c\pi_i + p u_i) u_k - p g_{ik}, \quad (2.4.81)$$

where

$$\pi_i = \frac{1}{c} T_{ik} u^k = \frac{\epsilon}{c} u_i \quad (2.4.82)$$

is equal to the four-momentum density in the locally Galilean rest frame (cf. (2.4.50)):

$$\pi_i = \frac{1}{c} T_{i0}. \quad (2.4.83)$$

We also have

$$\pi_i u^i = \frac{\epsilon}{c}. \quad (2.4.84)$$

The relation between ϵ and p is referred to as the *equation of state*.

In the Galilean frame of reference, combining (1.6.116), (2.4.71) and (2.4.77) gives

$$W = \frac{\epsilon + pv^2/c^2}{1 - v^2/c^2}, \quad (2.4.85)$$

$$\mathbf{S} = \frac{(\epsilon + p)\mathbf{v}}{1 - v^2/c^2}, \quad (2.4.86)$$

$$\sigma_{\alpha\beta} = -\frac{(\epsilon + p)v_\alpha v_\beta}{c^2 - v^2} - p\delta_{\alpha\beta}. \quad (2.4.87)$$

The relation (2.4.77) gives

$$T = T^i_i = \epsilon - 3p. \quad (2.4.88)$$

The component $T_{00} = \epsilon u_0^2 + p(u_0^2 - g_{00})$ is, using $u_0 = \frac{g_{00}dx^0 + g_{0\alpha}dx^\alpha}{ds}$, (1.4.97) and (1.4.98), equal to

$$T_{00} = \epsilon u_0^2 + pg_{00} \left(\frac{dl}{ds} \right)^2, \quad (2.4.89)$$

so it is positive under physical conditions $\epsilon > 0$, $p > 0$ and $g_{00} > 0$. If $\mathcal{T}_{ik}$ depends also on derivatives of u^i then matter described by the tensor (2.4.77) with the corresponding additional terms represents a *viscous fluid*.

In an arbitrary frame of reference, the tetrad dynamical energy-momentum density $\mathfrak{T}_{ik}$ describing isotropic matter with spin cannot be decomposed, because of its asymmetry, as in (2.4.77). However, it can be decomposed as in (2.4.81):

$$\mathfrak{T}_{ik} = \sqrt{-\mathfrak{g}} \left((c\mathfrak{p}_i + pu_i)u_k - pg_{ik} \right), \quad (2.4.90)$$

where

$$\mathfrak{p}_i = \frac{1}{c\sqrt{-\mathfrak{g}}} \mathfrak{T}_{ik} u^k \quad (2.4.91)$$

is the corresponding four-momentum density in the locally Galilean rest frame. The conservation law for the spin density (2.4.16) gives

$$c(\mathfrak{p}_i u_j - \mathfrak{p}_j u_i) = \nabla_k^* s_{ij}{}^k. \quad (2.4.92)$$

Defining

$$\epsilon = c\mathfrak{p}_i u^i \quad (2.4.93)$$

and multiplying (2.4.92) by u^j gives

$$\mathfrak{p}_i = \frac{\epsilon}{c} u_i + \frac{1}{c} \nabla_k^* s_{ij}{}^k u^j. \quad (2.4.94)$$

Thus we obtain

$$\mathfrak{T}_{ij} = \sqrt{-\mathfrak{g}} (\epsilon u_i u_j - ph_{ij} + \nabla_k^* s_{il}{}^k u^l u_j), \quad (2.4.95)$$

which, using the Belinfante-Rosenfeld relation (2.3.33), gives the metric dynamical energy-momentum tensor for isotropic matter with spin:

$$T_{ij} = \epsilon u_i u_j - ph_{ij} + \nabla_k^* s_{il}{}^k u^l u_j - \frac{1}{2} \nabla_k^* (s_{ij}{}^k + 2s^k{}_{(ij)}). \quad (2.4.96)$$

Substituting (2.4.94) into (2.4.92) gives the dynamical equation for the spin tensor:

$$\nabla_k^* s_{ij}{}^k - \nabla_k^* s_{il}{}^k u^l u_j + \nabla_k^* s_{jl}{}^k u^l u_i = 0. \quad (2.4.97)$$

If $s_{ijk} = 0$ then (2.4.94) gives $\mathbf{p}_i \propto u_i$. Multiplying (2.4.97) by u^j gives the identity, so any 3 components of (2.4.97) are linear combinations of the other components. Thus we can impose 3 constraints on $s_{ij}^{\quad k}$.

The spin density $\mathfrak{S}_{ijk}$ describing isotropic matter with spin can be decomposed as (cf. (2.4.180))

$$\mathfrak{S}_{ijk} = \sqrt{-g} s_{ij} u_k, \quad (2.4.98)$$

where

$$s_{ij} = \frac{1}{\sqrt{-g}} \mathfrak{S}_{ijk} u^k. \quad (2.4.99)$$

The 3 constraints on s_{ijk} , which are now the 3 constraints on s_{ij} , can be taken as $s_{ij} u^j = 0$ (cf. (2.4.182)). Substituting (2.4.90) and (2.4.98) into the conservation law for the energy-momentum density (2.4.32) gives

$$(c\mathbf{p}^i u^j)_{;j} + (pu^i u^j - pg^{ij})_{;j} = C_{jk}^{\quad i} c\mathbf{p}^j u^k + \frac{1}{2} s_{kl} u_j R^{klji}. \quad (2.4.100)$$

If $p = 0$ then integrating (2.4.100) over the volume gives

$$\frac{cD^{\{P^i}}{ds} = C_{jk}^{\quad i} cP^j u^k + \frac{1}{2} S_{kl} u_j R^{klji}, \quad (2.4.101)$$

where

$$P^i = \int \mathbf{p}^i dV, \quad S^{ij} = \int s^{ij} dV. \quad (2.4.102)$$

2.4.7 Mass and Papapetrou equations of motion

We consider matter which is distributed over a small region in space and consists of points with the coordinates x^i , forming an extended body whose motion is represented by a world tube in spacetime. The motion of the body as a whole is represented by an arbitrary timelike world line γ inside the world tube, which consists of points with the coordinates $X^i(\tau)$, where τ is the proper time on γ . We define

$$\delta x^i = x^i - X^i, \quad \delta x^0 = 0, \quad u^i = \frac{dX^i}{ds}. \quad (2.4.103)$$

Also define the following integrals:

$$M^{ik} = u^0 \int \mathfrak{T}^{ik} dV, \quad (2.4.104)$$

$$M^{ijk} = -u^0 \int \delta x^i \mathfrak{T}^{jk} dV, \quad (2.4.105)$$

$$N^{ijk} = u^0 \int \mathfrak{S}^{ijk} dV, \quad (2.4.106)$$

$$J^{ik} = \int (\delta x^i \mathfrak{T}^{k0} - \delta x^k \mathfrak{T}^{i0} + \mathfrak{S}^{ik0}) dV = \frac{1}{u^0} (M^{ki0} - M^{ik0} + N^{ik0}). \quad (2.4.107)$$

The quantity J^{ik} is equal to $\int (\delta x^i \mathfrak{T}^{kl} - \delta x^k \mathfrak{T}^{il} + \mathfrak{S}^{ikl}) dS_l$ taken for the volume hypersurface, so it is a tensor, which we call the *total spin tensor*. The quantity N^{ijk} is also a tensor. The relation $\delta x^0 = 0$ gives

$$M^{0jk} = 0. \quad (2.4.108)$$

We assume that the dimensions of the body are small, so integrals with two or more factors δx^i multiplying $\mathfrak{T}^{jk}$ and integrals with one or more factors δx^i multiplying $\mathfrak{S}^{jkl}$ can be neglected.

The conservation law for the tetrad energy-momentum density (2.4.32) is

$$\mathfrak{T}^{ji}_{\quad ;i} + \{^j_{ik}\} \mathfrak{T}^{ik} - C_{ik}^{\quad j} \mathfrak{T}^{ik} - \frac{1}{2} R_{ikl}^{\quad j} \mathfrak{S}^{ikl} = 0. \quad (2.4.109)$$

Integrating (2.4.109) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int \mathfrak{T}^{j0}{}_{,0} dV + \int \{\mathfrak{T}^j_{ik}\} \mathfrak{T}^{ik} dV - \int C_{ik}{}^j \mathfrak{T}^{ik} dV - \frac{1}{2} \int R_{ikl}{}^j \mathfrak{S}^{ikl} dV = 0. \quad (2.4.110)$$

Expanding

$$\Gamma_{jk}^i = \Gamma_{jk}^{i(0)} + \Gamma_{jk,l}^{i(0)} \delta x^l, \quad (2.4.111)$$

where the superscripts 0 denote the values at X^i , and substituting these expressions into (2.4.110) gives (omitting the superscripts)

$$\begin{aligned} & \left(\int \mathfrak{T}^{j0} dV \right)_{,0} + \{\mathfrak{T}^j_{ik}\} \int \mathfrak{T}^{ik} dV + \{\mathfrak{T}^j_{ik}\}_{,l} \int \delta x^l \mathfrak{T}^{ik} dV - C_{ik}{}^j \int \mathfrak{T}^{ik} dV \\ & - C_{ik,l}{}^j \int \delta x^l \mathfrak{T}^{ik} dV - \frac{1}{2} \int R_{ikl}{}^j \mathfrak{S}^{ikl} dV = 0 \end{aligned} \quad (2.4.112)$$

or, using the definitions (2.4.104), (2.4.105) and (2.4.106),

$$\frac{d}{ds} \left(\frac{M^{j0}}{u^0} \right) + \{\mathfrak{T}^j_{ik}\} M^{(ik)} - \{\mathfrak{T}^j_{ik}\}_{,l} M^{l(ik)} - C_{ik}{}^j M^{[ik]} + C_{ik,l}{}^j M^{l[ik]} - \frac{1}{2} R_{ikl}{}^j N^{ikl} = 0. \quad (2.4.113)$$

The conservation law (2.4.109) gives

$$(x^l \mathfrak{T}^{ji})_{,i} = \mathfrak{T}^{jl} - x^l \{\mathfrak{T}^j_{ik}\} \mathfrak{T}^{ik} + x^l C_{ik}{}^j \mathfrak{T}^{ik} + \frac{1}{2} x^l R_{ikm}{}^j \mathfrak{S}^{ikm}, \quad (2.4.114)$$

$$\begin{aligned} (x^l x^m \mathfrak{T}^{ji})_{,i} &= x^m \mathfrak{T}^{jl} + x^l \mathfrak{T}^{jm} - x^l x^m \{\mathfrak{T}^j_{ik}\} \mathfrak{T}^{ik} + x^l x^m C_{ik}{}^j \mathfrak{T}^{ik} \\ &+ \frac{1}{2} x^l x^m R_{ikn}{}^j \mathfrak{S}^{ikn}. \end{aligned} \quad (2.4.115)$$

Integrating (2.4.114) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int (x^l \mathfrak{T}^{j0})_{,0} dV = \int \mathfrak{T}^{jl} dV - \int x^l \{\mathfrak{T}^j_{ik}\} \mathfrak{T}^{ik} dV + \int x^l C_{ik}{}^j \mathfrak{T}^{ik} dV + \frac{1}{2} \int x^l R_{ikm}{}^j \mathfrak{S}^{ikm} dV. \quad (2.4.116)$$

Substituting (2.4.103) into (2.4.116) gives

$$\begin{aligned} & \frac{u^l}{u^0} \int \mathfrak{T}^{j0} dV + X^l \int \mathfrak{T}^{j0}{}_{,0} dV + \int (\delta x^l \mathfrak{T}^{j0})_{,0} dV = \int \mathfrak{T}^{jl} dV - X^l \int \{\mathfrak{T}^j_{ik}\} \mathfrak{T}^{ik} dV \\ & - \int \delta x^l \{\mathfrak{T}^j_{ik}\} \mathfrak{T}^{ik} dV + X^l \int C_{ik}{}^j \mathfrak{T}^{ik} dV + \int \delta x^l C_{ik}{}^j \mathfrak{T}^{ik} dV \\ & + \frac{1}{2} X^l \int R_{ikm}{}^j \mathfrak{S}^{ikm} dV, \end{aligned} \quad (2.4.117)$$

which reduces, due to (2.4.110), to

$$\frac{u^l}{u^0} \int \mathfrak{T}^{j0} dV + \left(\int (\delta x^l \mathfrak{T}^{j0} dV) \right)_{,0} = \int \mathfrak{T}^{jl} dV - \int \delta x^l \{\mathfrak{T}^j_{ik}\} \mathfrak{T}^{ik} dV + \int \delta x^l C_{ik}{}^j \mathfrak{T}^{ik} dV. \quad (2.4.118)$$

Substituting (2.4.111) into (2.4.118), omitting the superscripts and using the definitions (2.4.104), (2.4.105) and (2.4.106), turns (2.4.118) into

$$\frac{u^l}{u^0} M^{j0} - \frac{d}{ds} \left(\frac{M^{lj0}}{u^0} \right) = M^{jl} + \{\mathfrak{T}^j_{ik}\} M^{lik} - C_{ik}{}^j M^{lik}. \quad (2.4.119)$$

Putting $l = 0$ in (2.4.119) gives the identity because of (2.4.108).

Integrating (2.4.115) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\begin{aligned} \int (x^l x^m \mathfrak{T}^{j0})_{,0} dV &= \int x^m \mathfrak{T}^{jl} dV + \int x^l \mathfrak{T}^{jm} dV - \int x^l x^m \{^j_{ik}\} \mathfrak{T}^{ik} dV \\ &+ \int x^l x^m C_{ik}{}^j \mathfrak{T}^{ik} dV + \frac{1}{2} \int x^l x^m R_{ikn}{}^j \mathfrak{S}^{ikn} dV. \end{aligned} \quad (2.4.120)$$

Substituting (2.4.103) into (2.4.116) gives

$$\begin{aligned} X^l X^m \int \mathfrak{T}^{j0}{}_{,0} dV &+ \frac{u^l}{u^0} X^m \int \mathfrak{T}^{j0} dV + \frac{u^l}{u^0} \int \delta x^m \mathfrak{T}^{j0} dV + \frac{u^m}{u^0} X^l \int \mathfrak{T}^{j0} dV \\ &+ \frac{u^m}{u^0} \int \delta x^l \mathfrak{T}^{j0} dV + X^l \int \delta x^m \mathfrak{T}^{j0}{}_{,0} dV + X^m \int \delta x^l \mathfrak{T}^{j0}{}_{,0} dV \\ &= -X^l X^m \left(\int \{^j_{ik}\} \mathfrak{T}^{ik} dV - \int C_{ik}{}^j \mathfrak{T}^{ik} dV - \frac{1}{2} \int R_{ikl}{}^j \mathfrak{S}^{ikl} dV \right) \\ &+ X^l \left(\int \mathfrak{T}^{jm} dV - \int \delta x^m \{^j_{ik}\} \mathfrak{T}^{ik} dV + \int \delta x^m C_{ik}{}^j \mathfrak{T}^{ik} dV \right) \\ &+ X^m \left(\int \mathfrak{T}^{jl} dV - \int \delta x^l \{^j_{ik}\} \mathfrak{T}^{ik} dV + \int \delta x^l C_{ik}{}^j \mathfrak{T}^{ik} dV \right) \\ &+ \int \delta x^m \mathfrak{T}^{jl} dV + \int \delta x^l \mathfrak{T}^{jm} dV, \end{aligned} \quad (2.4.121)$$

which reduces, due to (2.4.110) and (2.4.118), to

$$\frac{u^l}{u^0} \int \delta x^m \mathfrak{T}^{j0} dV + \frac{u^m}{u^0} \int \delta x^l \mathfrak{T}^{j0} dV = \int \delta x^m \mathfrak{T}^{jl} dV + \int \delta x^l \mathfrak{T}^{jm} dV \quad (2.4.122)$$

or

$$\frac{u^l}{u^0} M^{mi0} + \frac{u^m}{u^0} M^{li0} = M^{mil} + M^{lim}. \quad (2.4.123)$$

The expressions analogous to (2.4.114) and (2.4.115) with higher multiples of x^i do not introduce new relations.

The conservation law for the angular momentum density (2.4.15) is

$$\mathfrak{S}^{ijk}{}_{,k} - \Gamma_{lk}^i \mathfrak{S}^{jlk} + \Gamma_{lk}^j \mathfrak{S}^{ilk} - 2\mathfrak{T}^{[ij]} = 0. \quad (2.4.124)$$

Integrating (2.4.124) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int \mathfrak{S}^{ij0}{}_{,0} dV - \int \Gamma_{lk}^i \mathfrak{S}^{jlk} dV + \int \Gamma_{lk}^j \mathfrak{S}^{ilk} dV - 2 \int \mathfrak{T}^{[ij]} dV = 0. \quad (2.4.125)$$

Substituting (2.4.111) into (2.4.125), omitting the superscripts and using the definitions (2.4.104) and (2.4.106), turns (2.4.125) into

$$\frac{d}{ds} \left(\frac{N^{ij0}}{u^0} \right) - \Gamma_{lk}^i N^{jlk} + \Gamma_{lk}^j N^{ilk} - 2M^{[ij]} = 0. \quad (2.4.126)$$

The conservation law (2.4.124) gives

$$(x^l \mathfrak{S}^{ijk})_{,k} = \mathfrak{S}^{ijl} + x^l \Gamma_{lk}^i \mathfrak{S}^{jlk} - x^l \Gamma_{lk}^j \mathfrak{S}^{ilk} + 2x^l \mathfrak{T}^{[ij]}. \quad (2.4.127)$$

Integrating (2.4.127) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int (x^l \mathfrak{S}^{ij0})_{,0} dV = \int \mathfrak{S}^{ijl} dV + \int x^l \Gamma_{mk}^i \mathfrak{S}^{jmk} dV - \int x^l \Gamma_{mk}^j \mathfrak{S}^{imk} dV + 2 \int x^l \mathfrak{T}^{[ij]} dV. \quad (2.4.128)$$

Substituting (2.4.103) into (2.4.128) gives

$$\begin{aligned} \frac{u^l}{u^0} \int \mathfrak{S}^{ij0} dV + X^l \int \mathfrak{S}^{ij0}{}_{,0} dV &= \int \mathfrak{S}^{ijl} dV + X^l \left(\int \Gamma_{m\ k}^i \mathfrak{S}^{jmk} dV - \int \Gamma_{m\ k}^j \mathfrak{S}^{imk} dV \right. \\ &\quad \left. + 2 \int \mathfrak{T}^{[ij]} dV \right) + 2 \int \delta x^l \mathfrak{T}^{[ij]} dV, \end{aligned} \quad (2.4.129)$$

which reduces, due to (2.4.125), to

$$\frac{u^l}{u^0} \int \mathfrak{S}^{ij0} dV = \int \mathfrak{S}^{ijl} dV + 2 \int \delta x^l \mathfrak{T}^{[ij]} dV \quad (2.4.130)$$

or

$$M^{l[ij]} = -\frac{1}{2} \left(\frac{u^l}{u^0} N^{ij0} - N^{ijl} \right). \quad (2.4.131)$$

Putting $l = 0$ in (2.4.131) gives the identity because of (2.4.108). The expressions analogous to (2.4.127) with higher multiples of x^i do not introduce new relations.

Taking the cyclic permutations of the indices i, l, m in (2.4.123), adding the first and second of these relations, and subtracting the third, gives

$$\frac{u^l}{u^0} M^{[mi]0} + \frac{u^i}{u^0} M^{[ml]0} + \frac{u^m}{u^0} M^{(li)0} = M^{l[im]} + M^{i[lm]} + M^{m(il)}. \quad (2.4.132)$$

Substituting (2.4.107) and (2.4.131) into (2.4.132) gives

$$M^{m(il)} = u^{(i} J^{l)m} + \frac{u^m}{u^0} M^{(il)0} + N^{m(il)}. \quad (2.4.133)$$

Putting $m = 0$ in (2.4.133), substituting it into (2.4.133) and using (2.4.108) gives

$$M^{m(il)} = u^{(i} J^{l)m} - \frac{u^m}{u^0} (u^{(i} J^{l)0} + N^{0(il)}) + N^{m(il)}. \quad (2.4.134)$$

Combining (2.4.131) and (2.4.134) gives

$$M^{mil} = u^{(i} J^{l)m} - \frac{u^m}{u^0} (u^{(i} J^{l)0} + N^{0(il)}) + N^{m(il)} - \frac{1}{2} \left(\frac{u^m}{u^0} N^{il0} - N^{ilm} \right). \quad (2.4.135)$$

Therefore

$$M^{(ik)0} = -(u^{(i} J^{k)0} + N^{0(ik)}). \quad (2.4.136)$$

Combining the antisymmetric part of (2.4.119) and (2.4.126) gives

$$\begin{aligned} M^{jl} - M^{lj} &= \frac{u^l}{u^0} M^{j0} - \frac{u^j}{u^0} M^{l0} - (\{i\}_k^j - C_{ik}^j) M^{lik} + (\{i\}_k^l - C_{ik}^l) M^{jik} \\ &\quad - \frac{d}{ds} \left(\frac{M^{lj0} - M^{jl0}}{u^0} \right). \end{aligned} \quad (2.4.137)$$

Using (2.4.107), (2.4.131) and (2.4.134) brings (2.4.137) to

$$\begin{aligned} -\Gamma_{i\ k}^j N^{lik} + \Gamma_{i\ k}^l N^{jik} &= \frac{u^l}{u^0} M^{j0} - \frac{u^j}{u^0} M^{l0} - \{i\}_k^j \left(u^i J^{kl} + N^{lik} - \frac{u^l}{u^0} (u^i J^{k0} + N^{0ik}) \right) \\ &\quad - \frac{1}{2} C_{ik}^j \left(\frac{u^l}{u^0} N^{ik0} - N^{ikl} \right) + \{i\}_k^l \left(u^i J^{kj} + N^{jik} - \frac{u^j}{u^0} (u^i J^{k0} + N^{0ik}) \right) \\ &\quad + \frac{1}{2} C_{ik}^l \left(\frac{u^j}{u^0} N^{ik0} - N^{ikj} \right) + \frac{d}{ds} J^{lj}, \end{aligned} \quad (2.4.138)$$

which, using $\frac{D\{\}}{ds} J^{lj} = \frac{d}{ds} J^{lj} + u^k \{i\}_k^l J^{ij} + u^k \{i\}_k^j J^{li}$, turns into

$$\frac{D\{\}}{ds} J^{lj} = \left[\frac{u^j}{u^0} M^{l0} + \frac{u^j}{u^0} \{i\}_k^l (u^i J^{k0} + N^{0ik}) - \frac{1}{2} C_{ik}^l \left(\frac{u^j}{u^0} N^{ik0} - N^{ikj} \right) + C_{ik}^l N^{jik} \right] - \left[l \leftrightarrow j \right] \quad (2.4.139)$$

or, using the four-momentum

$$P^l = \frac{1}{c} \int \mathfrak{T}^{l0} dV, \quad (2.4.140)$$

into

$$\frac{D\{\}}{ds} J^{lj} = \left[cu^j P^l + \frac{u^j}{u^0} \{^l_{ik}\} (u^i J^{k0} + N^{0ik}) - \frac{1}{2} C_{ik}^l \left(\frac{u^j}{u^0} N^{ik0} - N^{ikj} \right) + C_{ik}^l N^{jik} \right] - \left[l \leftrightarrow j \right]. \quad (2.4.141)$$

Therefore

$$\begin{aligned} \frac{D\{\}}{ds} J^{li} u_i &= cP^l + \frac{1}{u^0} \{^l_{ik}\} (u^i J^{k0} + N^{0ik}) - \frac{1}{2} C_{ik}^l \left(\frac{1}{u^0} N^{ik0} - N^{ikj} u_j \right) + C_{ik}^l N^{jik} u_j \\ &- cu^l u_j P^j - \frac{u^l u_j}{u^0} \{^j_{ik}\} (u^i J^{k0} + N^{0ik}) + \frac{1}{2} C_{ik}^j \left(\frac{u^l}{u^0} N^{ik0} - N^{ikl} \right) u_j \\ &- C_{ik}^j N^{lik} u_j, \end{aligned} \quad (2.4.142)$$

which gives with (2.4.141)

$$\begin{aligned} \frac{D\{\}}{ds} J^{lj} + u^l u_i \frac{D\{\}}{ds} J^{ji} - u^j u_i \frac{D\{\}}{ds} J^{li} &= C_{ik}^l N^{jik} + \frac{1}{2} C_{ik}^l N^{ikj} - C_{ik}^j N^{lik} \\ &- \frac{1}{2} C_{ik}^j N^{ikl} - u^j u_m \left(C_{ik}^l N^{mik} + \frac{1}{2} C_{ik}^l N^{ikm} - C_{ik}^m N^{lik} - \frac{1}{2} C_{ik}^m N^{ikl} \right) \\ &+ u^l u_m \left(C_{ik}^j N^{mik} + \frac{1}{2} C_{ik}^j N^{ikm} - C_{ik}^m N^{jik} - \frac{1}{2} C_{ik}^m N^{ikj} \right) \\ &= 2(\delta_{[n}^l \delta_{m]}^j - u^j u_{[m} \delta_{n]}^l + u^l u_{[m} \delta_{n]}^j) \left(C_{ik}^n N^{mik} + \frac{1}{2} C_{ik}^n N^{ikm} \right). \end{aligned} \quad (2.4.143)$$

This equation for J_{ik} , except for the terms with N^{ijk} , resembles (2.4.97) for $\nabla_k^* s_{ij}^k$.

Multiplying (2.4.143) by u_j gives the identity, so only 3 equations in (2.4.143) are independent. Thus 3 components of J^{ik} are arbitrary and we can impose 3 constraints on J^{ik} . A simple and natural choice is

$$J^{ik} u_k = 0, \quad (2.4.144)$$

which means that in the local rest frame $J^{\alpha 0} = 0$, so the three independent components of J^{ik} are the spatial $J^{\alpha\beta}$. Analogously to the Pauli-Lubański pseudovector (1.6.71), define the *four-spin pseudovector*

$$J^i = \frac{1}{2} e^{ijkl} u_j J_{kl}, \quad (2.4.145)$$

which is orthogonal to u^i ,

$$J^i u_i = 0. \quad (2.4.146)$$

The condition (2.4.144) gives the relation inverse to (2.4.145):

$$J^{ik} = -e^{ikjl} u_j J_l. \quad (2.4.147)$$

Differentiating (2.4.145) covariantly with respect to $\{^i_{jk}\}$ and using (2.4.143) gives

$$\begin{aligned} \frac{D\{\}}{ds} J^i &= -\frac{1}{2} e^{ijkl} \frac{D\{\}}{ds} u_j e_{klmn} u^m J^n + e^{ijkl} u_j \delta_{[n}^k \delta_{m]}^l \left(C_{pr}^n N^{mpr} + \frac{1}{2} C_{pr}^n N^{pr m} \right) \\ &= -u^i \frac{D\{\}}{ds} u^k J_k + e^{ij}_{nm} u_j \left(C_{pr}^n N^{mpr} + \frac{1}{2} C_{pr}^n N^{pr m} \right) \\ &= -u^i \frac{D\{\}}{ds} u^k J_k + \frac{D\{\}}{ds} u^i u^k J_k + e^{ij}_{nm} u_j \left(C_{pr}^n N^{mpr} + \frac{1}{2} C_{pr}^n N^{pr m} \right). \end{aligned} \quad (2.4.148)$$

Thus the covariant (with respect to the Levi-Civita connection) change of the spin pseudovector along the world line γ is the sum of the corresponding Fermi-Walker transport (with respect to the Levi-Civita connection) and a term which depends on the torsion and spin density.

In (2.4.142), the four-momentum P^l depends on terms proportional to the four-velocity u^l and terms in which the index l appears in other quantities. We define the *mass* of the system described by the energy-momentum density $\mathfrak{T}^{ik}$ as the coefficient m of u^l in the expansion for $\frac{P^l}{c}$,

$$\frac{P^l}{c} = mu^l + \text{terms not proportional to } u^l, \quad (2.4.149)$$

so

$$m = \frac{u_j}{c} P^j + \frac{u_j}{c^2 u^0} \{^j_{ik}\} (u^i J^{k0} + N^{0ik}) - \frac{u_j}{2c^2 u^0} C_{ik}{}^j N^{ik0} = \frac{u_j}{c} \Pi^j, \quad (2.4.150)$$

where

$$\Pi^j = P^j + \frac{1}{cu^0} \{^j_{ik}\} (u^i J^{k0} + N^{0ik}) - \frac{1}{2cu^0} C_{ik}{}^j N^{ik0} \quad (2.4.151)$$

is the *modified four-momentum*. Substituting (2.4.150) and (2.4.151) into (2.4.142) gives

$$\frac{D\{\}}{ds} J^{li} u_i = c\Pi^l - mc^2 u^l + u_j \left(C_{ik}^l N^{jik} + \frac{1}{2} C_{ik}{}^l N^{ikj} \right) - u_j \left(C_{ik}^j N^{lik} + \frac{1}{2} C_{ik}{}^j N^{ikl} \right), \quad (2.4.152)$$

so $\Pi^l - mcu^l = \Pi^j (\delta_j^l - u^l u_j)$ is a vector. Thus the modified four-momentum Π^i is a vector and the mass m is a scalar. Substituting (2.4.151) into (2.4.141) gives the *Papapetrou equation of motion for the spin*:

$$\frac{D\{\}}{ds} J^{lj} = cu^j \Pi^l - cu^l \Pi^j + C_{ik}^l N^{jik} + \frac{1}{2} C_{ik}{}^l N^{ikj} - C_{ik}^j N^{lik} - \frac{1}{2} C_{ik}{}^j N^{ikl}. \quad (2.4.153)$$

Putting (2.4.119), (2.4.126), (2.4.131), (2.4.134), (2.4.136) and (2.4.151) into (2.4.113) gives

$$\begin{aligned} & \frac{d}{ds} \left(c\Pi^j - \frac{1}{u^0} \{^j_{ik}\} (u^i J^{k0} + N^{0ik}) + \frac{1}{2u^0} C_{ik}{}^j N^{ik0} \right) + \{^j_{ik}\} u^k \left(c\Pi^i - \frac{1}{u^0} \{^i_{lm}\} (u^l J^{m0} + N^{0lm}) + \frac{1}{2u^0} C_{lm}{}^i N^{ilm0} \right) - \{^j_{ik}\} \frac{d}{ds} \left(\frac{M^{(ik)0}}{u^0} \right) \\ & - \{^j_{ik}\}_{,l} \left(u^{(i} J^{k)l} - \frac{u^l}{u^0} (u^{(i} J^{k)0} + N^{0(ik)}) + N^{l(ik)} \right) - \frac{1}{2} C_{ik}{}^j \frac{d}{ds} \left(\frac{N^{ik0}}{u^0} \right) \\ & - \frac{1}{2} C_{ik}{}^j (-\Gamma_{lm}^i N^{klm} + \Gamma_{lm}^k N^{ilm}) - \frac{1}{2} C_{ik}{}^j{}_{,l} \left(\frac{u^l}{u^0} N^{ik0} - N^{ikl} \right) - \frac{1}{2} R^{iklj} N_{ikl} \\ & = c \frac{D\{\}}{ds} \Pi^j - \{^j_{ik}\} \{^i_{lm}\} (u^l J^{mk} + N^{klm}) - \{^j_{ik}\}_{,m} (u^i J^{km} + N^{mik}) \\ & + \frac{1}{2} \{^j_{ik}\} C_{lm}{}^i N^{lmk} + C_{ik}{}^j \Gamma_{lm}^i N^{klm} + \frac{1}{2} C_{ik}{}^j{}_{,m} N^{ikm} - \frac{1}{2} R^{iklj} N_{ikl} = 0. \end{aligned} \quad (2.4.154)$$

Using (1.4.51) turns (2.4.154) into the *Papapetrou equation of motion for the momentum*:

$$\frac{D\{\}}{ds} \Pi^j = -\frac{1}{2c} P_{imk}^j u^i J^{mk} - \frac{1}{2c} N_{ikl} C^{ikl:j}. \quad (2.4.155)$$

If the spin density vanishes then the Einstein-Cartan gravitational field equations reduce to the Einstein-Hilbert gravitational field equations. The conservation law for the spin density (2.4.15) with the condition $\mathfrak{S}^{ijk} = 0$ gives the symmetry of the energy-momentum density, $\mathfrak{T}^{ik} = \mathfrak{T}^{ki}$. The relations (2.4.104), (2.4.105), (2.4.106) and (2.4.107) give then

$$M^{ik} = M^{ki}, \quad (2.4.156)$$

$$M^{ijk} = M^{ikj}, \quad (2.4.157)$$

$$N^{ijk} = 0, \quad (2.4.158)$$

$$J^{ik} = cL^{ik} = \int (\delta x^i \mathfrak{T}^{k0} - \delta x^k \mathfrak{T}^{i0}) dV = \frac{1}{u^0} (-M^{ik0} + M^{ki0}), \quad (2.4.159)$$

where L^{ik} is the angular momentum tensor, analogous to (2.4.58). The modified four-momentum (2.4.151) reduces to

$$\Pi^j = P^j + \frac{1}{u^0} \{^j_{ik}\} u^i L^{k0} \quad (2.4.160)$$

and (2.4.152) gives

$$\Pi^l = m c u^l + \frac{D^{\{\} L^l}{ds} u_i. \quad (2.4.161)$$

The relation (2.4.148) reduces to

$$\frac{D^{\{\} J^i}{ds} = -u^i \frac{D^{\{\} u^k}{ds} J_k + \frac{D^{\{\} u^i}{ds} u^k J_k, \quad (2.4.162)$$

so the covariant (with respect to the Levi-Civita connection) change of the spin pseudovector along the world line γ is equal to the corresponding Fermi-Walker transport. Multiplying (2.4.162) by J_i and using (2.4.146) gives

$$J^i J_i = \text{const}, \quad (2.4.163)$$

so the change of the spin pseudovector along a world line is a rotation, called *spin precession*. The Papapetrou equation of motion for the spin (2.4.153) reduces to

$$\frac{D^{\{\} L^{lj}}{ds} = u^j \Pi^l - u^l \Pi^j, \quad (2.4.164)$$

while the Papapetrou equation of motion for the momentum (2.4.155) reduces to

$$\frac{D^{\{\} \Pi^j}{ds} = -\frac{1}{2} P^j_{imk} u^i L^{mk}. \quad (2.4.165)$$

The change of the mass m along the world line γ is, using (1.4.48), (2.4.144), (2.4.150), (2.4.161) and (2.4.165),

$$\begin{aligned} \frac{dm}{ds} &= \frac{D^{\{\} m}{ds} = \frac{1}{c} u_j \frac{D^{\{\} \Pi^j}{ds} + \frac{1}{c} \Pi^j \frac{D^{\{\} u_j}{ds} = \frac{1}{c} \Pi^j \frac{D^{\{\} u_j}{ds} = \frac{1}{c} \frac{D^{\{\} L^{ji}}{ds} u_i \frac{D^{\{\} u_j}{ds} \\ &= -\frac{1}{c} L^{ji} \frac{D^{\{\} u_i}{ds} \frac{D^{\{\} u_j}{ds} = 0, \end{aligned} \quad (2.4.166)$$

so

$$m = \text{const}. \quad (2.4.167)$$

In the absence of the external gravitational field and neglecting the gravitational field of the body, the relation (2.4.160) gives

$$\Pi^j = P^j, \quad (2.4.168)$$

so (2.4.165) reduces to

$$\frac{dP^j}{ds} = 0, \quad (2.4.169)$$

whose integration gives the conservation of the four-momentum along a world line:

$$P^i = \text{const}. \quad (2.4.170)$$

The equation of motion for the spin (2.4.164) becomes

$$\frac{dL^{lj}}{ds} = u^j P^l - u^l P^j, \quad (2.4.171)$$

whose integration gives the conservation of the angular momentum along a world line:

$$L^{ik} + X^i P^k - X^k P^i = \text{const}. \quad (2.4.172)$$

The tensor L^{ik} is the intrinsic angular momentum of the body, while the tensor (in the absence of the gravitational field) $X^i P^k - X^k P^i$ is the orbital angular momentum associated with the motion of the body as a whole. If $L^{ik} = 0$ then (2.4.172) gives $P^i \propto u^i$, so (2.4.170) is equivalent to $u^i = \text{const}$ and thus X^i is a linear function of the proper time τ . If $L^{ik} \neq 0$ then X^i can be given by 3 arbitrary functions of τ (since $u^i u_i = 1$). In the *momentum rest frame*, in which $P^\alpha = 0$, $u^\alpha \neq 0$, so the body has an arbitrary internal motion. The 3 constraints (2.4.144) eliminate this arbitrariness, so the equations of motion entirely determine the motion of the body.

2.4.8 Spin tensor for particles

If the body is not spatially extended then it is referred to as a *particle*. The corresponding condition $\delta x^\alpha = 0$ gives

$$M^{ijk} = 0, \quad L^{ik} = 0. \quad (2.4.173)$$

Therefore (2.4.131) reduces to $\frac{u^l}{u^0} N^{ij0} - N^{ijl} = 0$, which with (2.4.107) gives

$$N^{ijl} = J^{ij} u^l, \quad (2.4.174)$$

so

$$J^{ij} = S^{ij} = N^{ijk} u_k, \quad (2.4.175)$$

where S^{ij} is the *intrinsic spin tensor*. If the body is spatially extended then the difference

$$R^{ik} = J^{ik} - S^{ik} \quad (2.4.176)$$

is the *rotational spin tensor*. The difference between the rotational spin tensor and angular momentum tensor is, due to (2.4.107) and (2.4.131),

$$R^{ik} - cL^{ik} = \frac{N^{ik0}}{u^0} - N^{ikl} u_l = -2M^{l[ik]} u_l. \quad (2.4.177)$$

This expression vanishes, because of (2.4.108), in the *velocity rest frame*, in which $u^\alpha = 0$, which is also locally Galilean, so $u_\alpha = 0$. The relation (2.4.152) becomes

$$c\Pi^l = mc^2 u^l + \frac{D\{\}}{ds} S^{li} u_i - u_j \left(C_{ik}^l S^{ji} u^k + \frac{1}{2} C_{ik}^l S^{ik} u^j \right) + u_j \left(C_{ik}^j S^{li} u^k + \frac{1}{2} C_{ik}^j S^{ik} u^l \right), \quad (2.4.178)$$

while the Papapetrou equation of motion for the spin (2.4.153) becomes

$$\frac{D\{\}}{ds} S^{lj} = c u^j \Pi^l - c u^l \Pi^j + C_{ik}^l S^{ji} u^k + \frac{1}{2} C_{ik}^l S^{ik} u^j - C_{ik}^j S^{li} u^k - \frac{1}{2} C_{ik}^j S^{ik} u^l. \quad (2.4.179)$$

The relation (2.4.174) implies that the spin tensor for a system of particles satisfies

$$s^{ijl} = s^{ij} u^l, \quad (2.4.180)$$

where

$$s^{ij} = s^{ijl} u_l \quad (2.4.181)$$

is the *spin-density tensor*, which is orthogonal to u^i (cf. (2.4.97) and (2.4.144)):

$$s^{ij} u_j = 0. \quad (2.4.182)$$

Thus s_{ij} has 3 independent components. A system satisfying (2.4.180) is referred to as a *spin fluid*. The spin tensor (2.4.180) is traceless, due to (2.4.182). The relation (2.4.92) gives

$$c(\mathbf{p}_i u_j - \mathbf{p}_j u_i) = u^k \nabla_k^* s_{ij} + s_{ij} \nabla_k^* u^k = \frac{D\{\}}{ds} s_{ij} + \text{other terms}, \quad (2.4.183)$$

which resembles (2.4.179) in which S^{ik} is replaced by s^{ik} and Π^i by $\mathbf{p}^i$. Moreover, (2.4.93) corresponds to (2.4.150) in which mc^2 is replaced by ϵ . Similarly, (2.4.94) gives

$$\mathbf{p}_i = \frac{\epsilon}{c}u_i + \frac{1}{c}(u^k \nabla_k^* s_{ij} + s_{ij} \nabla_k^* u^k)u^j = \frac{\epsilon}{c}u_i + \frac{u^j}{c} \frac{D\{ \}}{ds} s_{ij} + \text{other terms}, \quad (2.4.184)$$

which resembles (2.4.178). In a locally Galilean frame of reference which is also a rest frame, (2.4.182) becomes

$$s_{0\alpha} = 0. \quad (2.4.185)$$

In this frame, the 3 components of s_{ij} are spatial, $s_{\alpha\beta}$, and are equivalent to 3 components of the *spatial spin-density pseudovector*:

$$s^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} s_{\beta\gamma}. \quad (2.4.186)$$

2.4.9 Energy-momentum tensor for particles

If a particle is spinless then its four-momentum is proportional to its four-velocity due to (2.4.160) and (2.4.161):

$$P^l = mc u^l, \quad (2.4.187)$$

which gives

$$P^2 = m^2 c^2, \quad (2.4.188)$$

in agreement with (1.6.81). Equations (2.4.119), (2.4.140), (2.4.156), (2.4.158) and (2.4.187) give

$$M^{ik} = \frac{u^i}{u^0} M^{k0} = \frac{u^i u^k}{(u^0)^2} M^{00} = mc^2 u^i u^k, \quad (2.4.189)$$

so

$$\int \mathfrak{T}^{ik} dV = mc^2 \frac{u^i u^k}{u^0} \quad (2.4.190)$$

or

$$\mathfrak{T}^{ik}(\mathbf{x}) = mc^2 \delta(\mathbf{x} - \mathbf{x}_0) \frac{u^i u^k}{u^0}, \quad (2.4.191)$$

where $\delta(\mathbf{x} - \mathbf{x}_0)$ is the spatial Dirac delta representing a point mass located at $\mathbf{x}_0$. We define the *mass density* μ such that

$$\mu \sqrt{\mathfrak{s}} dV = dm, \quad (2.4.192)$$

where $\mathfrak{s}$ is given by (1.4.110). The mass density for a particle located at $\mathbf{x}_a$ is

$$\mu(\mathbf{x}) = \frac{m}{\sqrt{\mathfrak{s}}} \delta(\mathbf{x} - \mathbf{x}_a), \quad (2.4.193)$$

so (2.4.191) turns into

$$\mathfrak{T}^{ik} = \mu c^2 \sqrt{\mathfrak{s}} \frac{u^i u^k}{u^0}. \quad (2.4.194)$$

Thus the energy-momentum tensor for a spinless particle is given by

$$\begin{aligned} T^{ik}(x) &= \mu(\mathbf{x}) c^2 \frac{u^i u^k}{\sqrt{g_{00}} u^0} = \frac{\mu(\mathbf{x}) c}{\sqrt{g_{00}}} \frac{dx^i}{ds} \frac{dx^k}{dt} = mc^2 \delta(\mathbf{x} - \mathbf{x}_a) \frac{u^i u^k}{\sqrt{-g} u^0} \\ &= mc^2 \int \frac{u^i u^k}{\sqrt{-g}} \delta(x - x_a(\tau)) d\tau, \end{aligned} \quad (2.4.195)$$

where $x_a(\tau)$ is the particle's worldline as a function of its proper time τ . For a system of particles, we have

$$T^{ik}(\mathbf{x}) = \sum_a m_a c^2 \delta(\mathbf{x} - \mathbf{x}_a) \frac{u^i u^k}{\sqrt{-g} u^0}. \quad (2.4.196)$$

The Papapetrou equation of motion (2.4.165) for a spinless particle reduces to the metric geodesic equation (1.4.80),

$$\frac{D\mathfrak{U}u^i}{ds} = 0. \quad (2.4.197)$$

In the absence of torsion and in the locally Galilean frame of reference, the conservation law for the energy-momentum tensor is given by (2.4.34), so

$$T_{\alpha}{}^i{}_{,i} = 0. \quad (2.4.198)$$

We consider a closed system of particles which carry out a finite motion, in which all quantities vary over finite ranges. We define the average over a certain time interval τ of a function f of these quantities as $\bar{f} = \frac{1}{\tau} \int_0^\tau f dt$. The average of the derivative of a bounded quantity $\bar{f} = \frac{1}{\tau} (f(\tau) - f(0)) \rightarrow 0$ as $\tau \rightarrow \infty$. Thus averaging (2.4.198) over the time gives

$$\bar{T}_{\alpha}{}^{\beta}{}_{,\beta} = 0. \quad (2.4.199)$$

Multiplying (2.4.199) by x^α and integrating over the volume gives, omitting surface integrals,

$$\int x^\alpha \bar{T}_{\alpha}{}^{\beta}{}_{,\beta} dV = - \int \bar{T}_{\alpha}{}^{\alpha} dV = 0. \quad (2.4.200)$$

The average energy of the system (2.4.52) is thus

$$\bar{E} = \int \bar{T}_0{}^0 dV = \int \bar{T}_i{}^i dV. \quad (2.4.201)$$

Substituting (1.6.116) into (2.4.196) gives

$$T_i{}^i(\mathbf{x}) = \sum_a m_a c^2 \delta(\mathbf{x} - \mathbf{x}_a) \left(1 - \frac{v^2}{c^2}\right)^{1/2}, \quad (2.4.202)$$

so $T_i{}^i \geq 0$. Putting (2.4.202) into (2.4.201) gives

$$\bar{E} = \sum_a m_a c^2 \overline{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}, \quad (2.4.203)$$

which is referred to as the *virial theorem*.

Comparing (2.4.88) with (2.4.202) gives

$$\epsilon - 3p = \sum_a m_a c^2 \left(1 - \frac{v^2}{c^2}\right)^{1/2}, \quad (2.4.204)$$

where the summation extends over all particles in unit volume, so $p \leq \epsilon/3$. In the nonrelativistic limit $p \approx 0$, while in the ultrarelativistic limit ($v \sim c$) $p \approx \epsilon/3$. We consider a system of noninteracting identical particles of mass m , which we call an *ideal gas*, with the number of particles in unit volume (*concentration*) n , so

$$\mu = nm. \quad (2.4.205)$$

Comparing (2.4.77) in the locally Galilean rest frame with (2.4.195) gives the kinetic formulae for ideal gases:

$$\epsilon = nm c^2 \bar{\gamma}, \quad (2.4.206)$$

$$p = \frac{nm}{3} \bar{\gamma} v^2. \quad (2.4.207)$$

In a locally inertial frame of reference, (1.6.116) and (2.4.187) give

$$P^i = mc\gamma \left(1, \frac{\mathbf{v}}{c}\right), \quad (2.4.208)$$

so the energy and momentum of the particle are

$$E = mc^2\gamma, \quad (2.4.209)$$

$$\mathbf{P} = m\gamma\mathbf{v}. \quad (2.4.210)$$

Thus (2.4.188) gives

$$E^2 = (Pc)^2 + (mc^2)^2. \quad (2.4.211)$$

In the rest frame of the particle, $\mathbf{P} = 0$, (2.4.211) reduces to Einstein's formula for the *rest energy*,

$$E = mc^2. \quad (2.4.212)$$

The formulae (2.4.209) and (2.4.210) give

$$\mathbf{v} = \frac{\mathbf{P}c^2}{E}. \quad (2.4.213)$$

Taking the differential of (2.4.211) gives $E dE = c^2 \mathbf{P} \cdot d\mathbf{P}$, from which we obtain, using (2.4.213),

$$dE = \mathbf{v} \cdot d\mathbf{P}. \quad (2.4.214)$$

If a particle is massless, $m = 0$, then (2.4.211) and (2.4.213) give

$$E = Pc, \quad v = c. \quad (2.4.215)$$

References: [2, 3, 4, 5, 6, 8, 9].

2.5 Gravitational field equations

2.5.1 Einstein-Hilbert action and Einstein equations

The *Einstein-Hilbert action* for the gravitational field and matter is, due to (2.2.2),

$$S = -\frac{1}{2\kappa c} \int P \sqrt{-\mathbf{g}} d\Omega + S_m, \quad (2.5.1)$$

where the metric tensor is regarded as a variational variable and the affine connection is the Levi-Civita connection. Varying (2.5.1) with respect to the metric tensor gives, using (2.3.1) and the identity $\delta \sqrt{-\mathbf{g}} = -\frac{1}{2} \sqrt{-\mathbf{g}} g_{ik} \delta g^{ik}$ (which results from $\delta \mathbf{g} = \mathbf{g} g^{ik} \delta g_{ik} = -\mathbf{g} g_{ik} \delta g^{ik}$),

$$\delta S = -\frac{1}{2\kappa c} \int \left(\delta P_{ik} g^{ik} \sqrt{-\mathbf{g}} + P_{ik} \delta g^{ik} \sqrt{-\mathbf{g}} - \frac{1}{2} P \sqrt{-\mathbf{g}} g_{ik} \delta g^{ik} \right) d\Omega + \frac{1}{2c} \int T_{ik} \sqrt{-\mathbf{g}} \delta g^{ik} d\Omega. \quad (2.5.2)$$

Partial integration of the first term on the right-hand side of (2.5.2), using (1.4.57), brings this term to zero:

$$\int \delta P_{ik} \mathbf{g}^{ik} d\Omega = \int \left((\delta \{^l_{ik}\})_{:l} - (\delta \{^l_{il}\})_{:k} \right) \mathbf{g}^{ik} d\Omega = - \int (\mathbf{g}^{ik}_{:l} \delta \{^l_{ik}\} - \mathbf{g}^{ik}_{:k} \delta \{^l_{il}\}) d\Omega = 0, \quad (2.5.3)$$

where

$$\mathbf{g}^{ik} = \sqrt{-\mathbf{g}} g^{ik} \quad (2.5.4)$$

is the contravariant *metric density*, whose covariant derivative with respect to the Christoffel symbols vanishes, $\mathbf{g}^{ik}_{:l} = 0$. Equating $\delta S = 0$ in (2.5.2) gives the *Einstein equations* of the general theory of relativity:

$$G_{ik} = \kappa T_{ik} \quad (2.5.5)$$

or

$$P_{ik} = \kappa \left(T_{ik} - \frac{1}{2} T g_{ik} \right). \quad (2.5.6)$$

Because $\delta \int P \sqrt{-g} d\Omega = \delta \int G \sqrt{-g} d\Omega$, where the noncovariant quantity G is given by (2.2.5), the left-hand side of the Einstein equations is

$$G_{ik} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}G)}{\partial g^{ik}}. \quad (2.5.7)$$

The covariant conservation of the Einstein tensor (1.4.67) imposes the conservation of the metric dynamical energy-momentum tensor (2.4.23). Therefore the gravitational field equations contain the equations of motion of matter. In *vacuum*, where $T_{ik} = 0$, the Ricci tensor in (2.5.6) vanishes:

$$P_{ik} = 0. \quad (2.5.8)$$

Thus vanishing of P_{ik} at a given point in spacetime is a covariant criterion of whether matter is present or absent at this point.

The Einstein equations (2.5.5) are 10 second-order partial differential equations for: $10 - 4 = 6$ independent components of the metric tensor g_{ik} (the factor 4 is the number of the coordinates which can be chosen arbitrarily), 3 independent components of the four-velocity u^i , and either ϵ or p (which are related to each other by the equation of state). The contracted Bianchi identity (1.4.67) gives the equations of motion of matter. In vacuum, the Einstein equations are $10 - 4 = 6$ independent equations (the factor 4 is the number of constraints from the contracted Bianchi identity) for 6 independent components of the metric tensor g_{ik} .

In the Einstein equations, the only second time-derivatives of g_{ik} are the derivatives of the spatial components of the metric tensor, $\ddot{g}_{\alpha\beta}$, and they appear only in the $\alpha\beta$ components of the field equations (2.5.5). Therefore the initial values (at $t = 0$) for $g_{\alpha\beta}$ and $\dot{g}_{\alpha\beta}$ can be chosen arbitrarily. The first time-derivatives $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ appear only in the $\alpha\beta$ components of the field equations (2.5.5). The 0α and 00 components of the field equations (2.5.5) give the initial values for $g_{0\alpha}$ and g_{00} . The undetermined initial values for $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ correspond to 4 degrees of freedom for a free gravitational field. A general gravitational field has 8 degrees of freedom: 4 degrees of freedom for a free gravitational field, 3 related to the four-velocity, and 1 related to ϵ (or p).

2.5.2 Einstein pseudotensor and principle of equivalence

We define

$$\mathcal{G} = \sqrt{-g}G, \quad (2.5.9)$$

where G is the noncovariant quantity (2.2.5). The action for the gravitational field and matter,

$$S = -\frac{1}{2\kappa c} \int \mathcal{G} d\Omega + S_m = \frac{1}{c} \int \left(-\frac{1}{2\kappa} \mathcal{G} + \mathfrak{L}_m \right) d\Omega, \quad (2.5.10)$$

produces the Einstein field equations by varying the metric tensor, because $\mathcal{G}$ differs from $\sqrt{-g}P$ by a total divergence:

$$\frac{\delta}{\delta g^{ik}} \left(-\frac{1}{2\kappa} \mathcal{G} + \mathfrak{L}_m \right) = 0. \quad (2.5.11)$$

Construct a canonical energy-momentum density (2.4.35) corresponding to the gravitational field, treating $-\frac{1}{2\kappa} \mathcal{G}$ (which depends only on g^{ij} and its first derivatives $g^{ij}_{,k}$) like $\mathfrak{L}_m$ and g^{ik} like a matter field ϕ :

$$\mathfrak{t}_k^i = -\frac{1}{2\kappa} \left(\frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} g^{jl}_{,k} - \delta_k^i \mathfrak{G} \right). \quad (2.5.12)$$

This quantity is not a tensor density since $\mathfrak{G}$ is not a scalar density and its division by $\sqrt{-g}$, $\frac{\mathfrak{t}_k^i}{\sqrt{-g}}$, is referred to as the *Einstein energy-momentum pseudotensor* for the gravitational field. The four-momentum corresponding to the total energy-momentum density for the gravitational field and matter (which is not a vector) is then

$$P_i = \frac{1}{c} \int (\mathfrak{t}_i^k + \mathcal{T}_i^k) dS_k, \quad (2.5.13)$$

where the sum $\mathfrak{t}_i^k + \mathcal{T}_i^k$ is called the *Einstein energy-momentum complex*. The definition (2.5.12) gives

$$\begin{aligned} 2\kappa \mathfrak{t}_{k,i}^i &= \partial_i \frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} g^{jl}_{,k} - \frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} g^{jl}_{,ki} + \mathfrak{G}_{,k} = \partial_i \frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} g^{jl}_{,k} - \frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} g^{jl}_{,ki} + \frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} g^{jl}_{,k} \\ &+ \frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} g^{jl}_{,ik} = \left(\frac{\partial \mathfrak{G}}{\partial g^{jl}} - \partial_i \frac{\partial \mathfrak{G}}{\partial g^{jl}_{,i}} \right) g^{jl}_{,k} = \frac{\delta \mathfrak{G}}{\delta g^{jl}} g^{jl}_{,k}, \end{aligned} \quad (2.5.14)$$

which, using (2.5.11), gives

$$\mathfrak{t}_{k,i}^i = \frac{\delta \mathfrak{L}_m}{\delta g^{jl}} g^{jl}_{,k} = \frac{1}{2} \mathcal{T}_{jl} g^{jl}_{,k}. \quad (2.5.15)$$

The covariant conservation (2.4.23) gives

$$\mathcal{T}_{k,i}^i = \{^l_{ki}\} \mathcal{T}_l^i = \frac{1}{2} g^{lm} g_{im,k} \mathcal{T}_l^i = -\frac{1}{2} g^{lm}_{,k} \mathcal{T}_{lm}, \quad (2.5.16)$$

so the total energy-momentum density for the gravitational field and matter is ordinarily conserved:

$$(\mathfrak{t}_k^i + \mathcal{T}_k^i)_{,i} = 0. \quad (2.5.17)$$

Integrating (2.5.17) over the four-dimensional volume and using the Gauß-Stokes theorem gives

$$\oint (\mathfrak{t}_k^i + \mathcal{T}_k^i) dS_i = 0, \quad (2.5.18)$$

so the four-momentum (2.5.13) is conserved, $P_i = \text{const.}$ Because the quantity $\mathfrak{t}_{ik}$ is not symmetric in the indices i, k , the total angular momentum constructed from P^i as in (2.4.58),

$$M^{ik} = \int (x^i dP^k - x^k dP^i) = \frac{1}{c} \int (x^i (\mathfrak{t}^{kl} + \mathcal{T}^{kl}) - x^k (\mathfrak{t}^{il} + \mathcal{T}^{il})) dS_l, \quad (2.5.19)$$

is not conserved. The conservation law (2.5.17) gives $\mathfrak{t}_k^l + \mathcal{T}_k^l = \eta_k^{li}_{,i}$, where $\eta^{kli} = -\eta^{kil}$, so $\mathfrak{t}_k^l - \mathfrak{t}_k^l = (\eta_k^{li} - \eta_k^{li})_{,i}$. Analogously to (2.4.44) and (2.4.45), we could bring $\mathfrak{t}_k^l + \mathcal{T}_k^l$ to a symmetric form. However, using $(\eta_k^{li} - \eta_k^{li})$ instead of Σ_k^{li} in (2.4.45), where θ_i^k is replaced by $\mathfrak{t}_i^k + \mathcal{T}_i^k$, gives $\tau_i^k = 0$, so this symmetrization procedure does not work for the Einstein pseudotensor.

The Einstein pseudotensor (2.5.12) can be explicitly written as

$$\mathfrak{t}_k^i = \frac{1}{2\kappa} (\{^i_{lm}\} \mathfrak{g}^{lm}_{,k} - \{^l_{ml}\} \mathfrak{g}^{mi}_{,k} + \delta_k^i \mathfrak{G}), \quad (2.5.20)$$

so it is a homogeneous quadratic function of the Christoffel symbols. Thus it vanishes in the local Galilean frame of reference. It can also differ from zero in the Minkowski spacetime (in the absence of the gravitational field) if we choose the coordinates such that the Christoffel symbols do not vanish. Therefore the energy of the gravitational field is not absolutely localized in spacetime; it depends on the choice of the coordinates. The gravitational field can be always eliminated locally by transforming the coordinate system to the local Galilean frame of reference in which the Einstein pseudotensor vanishes. This property of the gravitational field is referred to as the *principle of equivalence*.

The construction of a conserved four-momentum for the gravitational field and matter is possible because the Lagrangian density for the gravitational field $\mathfrak{L}_g$ is linear in the second derivatives of the metric tensor. The Lagrangian density (2.2.2) can be generalized to

$$\mathfrak{L}_g = -\frac{1}{2\kappa} \sqrt{-\mathfrak{g}} (P + 2\Lambda), \quad (2.5.21)$$

where Λ is referred to as the *cosmological constant*, without altering the Einstein energy-momentum pseudotensor (2.5.12). Another scalar density which is linear in curvature is $\epsilon^{ijkl} P_{ijkl}$, but this parity-violating expression vanishes due to the cyclic identity (1.4.64). Therefore the simplest choice for

a gravitational Lagrangian density, linear in P , is the only one which admits ordinary conservation laws for the gravitational field and matter, and thus it is physical. Other possible gravitational Lagrangians can contain terms which depend explicitly on the torsion tensor, but such modifications of general relativity or the Einstein-Cartan gravity would involve free parameters and thus they would be less fundamental.

2.5.3 Landau-Lifshitz energy-momentum pseudotensor

The covariant conservation (2.4.23) in the local Galilean frame of reference is

$$\mathcal{T}_k{}^i{}_{,i} = 0, \quad (2.5.22)$$

so $\mathcal{T}^{ik}$ can be expressed as $\mathcal{T}^{ik} = \eta^{ikl}{}_{,l}$, where $\eta^{ikl} = -\eta^{ilk}$. The Einstein equations (2.5.5) in the Galilean frame are

$$(-\mathbf{g})T^{ik} = h^{ikl}{}_{,l}, \quad (2.5.23)$$

where

$$h^{ikl} = \lambda^{iklm}{}_{,m} = -h^{ilk}, \quad (2.5.24)$$

$$\lambda^{iklm} = \frac{1}{2\kappa}(-\mathbf{g})(g^{ik}g^{lm} - g^{il}g^{km}). \quad (2.5.25)$$

In an arbitrary frame of reference, (2.5.23) is not valid. We define t^{ik} such that

$$(-\mathbf{g})(t^{ik} + T^{ik}) = h^{ikl}{}_{,l}. \quad (2.5.26)$$

Therefore

$$((- \mathbf{g})(t^{ik} + T^{ik}))_{,k} = 0, \quad (2.5.27)$$

so there is a conservation of the four-momentum of the gravitational field and matter,

$$P^i = \frac{1}{c} \int (-\mathbf{g})(t^{ik} + T^{ik}) dS_k. \quad (2.5.28)$$

The quantity t^{ik} is not a tensor density, so the conserved four-momentum P^i (2.5.28) is not a vector. The four-momentum P^i is not a vector even for Lorentz transformations, because of the factor $-\mathbf{g}$ instead of the correct (weight 1) density $\sqrt{-\mathbf{g}}$ in (2.5.28). Dividing P^i by $\sqrt{-\mathbf{g}}$ at some fixed point (a natural choice is infinity) turns it into a vector under Lorentz transformations. Using (2.5.26) turns (2.5.28), for the hypersurface $dS_0 = dV$, into

$$P^i = \frac{1}{c} \int h^{ikl}{}_{,l} dS_k = \frac{1}{2c} \oint h^{ikl} df_{kl}^* = \frac{1}{c} \oint h^{i0\alpha} df_\alpha. \quad (2.5.29)$$

The quantity t^{ik} is referred to as the *Landau-Lifshitz energy-momentum pseudotensor* for the gravitational field, and the sum $(-\mathbf{g})(t^{ik} + T^{ik})$ is called the *Landau-Lifshitz complex*.

The explicit expression for the Landau-Lifshitz pseudotensor is

$$\begin{aligned} t^{ik} = & \frac{1}{2\kappa} \left((g^{il}g^{km} - g^{ik}g^{lm})(2\{l_m^n\}\{n_p^p\} - \{l_p^n\}\{m_n^p\} - \{l_n^n\}\{m_p^p\}) \right. \\ & + g^{il}g^{mn}(\{l_p^k\}\{m_n^p\} + \{m_n^k\}\{l_p^p\} - \{n_p^k\}\{l_m^p\} - \{l_m^k\}\{n_p^p\}) \\ & + g^{kl}g^{mn}(\{l_p^i\}\{m_n^p\} + \{m_n^i\}\{l_p^p\} - \{n_p^i\}\{l_m^p\} - \{l_m^i\}\{n_p^p\}) \\ & \left. + g^{lm}g^{np}(\{l_n^i\}\{m_p^k\} - \{l_m^i\}\{n_p^k\}) \right) \end{aligned} \quad (2.5.30)$$

or

$$\begin{aligned} (-\mathbf{g})t^{ik} = & \frac{1}{2\kappa} \left(\mathbf{g}^{ik}{}_{,l} \mathbf{g}^{lm}{}_{,m} - \mathbf{g}^{il}{}_{,l} \mathbf{g}^{km}{}_{,m} + \frac{1}{2} g^{ik} g_{lm} \mathbf{g}^{ln}{}_{,p} \mathbf{g}^{pm}{}_{,n} \right. \\ & - (g^{il} g_{mn} \mathbf{g}^{kn}{}_{,p} \mathbf{g}^{mp}{}_{,l} + g^{kl} g_{mn} \mathbf{g}^{in}{}_{,p} \mathbf{g}^{mp}{}_{,l}) + g_{lm} g^{np} \mathbf{g}^{il}{}_{,n} \mathbf{g}^{km}{}_{,p} \\ & \left. + \frac{1}{8} (2g^{il} g^{km} - g^{ik} g^{lm}) (2g_{np} g_{qr} - g_{pq} g_{nr}) \mathbf{g}^{nr}{}_{,l} \mathbf{g}^{pq}{}_{,m} \right). \end{aligned} \quad (2.5.31)$$

This pseudotensor is symmetric in the indices i, k , so there is a conservation of the total angular momentum constructed from P^i as in (2.4.58),

$$M^{ik} = \int (x^i dP^k - x^k dP^i) = \frac{1}{c} \int (x^i (t^{kl} + \mathcal{T}^{kl}) - x^k (t^{il} + \mathcal{T}^{il})) (-\mathbf{g}) dS_l. \quad (2.5.32)$$

Dividing M^{ik} by $\sqrt{-\mathbf{g}}$ at infinity turns it into an antisymmetric tensor under Lorentz transformations. Using (2.5.24) and (2.5.26) turns (2.5.28), for the hypersurface $dS_0 = dV$, into

$$\begin{aligned} M^{ik} &= \frac{1}{c} \int (x^i \lambda^{klmn}_{,nm} - x^k \lambda^{ilmn}_{,nm}) dS_l = \frac{1}{2c} \oint (x^i \lambda^{klmn}_{,n} - x^k \lambda^{ilmn}_{,n}) df_{lm}^* \\ &\quad - \frac{1}{c} \oint (\lambda^{klin} - \lambda^{ilkn})_{,n} dS_l = \frac{1}{c} \oint (x^i h^{k0\alpha} - x^k h^{i0\alpha} + \lambda^{i0\alpha k}) df_\alpha. \end{aligned} \quad (2.5.33)$$

Choosing the volume hypersurface $dV = dS_0$ gives

$$M^{ik} = \frac{1}{c} \int (x^i (t^{k0} + \mathcal{T}^{k0}) - x^k (t^{i0} + \mathcal{T}^{i0})) (-\mathbf{g}) dV. \quad (2.5.34)$$

The conservation of $M^{0\alpha}$ in (2.5.34) divided by the conservation of P^0 in (2.5.28) gives a uniform motion (2.4.61) with velocity (2.4.62) of the center of inertia for the gravitational field and matter, with the coordinates

$$X^\alpha = \frac{\int x^\alpha (t^{00} + \mathcal{T}^{00}) (-\mathbf{g}) dV}{\int (t^{00} + \mathcal{T}^{00}) (-\mathbf{g}) dV}. \quad (2.5.35)$$

The coordinates of the center of inertia (2.5.35), like (2.4.63), are not the spatial components of a four-dimensional vector.

2.5.4 Utiyama action

The *Utiyama action* for the gravitational field and matter is equal to (2.5.1), where the tetrad is regarded as a variational variable and the spin connection is the Levi-Civita spin connection (1.5.31). Thus (2.3.5) gives

$$\delta S = -\frac{1}{2\kappa c} \int \delta(\mathbf{e}P) d\Omega + \frac{1}{c} \int \mathfrak{T}_i^a \delta e_a^i d\Omega. \quad (2.5.36)$$

The Lagrangian density for the gravitational field is given by (2.2.2), with the Riemann scalar P given by (1.5.39) and (1.5.41):

$$\mathbf{e}P = \mathbf{e}e_a^i e^{jb} (\varpi_{bj,i}^a - \varpi_{bi,j}^a + \varpi_{ci}^a \varpi_{bj}^c - \varpi_{cj}^a \varpi_{bi}^c) = 2\mathbf{e}_{ab}^{ij} (\varpi_{j,i}^{ab} + \varpi_{ci}^a \varpi_{cj}^{cb}), \quad (2.5.37)$$

where

$$\mathbf{e}_{ab}^{ij} = \mathbf{e}e_a^{[i} e_b^{j]}. \quad (2.5.38)$$

Varying $\mathbf{e}P$ and omitting total derivatives gives in the absence of torsion, using $\delta \mathbf{e} = \mathbf{e}e_a^i \delta e_i^a$ and $\mathbf{e}_{ab|j}^{ij} = \mathbf{e}_{ab,j}^{ij} - \varpi_{aj}^c \mathbf{e}_{cb}^{ij} - \varpi_{bj}^c \mathbf{e}_{ac}^{ij} = 0$ (which results from (1.5.22)),

$$\begin{aligned} \delta(\mathbf{e}P) &= (2P_i^a - Pe_i^a) \mathbf{e} \delta e_a^i + 2\mathbf{e}_{ab}^{ij} \delta (\varpi_{j,i}^{ab} + \varpi_{ci}^a \varpi_{cj}^{cb}) = (2P_i^a - Pe_i^a) \mathbf{e} \delta e_a^i \\ &\quad + 2(\mathbf{e}_{ab,j}^{ij} - \varpi_{aj}^c \mathbf{e}_{cb}^{ij} - \varpi_{bj}^c \mathbf{e}_{ac}^{ij}) \delta \varpi_{ij}^{ab} = (2P_i^a - Pe_i^a) \mathbf{e} \delta e_a^i. \end{aligned} \quad (2.5.39)$$

Equating $\delta S = 0$ gives the tetrad Einstein equations:

$$P_i^a - \frac{1}{2} Pe_i^a = \frac{\kappa}{\mathbf{e}} \mathfrak{T}_i^a, \quad (2.5.40)$$

equivalent to the metric Einstein equations (2.5.5) because of (2.3.4) and (2.3.32) (in the absence of torsion).

2.5.5 Møller pseudotensor

The Riemann scalar P is linear in derivatives of ϖ^a_{bi} :

$$\begin{aligned} \epsilon P = & (\epsilon e_a^i e_b^j \varpi^{ab}_{j,i}) - (\epsilon e_a^i e_b^j)_{,i} \varpi^{ab}_j - (\epsilon e_a^i e_b^j \varpi^{ab}_i)_{,j} + (\epsilon e_a^i e_b^j)_{,j} \varpi^{ab}_i + \epsilon e_a^i e_b^j \varpi^{ac}_i \varpi_c^b{}_j \\ & - \epsilon e_a^i e_b^j \varpi^{ac}_j \varpi_c^b{}_i = 2(\epsilon e_a^i e_b^j \varpi^{ab}_{j,i}) - 2(\epsilon e_a^i e_b^j)_{,i} \varpi^{ab}_j + \epsilon e_a^i e_b^j \varpi^{ac}_i \varpi_c^b{}_j \\ & - \epsilon e_a^i e_b^j \varpi^{ac}_j \varpi_c^b{}_i. \end{aligned} \quad (2.5.41)$$

Thus we can subtract from ϵP total derivatives without altering the field equations, replacing P by a noncovariant quantity M :

$$\begin{aligned} \epsilon M = & -2(\epsilon e_a^i e_b^j)_{,i} \varpi^{ab}_j + \epsilon e_a^i e_b^j \varpi^{ac}_i \varpi_c^b{}_j - \epsilon e_a^i e_b^j \varpi^{ac}_j \varpi_c^b{}_i \\ = & -2\epsilon(\{^k_i\} \varpi^{ij}_j + \varpi^i_{ai} \varpi^{aj}_j - \{^i_k\} \varpi^{kj}_j + \varpi^j_{bi} \varpi^{ib}_j - \{^j_k\} \varpi^{ik}_j) \\ & + \epsilon(\omega^{ic}_i \omega_c^j{}_j - \omega^{ic}_j \omega_c^j{}_i) = \epsilon(\omega^{ia}_i \omega^j_{aj} - \omega^{ia}_j \omega^j_{ai}), \end{aligned} \quad (2.5.42)$$

using (1.4.35) and (1.5.31). Therefore

$$M = \omega^{ia}_i \omega^j_{aj} - \omega^{ia}_j \omega^j_{ai}. \quad (2.5.43)$$

Because of (1.5.15), the quantity (2.5.43) depends on the tetrad e^i_a and its first derivatives $e^i_{a,j}$. Therefore, analogously to (2.5.12), we can construct a canonical energy-momentum density corresponding to the gravitational field, treating $-\frac{1}{2\kappa}\epsilon M$:

$$\mathfrak{m}_k{}^i = -\frac{1}{2\kappa} \left(\frac{\partial(\epsilon M)}{\partial e^j_{a,i}} e^j_{a,k} - \delta^i_k \epsilon M \right). \quad (2.5.44)$$

This quantity is not a tensor density since ϵM is not a scalar density and its division by ϵ , $\frac{\mathfrak{m}_k{}^i}{\epsilon}$, is referred to as the *Møller energy-momentum pseudotensor* for the gravitational field. One can show, analogously to the steps leading to (2.5.17), that the total energy-momentum density for the gravitational field and matter is ordinarily conserved:

$$(\mathfrak{m}_k{}^i + \mathcal{T}_k{}^i)_{,i} = 0. \quad (2.5.45)$$

Thus the corresponding total four-momentum is conserved:

$$P_i = \frac{1}{c} \int (\mathfrak{m}_i{}^k + \mathcal{T}_i{}^k) dS_k = \text{const}, \quad (2.5.46)$$

where the sum $\mathfrak{m}_i{}^k + \mathcal{T}_i{}^k$ is called the *Møller energy-momentum complex*. The Møller pseudotensor depends on the choice of both the coordinates and the tetrad. To fix the tetrad, one can impose on it 6 constraints which are covariant under constant Lorentz transformations but not under general Lorentz transformations (otherwise these constraints would not fix the tetrad since Lorentz transformations are tetrad rotations). A natural choice is to constrain the 6 components of the spin connection ω_{ijk} in which the last index is contracted with a covariant derivative or the trace of the spin connection.

2.5.6 Einstein-Cartan action

If we regard the torsion tensor as a variational variable (in addition to the metric tensor) then the action for the gravitational field and matter is, due to (2.2.1),

$$S = -\frac{1}{2\kappa c} \int R \sqrt{-g} d\Omega + S_m, \quad (2.5.47)$$

and it is referred to as the *Einstein-Cartan action*. Using (1.4.54) gives

$$S = -\frac{1}{2\kappa c} \int \left(P - g^{ik} (2C^l_{il;k} + C^j_{ij} C^l_{kl} - C^l_{im} C^m_{kl}) \right) \sqrt{-g} d\Omega + S_m. \quad (2.5.48)$$

Partial integration of the terms with covariant derivatives : and omitting total derivatives (which do not contribute to the field equations) reduces (2.5.48) to

$$S = -\frac{1}{2\kappa c} \int \left(P - g^{ik} (C^j_{ij} C^l_{kl} - C^l_{im} C^m_{kl}) \right) \sqrt{-g} d\Omega + S_m. \quad (2.5.49)$$

Varying (2.5.49) with respect to the metric tensor and contortion tensor (which is equivalent to varying with respect to the torsion tensor) gives, using (2.3.19), (2.3.26) and (2.5.3),

$$\begin{aligned} \delta S = & -\frac{1}{2\kappa c} \int \left(P_{ik} - \frac{1}{2} P g_{ik} - C^j_{ij} C^l_{kl} + C^l_{im} C^m_{kl} + \frac{1}{2} g_{ik} (C^{jm}_j C^l_{ml} - C^{lj}_m C^m_{jl}) \right) \\ & \times \sqrt{-g} \delta g^{ik} d\Omega - \frac{1}{\kappa c} \int (C^{kj}_i - C^{lj}_i \delta^k_i) \sqrt{-g} \delta C^i_{jk} d\Omega + \frac{1}{2c} \int T_{ik} \sqrt{-g} \delta g^{ik} d\Omega \\ & + \frac{1}{2c} \int s_j^{ik} \sqrt{-g} \delta C^j_{ik} d\Omega. \end{aligned} \quad (2.5.50)$$

For variations δg^{ik} , $\delta S = 0$ gives the *first Einstein-Cartan equation*

$$G_{ik} = \kappa (T_{ik} + U_{ik}), \quad (2.5.51)$$

where

$$U_{ik} = \frac{1}{\kappa} \left(C^j_{ij} C^l_{kl} - C^l_{ij} C^j_{kl} - \frac{1}{2} g_{ik} (C^{jm}_j C^l_{ml} - C^{mj}_l C^l_{jm}) \right) \quad (2.5.52)$$

or

$$\begin{aligned} U_{ik} = & \frac{1}{\kappa} \left(-(S^l_{ij} + 2S_{(ij)}^l) (S^j_{kl} + 2S_{(kl)}^j) + 4S_i S_k + \frac{1}{2} g_{ik} (S^{mj}_l + 2S^{(jl)m}) \right. \\ & \left. \times (S_{ljm} + 2S_{(jm)l}) - 2g_{ik} S^j S_j \right). \end{aligned} \quad (2.5.53)$$

For variations δC^j_{ik} , $\delta S = 0$ gives the *second Einstein-Cartan equation*

$$C^k_{[ji]} - \delta^k_{[i} C^l_{j]l} = \frac{\kappa}{2} s_{ij}^k \quad (2.5.54)$$

or

$$T^j_{ik} = -\frac{\kappa}{2} s_{ik}^j, \quad (2.5.55)$$

where

$$T^j_{ik} = S^j_{ik} - S_i \delta^j_k + S_k \delta^j_i \quad (2.5.56)$$

is the *modified torsion tensor*. The relation (2.5.55) is equivalent to

$$S^k_{ij} = -\frac{\kappa}{2} (s_{ij}^k + \delta^k_{[i} s_{j]l}^l). \quad (2.5.57)$$

This relation between the torsion and spin tensors is algebraic: torsion at a given point in spacetime does not vanish only if there is matter at this point, represented in the Lagrangian density by a function which depends on torsion. Unlike the metric, which is related to matter through a differential field equation, torsion does not propagate. Combining (2.5.52) and (2.5.57) gives

$$U^{ik} = \kappa \left(-s^{ij}_{[l} s^{kl}_{j]} - \frac{1}{2} s^{ijl} s^k_{jl} + \frac{1}{4} s^{jli} s_{jl}^k + \frac{1}{8} g^{ik} (-4s^l_{j[m} s^{jm}_{l]} + s^{jlm} s_{jlm}) \right). \quad (2.5.58)$$

The tensor (2.5.58) represents a correction to the dynamical energy-momentum tensor from the spin contributions to the geometry of spacetime, quadratic in the spin density (so the sign of the spin density does not affect this correction) and corresponding to a spin-spin contact interaction. If matter fields do not depend on torsion then $U_{ik} = 0$ and the first Einstein-Cartan equation (2.5.51) reduces to the Einstein equations (2.5.5).

2.5.7 Sciama-Kibble action

The *Sciama-Kibble action* for the gravitational field and matter is equal to (2.5.47), where both the tetrad and spin connection are regarded as variational variables. Thus (2.3.27) gives

$$\delta S = -\frac{1}{2\kappa c} \int \delta(\epsilon R) d\Omega + \frac{1}{c} \int \mathfrak{T}_i^a \delta e_a^i d\Omega + \frac{1}{2c} \int \mathfrak{S}_{ab}^i \delta \omega^{ab}_i d\Omega. \quad (2.5.59)$$

The Lagrangian density for the gravitational field is given by (2.2.1), with the curvature scalar R given by (1.5.36) and (1.5.38):

$$\epsilon R = \epsilon e_a^i e^{jb} (\omega_{bj,i}^a - \omega_{bi,j}^a + \omega_{ci}^a \omega_{bj}^c - \omega_{cj}^a \omega_{bi}^c) = 2\epsilon_{ab}^{ij} (\omega_{j,i}^{ab} + \omega_{ci}^a \omega_{bj}^{cb}). \quad (2.5.60)$$

Varying ϵR and omitting total derivatives gives, using $\epsilon_{ab|j}^{ij} = \epsilon_{ab,j}^{ij} - \omega_{aj}^c \epsilon_{cb}^{ij} - \omega_{bj}^c \epsilon_{ac}^{ij} + \Gamma_{kj}^i \epsilon_{ab}^{kj} + \Gamma_{kj}^j \epsilon_{ab}^{ik} - \Gamma_{kj}^k \epsilon_{ab}^{ij} = 0$ (which results from (1.5.22)),

$$\begin{aligned} \delta(\epsilon R) &= (2R_i^a - Re_i^a) \epsilon \delta e_a^i + 2\epsilon_{ab}^{ij} \delta(\omega_{j,i}^{ab} + \omega_{ci}^a \omega_{bj}^{cb}) \\ &= (2R_i^a - Re_i^a) \epsilon \delta e_a^i + 2(\epsilon_{ab,j}^{ij} - \omega_{aj}^c \epsilon_{cb}^{ij} - \omega_{bj}^c \epsilon_{ac}^{ij}) \delta \omega^{ab}_i \\ &= (2R_i^a - Pe_i^a) \epsilon \delta e_a^i - 2(S_{kj}^i \epsilon_{ab}^{kj} + 2S_j \epsilon_{ab}^{ij}) \delta \omega^{ab}_i. \end{aligned} \quad (2.5.61)$$

For variations $\delta \omega^{ab}_i$, $\delta S = 0$ gives

$$S_{ab}^i - S_a e_b^i + S_b e_a^i = -\frac{\kappa}{2\epsilon} \mathfrak{S}_{ab}^i, \quad (2.5.62)$$

equivalent to the second Einstein-Cartan equation (2.5.55). For variations δe_a^i , $\delta S = 0$ gives

$$R_i^a - \frac{1}{2} Re_i^a = \frac{\kappa}{\epsilon} \mathfrak{T}_i^a \quad (2.5.63)$$

or

$$R_{ki} - \frac{1}{2} R g_{ik} = \frac{\kappa}{\sqrt{-g}} \mathfrak{T}_{ik}. \quad (2.5.64)$$

Substituting (2.5.55) and (2.5.64) into the conservation law for the spin density (2.4.16) gives

$$-2(S_{ij;k}^k - S_{i;j} + S_{j;i}) = R_{ji} - R_{ij} - 4S_k (S_{ij}^k - S_i \delta_j^k + S_j \delta_i^k), \quad (2.5.65)$$

which is equivalent to the contracted cyclic identity (1.4.61). Thus the contracted cyclic identity imposes the conservation law for the spin density in the Einstein-Cartan gravity. Substituting (2.5.55) and (2.5.64) into the conservation law for the energy-momentum density (2.4.31) gives

$$R_{ij}^j - \frac{1}{2} R_{;i} = 2S_j \left(R_{ij}^j - \frac{1}{2} R \delta_i^j \right) + 2S_{ki}^j \left(R_{ij}^k - \frac{1}{2} R \delta_j^k \right) - (S_{kl}^j - S_k \delta_l^j + S_l \delta_k^j) R_{ji}^{kl}, \quad (2.5.66)$$

which is equivalent to the contracted Bianchi identity (1.4.62). Thus the contracted Bianchi identity imposes the conservation law for the energy-momentum density in the Einstein-Cartan gravity. Substituting (2.5.55) and (2.5.64) into the Belinfante-Rosenfeld relation (2.3.33) gives

$$\begin{aligned} \kappa T_{ik} &= R_{ki} - \frac{1}{2} R g_{ik} + \nabla_j^* (S_{ik}^j + 2S_k \delta_i^j - 2S_{(ik)}^j - 2S^j g_{ik}) = R_{ki} - \frac{1}{2} R g_{ik} \\ &+ \nabla_j^* (-C_{ki}^j + C_{kl}^l \delta_i^j - C_{lj}^l g_{ik}). \end{aligned} \quad (2.5.67)$$

Combining (1.4.52), (1.4.54) and (2.5.67) gives

$$\begin{aligned} \kappa T_{ik} &= P_{ik} - \frac{1}{2} P g_{ik} + C_{ki;l}^l - C_{kl;i}^l + C_{ki}^j C_{jl}^l - C_{kl}^j C_{ji}^l - \frac{1}{2} g_{ik} (-2C_{l;j}^{lj} \\ &- C_{lj}^{lj} C_{jm}^m + C_{mj}^{mj} C_{lj}^l) - C_{ki;j}^j - C_{lj}^j C_{ki}^l + C_{kj}^l C_{li}^j + C_{ij}^l C_{kl}^j + C_{kj;i}^j \\ &- C_{ki}^l C_{lj}^j - g_{ik} (C_{lj}^{lj} + C_{lj}^j C_{lm}^m) - C_j (-C_{ki}^j + C_{kl}^l \delta_i^j - C_{lj}^{lj} g_{ik}), \end{aligned} \quad (2.5.68)$$

which is equivalent to the first Einstein-Cartan equation (2.5.51). Thus the relation between the Ricci tensor and the Riemannian Ricci tensor is equivalent to the Belinfante-Rosenfeld relation in the Einstein-Cartan gravity, and (2.5.64) is another form of the first Einstein-Cartan equation.

2.5.8 Einstein-Cartan pseudotensor

Replacing the action for the gravitational field and matter (2.5.49) by

$$S = -\frac{1}{2\kappa c} \int \left(G - g^{ik} (C^j_{ij} C^l_{kl} - C^l_{im} C^m_{kl}) \right) \sqrt{-g} d\Omega + S_m \quad (2.5.69)$$

produces the first Einstein-Cartan equation by varying the metric tensor, because $\sqrt{-g}G$ differs from $\sqrt{-g}P$ by a total divergence:

$$\frac{\delta}{\delta g^{ik}} \left(-\frac{1}{2\kappa} \left(\mathcal{G} - \mathbf{g}^{np} (C^j_{nj} C^l_{pl} - C^l_{nm} C^m_{pl}) \right) + \mathfrak{L}_m \right) = 0. \quad (2.5.70)$$

The canonical energy-momentum density for the gravitational field is also given by (2.5.12). The relations (2.5.14) and (2.5.70) give

$$\mathfrak{t}^i_{k,i} = \frac{\delta(\mathfrak{L}_m + 2\kappa \mathbf{g}^{np} (C^j_{nj} C^l_{pl} - C^l_{nm} C^m_{pl}))}{\delta g^{jl}} g^{jl}_{,k} = \frac{1}{2} (\mathcal{T}_{jl} + \sqrt{-g} U_{jl}) g^{jl}_{,k}. \quad (2.5.71)$$

The covariant conservation (2.4.23) gives

$$\begin{aligned} (\mathcal{T}_k^i + \sqrt{-g} U_k^i)_{,i} &= \{^l_k\} (\mathcal{T}_l^i + \sqrt{-g} U_l^i) = \frac{1}{2} g^{lm} g_{im,k} (\mathcal{T}_l^i + \sqrt{-g} U_l^i) \\ &= -\frac{1}{2} g^{lm}_{,k} (\mathcal{T}_{lm} + \sqrt{-g} U_{lm}), \end{aligned} \quad (2.5.72)$$

so the total energy-momentum density for the gravitational field and matter is ordinarily conserved:

$$(\mathfrak{t}_k^i + \mathcal{T}_k^i + \sqrt{-g} U_k^i)_{,i} = 0. \quad (2.5.73)$$

Thus the corresponding four-momentum is conserved:

$$P_i = \frac{1}{c} \int (\mathfrak{t}_i^k + \mathcal{T}_i^k + \sqrt{-g} U_i^k) dS_k = \text{const}. \quad (2.5.74)$$

The quantity $\frac{\mathfrak{t}_i^k}{\sqrt{-g}} + U_i^k$ is referred to as the *Einstein-Cartan energy-momentum pseudotensor* for the gravitational field, and the sum $\mathfrak{t}_i^k + \mathcal{T}_i^k + \sqrt{-g} U_i^k$ is called the *Einstein-Cartan energy-momentum complex*.

2.5.9 Palatini variation

If the matter action S_m does not depend on the affine connection, its variation with respect to the metric and connection ($\delta\Gamma_{ik}^j$ is a tensor) is referred to as the *Palatini variation*. Varying (2.5.47) with respect to Γ_{ij}^k gives, due to (1.3.39),

$$\delta S = -\frac{1}{2\kappa c} \int \delta R_{ik} \mathbf{g}^{ik} d\Omega = -\frac{1}{2\kappa c} \int ((\delta\Gamma_{ik}^l)_{;l} - (\delta\Gamma_{il}^l)_{;k} - 2S_{lk}^j \delta\Gamma_{ij}^l) g^{ik} \sqrt{-g} d\Omega. \quad (2.5.75)$$

Partial integration and omitting total derivatives in (2.5.75) gives, using (1.2.34),

$$\delta S = \frac{1}{2\kappa c} \int (\delta\Gamma_{ik}^l \mathbf{g}^{ik}_{;l} - 2S_l \delta\Gamma_{ik}^l \mathbf{g}^{ik} - \delta\Gamma_{il}^l \mathbf{g}^{ik}_{;k} + 2S_k \delta\Gamma_{il}^l \mathbf{g}^{ik} + 2S_{lk}^j \delta\Gamma_{ij}^l \mathbf{g}^{ik}) d\Omega. \quad (2.5.76)$$

Since the affine connection is metric-compatible, $g_{ij;k} = 0$, $\delta S = 0$ turns the torsion tensor into zero, so the connection is formed by the Christoffel symbols and the field equations are the Einstein equations (2.5.5). Thus varying the action for matter fields, which do not depend on the affine

connection, with respect to the connection is equivalent to varying it with respect to the torsion tensor. However, if the matter action S_m depends on the affine connection then (2.5.76) becomes

$$\begin{aligned} \delta S = & \frac{1}{2\kappa c} \int (\delta \Gamma_{ik}^l \mathbf{g}^{ik}_{;l} - 2S_l \delta \Gamma_{ik}^l \mathbf{g}^{ik} - \delta \Gamma_{il}^l \mathbf{g}^{ik}_{;k} + 2S_k \delta \Gamma_{il}^l \mathbf{g}^{ik} + 2S_{lk}^j \delta \Gamma_{ij}^l \mathbf{g}^{ik}) d\Omega \\ & + \frac{1}{2c} \int \Pi_j^{ik} \delta \Gamma_{ik}^j d\Omega, \end{aligned} \quad (2.5.77)$$

where the *hypermomentum density* is defined as

$$\Pi_j^{ik} = 2 \frac{\delta \mathfrak{L}_m}{\delta \Gamma_{ik}^j}. \quad (2.5.78)$$

Since the connection is metric-compatible, $\delta S = 0$ gives

$$g^{ik} S_j - \delta_j^k S^i - S^{ki}_j = \frac{\kappa}{2\sqrt{-g}} \Pi_j^{ik}. \quad (2.5.79)$$

Contracting the indices i, j gives

$$\Pi_i^{ik} = 0, \quad (2.5.80)$$

which also results from the invariance of the Lagrangian density under a projective transformation (1.2.50) (the symmetric part of the Ricci tensor is invariant under this transformation):

$$\delta \mathfrak{L} = \delta \mathfrak{L}_m = \frac{1}{2} \Pi_j^{ik} \delta \Gamma_{ik}^j = \frac{1}{2} \Pi_j^{ik} \delta_i^j \delta A_k = 0. \quad (2.5.81)$$

The relation (2.5.80) constrains possible forms of matter Lagrangians algebraically, so it is not a conservation law. Therefore varying the action with respect to the affine connection, unlike that with respect to the torsion (or spin connection), does not constitute a physical variational principle. Only the antisymmetric part of the connection (torsion) can be regarded as a dynamical variable; its symmetric part can always be brought locally to zero by a suitable transformation of the coordinates.

2.5.10 Gravitational potential

If the metric tensor g_{ij} is approximately equal to the Minkowski metric tensor η_{ij} then the corresponding gravitational field is *weak*. We can write

$$g_{00} \approx 1 + \frac{2\phi}{c^2}, \quad (2.5.82)$$

where ϕ is referred to as the *gravitational potential*. Thus nonrelativistic gravitational fields, corresponding to the limit $c \rightarrow \infty$, are weak. Also $u^0 \approx 1$ and $u^\alpha \approx 0$. In this limit, the leading component of the Levi-Civita connection is

$$\{^{\alpha}_{00}\} \approx -\frac{1}{2} g^{\alpha\beta} \frac{\partial g_{00}}{\partial x^\beta} = \frac{1}{c^2} \frac{\partial \phi}{\partial x^\alpha}, \quad (2.5.83)$$

so the metric geodesic equation (1.4.80) reduces to

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} = -\nabla\phi, \quad (2.5.84)$$

where $\mathbf{g}$ is the acceleration due to gravity. The quantity $\mathbf{G}$ in (2.2.5) reduces to

$$\mathbf{G} = \frac{2}{c^4} (\nabla\phi)^2. \quad (2.5.85)$$

The leading component of the Riemannian Ricci tensor is

$$P_{00} \approx \frac{\partial \{^{\alpha}_{00}\}}{\partial x^\alpha} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial x^{\alpha 2}} = \frac{1}{c^2} \Delta\phi. \quad (2.5.86)$$

The leading component of the energy-momentum tensor (2.4.195) is

$$T_{00} = \mu c^2. \quad (2.5.87)$$

Therefore the Einstein equations in the nonrelativistic limit reduce to the *Poisson equation*:

$$\Delta\phi = 4\pi G\mu, \quad (2.5.88)$$

where

$$G = \frac{c^4\kappa}{8\pi} \quad (2.5.89)$$

is *Newton's gravitational constant*. In vacuum, where $\mu = 0$, the Poisson equation reduces to the Laplace equation:

$$\Delta\phi = 0. \quad (2.5.90)$$

2.5.11 Relativistic ideal fluids

The covariant conservation (2.4.24) of the metric energy-momentum tensor (2.4.77) gives

$$((\epsilon + p)u^k)_{;k} u^i + (\epsilon + p)u^k u^i_{;k} = p_{,k} g^{ik}. \quad (2.5.91)$$

Multiplying (2.5.91) by u_i gives

$$((\epsilon + p)u^k)_{;k} = p_{,k} u^k, \quad (2.5.92)$$

which, upon substituting into (2.5.91) yields the *Euler equation*:

$$(\epsilon + p) \frac{D^{\{\} u^i}{ds} = p_{,k} h^{ik}. \quad (2.5.93)$$

If $p_{,i} \propto u_i$ (which includes the case $p = \text{const}$) then (2.5.93) reduces to the metric geodesic equation (1.4.80). Defining a quantity n such that

$$\frac{dn}{n} = \frac{d\epsilon}{\epsilon + p} \quad (2.5.94)$$

brings (2.5.92) to the conservation law

$$(nu^i)_{;i} = 0. \quad (2.5.95)$$

The quantity n may thus represent the proper (in the rest frame) *number density* of particles composing the fluid.

In the nonrelativistic limit, $c \rightarrow \infty$, $u^0 \sim 1$, $u^\alpha \approx \frac{v^\alpha}{c}$, $\epsilon \approx \mu c^2$ and $p \ll \epsilon$, so (2.5.92) reduces to the *equation of continuity*:

$$\frac{\partial \mu}{\partial t} + \text{div } \mathbf{s} = 0, \quad (2.5.96)$$

where

$$\mathbf{s} = \mu \mathbf{v} \quad (2.5.97)$$

is referred to as the *mass current*. Integrating (2.5.96) over the volume gives

$$\frac{\partial}{\partial t} \int \mu dV + \oint \mathbf{s} \cdot d\mathbf{f} = 0, \quad (2.5.98)$$

which means that the change in time of the total mass inside a volume, $m = \int \mu dV$, is balanced by the *mass flux* through the surface bounding this volume, representing the conservation of the total mass of a fluid. The Euler equation (2.5.93) reduces in this limit to

$$\mu \left(\frac{\partial v^\alpha}{\partial t} + v^\alpha_{;\beta} v^\beta \right) = \mu \phi_{,\beta} \eta^{\alpha\beta} + p_{,\beta} \eta^{\alpha\beta} \quad (2.5.99)$$

or

$$\mu \frac{d\mathbf{v}}{dt} = \mu \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\mu \nabla \phi - \nabla p. \quad (2.5.100)$$

Integrating (2.5.100) over the volume gives, using $\mathbf{P} = \int \mu \mathbf{v} dV$, the change in time of the total momentum of a fluid:

$$\frac{d\mathbf{P}}{dt} = - \int \nabla \phi dm - \oint p d\mathbf{f}. \quad (2.5.101)$$

Without pressure gradients, (2.5.100) reduces to (2.5.84).

The number of particles dN in a volume element dV in the rest frame of reference is equal to

$$dN = n dV, \quad (2.5.102)$$

where n is the proper number density. In a frame of reference moving relative to the rest frame with velocity $\mathbf{v}$, the same volume element is given by $dV' = dV \sqrt{1 - v^2/c^2}$ (1.6.107), and the number density is n' . Since $dN' = n' dV'$ and $dN' = dN$ is an invariant, we have

$$n' = \frac{n}{\sqrt{1 - v^2/c^2}}. \quad (2.5.103)$$

2.5.12 Relativistic spin fluids

For a spin fluid, substituting (2.4.180) into the second Einstein-Cartan equation (2.5.57) and using (2.4.182) gives

$$S_i = 0, \quad \nabla_i^* = \nabla_i. \quad (2.5.104)$$

Thus the corresponding metric dynamical energy-momentum tensor (2.4.96) reduces to

$$T_{ij} = \epsilon u_i u_j - p h_{ij} - \nabla_k s_{(ij)}^k + \nabla_k s_{il}^k u^l u_j - \frac{1}{2} \nabla_k s_{ij}^k. \quad (2.5.105)$$

Accordingly, the torsion tensor is

$$S_{ik}^j = -\frac{1}{2} \kappa s_{ik} u^j. \quad (2.5.106)$$

The last two terms on the right of the second line of (2.5.105) can be written, using (2.4.92), (2.4.94) and (2.4.182), as

$$\begin{aligned} \nabla_k s_{il}^k u^l u_j - \frac{1}{2} \nabla_k s_{ij}^k &= c(\mathbf{p}_i u_l - \mathbf{p}_l u_i) u^l u_j - \frac{1}{2} c(\mathbf{p}_i u_j - \mathbf{p}_j u_i) \\ &= \frac{1}{2} c(\mathbf{p}_i u_j + \mathbf{p}_j u_i) - \epsilon u_i u_j = \frac{1}{2} (\nabla_k s_{il}^k u^l u_j + \nabla_k s_{jl}^k u^l u_i) = -\nabla_l s_{k(i}^l u^k u_{j)} \\ &= -\nabla_l (s_{k(i} u^l) u^k u_{j)}) = -\nabla_l s_{k(i} u^l u^k u_{j)}) = -\nabla_l (s_{k(i} u_{j)}) u^l u^k \\ &= -\nabla_l (s_{(i}^k u_{j)}) u_k u^l. \end{aligned} \quad (2.5.107)$$

The metric dynamical energy-momentum tensor (2.5.105) is then

$$T^{ij} = \epsilon u^i u^j - p h^{ij} - (\delta_k^l + u_k u^l) \nabla_l (s^{k(i} u^{j)}). \quad (2.5.108)$$

The last term on the right of (2.5.108) can be decomposed according to (1.4.28) into

$$\begin{aligned} -(\delta_k^l + u_k u^l) \nabla_l (s^{k(i} u^{j)}) &= -(\delta_k^l + u_k u^l) \nabla_l^{\{\}} (s^{k(i} u^{j)}) - (\delta_k^l + u_k u^l) (C_{ml}^k s^{m(i} u^{j)}) \\ &\quad + C_{ml}^i s^{k(m} u^{j)} + C_{ml}^j s^{k(i} u^{m)}). \end{aligned} \quad (2.5.109)$$

This term reduces, using (2.4.182) and (2.5.54), to

$$\begin{aligned}
& -(\delta_k^l + u_k u^l) \nabla_l^{\{\}} (s^{k(i} u^{j)}) - \delta_k^l (C_{ml}^i s^{k(m} u^{j)} + C_{ml}^j s^{k(i} u^{m)}) - u_k u^l C_{ml}^k s^{m(i} u^{j)}) \\
& = -(\delta_k^l + u_k u^l) \nabla_l^{\{\}} (s^{k(i} u^{j)}) - C_{mk}^i s^{k(m} u^{j)} - C_{mk}^j s^{k(i} u^{m}) \\
& = -(\delta_k^l + u_k u^l) \nabla_l^{\{\}} (s^{k(i} u^{j)}) + \frac{1}{2} \kappa (s_{mk} u^i + s_k^i u_m + s_m^i u_k) s^{k(m} u^{j)} \\
& \quad + \frac{1}{2} \kappa (s_{mk} u^j + s_k^j u_m + s_m^j u_k) s^{k(m} u^{i)} \\
& = -(\delta_k^l + u_k u^l) \nabla_l^{\{\}} (s^{k(i} u^{j)}) - \frac{1}{2} \kappa (s_{kl} s^{kl} u^i u^j - s^{ik} s_k^j). \tag{2.5.110}
\end{aligned}$$

Thus (2.5.108) becomes

$$T^{ij} = \epsilon u^i u^j - p h^{ij} - (\delta_k^l + u_k u^l) \nabla_l^{\{\}} (s^{k(i} u^{j)}) - \kappa s^2 u^i u^j + \frac{1}{2} \kappa s^{ik} s_k^j, \tag{2.5.111}$$

where

$$s^2 = \frac{1}{2} s^{ij} s_{ij} > 0. \tag{2.5.112}$$

Substituting (2.4.180) into (2.5.58) and using (2.4.182) gives

$$U^{ij} = \frac{1}{2} \kappa s^2 u^i u^j + \frac{1}{4} \kappa s^2 g^{ij} - \frac{1}{2} \kappa s^{ik} s_k^j. \tag{2.5.113}$$

Adding (2.5.111) and (2.5.113) brings the combined energy-momentum tensor $T^{ij} + U^{ij}$ in the first Einstein-Cartan equation (2.5.51) to

$$\begin{aligned}
T^{ij} + U^{ij} &= \left(\epsilon - \frac{1}{4} \kappa s^2 \right) u^i u^j - \left(p - \frac{1}{4} \kappa s^2 \right) h^{ij} \\
&\quad - (\delta_k^l + u_k u^l) \nabla_l^{\{\}} (s^{k(i} u^{j)}). \tag{2.5.114}
\end{aligned}$$

If the spin orientation of particles (cf. (2.6.60)) in a spin fluid is random then the macroscopic spacetime average of s_{ij} and of its gradients, such as of the last term on the right of (2.5.114), vanish. On the contrary, the terms that are quadratic in the spin tensor do not vanish after averaging. Thus the combined energy-momentum tensor of a macroscopic spin fluid describes a perfect fluid with the effective energy density

$$\tilde{\epsilon} = (T_{ij} + U_{ij}) u^i u^j = \epsilon - \frac{1}{4} \kappa s^2 \tag{2.5.115}$$

and the effective pressure

$$\tilde{p} = -\frac{1}{3} (T_{ij} + U_{ij}) h^{ij} = p - \frac{1}{4} \kappa s^2. \tag{2.5.116}$$

If the spin orientation of particles (cf. (2.6.60)) in a spin fluid is not random then the combined energy density of a macroscopic spin fluid is

$$\begin{aligned}
\tilde{\epsilon} &= \epsilon - \frac{1}{4} \kappa s^2 - (\delta_k^l + u_k u^l) u^i \nabla_l^{\{\}} s^k_i = \epsilon - \frac{1}{4} \kappa s^2 - \nabla_k^{\{\}} s^k_i u^i \\
&= \epsilon - \frac{1}{4} \kappa s^2 + s^{ki} \nabla_{[k}^{\{\}} u_{i]} = \epsilon - \frac{1}{4} \kappa s^2 + s^{ki} \partial_{[k} u_{i]}. \tag{2.5.117}
\end{aligned}$$

In a locally Galilean frame of reference which is also a rest frame, (2.5.117) becomes

$$\tilde{\epsilon} = \epsilon - \frac{1}{4} \kappa s^2 + \frac{1}{2} \mathbf{s} \cdot \text{curl } \mathbf{v}. \tag{2.5.118}$$

where $\mathbf{s}$ is the spatial spin-density pseudovector (2.4.186). The quantity $\sqrt{s^2}$ is proportional to the square of the proper particle number density, n^2 .

2.5.13 Raychaudhuri equation

We consider a congruence of particles with four-velocity u^i . We define the *expansion/contraction scalar* θ , the traceless *shear tensor* σ_{ik} , and the antisymmetric *rotation/vorticity tensor* ω_{ik} according to

$$\theta = u^i{}_{;i}, \quad (2.5.119)$$

$$\sigma_{ik} = u_{(i;k)} - \frac{1}{3}\theta h_{ik} - w_{(i}u_{k)}, \quad (2.5.120)$$

$$\omega_{ik} = u_{[i;k]} - w_{[i}u_{k]}, \quad (2.5.121)$$

where w^i is the four-acceleration (1.6.117). Expansion has $\theta > 0$ and contraction has $\theta < 0$. These definitions give

$$u_{i;j}h_k^j = \sigma_{ik} + \omega_{ik} + \frac{1}{3}\theta h_{ik}. \quad (2.5.122)$$

Contracting $u^i{}_{;jk} - u^i{}_{;kj} = R^i{}_{lkj}u^l$ with respect to the indices i, j gives $\theta_{;k}u^k - u^j{}_{;kj}u^k = -R_{kl}u^k u^l$ or

$$-R_{kl}u^k u^l = \theta_{;k}u^k - w^i{}_{;i} + u_{i;k}u^{k;i}. \quad (2.5.123)$$

Defining

$$\sigma^2 = \frac{1}{2}\sigma_{ik}\sigma^{ik}, \quad \omega^2 = \frac{1}{2}\omega_{ik}\omega^{ik} \quad (2.5.124)$$

gives

$$u_{i;k}u^{k;i} = 2(\sigma^2 - \omega^2) + \frac{1}{3}\theta^2, \quad (2.5.125)$$

which brings (2.5.123) to the *Raychaudhuri equation*:

$$\frac{d\theta}{ds} = -2(\sigma^2 - \omega^2) - \frac{1}{3}\theta^2 + w^i{}_{;i} - R_{ik}u^i u^k. \quad (2.5.126)$$

For a perfect fluid, the Einstein equations give

$$R_{ik}u^i u^k = \frac{\kappa}{2}(\epsilon + 3p). \quad (2.5.127)$$

We define four energy conditions. The *null energy condition* is satisfied if

$$T_{ij}k^i k^j \geq 0 \quad (2.5.128)$$

for any null, future-pointing vector k^i . For a perfect fluid, this condition gives

$$\epsilon + p \geq 0. \quad (2.5.129)$$

The *weak energy condition* is satisfied if

$$T_{ij}u^i u^j \geq 0 \quad (2.5.130)$$

for any causal (null or timelike), future-pointing vector u^i . For a perfect fluid, this condition gives

$$\epsilon + p \geq 0, \quad \epsilon \geq 0. \quad (2.5.131)$$

The *strong energy condition* is satisfied if

$$\left(T_{ij} - \frac{1}{2}Tg_{ij}\right)u^i u^j \geq 0 \quad (2.5.132)$$

for any causal, future-pointing vector u^i . For a perfect fluid, this condition gives

$$\epsilon + p \geq 0, \quad \epsilon + 3p \geq 0. \quad (2.5.133)$$

The *dominant energy condition* is satisfied if the weak condition is satisfied and $T_{ij}u^j$ is a causal, future-pointing vector. For a perfect fluid, this condition gives

$$\epsilon + p \geq 0, \quad \epsilon \geq |p|. \quad (2.5.134)$$

This condition guarantees that particles in a congruence do not move faster than light. The dominant condition implies the weak condition. The weak condition implies the null condition. The strong condition implies the null condition.

If the strong condition is satisfied and particles in a congruence move without rotation and acceleration ($\omega_{ik} = w^i = 0$) then (2.5.126) and the Einstein equations give

$$\frac{d\theta}{d\tau} \leq -\frac{c}{3}\theta^2. \quad (2.5.135)$$

If θ_0 is θ at $\tau = 0$ then

$$\theta^{-1} \geq \theta_0^{-1} + \frac{c\tau}{3}. \quad (2.5.136)$$

Therefore, if $\theta_0 < 0$ (initial contraction) then θ diverges to a curvature *singularity* (a point in spacetime where the density of matter and curvature are infinite) as τ increases: $\theta \rightarrow -\infty$.

2.5.14 Event horizon

If a hypersurface S in spacetime is given by an equation of constraint,

$$f(x^i) = 0, \quad (2.5.137)$$

then the vector normal to this hypersurface is

$$n_i = \frac{\partial f}{\partial x^i}. \quad (2.5.138)$$

If such a vector is a null vector,

$$n_i n^i = 0, \quad (2.5.139)$$

then S is a null hypersurface. All infinitesimal displacements dx^i along such a hypersurface satisfy, according to (2.5.138),

$$df = n_i dx^i = 0. \quad (2.5.140)$$

Equations (2.5.139) and (2.5.140) indicate that n^i lies itself on the null hypersurface to which it is normal,

$$dx^i \propto n^i, \quad (2.5.141)$$

which also gives

$$ds^2 = dx_i dx^i \propto n_i n^i = 0. \quad (2.5.142)$$

Thus all world lines on a null hypersurface are null. The light cones at the points of such a hypersurface are tangent to this hypersurface. Since all physical world lines must lie within the local light cones, the forward-time motion through a null hypersurface can occur in only one direction. To avoid any discontinuities, this direction is the same for all points on such a hypersurface. A null hypersurface is therefore an *event horizon*: a boundary in spacetime beyond which events cannot affect events on the other side. All laws of classical physics are known to be *time-symmetric*, that is, symmetric under the transformation $t \rightarrow -t$. However, the existence of event horizons, which are solutions to these laws and provide boundary conditions for spacetime, violates this symmetry. The unidirectional character of the motion of matter through an event horizon can be used to define the past and the future: *the arrow of time*.

References: [1, 2, 3, 4, 5, 6, 8].

2.6 Spinor fields

2.6.1 Dirac matrices

The Dirac matrices defined by (1.7.1) are complex. A particular solution of (1.7.1) is given by the *Dirac representation*:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}, \quad (2.6.1)$$

where I is the unit 2×2 matrix and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.6.2)$$

are the *Pauli matrices* (all indices are coordinate invariant). The Pauli matrices are traceless $\text{tr}(\sigma^\alpha) = 0$ and Hermitian $\sigma^{\alpha\dagger} = \sigma^\alpha$ (the Hermitian conjugation of a matrix A is the combination of the complex conjugation and transposition, $A^\dagger = A^{*T}$), satisfy

$$\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\epsilon_{\alpha\beta\gamma} \sigma_\gamma, \quad (2.6.3)$$

and their square is I . The identity (2.6.3) gives the anticommutation relation

$$\left[\frac{\sigma_\alpha}{2}, \frac{\sigma_\beta}{2} \right] = i\epsilon_{\alpha\beta\gamma} \frac{\sigma_\gamma}{2}, \quad (2.6.4)$$

so $\frac{\gamma_\alpha}{2}$ form the lowest, two-dimensional representation of the angular momentum operator M_α (1.6.77). The properties of σ^α imply that the Dirac matrices are traceless $\text{tr}(\gamma^i) = 0$ and satisfy

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{\alpha\dagger} = -\gamma^\alpha, \quad \gamma^{i\dagger} = \gamma^0 \gamma^i \gamma^0, \quad \gamma^{i*} = \gamma^2 \gamma^i \gamma^2. \quad (2.6.5)$$

The relation (1.7.1) yields the total antisymmetry of $\gamma^0 \gamma^1 \gamma^2 \gamma^3$:

$$\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^{[0} \gamma^1 \gamma^2 \gamma^3]. \quad (2.6.6)$$

We define

$$\gamma^5 = -\frac{i}{24} \epsilon_{ijkl} \gamma^i \gamma^j \gamma^k \gamma^l = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad (2.6.7)$$

which is traceless $\text{tr}(\gamma^5) = 0$ and Hermitian $\gamma^{5\dagger} = \gamma^5$, and satisfies

$$\{\gamma^i, \gamma^5\} = 0, \quad (\gamma^5)^2 = 1. \quad (2.6.8)$$

In the Dirac representation,

$$\gamma^5 = \gamma^{5*} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (2.6.9)$$

The anticommutation relation (1.7.1) gives

$$\gamma^i \gamma_i = 4, \quad (2.6.10)$$

$$\gamma^i \gamma^j \gamma_i = -2\gamma^j, \quad (2.6.11)$$

$$\gamma^i \gamma^j \gamma^k \gamma_i = 4\eta^{jk} I, \quad (2.6.12)$$

$$\gamma^i \gamma^j \gamma^k \gamma^l \gamma_i = -2\gamma^l \gamma^k \gamma^j, \quad (2.6.13)$$

$$\gamma^i \gamma^j \gamma^k = \eta^{ij} \gamma^k + \eta^{jk} \gamma^i - \eta^{ik} \gamma^j + i\epsilon^{ijkl} \gamma_l \gamma^5. \quad (2.6.14)$$

The Dirac representation is not unique; the relation (1.7.1) is invariant under a similarity transformation $\gamma^i \rightarrow S \gamma^i S^{-1}$, where S is a nondegenerate ($\det S \neq 0$) matrix. Accordingly, $\psi \rightarrow S\psi$ and $\bar{\psi} \rightarrow \bar{\psi} S^{-1}$. Taking $S = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ turns the Dirac representation into the *chiral* or *Weyl representation*, in which

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \quad (2.6.15)$$

For an infinitesimal Lorentz transformation (1.6.7), the relations (1.7.5) and (1.7.6) give $L = I + \frac{1}{8}\epsilon_{ab}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, so

$$L^\dagger = I + \frac{1}{8}\epsilon_{ab}(\gamma^{b\dagger}\gamma^{a\dagger} - \gamma^{a\dagger}\gamma^{b\dagger}) \quad (2.6.16)$$

is equal to L^{-1} (so L is unitary) for rotations and equal to L for boosts. The relation (2.6.5) gives then

$$L^\dagger\gamma^0 = \gamma^0 + \frac{1}{8}\epsilon_{ab}(\gamma^{b\dagger}\gamma^{a\dagger} - \gamma^{a\dagger}\gamma^{b\dagger})\gamma^0 = \gamma^0 - \frac{1}{8}\epsilon_{ab}\gamma^0(\gamma^a\gamma^b - \gamma^b\gamma^a) = \gamma^0 L^{-1}. \quad (2.6.17)$$

Thus the quantity $\psi^\dagger\gamma^0$ transforms under (1.7.7) like an adjoint spinor:

$$\psi^\dagger\gamma^0 \rightarrow \psi^\dagger L^\dagger\gamma^0 = \psi^\dagger\gamma^0 L^{-1}. \quad (2.6.18)$$

The spinors ψ and $\psi^\dagger\gamma^0$ can be used to construct tensors, as in (1.7.11): $\psi^\dagger\gamma^0\psi$ transforms like a scalar, $\psi^\dagger\gamma^0\gamma^i\psi$ is a vector, $\psi^\dagger\gamma^0\gamma^{[i}\gamma^{j]}\psi$ is an antisymmetric tensor, $\psi^\dagger\gamma^0\gamma^5\psi$ is a pseudoscalar (dual scalar density), and $\psi^\dagger\gamma^0\gamma^i\gamma^5\psi$ is a pseudovector (dual vector density). Higher-rank tensors constructed from ψ and $\psi^\dagger\gamma^0$ reduce to the above 5 kinds of tensors because of (2.6.14). Hereinafter, we will use $\bar{\psi}$ to denote $\psi^\dagger\gamma^0$. To show that $\bar{\psi}\gamma^5\psi$ transforms like a pseudoscalar, we substitute (1.7.4) into (2.6.7) and use (2.6.6), which gives

$$\gamma^5 = i\Lambda^0{}_a\Lambda^1{}_b\Lambda^2{}_c\Lambda^3{}_d L\gamma^a\gamma^b\gamma^c\gamma^d L^{-1} = i\Lambda^0{}_a\Lambda^1{}_b\Lambda^2{}_c\Lambda^3{}_d L\gamma^{[a}\gamma^b\gamma^c\gamma^{d]} L^{-1}. \quad (2.6.19)$$

Using (1.1.22) and

$$\gamma^{[a}\gamma^b\gamma^c\gamma^{d]} = -ie^{abcd}\gamma^5, \quad (2.6.20)$$

which results from (2.6.7), gives

$$\gamma^5 = e^{abcd}\Lambda^0{}_a\Lambda^1{}_b\Lambda^2{}_c\Lambda^3{}_d L\gamma^5 L^{-1} = \det(\Lambda^a{}_b)L\gamma^5 L^{-1}. \quad (2.6.21)$$

Therefore we have

$$\bar{\psi}\gamma^5\psi \rightarrow \bar{\psi}L^{-1}\det(\Lambda^a{}_b)L\gamma^5 L^{-1}L\psi = \det(\Lambda^a{}_b)\bar{\psi}\gamma^5\psi, \quad (2.6.22)$$

which is the transformation law for a Lorentz scalar density. Similarly,

$$\bar{\psi}\gamma^c\gamma^5\psi \rightarrow \bar{\psi}L^{-1}\Lambda^c{}_d\det(\Lambda^a{}_b)L\gamma^d\gamma^5 L^{-1}L\psi = \det(\Lambda^a{}_b)\Lambda^c{}_d\bar{\psi}\gamma^d\gamma^5\psi, \quad (2.6.23)$$

which is the transformation law for a Lorentz vector density.

We define the *chirality projection operators*

$$P_\pm = \frac{I \pm \gamma^5}{2}, \quad P_+ + P_- = I, \quad P_\pm^2 = I, \quad P_+P_- = P_-P_+ = 0. \quad (2.6.24)$$

They project a spinor ψ into the *right-handed* spinor ψ_R and *left-handed* spinor ψ_L ,

$$\psi_R = P_+\psi, \quad \psi_L = P_-\psi, \quad \psi = \psi_R + \psi_L. \quad (2.6.25)$$

If we split a spinor ψ into two two-component parts,

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (2.6.26)$$

then, in the chiral representation:

$$\psi_L = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (2.6.27)$$

An infinitesimal rotation is described by a Lorentz matrix (1.6.7) with $\epsilon_{0\alpha} = 0$. The corresponding spinor transformation matrix (1.7.5) in the Dirac representation is, using (1.6.25),

$$L = I + \frac{1}{4}\epsilon_{\alpha\beta}\gamma^\alpha\gamma^\beta = I + \frac{1}{4}e_{\alpha\beta}\gamma^\alpha\gamma^\beta\vartheta_\gamma = I - \frac{i}{2}\vartheta_\alpha \begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\alpha \end{pmatrix}. \quad (2.6.28)$$

Thus the (unitary) spinor transformation matrix for a finite rotation by an angle ϑ about an axis parallel to a unit vector $\mathbf{n}$, $\boldsymbol{\vartheta} = \vartheta \mathbf{n}$, is

$$L = \exp \left[-\frac{i}{2} \boldsymbol{\vartheta} \cdot \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \right] = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} \mathbf{n} \cdot \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \quad (2.6.29)$$

If we split a spinor ψ into two parts u and v (2.6.26), then both u and v transform under rotations according to

$$u \rightarrow Su, \quad v \rightarrow Sv, \quad (2.6.30)$$

where

$$S = \cos \frac{\vartheta}{2} I - i \sin \frac{\vartheta}{2} \mathbf{n} \cdot \boldsymbol{\sigma}. \quad (2.6.31)$$

A rotation by a full angle 2π changes the sign of a spinor. A rotation by 4π brings a spinor into its original position.

An infinitesimal boost is described by a Lorentz matrix (1.6.7) with $\epsilon_{\alpha\beta} = 0$. The corresponding spinor transformation matrix (1.7.5) in the Dirac representation is, using (1.6.26),

$$L = I + \frac{1}{2} \epsilon_{0\alpha} \gamma^0 \gamma^\alpha = I + \frac{1}{2} \eta_\alpha \gamma^0 \gamma^\alpha = I + \frac{1}{2} \eta_\alpha \begin{pmatrix} 0 & \sigma_\alpha \\ \sigma_\alpha & 0 \end{pmatrix}. \quad (2.6.32)$$

Thus the spinor transformation matrix for a finite boost with a rapidity η along an axis parallel to a unit vector $\mathbf{n}$, $\boldsymbol{\eta} = \eta \mathbf{n}$, is

$$L = \exp \left[\frac{1}{2} \boldsymbol{\eta} \cdot \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \right] = \cosh \frac{\eta}{2} I + \sinh \frac{\eta}{2} \mathbf{n} \cdot \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \quad (2.6.33)$$

Using (1.6.90) gives

$$L = \frac{1}{\sqrt{2(1+\gamma)}} \begin{pmatrix} 1+\gamma & \gamma \boldsymbol{\beta} \cdot \boldsymbol{\sigma} \\ \gamma \boldsymbol{\beta} \cdot \boldsymbol{\sigma} & 1+\gamma \end{pmatrix}. \quad (2.6.34)$$

If this boost transforms a particle of mass m from rest to a motion with momentum $\mathbf{p}$ and energy E then (2.6.34) is equivalent, due to (2.4.209) and (2.4.210), to

$$L = \frac{1}{\sqrt{2mc^2(E+mc^2)}} \begin{pmatrix} E+mc^2 & c\boldsymbol{\sigma} \cdot \mathbf{p} \\ c\boldsymbol{\sigma} \cdot \mathbf{p} & E+mc^2 \end{pmatrix}. \quad (2.6.35)$$

2.6.2 Dirac equation

A Lagrangian density for dynamical spinor fields must contain first derivatives of spinors. The simplest scalar containing derivatives of spinors is quadratic in ψ , $\bar{\psi} \gamma^i \psi_{;i}$, where $\psi_{;i}$ is the covariant derivative of ψ (1.7.14). This quantity is complex. In the locally inertial frame of reference, its complex conjugate is

$$(\bar{\psi} \gamma^i \psi_{;i})^* = (\bar{\psi} \gamma^i \psi_{;i})^\dagger = \psi_{;i}^\dagger \gamma^{i\dagger} \bar{\psi}^\dagger = \bar{\psi}_{;i} \gamma^0 \gamma^{i\dagger} \gamma^0 \psi = \bar{\psi}_{;i} \gamma^i \psi, \quad (2.6.36)$$

so both $\bar{\psi} \gamma^i \psi_{;i} + \bar{\psi}_{;i} \gamma^i \psi$ and $i(\bar{\psi} \gamma^i \psi_{;i} - \bar{\psi}_{;i} \gamma^i \psi)$ are real. The former is, however, equal to a total divergence $(\bar{\psi} \gamma^i \psi)_{;i}$, so a Lagrangian density proportional to such term does not contribute to field equations. Thus the simplest dynamical part of a spinor Lagrangian density is proportional to $i(\bar{\psi} \gamma^i \psi_{;i} - \bar{\psi}_{;i} \gamma^i \psi)$. Another scalar that can be used in a spinor Lagrangian is proportional to $\bar{\psi} \psi$. Therefore the simplest Lagrangian density for spinor fields, in the locally Galilean frame of reference, has the form

$$\mathfrak{L}_\psi = \frac{i}{2} (\bar{\psi} \gamma^i \psi_{;i} - \bar{\psi}_{;i} \gamma^i \psi) - m \bar{\psi} \psi, \quad (2.6.37)$$

where m is a real scalar constant called the *spinor mass*, and it is referred to as the *Dirac Lagrangian density*. For any frame of reference,

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2} (\bar{\psi} \gamma^i \psi_{;i} - \bar{\psi}_{;i} \gamma^i \psi) - m \mathfrak{e} \bar{\psi} \psi = \frac{i\mathfrak{e}}{2} e_a^i (\bar{\psi} \gamma^a \psi_{;i} - \bar{\psi}_{;i} \gamma^a \psi) - m \mathfrak{e} \bar{\psi} \psi. \quad (2.6.38)$$

We consider the metric formulation of gravity with the Einstein-Hilbert action (2.5.1). Therefore spacetime has the Riemannian geometry, so $\psi_{;i} = \psi_{:i}$. Varying (2.6.38) with respect to $\bar{\psi}$ and omitting total derivatives gives

$$\delta \mathfrak{L}_\psi = \delta \bar{\psi} (i\gamma^i \psi_{:i} - m\psi), \quad (2.6.39)$$

so the stationarity of the action $\delta S = 0$ under $\delta \bar{\psi}$ gives the *Dirac equation*:

$$i\gamma^i \psi_{:i} = m\psi. \quad (2.6.40)$$

Varying (2.6.38) with respect to ψ and omitting total derivatives gives the adjoint conjugate of (2.6.40):

$$-i\bar{\psi}_{:i} \gamma^i = m\bar{\psi}. \quad (2.6.41)$$

The Dirac equation is linear in ψ , so ψ can be multiplied by an arbitrary constant without altering (2.6.40). Varying (2.6.38) with respect to e_a^i gives the tetrad energy-momentum density for the spinor field,

$$\mathfrak{T}_i^a = \frac{i\mathfrak{e}}{2} (\bar{\psi} \gamma^a \psi_{:i} - \bar{\psi}_{:i} \gamma^a \psi - e_i^a \bar{\psi} \gamma^j \psi_{:j} + e_i^a \bar{\psi}_{:j} \gamma^j \psi) + m\mathfrak{e} e_i^a \bar{\psi} \psi, \quad (2.6.42)$$

so

$$T_{ik} = \frac{i}{2} (\bar{\psi} \gamma_{(k} \psi_{:i)} - \bar{\psi}_{:(i} \gamma_{k)} \psi - g_{ik} \bar{\psi} \gamma^j \psi_{:j} + g_{ik} \bar{\psi}_{:j} \gamma^j \psi) + m g_{ik} \bar{\psi} \psi. \quad (2.6.43)$$

The conservation law (2.4.24) applied to the energy-momentum tensor (2.6.43) gives the Dirac equations (2.6.40) and (2.6.41).

Subtracting (2.6.41) multiplied by ψ from (2.6.40) multiplied by $\bar{\psi}$ gives, using (1.7.33) and $\psi|_i = \psi_{:i}$,

$$(\bar{\psi} \gamma^i \psi)|_i = (\bar{\psi} \gamma^i \psi)_{:i} = 0, \quad (2.6.44)$$

so the vector density

$$\mathbf{j}_V^i = \mathfrak{e} \bar{\psi} \gamma^i \psi, \quad (2.6.45)$$

called the *vector Dirac current*, is conserved: $\mathbf{j}_{V,i}^i = 0$ or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (2.6.46)$$

where

$$c\rho = \mathfrak{e} \psi^\dagger \psi, \quad \mathbf{j} = \mathfrak{e} \bar{\psi} \gamma \psi. \quad (2.6.47)$$

The vector Dirac current is real because

$$(\bar{\psi} \gamma^i \psi)^* = (\psi^\dagger \gamma^0 \gamma^i \psi)^\dagger = \psi^\dagger \gamma^{i\dagger} \gamma^0 \psi = \psi^\dagger \gamma^0 \gamma^i \psi = \bar{\psi} \gamma^i \psi. \quad (2.6.48)$$

The spinor density ρ is real and positive. The conservation law (2.6.46) is referred to as the equation of continuity, like (2.5.96).

The Dirac equation (2.6.40) gives

$$-\gamma^j (\gamma^i \psi_{:i})|_j = im \gamma^j \psi_{:j} \quad (2.6.49)$$

or, due to (1.7.33),

$$-\gamma^j \gamma^i \psi_{:ij} = m^2 \psi. \quad (2.6.50)$$

Using (1.7.30) (in the absence of torsion) and (1.7.39) turns (2.6.50) into the *Klein-Gordon-Fock equation*:

$$\psi_{:i}^{:i} + m^2 \psi = \frac{1}{8} P_{kl ij} \gamma^k \gamma^l \gamma^i \gamma^j \psi, \quad (2.6.51)$$

where P_{ijkl} is the Riemann curvature tensor. If a spinor is equal to its either left- or right-handed projection, $\psi = \psi_L$ or $\psi = \psi_R$, then it is called a *Weyl spinor*. Multiplying (2.6.40) by $P_\pm$ gives

$$iP_\pm \gamma^i \psi_{:i} = i\gamma^i P_\mp \psi_{:i} = mP_\pm \psi \quad (2.6.52)$$

or

$$i\gamma^i \psi_{:i}^{L(R)} = m\psi^{R(L)}. \quad (2.6.53)$$

Thus if ψ is a Weyl spinor then $m = 0$.

2.6.3 Spinors in Einstein-Cartan-Sciama-Kibble gravity

We consider the metric-affine formulation of gravity with the Einstein-Cartan action (2.5.47), in which spacetime has the Riemann-Cartan geometry. Varying (2.6.38) with respect to e_a^i gives the tetrad energy-momentum density for the spinor field,

$$\mathfrak{T}_i^a = \frac{i\epsilon}{2}(\bar{\psi}\gamma^a\psi_{;i} - \bar{\psi}_{;i}\gamma^a\psi - e_i^a\bar{\psi}\gamma^j\psi_{;j} + e_i^a\bar{\psi}_{;j}\gamma^j\psi) + m\epsilon e_i^a\bar{\psi}\psi. \quad (2.6.54)$$

Putting the definition of the covariant derivative of a spinor (1.7.14) into (2.6.38) gives

$$\mathfrak{L}_\psi = \frac{i\epsilon}{2}(\bar{\psi}\gamma^i\psi_{;i} - \bar{\psi}_{;i}\gamma^i\psi) - \frac{i\epsilon}{2}\bar{\psi}\{\gamma^i, \Gamma_i\}\psi - m\epsilon\bar{\psi}\psi. \quad (2.6.55)$$

Using the Fock-Ivanenko coefficients (1.7.28) as the spinor connection Γ_i turns (2.6.55) into

$$\mathfrak{L}_\psi = \frac{i\epsilon}{2}(\bar{\psi}\gamma^i\psi_{;i} - \bar{\psi}_{;i}\gamma^i\psi) + \frac{i\epsilon}{8}\omega_{abi}\bar{\psi}\{\gamma^i, \gamma^a\gamma^b\}\psi - m\epsilon\bar{\psi}\psi. \quad (2.6.56)$$

The spin density (2.3.17) corresponding to the Lagrangian density (2.6.56) is, due to the identity $\{\gamma^i, \gamma^j\gamma^k\} = 2\gamma^{[i}\gamma^j\gamma^{k]}$,

$$\mathfrak{S}^{ijk} = \frac{i\epsilon}{2}\bar{\psi}\gamma^{[i}\gamma^j\gamma^{k]}\psi. \quad (2.6.57)$$

The spin density (2.6.57) is totally antisymmetric,

$$\mathfrak{S}^{ijk} = \mathfrak{S}^{[ijk]}, \quad (2.6.58)$$

and independent of the spinor mass m . The corresponding spin tensor is also totally antisymmetric,

$$s^{ijk} = \frac{i}{2}\bar{\psi}\gamma^{[i}\gamma^j\gamma^{k]}\psi = s^{[ijk]} = -e^{ijkl}s_l, \quad (2.6.59)$$

where

$$s^i = \frac{1}{2}\bar{\psi}\gamma^i\gamma^5\psi. \quad (2.6.60)$$

The second Einstein-Cartan equation (2.5.57) for the spin tensor (2.6.59) gives a totally antisymmetric torsion tensor,

$$S_{ijk} = -\frac{i\kappa}{4}\bar{\psi}\gamma_{[i}\gamma_j\gamma_{k]}\psi, \quad (2.6.61)$$

so $S_i = 0$. Thus the contortion tensor is, using (2.6.14),

$$C_{ijk} = \frac{\kappa}{4}\epsilon_{ijkl}\bar{\psi}\gamma^l\gamma^5\psi. \quad (2.6.62)$$

The pseudovector density

$$j_A^i = \epsilon\bar{\psi}\gamma^i\gamma^5\psi = 2\epsilon s^i \quad (2.6.63)$$

is called the *axial Dirac current*. The axial Dirac current and the pseudovector (2.6.60) are real because

$$(\bar{\psi}\gamma^i\gamma^5\psi)^* = (\psi^\dagger\gamma^0\gamma^i\gamma^5\psi)^\dagger = \psi^\dagger\gamma^5\gamma^i\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^i\psi = \psi^\dagger\gamma^0\gamma^i\gamma^5\psi = \bar{\psi}\gamma^i\gamma^5\psi. \quad (2.6.64)$$

Thus the spin tensor (2.6.59) is also real.

Varying (2.6.56) with respect to $\bar{\psi}$ gives, after omitting total divergences,

$$\frac{i}{2}(\epsilon\gamma^k\psi_{;k} + (\epsilon\gamma^k\psi)_{;k} - \epsilon\{\Gamma_k, \gamma^k\}\psi) - \epsilon m\psi = 0. \quad (2.6.65)$$

Substituting

$$(\epsilon\gamma^k\psi)_{;k} = \epsilon\gamma^k\psi_{;k} + \epsilon\gamma^k_{;k}\psi - 2\epsilon S_k\gamma^k\psi = \epsilon\gamma^k\psi_{;k} + \epsilon[\Gamma_k, \gamma^k]\psi \quad (2.6.66)$$

into (2.6.65) gives

$$i\gamma^k\psi_{;k} - i\gamma^k\Gamma_k\psi - m\psi = i\gamma^k\psi_{;k} - m\psi = 0. \quad (2.6.67)$$

The relation (1.5.33) gives

$$\psi_{;k} = \psi_{:k} + \frac{1}{4}C_{ijk}\gamma^i\gamma^j\psi, \quad (2.6.68)$$

from which we obtain, upon substituting (2.6.62),

$$\begin{aligned} \gamma^k\psi_{;k} &= \gamma^k\psi_{:k} + \frac{\kappa}{16}\epsilon_{ijkl}(\bar{\psi}\gamma^l\gamma^5\psi)\gamma^k\gamma^i\gamma^j\psi = \gamma^k\psi_{:k} + \frac{i\kappa}{16}\epsilon_{ijkl}(\bar{\psi}\gamma^l\gamma^5\psi)\epsilon^{ijkm}\gamma_m\gamma^5\psi \\ &= \gamma^k\psi_{:k} - \frac{3i\kappa}{8}(\bar{\psi}\gamma^l\gamma^5\psi)\gamma_l\gamma^5\psi. \end{aligned} \quad (2.6.69)$$

Therefore (2.6.67) becomes the *Hehl-Datta equation*:

$$i\gamma^k\psi_{;k} + \frac{3\kappa}{8}(\bar{\psi}\gamma_k\gamma^5\psi)\gamma^k\gamma^5\psi = m\psi. \quad (2.6.70)$$

Varying (2.6.56) with respect to ψ gives the adjoint conjugate of (2.6.70),

$$-i\bar{\psi}_{;k}\gamma^k + \frac{3\kappa}{8}(\bar{\psi}\gamma_k\gamma^5\psi)\bar{\psi}\gamma^k\gamma^5 = m\bar{\psi}. \quad (2.6.71)$$

The Hehl-Datta equation (2.6.70) differs from the Dirac equation (2.6.40) by a nonlinear term, cubic in the spinor field and representing a spinor self-interaction, corresponding to a spin-spin interaction in the tensor (2.5.58). The conservation law (2.4.32) applied to the energy-momentum density (2.6.54) gives the Hehl-Datta equations (2.6.70) and (2.6.71). Subtracting (2.6.71) multiplied by ψ from (2.6.70) multiplied by $\bar{\psi}$ gives the conservation of the vector Dirac current (2.6.45). In the metric formulation of gravity, (2.6.70) and (2.6.71) are the field equations corresponding to the *Hehl-Datta Lagrangian density*:

$$\mathfrak{L}_\psi = \frac{i\epsilon}{2}(\bar{\psi}\gamma^i\psi_{;i} - \bar{\psi}_{;i}\gamma^i\psi) - m\epsilon\bar{\psi}\psi + \frac{3\kappa\epsilon}{16}(\bar{\psi}\gamma_k\gamma^5\psi)(\bar{\psi}\gamma^k\gamma^5\psi). \quad (2.6.72)$$

The total antisymmetry of the spin density (2.6.58) implies

$$N^{ijk} = N^{[ijk]}, \quad (2.6.73)$$

where N^{ijk} is given by (2.4.106). We also have

$$N^{ijk} = 3S^{[ij}u^{k]}, \quad (2.6.74)$$

where S^{ij} is the intrinsic spin tensor (2.4.175). The covariant (with respect to the Levi-Civita connection) change (2.4.148) of the spin pseudovector along a world line becomes

$$\frac{D\{J^i\}}{ds} = -u^i\frac{D\{u^k\}}{ds}J_k + \frac{D\{u^i\}}{ds}u^kJ_k + \frac{3}{2}e^{ij}_{nm}u_jS^n_{ik}N^{ikm} = 3S^i_{jk}u^jN^k, \quad (2.6.75)$$

where

$$N^i = \frac{1}{6}e^{ijkl}N_{jkl}. \quad (2.6.76)$$

If $N^i \propto J^i$ then (2.6.75) gives $J^iJ_i = \text{const.}$ For a point or a system of points, $M^{i[jk]}$ given by (2.4.105) vanishes. Thus (2.4.131) reduces to

$$N^{ijl} = \frac{u^l}{u^0}N^{ij0}, \quad (2.6.77)$$

which for a spinor particle gives $N^{i0} = -\frac{u^i}{u^0}N^{i00}$ and thus $N^{ijk} = 0$ or

$$\psi = 0. \quad (2.6.78)$$

Therefore a spinor field in the Einstein-Cartan-Sciama-Kibble gravity cannot be approximated as a point particle or a system of point particles.

For a totally antisymmetric spin tensor, the correction to the dynamical energy-momentum tensor from the spin contributions to the geometry of spacetime (2.5.58) reduces to

$$U^{ik} = \frac{\kappa}{4} \left(s^{ijl} s^k{}_{jl} - \frac{1}{2} g^{ik} s^{jlm} s_{jlm} \right). \quad (2.6.79)$$

The relation (2.6.74) suggests that

$$s^{ijk} = 3s^{[ij}u^{k]}, \quad (2.6.80)$$

where s^{ij} is the spin-density tensor (2.4.181). Analogously to (2.4.145), we can define the *spin-density pseudovector*

$$s^i = \frac{1}{2} e^{ijkl} u_j s_{kl}, \quad (2.6.81)$$

which is orthogonal to u^i ,

$$s^i u_i = 0. \quad (2.6.82)$$

The inverse relation is

$$s^{ik} = -e^{ikjl} u_j s_l. \quad (2.6.83)$$

The spin-density pseudovector (2.6.81) coincides with the pseudovector (2.6.60).

2.6.4 Discrete symmetries of spinors

The spinor representation of the parity transformation (1.6.3) is given by

$$L_P = C\gamma^0, \quad (2.6.84)$$

where $C = \text{const.}$ Indeed, substituting (3.1.170) into (1.7.4) gives, using (2.6.5),

$$\gamma^a = \Lambda^a{}_b \gamma^0 \gamma^b \gamma^0 = \Lambda^a{}_b \gamma^{b\dagger}, \quad (2.6.85)$$

which holds if $\Lambda^a{}_b = \Lambda^a{}_b(P)$. Since the double parity transformation is equivalent to the identity transformation, $L_P^2 = I$, we have $C = \pm 1$. The spinor representation of the time-reversal transformation (1.6.4) is given by

$$L_T = C\gamma^0\gamma^5, \quad (2.6.86)$$

where $C = \text{const.}$ Indeed, substituting (3.1.172) into (1.7.4) gives, using (2.6.5) and (2.6.8),

$$\gamma^a = \Lambda^a{}_b \gamma^0 \gamma^5 \gamma^b \gamma^5 \gamma^0 = -\Lambda^a{}_b \gamma^{b\dagger}, \quad (2.6.87)$$

which holds if $\Lambda^a{}_b = \Lambda^a{}_b(T)$. Since the double time-reversal transformation is equivalent to the identity transformation, $L_T^2 = I$, we have $C = \pm i$.

The *charge conjugation* of a spinor ψ is defined as

$$\psi^c = -i\gamma^2\psi^*, \quad \psi^* = -i\gamma^2\psi^c. \quad (2.6.88)$$

The double charge-conjugation transformation is equivalent to the identity transformation:

$$(\psi^c)^c = -i\gamma^2(\psi^c)^* = -i\gamma^2(-i\gamma^2\psi^*)^* = \gamma^2\gamma^{2*}\psi = \psi. \quad (2.6.89)$$

The charge conjugation of the left-handed projection of a spinor is the right-handed projection of the charge conjugation of the spinor and vice versa (cf. (2.6.25)):

$$\begin{aligned} \left((I \mp \gamma^5)\psi \right)^c &= -i\gamma^2 \left((I \mp \gamma^5)\psi \right)^* = -i\gamma^2 (I \mp \gamma^5)\psi^* = -i(I \pm \gamma^5)\gamma^2\psi^* \\ &= (I \pm \gamma^5)\gamma^c. \end{aligned} \quad (2.6.90)$$

2.7 Electromagnetic field

2.7.1 Gauge invariance and electromagnetic potential

The Lagrangian density (2.6.38) is a real combination of the complex Dirac matrices γ^i and spinors $\psi, \bar{\psi}$. It is invariant under a *gauge transformation of the first type* of the spinor fields,

$$\psi \rightarrow \psi' = e^{ie\alpha}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = e^{-ie\alpha}\bar{\psi}, \quad (2.7.1)$$

if $e\alpha$ is a real scalar constant, but it is not invariant for $e\alpha(x^i)$, because

$$\psi'_{;\mu} = e^{ie\alpha}(\psi_{;\mu} + ie\alpha_{;\mu}\psi). \quad (2.7.2)$$

Introduce a compensating vector field A_μ , called the *electromagnetic potential*, such that the Weyl or *electromagnetic covariant derivative*

$$D_\mu = \nabla_\mu - ieA_\mu \quad (2.7.3)$$

of a spinor ψ ,

$$D_\mu\psi = \psi_{;\mu} - ieA_\mu\psi, \quad (2.7.4)$$

transforms under (2.7.1) like ψ :

$$D_\mu\psi' = e^{ie\alpha}D_\mu\psi. \quad (2.7.5)$$

This requirement gives

$$\psi'_{;\mu} - ieA'_\mu\psi' = e^{ie\alpha}(\psi_{;\mu} - ieA_\mu\psi), \quad (2.7.6)$$

which, with (2.7.1) and (2.7.2), yields the transformation law for the electromagnetic potential,

$$A'_\mu = A_\mu + \alpha_{;\mu}, \quad (2.7.7)$$

called a *gauge transformation of the second type*. The real scalar constant e is called the *spinor electric charge*. The adjoint conjugation of (2.7.4) is

$$D_\mu\bar{\psi} = \bar{\psi}_{;\mu} + ieA_\mu^*\bar{\psi}. \quad (2.7.8)$$

The scalar $\bar{\psi}\psi$ is invariant under (2.7.1), so

$$D_\mu(\bar{\psi}\psi) = \partial_\mu(\bar{\psi}\psi), \quad (2.7.9)$$

which constraints the electromagnetic potential to be real:

$$A_\mu^* = A_\mu. \quad (2.7.10)$$

The time component of A^μ , $\phi = A^0$, is called the *electric potential* and the spatial components A^α form the *magnetic potential* $\mathbf{A}$:

$$A^\mu = (\phi, \mathbf{A}). \quad (2.7.11)$$

The gauge transformation (2.7.7) reads

$$\phi' = \phi + \frac{\partial\alpha}{c\partial t}, \quad \mathbf{A}' = \mathbf{A} - \nabla\alpha. \quad (2.7.12)$$

In the local Minkowski spacetime, A^μ transforms according to (1.6.99),

$$\begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta} \\ \gamma\boldsymbol{\beta} & 1 + \frac{(\gamma-1)\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2} \end{pmatrix} \begin{pmatrix} \phi' \\ \mathbf{A}' \end{pmatrix}. \quad (2.7.13)$$

The gauge-invariant modification of the Dirac Lagrangian density (2.6.38) is

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2}e_a^i(\bar{\psi}\gamma^a D_i\psi - D_i\bar{\psi}\gamma^a\psi) - m\mathfrak{e}\bar{\psi}\psi. \quad (2.7.14)$$

The spin density corresponding to the Lagrangian density (2.7.14) remains equal to (2.6.57); it is independent of the electric charge e . The electromagnetic potential corresponds, up to the multiplication by an arbitrary constant, to the vector multiple of I in the formula for the spinor connection (1.7.26). The electromagnetic potential is analogous to the affine connection: it modifies a derivative of a spinor so such derivative transforms like a spinor under unitary gauge transformations of the first type, while the connection modifies a derivative of a tensor so such derivative transforms like a tensor under coordinate transformations.

The gauge-invariant modification of the Dirac equation (2.6.40) is

$$i\gamma^k\psi_{;k} + eA_k\gamma^k\psi = m\psi, \quad (2.7.15)$$

whose adjoint conjugate is

$$-i\bar{\psi}_{;k}\gamma^k + eA_k\bar{\psi}\gamma^k = m\bar{\psi}. \quad (2.7.16)$$

The gauge-invariant modification of the Hehl-Datta equation (2.6.70) is

$$i\gamma^k\psi_{;k} + eA_k\gamma^k\psi = m\psi - \frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^5\gamma^k\psi, \quad (2.7.17)$$

whose adjoint conjugate is

$$-i\bar{\psi}_{;k}\gamma^k + eA_k\bar{\psi}\gamma^k = m\bar{\psi} - \frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\bar{\psi}\gamma^5\gamma^k. \quad (2.7.18)$$

Taking the complex conjugate of (2.7.17) gives, using (2.6.64),

$$-i\gamma^{k*}\psi_{;k}^* + eA_k\gamma^{k*}\psi^* = m\psi^* - \frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^{5*}\gamma^{k*}\psi^*. \quad (2.7.19)$$

Using (2.6.5), (2.6.8), (2.6.9) and (2.6.88) turns (2.7.19) into

$$i\gamma^k\psi_{;k}^c - eA_k\gamma^k\psi^c = m\psi^c - \frac{3\kappa}{8}(\bar{\psi}\gamma^5\gamma_k\psi)\gamma^5\gamma^k\psi^c. \quad (2.7.20)$$

Finally, one can show that

$$\bar{\psi}\gamma^5\gamma_k\psi = (\bar{\psi}\gamma^5\gamma_k\psi)^* = -\bar{\psi}^c\gamma^5\gamma_k\psi^c, \quad (2.7.21)$$

so (2.7.20) reads

$$i\gamma^k\psi_{;k}^c - eA_k\gamma^k\psi^c = m\psi^c + \frac{3\kappa}{8}(\bar{\psi}^c\gamma^5\gamma_k\psi^c)\gamma^5\gamma^k\psi^c. \quad (2.7.22)$$

Comparing (2.7.22) with (2.6.70) shows that ψ and ψ^c satisfy, apart from the cubic term, the same equation for e and $-e$, respectively. Thus the charge-conjugation transformation changes sign of the spinor electric charge. In the Einstein gravity, the cubic term vanishes; (ψ, A_i) and $(\psi^c, -A_i)$ are symmetric under the charge-conjugation transformation. In the Einstein-Cartan gravity, the cubic term does not vanish and it changes sign under the charge-conjugation transformation. In this case, (ψ, A_i) and $(\psi^c, -A_i)$ are not symmetric under the charge-conjugation transformation: the torsion generates an asymmetry between a spinor and its charge conjugate.

2.7.2 Electromagnetic field tensor

The commutator of total covariant derivatives of a spinor is given by (1.7.30) with the curvature spinor K_{ij} given by (1.7.36), where the tensor B_{ij} is related to the vector A_i in (1.7.26) by (1.7.38). Therefore the commutator of the electromagnetic covariant derivatives of a spinor, $[D_i, D_j]\psi$, is given by (1.7.30) with the curvature spinor

$$K_{ij} = \frac{1}{4}R_{kl ij}\gamma^k\gamma^l + ieF_{ij}I, \quad (2.7.23)$$

where the antisymmetric tensor

$$F_{ij} = A_{j,i} - A_{i,j} = A_{j;i} - A_{i;j} \quad (2.7.24)$$

is referred to as the *electromagnetic field tensor*. The electromagnetic field tensor is analogous to the curvature tensor: it appears in the expression for the commutator of electromagnetic covariant derivatives of a spinor, while the curvature tensor appears in the expression for the commutator of coordinate-covariant derivatives of a tensor. Substituting (2.7.7) into (2.7.24) gives

$$F'_{ij} = F_{ij}, \quad (2.7.25)$$

so the electromagnetic field tensor is gauge invariant. The definition (2.7.24) is equivalent to the *first Maxwell-Minkowski equation*

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \quad (2.7.26)$$

or

$$\epsilon^{ijkl} F_{jk,l} = \epsilon^{ijkl} F_{j;k;l} = 0. \quad (2.7.27)$$

We define a spatial vector $\mathbf{E}$ whose covariant components are related to the 0α components of the electromagnetic field tensor (2.7.24):

$$E_\alpha = F_{0\alpha}, \quad (2.7.28)$$

and a spatial tensor $B_{\alpha\beta}$ equal to the spatial part of (2.7.24):

$$B_{\alpha\beta} = F_{\alpha\beta}, \quad B^\alpha = -\frac{1}{2\sqrt{\mathfrak{s}}} \epsilon^{\alpha\beta\gamma} B_{\beta\gamma}, \quad B_{\alpha\beta} = -\sqrt{\mathfrak{s}} \epsilon_{\alpha\beta\gamma} B^\gamma, \quad (2.7.29)$$

where $\mathfrak{s}$ is given by (1.4.110). The component of (2.7.26) with all spatial indices, $B_{\alpha\beta,\gamma} + B_{\beta\gamma,\alpha} + B_{\gamma\alpha,\beta} = 0$, gives, using (1.4.126),

$$\text{div } \mathbf{B} = 0. \quad (2.7.30)$$

The components of (2.7.26) with one temporal index, $B_{\alpha\beta,0} + E_{\alpha,\beta} - E_{\beta,\alpha} = 0$, gives, using (1.4.127),

$$\text{curl } \mathbf{E} = -\frac{1}{c\sqrt{\mathfrak{s}}} \frac{\partial(\sqrt{\mathfrak{s}}\mathbf{B})}{\partial t}. \quad (2.7.31)$$

The spatial vector $\mathbf{E}$ is called the *electric field* and the spatial pseudovector $\mathbf{B}$ is the *magnetic field*.

In the locally geodesic and Galilean frame of reference, these fields depend on the components of the electromagnetic potential (2.7.11) according to (2.7.24):

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{c\partial t} - \nabla \phi, \quad (2.7.32)$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (2.7.33)$$

and they are invariant under (2.7.12). The tensor F_{ij} is given by

$$F_{ij} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad \mathbf{E} = (F_{01}, F_{02}, F_{03}), \quad \mathbf{B} = (F_{32}, F_{13}, F_{21}), \quad (2.7.34)$$

and transforms according to (1.6.100). Thus the electric and magnetic fields transform according to

$$\mathbf{E} = \gamma(\mathbf{E}' - \boldsymbol{\beta} \times \mathbf{B}') + \frac{1-\gamma}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{E}') \boldsymbol{\beta}, \quad (2.7.35)$$

$$\mathbf{B} = \gamma(\mathbf{B}' + \boldsymbol{\beta} \times \mathbf{E}') + \frac{1-\gamma}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{B}') \boldsymbol{\beta}. \quad (2.7.36)$$

In this frame, (2.7.30) and (2.7.31) become the *first pair of the Maxwell equations*:

$$\text{div } \mathbf{B} = 0, \quad (2.7.37)$$

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{c\partial t}. \quad (2.7.38)$$

Applying the div operator to (2.7.33) gives (2.7.37) and applying the curl operator to (2.7.32) gives (2.7.38). Applying the div operator to (2.7.38) gives (2.7.37). Integrating the first pair of the Maxwell equations over the volume and surface area, respectively, gives

$$\oint \mathbf{B} \cdot d\mathbf{f} = 0, \quad (2.7.39)$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \left(\int \mathbf{B} \cdot d\mathbf{f} \right). \quad (2.7.40)$$

The integral $\oint \mathbf{A} \cdot d\mathbf{f}$ is the flux of a vector $\mathbf{A}$ through the surface $\mathbf{f}$ and the integral $\oint \mathbf{A} \cdot d\mathbf{l}$ is called the *circulation* of $\mathbf{A}$ along the contour $\mathbf{l}$. Thus the flux of the magnetic field through a closed surface vanishes and the circulation of the electric field along a contour, which is called the *electromotive force*, is equal to the minus time derivative of the flux of the magnetic field through the surface enclosed by this contour (*Faraday's law*).

In a locally Galilean frame of reference, the simplest invariants (under proper Lorentz transformations) of the electromagnetic field are quadratic in F_{ij} :

$$F_{ij}F^{ij} = 2(B^2 - E^2) = \text{const}, \quad \epsilon^{ijkl}F_{ij}F_{kl} = 8\mathbf{E} \cdot \mathbf{B} = \text{const}. \quad (2.7.41)$$

If the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular in frame K , $\mathbf{E} \cdot \mathbf{B} = 0$, then they are mutually perpendicular in other inertial frames. If $\mathbf{E}$ and $\mathbf{B}$ are equal in magnitude in K , $B^2 - E^2 = 0$, then they are equal in magnitude in other inertial frames. The transformation laws (2.7.35) and (2.7.36) imply that if $\mathbf{E}' = 0$ in the frame K' then in the frame K

$$\mathbf{E} = -\gamma\boldsymbol{\beta} \times \mathbf{B}' = -\boldsymbol{\beta} \times \mathbf{B}, \quad (2.7.42)$$

and if $\mathbf{B}' = 0$ in the frame K' then in the frame K

$$\mathbf{B} = \gamma\boldsymbol{\beta} \times \mathbf{E}' = \boldsymbol{\beta} \times \mathbf{E}. \quad (2.7.43)$$

If the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular in K , but not equal in magnitude, then there exists a frame K' in which the field is either electric, $\mathbf{B}' = 0$ (if $E > B$), or magnetic, $\mathbf{E}' = 0$ (if $E < B$). The velocity of K' relative to K is perpendicular to $\mathbf{E}$ and $\mathbf{B}$, and it is equal in magnitude to respectively either $c\frac{B}{E}$ or $c\frac{E}{B}$. Equivalently, if one of the vectors $\mathbf{E}, \mathbf{B}$ vanishes in one frame of reference then these vectors are mutually perpendicular in other inertial frames. Except for the case where the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular and equal in magnitude, there exist frames in which these vectors are parallel to each other at a given point. These frames move relative to one another with velocities parallel to both vectors. One of such frames, K' (in which $\mathbf{E}' \parallel \mathbf{B}'$), has a velocity $\mathbf{V}$ relative to K which is perpendicular to both vectors $\mathbf{E}$ and $\mathbf{B}$. Substituting the formulae

$$\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) + \frac{1-\gamma}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{E})\boldsymbol{\beta}, \quad (2.7.44)$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) + \frac{1-\gamma}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{B})\boldsymbol{\beta}, \quad (2.7.45)$$

which are inverse to (2.7.35) and (2.7.36), into the condition $\mathbf{E}' \times \mathbf{B}' = 0$ and using $\boldsymbol{\beta} = k\mathbf{E} \times \mathbf{B}$, where k is a constant of proportionality, gives $(\mathbf{E} + k(\mathbf{E} \times \mathbf{B}) \times \mathbf{B}) \times (\mathbf{B} - k(\mathbf{E} \times \mathbf{B}) \times \mathbf{E}) = 0$ or $k = \frac{1+\beta^2}{E^2+B^2}$, so

$$\frac{\frac{\mathbf{V}}{c}}{1 + \frac{V^2}{c^2}} = \frac{\mathbf{E} \times \mathbf{B}}{E^2 + B^2}. \quad (2.7.46)$$

2.7.3 Lagrangian density for electromagnetic field

The simplest gauge-invariant Lagrangian density representing the electromagnetic field is a linear combination of terms quadratic in F_{ij} : $\sqrt{-g}F_{ij}F^{ij}$ and $\epsilon^{ijkl}F_{ij}F_{kl}$, which in a locally Galilean frame of reference reduce to (2.7.41). The second term is a total divergence because of (2.7.27):

$$\epsilon^{ijkl}F_{ij}F_{kl} = 2(\epsilon^{ijkl}F_{ij}A_{l,k}), \quad (2.7.47)$$

so it does not contribute to the field equations. Thus the Lagrangian density for the electromagnetic field is given by

$$\mathfrak{L}_{EM} = -\frac{1}{16\pi}\sqrt{-\mathfrak{g}}F_{ij}F^{ij}, \quad (2.7.48)$$

where the Gaußian factor $\frac{1}{16\pi}$ sets the units of A_i . In the locally geodesic and Galilean frame of reference, (2.7.48) becomes

$$\mathfrak{L}_{EM} = \frac{1}{8\pi}(E^2 - B^2). \quad (2.7.49)$$

Therefore in order for the action S to have a minimum, there must be the minus sign in front of the right-hand side of (2.7.48). Otherwise an arbitrarily rapid change of $\mathbf{A}$ in time would result in an arbitrarily large value of $\mathbf{E}$, according to (2.7.32), and thus an arbitrarily low value of S , so the action would have no minimum. A generalization of the tensor (2.7.24) to a covariant derivative with respect to the affine connection Γ_{ij}^k , $A_{j;i} - A_{i;j} = F_{ij} + 2S_{ij}^k A_k$, is not gauge invariant, so the torsion tensor cannot appear in a gauge-invariant Lagrangian density which is quadratic in F_{ij} . Thus the electromagnetic field, unlike spinor fields, does not couple to torsion.

2.7.4 Electromagnetic current

We define the *electromagnetic current density*

$$\mathbf{j}^i = -\frac{c\delta\mathfrak{L}_m}{\delta A_i}, \quad (2.7.50)$$

and the *electromagnetic current vector*

$$j^i = \frac{\mathbf{j}^i}{\sqrt{-\mathfrak{g}}}. \quad (2.7.51)$$

The invariance of the action under an arbitrary infinitesimal gauge transformation $\delta A_i = A'_i - A_i = \phi_{,i}$ gives, upon partial integration and omitting a total divergence,

$$\delta S = -\frac{1}{c^2} \int \mathbf{j}^i \delta A_i d\Omega = -\frac{1}{c^2} \int \mathbf{j}^i \phi_{,i} d\Omega = \frac{1}{c^2} \int \mathbf{j}^i_{,i} \phi d\Omega = 0, \quad (2.7.52)$$

so the electromagnetic current is conserved,

$$\mathbf{j}^i_{,i} = 0, \quad j^i_{;i} = 0. \quad (2.7.53)$$

The gauge-invariant Lagrangian density (2.7.14) for spinor matter is

$$\mathfrak{L}_\psi = \frac{i\mathfrak{e}}{2} e_a^i (\bar{\psi} \gamma^a \psi_{;i} - \bar{\psi}_{;i} \gamma^a \psi) - m\mathfrak{e} \bar{\psi} \psi - e A_i \mathfrak{e} \bar{\psi} \gamma^i \psi, \quad (2.7.54)$$

so the electromagnetic current for the spinor field is

$$\mathbf{j}^i = e\mathfrak{e} \bar{\psi} \gamma^i \psi, \quad (2.7.55)$$

which is proportional to the conserved vector Dirac current (2.6.45). The electromagnetic current density (2.7.50) corresponds to the current (2.4.7) with $\xi^i=0$ and $\bar{\delta}\phi = ie\alpha\phi$ (which is equal to the infinitesimal $\phi' - \phi$ due to (2.7.1)).

We consider matter which is distributed over a small region in space, as in section (2.4.7). Integrating (2.7.53) over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals gives

$$\int \mathbf{j}^0_{,0} dV = 0. \quad (2.7.56)$$

The conservation law (2.7.53) also gives

$$(x^k \mathbf{j}^i)_{,i} = x^k_{,i} \mathbf{j}^i + x^k \mathbf{j}^i_{,i} = \delta_i^k \mathbf{j}^i = \mathbf{j}^k, \quad (2.7.57)$$

which, upon integrating over the volume hypersurface and using Gauß-Stokes theorem to eliminate surface integrals, gives

$$\left(\int x^k j^0 dV \right)_{,0} = \int j^k dV. \quad (2.7.58)$$

Using (2.4.103) turns (2.7.58) into

$$\frac{u^k}{u^0} \int j^0 dV + \left(\int \delta x^k j^0 dV \right)_{,0} = \int j^k dV. \quad (2.7.59)$$

For a particle located at $\mathbf{x}_a$, $\int \delta x^k j^0 dV = 0$ and $j^i(\mathbf{x})$ is thus proportional to $\delta(\mathbf{x} - \mathbf{x}_a)$, so

$$j^k = \frac{u^k}{u^0} j^0. \quad (2.7.60)$$

We define the *electric charge density* ρ such that

$$j^0 = \frac{c\rho}{\sqrt{g_{00}}}. \quad (2.7.61)$$

The electric charge density is not a tensor density. We define the *electric charge* e such that

$$\rho \sqrt{5} dV = de. \quad (2.7.62)$$

The electric charge density for particles with charges e_a located at $\mathbf{x}_a$ is

$$\rho(\mathbf{x}) = \sum_a \frac{e_a}{\sqrt{5}} \delta(\mathbf{x} - \mathbf{x}_a), \quad (2.7.63)$$

and $\int j^0 dV$ (which is equal to $\int j^i dS_i$ for a volume hypersurface, so it is a scalar) is

$$\int j^0 dV = \sum_a \int \sqrt{-g} \frac{ce_a}{\sqrt{g_{00}} \sqrt{5}} \delta(\mathbf{x} - \mathbf{x}_a) dV = c \sum_a e_a, \quad (2.7.64)$$

so the electric charge is a scalar. Thus the electromagnetic current vector for a system of charged particles is

$$j^k(\mathbf{x}) = \sum_a \frac{cu^k}{u^0} \frac{e_a}{\sqrt{-g}} \delta(\mathbf{x} - \mathbf{x}_a), \quad (2.7.65)$$

analogously to (2.4.201). The relation (2.7.56) represents the conservation of the total electric charge of a physical system. For a particle moving along a worldline $x_a(\tau)$, we have

$$j^k(x) = \frac{cu^k}{u^0} \frac{e}{\sqrt{-g}} \delta(\mathbf{x} - \mathbf{x}_a) = ec \int \frac{u^k}{\sqrt{-g}} \delta(x - x_a(\tau)) d\tau. \quad (2.7.66)$$

In the locally geodesic and Galilean frame of reference, $\frac{u^i}{u^0} = (1, \mathbf{v}/c)$, so

$$j^i = (c\rho, \mathbf{j}), \quad (2.7.67)$$

where $\mathbf{j}$ is the *spatial current vector*,

$$\mathbf{j} = \rho \mathbf{v}. \quad (2.7.68)$$

The conservation law (2.7.53) in this frame, $j^i_{,i} = 0$, has the form of the equation of continuity (2.6.46). For one particle located at $\mathbf{x}_0(t)$, $\rho(\mathbf{x}) = e\delta(\mathbf{x} - \mathbf{x}_0)$, (2.6.46) is explicitly satisfied since

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= e \frac{\partial}{\partial t} \delta(\mathbf{x} - \mathbf{x}_0) = e \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}_0} \delta(\mathbf{x} - \mathbf{x}_0) = -e \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \delta(\mathbf{x} - \mathbf{x}_0) \\ &= -\frac{\partial}{\partial \mathbf{x}} \cdot (e \mathbf{v} \delta(\mathbf{x} - \mathbf{x}_0)) = -\nabla \cdot \mathbf{j}, \end{aligned} \quad (2.7.69)$$

where $\mathbf{v} = \frac{d\mathbf{x}_0}{dt}$. For a system of charged particles, we also have

$$\int \mathbf{j} dV = \sum_a e_a \mathbf{v}_a. \quad (2.7.70)$$

The equation of continuity (2.6.46) represents, upon integrating over the volume, the conservation of the total electric charge:

$$\frac{\partial}{\partial t} \left(\int \rho dV \right) + \oint \mathbf{j} \cdot d\mathbf{f} = 0. \quad (2.7.71)$$

2.7.5 Maxwell equations

The total Lagrangian density for the electromagnetic field and matter is the sum of (2.7.48) and the term $-\sqrt{-\mathbf{g}} A_i j^i$ due to (2.7.50):

$$\mathfrak{L}_{EM} = -\frac{1}{16\pi} \sqrt{-\mathbf{g}} F_{ik} F^{ik} - \frac{1}{c} \sqrt{-\mathbf{g}} A_k j^k, \quad (2.7.72)$$

where we omit the terms corresponding to the gravitational field and matter which does not depend on A_k . Varying (2.7.72) with respect to A_k , integrating partially and omitting total divergences gives

$$\begin{aligned} \delta \mathfrak{L}_{EM} &= -\frac{1}{8\pi} \sqrt{-\mathbf{g}} F^{ik} \delta F_{ik} - \frac{j^k}{c} \delta A_k = -\frac{1}{8\pi} \sqrt{-\mathbf{g}} F^{ik} (\delta A_{k,i} - \delta A_{i,k}) - \frac{j^k}{c} \delta A_k \\ &= \frac{1}{4\pi} \sqrt{-\mathbf{g}} F^{ik} \delta A_{k,i} - \frac{j^k}{c} \delta A_k = \frac{1}{4\pi} (\sqrt{-\mathbf{g}} F^{ik})_{,i} \delta A_k - \frac{1}{c} \sqrt{-\mathbf{g}} j^k \delta A_k, \end{aligned} \quad (2.7.73)$$

so the principle of least action $\delta S = 0$ for arbitrary variations δA_k yields the *second Maxwell-Minkowski equation*

$$(\sqrt{-\mathbf{g}} F^{ik})_{,i} = \frac{4\pi}{c} j^k \quad (2.7.74)$$

or

$$F^{ik}{}_{;i} = \frac{4\pi}{c} j^k. \quad (2.7.75)$$

The electromagnetic field equation (2.7.74) implies that j^i is conserved, $j^i{}_{;i} = 0$, which corresponds to the conservation of the total electric charge, but does not constrain the motion of particles. Therefore a configuration of charged particles producing the electromagnetic field can be arbitrary, subject only to the condition that the total charge be conserved, unlike a configuration of particles producing the gravitational field which is not arbitrary but constrained by the gravitational field equations.

We define

$$D^\alpha = -\sqrt{g_{00}} F^{0\alpha}, \quad (2.7.76)$$

$$H^{\alpha\beta} = \sqrt{g_{00}} F^{\alpha\beta}, \quad H_\alpha = -\frac{1}{2} \sqrt{s} \epsilon_{\alpha\beta\gamma} H^{\beta\gamma}, \quad H^{\alpha\beta} = -\frac{1}{\sqrt{s}} \epsilon^{\alpha\beta\gamma} H_\gamma. \quad (2.7.77)$$

The relations $F_{0\alpha} = g_{0i} g_{\alpha j} F^{ij}$ and $F^{\alpha\beta} = g^{\alpha i} g^{\beta j} F_{ij}$ give then

$$D_\alpha = \frac{E_\alpha}{\sqrt{g_{00}}} + g^\beta H_{\alpha\beta}, \quad (2.7.78)$$

$$B^{\alpha\beta} = \frac{H^{\alpha\beta}}{\sqrt{g_{00}}} - g^\alpha E^\beta + g^\beta E^\alpha, \quad (2.7.79)$$

or, in the spatial-vector notation,

$$\mathbf{D} = \frac{\mathbf{E}}{\sqrt{g_{00}}} - \mathbf{g} \times \mathbf{H}, \quad (2.7.80)$$

$$\mathbf{B} = \frac{\mathbf{H}}{\sqrt{g_{00}}} + \mathbf{g} \times \mathbf{E}. \quad (2.7.81)$$

Using (1.4.107) brings the temporal component of (2.7.74) to

$$\frac{1}{\sqrt{\mathfrak{s}}}(\sqrt{\mathfrak{s}}D^\alpha)_{,\alpha} = 4\pi\rho \quad (2.7.82)$$

or

$$\operatorname{div} \mathbf{D} = 4\pi\rho. \quad (2.7.83)$$

The spatial components of (2.7.74) read

$$\frac{1}{\sqrt{\mathfrak{s}}}(\sqrt{\mathfrak{s}}H^{\alpha\beta})_{,\beta} + \frac{1}{\sqrt{\mathfrak{s}}}(\sqrt{\mathfrak{s}}D^\alpha)_{,0} = -4\pi\rho \frac{dx^\alpha}{dx^0} \quad (2.7.84)$$

or

$$\operatorname{curl} \mathbf{H} = \frac{1}{c\sqrt{\mathfrak{s}}} \frac{\partial(\sqrt{\mathfrak{s}}\mathbf{D})}{\partial t} + \frac{4\pi}{c} \mathbf{j}. \quad (2.7.85)$$

The conservation law (2.7.53) reads

$$\frac{1}{\sqrt{\mathfrak{s}}} \frac{\partial(\sqrt{\mathfrak{s}}\rho)}{\partial t} + \operatorname{div} \mathbf{j} = 0. \quad (2.7.86)$$

In the locally geodesic and Galilean frame of reference, (2.7.80) and (2.7.81) reduce to

$$\mathbf{D} = \mathbf{E}, \quad (2.7.87)$$

$$\mathbf{B} = \mathbf{H}. \quad (2.7.88)$$

In this frame, (2.7.83) and (2.7.85) become the *second pair of the Maxwell equations*:

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \quad (2.7.89)$$

$$\operatorname{curl} \mathbf{B} = \frac{\partial \mathbf{E}}{c\partial t} + \frac{4\pi}{c} \mathbf{j}. \quad (2.7.90)$$

Applying the div operator to (2.7.90) and using (2.7.89) gives (2.6.46). Integrating the second pair of the Maxwell equations over the volume and surface area, respectively, gives

$$\oint \mathbf{E} \cdot d\mathbf{f} = 4\pi q, \quad (2.7.91)$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\partial}{c\partial t} \left(\int \mathbf{E} \cdot d\mathbf{f} \right) + \frac{4\pi}{c} \int \mathbf{j} \cdot d\mathbf{f}. \quad (2.7.92)$$

Thus the flux of the electric field through a closed surface is proportional to the total charge inside the volume enclosed by the surface $\mathbf{f}$ (*Gauß' law*) and the circulation of the magnetic field along a contour is equal to the time derivative of the flux of the electric field through the surface enclosed by this contour, called the displacement current, plus the surface integral of the current vector (the *Ampère-Ørsted law*).

The two pairs of the Maxwell equations are linear in the fields $\mathbf{E}$ and $\mathbf{B}$. The sum of any two solutions of the Maxwell equations is also a solution of these equations. Thus the electromagnetic field of a system of sources (particles) is the sum of the fields from each source. The additivity of the electromagnetic field is referred to as the *principle of superposition*.

2.7.6 Energy-momentum tensor for electromagnetic field

The metric energy-momentum tensor (2.3.3) for the electromagnetic field T_{ik}^{EM} is given by the Lagrangian density (2.7.48):

$$\begin{aligned} \delta \mathfrak{L}_{EM} &= \frac{1}{32\pi} \sqrt{-\mathfrak{g}} g_{lm} F_{ik} F^{ik} \delta g^{lm} - \frac{1}{8\pi} \sqrt{-\mathfrak{g}} F_{ik} F_{lm} g^{il} \delta g^{km} \\ &= \frac{1}{8\pi} \sqrt{-\mathfrak{g}} \left(\frac{1}{4} g_{ik} F_{lm} F^{lm} - F_i^j F_{kj} \right) \delta g^{ik}, \end{aligned} \quad (2.7.93)$$

so

$$T_{ik}^{EM} = \frac{1}{4\pi} \left(\frac{1}{4} g_{ik} F_{lm} F^{lm} - F_i^j F_{kj} \right). \quad (2.7.94)$$

The corresponding energy density W , energy current $\mathbf{S}$ called the *Poynting vector*, and stress tensor $\sigma_{\alpha\beta}$ called the *Maxwell stress tensor*, are given in the locally geodesic and Galilean frame of reference, due to (2.4.71), by

$$W = \frac{1}{8\pi} (E^2 + B^2), \quad (2.7.95)$$

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}, \quad (2.7.96)$$

$$\sigma_{\alpha\beta} = \frac{1}{4\pi} \left(E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2} \delta_{\alpha\beta} (E^2 + B^2) \right). \quad (2.7.97)$$

Multiplying (2.7.38) by $\mathbf{B}$ and (2.7.90) by $\mathbf{E}$ and adding these scalar products gives

$$\frac{1}{c} \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{4\pi}{c} \mathbf{j} \cdot \mathbf{E} - (\mathbf{B} \cdot \text{curl} \mathbf{E} - \mathbf{E} \cdot \text{curl} \mathbf{B}), \quad (2.7.98)$$

from which we obtain

$$\frac{1}{2c} \frac{\partial}{\partial t} (E^2 + B^2) = -\frac{4\pi}{c} \mathbf{j} \cdot \mathbf{E} - \text{div}(\mathbf{E} \times \mathbf{B}) \quad (2.7.99)$$

or

$$\frac{\partial W}{\partial t} + \mathbf{j} \cdot \mathbf{E} + \text{div} \mathbf{S} = 0. \quad (2.7.100)$$

Under Lorentz transformations, W , $\mathbf{S}$ and $\sigma_{\alpha\beta}$ transform like the corresponding components of a tensor of rank (0,2) (2.4.71), according to (1.6.100).

The energy-momentum tensor for the electromagnetic field is traceless,

$$T_{ik}^{EM} g^{ik} = 0, \quad (2.7.101)$$

so (2.4.202) and the virial theorem (2.4.203) remain unchanged if the particles interact electromagnetically. The condition (2.7.101) also gives, using (2.4.88),

$$\epsilon_{EM} = 3p_{EM}, \quad (2.7.102)$$

so (2.4.204) implies that the free electromagnetic field is ultrarelativistic. In a frame of reference, in which the vectors $\mathbf{E}$ and $\mathbf{B}$ are parallel to one another or one of them vanishes (and the x -axis is along the direction of these vectors), the nonzero components of the tensor T_{ik} are

$$T_{00} = -T_{11} = T_{22} = T_{33} = W. \quad (2.7.103)$$

If the vectors $\mathbf{E}$ (along the x -axis) and $\mathbf{B}$ (along the y -axis) are mutually perpendicular and equal in magnitude then

$$T_{00} = T_{03} = T_{33} = W. \quad (2.7.104)$$

2.7.7 Lorentz force

We consider a charge particle interacting with the electromagnetic field. The total energy-momentum tensor for the particle and electromagnetic field is covariantly conserved, which gives the motion of the particle. The electromagnetic part yields, using (2.7.26) and (2.7.75),

$$\begin{aligned} T_i^k{}_{;k} &= \frac{1}{4\pi} \left(\frac{1}{2} F_{lm;i} F^{lm} - F_{il;k} F^{kl} - F_{il} F^{kl}{}_{;k} \right) = \frac{1}{4\pi} \left(-\frac{1}{2} F_{mi;l} F^{lm} - \frac{1}{2} F_{il;m} F^{lm} \right. \\ &\quad \left. - F_{il;k} F^{kl} - F_{il} F^{kl}{}_{;k} \right) = \frac{1}{4\pi} F_{il} F^{kl}{}_{;k} = -\frac{1}{c} F_{il} j^l. \end{aligned} \quad (2.7.105)$$

The particle part gives, using (2.4.195),

$$T_i{}^k{}_{:k} = \left(\mu c^2 \frac{u_i u^k}{\sqrt{g_{00}} u^0} \right)_{:k}, \quad (2.7.106)$$

so we obtain

$$\left(\mu c^2 \frac{u_i u^k}{\sqrt{g_{00}} u^0} \right)_{:k} - \frac{1}{c} F_{il} j^l = 0. \quad (2.7.107)$$

Multiplying (2.7.107) by u^i and using (2.7.60) gives

$$\left(\mu c^2 \frac{u^k}{\sqrt{g_{00}} u^0} \right)_{:k}, \quad (2.7.108)$$

which turns (2.7.107) into

$$\mu c^2 \frac{u^k}{\sqrt{g_{00}} u^0} u_{i:k} = \frac{1}{c} F_{il} \rho \frac{u^l}{\sqrt{g_{00}} u^0} \quad (2.7.109)$$

or

$$mc \frac{D^{\{3\}} u^i}{ds} = \frac{e}{c} F^{ij} u^j, \quad (2.7.110)$$

which is the equation of motion of a particle of mass m and charge e in the electromagnetic field F_{ij} . Multiplying (2.7.110) by u_i gives the identity, so (2.7.110) has 3 independent components. The right-hand side of (2.7.110) is referred to as the *Lorentz force*.

In the locally geodesic and Galilean frame of reference, $\frac{D^{\{3\}}}{ds} = \frac{d}{ds} = \frac{u^0}{c} \frac{d}{dt}$ and $u^i = (\gamma, \gamma \mathbf{v}/c)$, so (2.7.110) reads (we choose the spatial components as the 3 independent ones)

$$mc \frac{du^\alpha}{dt} = e F^{\alpha 0} + \frac{e}{c} F^{\alpha \beta} v_\beta \quad (2.7.111)$$

or, using (2.4.187),

$$\frac{d\mathbf{P}}{dt} = e \mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{B}. \quad (2.7.112)$$

The temporal component of (2.7.110) is

$$mc \frac{du_0}{dt} = \frac{e}{c} F_{0\alpha} v^\alpha \quad (2.7.113)$$

or

$$\frac{dE}{dt} = e \mathbf{v} \cdot \mathbf{E}, \quad (2.7.114)$$

which also results from multiplying (2.7.112) by $\mathbf{v}$ and using (2.4.214). Integrating (2.7.100) over the volume gives

$$\frac{\partial}{\partial t} \int W dV + \int \mathbf{j} \cdot \mathbf{E} dV + \oint \mathbf{S} \cdot d\mathbf{f} = 0, \quad (2.7.115)$$

which, with (2.7.70) and (2.7.114), yields the conservation of the total energy (2.4.66) of the electromagnetic field and particles:

$$\frac{\partial}{\partial t} \left(\int W dV + \sum_a E_a \right) + \oint \mathbf{S} \cdot d\mathbf{f} = 0. \quad (2.7.116)$$

References: [2, 3].

3 Particles

3.1 Lagrangian mechanics

3.1.1 Coordinates, velocities and accelerations

For a system of particles, which is localized at a finite number of points in space, spatial coordinates q_s of particles can be taken for physical fields ϕ . Any quantities q_s which completely define the position of a system of particles are called *generalized coordinates* of the system. They are differentiable functions of the time coordinate t , the first derivatives $\dot{q}_s$ are called *generalized velocities* of the system, and the second derivatives $\ddot{q}_s$ are called *generalized accelerations*.

We consider the frame of reference which is locally geodesic and Galilean. The position of the particle in space is defined by its three-dimensional *radius* vector $\mathbf{r}$, whose components are its Cartesian coordinates x, y, z . The velocity of the particle is $\mathbf{v} = \dot{\mathbf{r}}$ and its acceleration is $\mathbf{a} = \ddot{\mathbf{r}}$. We consider a particle moving along a curve $\mathbf{r}(t)$. We define a *tangent* vector,

$$\mathbf{t} = \frac{d\mathbf{r}}{ds}, \quad (3.1.1)$$

where

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2 + dz^2} \quad (3.1.2)$$

is the length of an infinitesimal arc of the curve. The tangent vector is a unit vector: $|\mathbf{t}| = 1$. The velocity is thus tangent to the path of the particle:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \dot{s} = \dot{s} \mathbf{t} = v \mathbf{t}. \quad (3.1.3)$$

We define a *normal* vector,

$$\mathbf{n} = \frac{\frac{d\mathbf{t}}{ds}}{\kappa} \quad (3.1.4)$$

where

$$\kappa = \left| \frac{d\mathbf{t}}{ds} \right| \quad (3.1.5)$$

is the *curvature* of the curve. The normal vector is also a unit vector: $|\mathbf{n}| = 1$, and it is orthogonal to the tangent vector: $\mathbf{n} \cdot \mathbf{t} = 0$, which follows from differentiating $\mathbf{t} \cdot \mathbf{t} = 1$ with respect to s . The inverse of the curvature is called the *radius of curvature* of the curve,

$$\rho = \kappa^{-1}. \quad (3.1.6)$$

The acceleration of the particle is then, using (3.1.3),

$$\mathbf{a} = \ddot{s} \mathbf{t} + \dot{s} \dot{\mathbf{t}} = \ddot{s} \mathbf{t} + \dot{s} \frac{d\mathbf{t}}{ds} \frac{ds}{dt} = \ddot{s} \mathbf{t} + \frac{\dot{s}^2}{\rho} \mathbf{n}, \quad (3.1.7)$$

so it is the sum of the *tangent acceleration* $\ddot{s} \mathbf{t} = \dot{v} \mathbf{t}$ and the *normal* or *centripetal acceleration* $\frac{\dot{s}^2}{\rho} \mathbf{n} = \frac{v^2}{\rho} \mathbf{n}$.

Higher time derivatives of the radius vector, such as the *jerk* vector

$$\mathbf{j} = \dot{\mathbf{r}}, \quad (3.1.8)$$

involve another unit vector,

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}, \quad (3.1.9)$$

called the *binormal* vector. The three mutually perpendicular unit vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ constitute a *Frenet trihedron*, which can be used as a local basis of a spatial frame of reference associated with a given particle. We define the *torsion* of a curve,

$$\tau = \frac{d\mathbf{n}}{ds} \cdot \mathbf{b}, \quad (3.1.10)$$

whose inverse is called the *radius of torsion* of the curve,

$$\sigma = \tau^{-1}. \quad (3.1.11)$$

The derivatives of the three vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ with respect to s depend on these vectors according to the *Frenet-Serret formulae*:

$$\begin{pmatrix} d\mathbf{t}/ds \\ d\mathbf{n}/ds \\ d\mathbf{b}/ds \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \quad (3.1.12)$$

3.1.2 Hamilton principle and Lagrange equations

According to the principle of least action or *Hamilton's principle*, the dynamics of every classical system is governed by minimizing the action of the system (2.1.1). We consider a locally Galilean frame of reference, in which generalized coordinates coincide with Cartesian radius vectors of particles. The energy-momentum tensor $T_{ik}(\mathbf{x})$ as a function of $\mathbf{x}$ for one particle located at the position $\mathbf{x}_0$ is proportional to $\delta(\mathbf{x} - \mathbf{x}_0)$, and so is the Lagrangian density $\mathfrak{L}$. Thus we can use, instead of $\mathfrak{L}$, the Lagrangian $L = \int \mathfrak{L} dV$ (2.4.53). The action (2.1.1) becomes

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt, \quad (3.1.13)$$

where the integration over t is between the instants t_1 and t_2 , and q denotes the generalized coordinates q_s .

The principle of least action states that the dynamics of a mechanical system is given by the condition (2.1.3): the system moves between the positions $q(t_1)$ and $q(t_2)$ at these instants in such a way that the integral (3.1.13) takes the least possible value. In most cases L depends only q and $\dot{q}$, but not higher derivatives with respect to t , so the mechanical state of the system is completely defined by the coordinates and velocities (a Lagrangian containing higher derivatives can always be written in terms of first derivatives by increasing the number of the coordinates q_s). The condition $\delta S = 0$ gives, using $\delta \dot{q} = \frac{d\delta q}{dt}$ and integrating by parts,

$$\begin{aligned} \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt &= \delta \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt \\ &+ \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} = 0. \end{aligned} \quad (3.1.14)$$

Since the positions of the system are fixed at t_1 and t_2 , the second term vanishes, so $\delta S = 0$ for arbitrary variations δq gives the Euler-Lagrange or *Lagrange equations* for each degree of freedom $s = 1..n$:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0. \quad (3.1.15)$$

These n second-order differential equations give the relations between accelerations, velocities and coordinates, i.e. they are the equations of motion of the system. The general solution of the equations of motion contains thus $2n$ arbitrary constants, related to the initial velocities and coordinates. If we do not fix $\delta q = 0$ at t_1 and t_2 then the variation of the action (3.1.14) is, using the Lagrange equations,

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}_s} \delta q_s \right]_{t_1}^{t_2}. \quad (3.1.16)$$

Adding the total derivative with respect to time of some function $f(q, t)$ to a Lagrangian L gives a new Lagrangian L' ,

$$L' = L + \frac{df}{dt}, \quad (3.1.17)$$

the new action is thus

$$S' = S + f(q(t_1), t_1) - f(q(t_2), t_2), \quad (3.1.18)$$

so the conditions $\delta S = 0$ and $\delta S' = 0$ are equivalent, and the Lagrange equations do not change. Thus the Lagrangian of a system is defined up to the additive total time derivative of an arbitrary function of the coordinates and time. If a system consists of two noninteracting parts A and B , with corresponding Lagrangian $L_A(q_A)$ and $L_B(q_B)$, then the Lagrangian for this system is the sum $L_A + L_B$. This additivity of the Lagrangian means that the field equations for either of the two parts do not involve quantities pertaining to the other part. If L_A also depends on q_B and/or $\dot{q}_B$, and/or L_B depends on q_A and/or $\dot{q}_A$, then the subsystems A and B interact. Multiplying the Lagrangian L by an arbitrary constant does not change the Lagrange equations for the corresponding system. The additive property of Lagrangians admits only the simultaneous multiplication of the Lagrangians of all the systems by the same constant, corresponding to choosing the unit of the Lagrangian.

If we vary the time, in addition to varying q , then the variation δf of any function $f(q, t)$ can be decomposed into the part $\dot{f}\delta t$ resulting from varying t and the part $\bar{\delta}f$ which equals δf if $\delta t = 0$:

$$\delta f = \dot{f}\delta t + \bar{\delta}f. \quad (3.1.19)$$

Thus the variation of the action is, using $\frac{d}{dt}\bar{\delta}f = \bar{\delta}\dot{f}$ and (3.1.19) for $f = q$,

$$\begin{aligned} \delta S &= \dot{S}\delta t + \bar{\delta}S = L\delta t|_{t_1}^{t_2} + \int_{t_1}^{t_2} \bar{\delta}L dt = L\delta t|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\bar{\delta}q + \frac{\partial L}{\partial \dot{q}}\bar{\delta}\dot{q} \right) dt \\ &= L\delta t|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\bar{\delta}q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\bar{\delta}q \right) dt = L\delta t|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \bar{\delta}q dt \\ &\quad + \frac{\partial L}{\partial \dot{q}}\bar{\delta}q \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \bar{\delta}q dt + \left[\frac{\partial L}{\partial \dot{q}}\delta q - \left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L \right) \delta t \right]_{t_1}^{t_2}. \end{aligned} \quad (3.1.20)$$

If $\delta q = 0$ and $\delta t = 0$ at t_1 and t_2 then (3.1.20) gives for arbitrary variations $\bar{\delta}q$ the Lagrange equations (3.1.15). If we do not fix the endpoints t_1 and t_2 and the positions q at these instants then the variation of the action (3.1.20) is, using the Lagrange equations,

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q - \left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L \right) \delta t \right]_{t_1}^{t_2}. \quad (3.1.21)$$

3.1.3 Action for particles

The Lagrange equations (3.1.15) in the Cartesian coordinates are

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} - \frac{\partial L}{\partial \mathbf{r}} = 0. \quad (3.1.22)$$

The energy-momentum tensor for a point particle of mass m located at the radius vector $\mathbf{r}_0$ is given by (2.4.196):

$$T^{ik}(\mathbf{r}) = mc^2\delta(\mathbf{r} - \mathbf{r}_0)\frac{u^i u^k}{\sqrt{-\mathbf{g}}u^0}, \quad (3.1.23)$$

so the variation of the action with respect to the metric tensor (2.3.1) gives

$$\begin{aligned} \delta S &= -\frac{1}{2c} \int T^{ik}\delta g_{ik}\sqrt{-\mathbf{g}}d\Omega = -\frac{mc}{2} \int \frac{u^i u^k}{u^0}\delta g_{ik}dx^0 = -\frac{mc}{2} \int u^i u^k \delta g_{ik}ds \\ &= -\frac{mc}{2} \int \frac{\delta g_{ik}dx^i dx^k}{ds} = -mc \int \delta \sqrt{g_{ik}dx^i dx^k} = -mc\delta \int ds. \end{aligned} \quad (3.1.24)$$

Thus the action for a *free* (interacting only with the gravitational field) particle is

$$S = -mc \int_1^2 ds, \quad (3.1.25)$$

where 1 and 2 denote the world points corresponding to the arrival of the particle at the initial and final position. The equation of motion of a free particle is thus the metric geodesic equation (1.4.80). The action for a system of noninteracting particles is the sum of the actions corresponding to each particle:

$$S = - \sum_a m_a c \int ds_a. \quad (3.1.26)$$

If the particles interact with each other then we must add to (3.1.26) the action describing fields which carry this interaction.

Multiplying the action (3.1.26) by an arbitrary constant does not change the Lagrange equations for the corresponding system. The additive property of actions admits only the simultaneous multiplication of the masses of all the particles by the same constant, corresponding to choosing the unit of the mass. The integral $\int_1^2 ds$ has its maximum value along a straight world line (cf. text after (1.6.110)). By integrating ds along a curved world line we can make $\int ds$ arbitrarily small. Thus if $m < 0$ then the action (3.1.25) cannot have a minimum, so the mass of a particle must be a positive quantity.

The form of the action for a free particle (3.1.25) can also be obtained from general considerations. The action is a scalar integral whose integrand is a differential of first order. The only scalar of this kind that one can construct for a free particle is proportional to the integral of the interval ds along the world line of the particle between two given world points. The constant of proportionality characterizes the particle and we denote it $-mc$, calling the scalar m the mass of the particle.

In a frame of reference which is inertial (locally geodesic and Galilean), space is homogeneous and isotropic and time is homogeneous. The homogeneity of space and time implies that the Lagrangian of a particle cannot depend explicitly on either $\mathbf{r}$ or t , so it must be a function of $\mathbf{v}$ only. The isotropy of space implies that this Lagrangian cannot depend on the direction of $\mathbf{v}$, so it must be a function of v^2 . In an inertial frame, the action (3.1.25) is

$$S = -mc^2 \int \gamma^{-1} dt = -mc^2 \int \sqrt{1 - \frac{v^2}{c^2}} dt. \quad (3.1.27)$$

Thus the corresponding Lagrangian is

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}. \quad (3.1.28)$$

For a free particle, the Lagrange equations in the Cartesian coordinates (3.1.22) give

$$\frac{\partial L}{\partial \mathbf{v}} = \text{const.} \quad (3.1.29)$$

Since L is a function of the velocity only, we obtain

$$\mathbf{v} = \text{const.}, \quad (3.1.30)$$

so a free particle moves with a velocity which is constant in both magnitude and direction (uniform in a straight line). This is referred to as the *law of inertia* or *Newton's first law of dynamics*. This law is invariant under Lorentz transformations which preserve the Minkowski metric tensor. Thus there exists an infinity of inertial frames, moving relative to one another uniformly in a straight line, in which the properties of space and time and the laws of mechanics are the same (Einstein's principle of relativity).

The electromagnetic current vector for a point particle of charge e located at the radius vector $\mathbf{r}_0$ is given by (2.7.65):

$$j^k(\mathbf{r}) = \frac{cu^k}{u^0} \frac{e}{\sqrt{-g}} \delta(\mathbf{r} - \mathbf{r}_0). \quad (3.1.31)$$

Substituting (3.1.31) into the second term of (2.7.72) gives

$$S = -\frac{1}{c^2} \int \sqrt{-g} A_k j^k dV dx^0 = -\frac{e}{c} \int A_k \frac{u^k}{u^0} dx^0 = -\frac{e}{c} \int A_k dx^k, \quad (3.1.32)$$

so the total action for a particle of mass m and charge e interacting with the electromagnetic potential A_i is

$$S = -mc \int ds - \frac{e}{c} \int A_i dx^i. \quad (3.1.33)$$

For a system of particles, the total action is the sum of the actions (3.1.33) for each particle. Under the gauge transformation (2.7.7), the action (3.1.33) changes by the integral of a total differential:

$$S' = -mc \int ds - \frac{e}{c} \int A_i dx^i - \frac{e}{c} \int d\alpha, \quad (3.1.34)$$

so the conditions $\delta S = 0$ and $\delta S' = 0$ are equivalent, and the corresponding Lagrange equations are gauge invariant.

The form of the term $-\frac{e}{c} \int A_i dx^i$ in (3.1.33) can also be obtained from general considerations. The action is a scalar integral whose integrand is a differential of first order, and the only gauge-invariant scalar of this kind that one can construct from the electromagnetic potential is proportional to the integral of the product $A_i dx^i$ along the world line of the particle between two given world points. The constant of proportionality characterizes the interaction of the particle with the electromagnetic field and we denote it $-\frac{e}{c}$, calling the scalar e the electric charge of the particle.

The variation of (3.1.25) with respect to the coordinates x^i gives, using (1.4.78), (1.4.79) and (1.4.80),

$$\delta S = mc \left(\int \frac{D^{\{u_i\}}}{ds} \delta x^i ds - \int d(u_i \delta x^i) \right). \quad (3.1.35)$$

The variation of $\int A_i dx^i$ is

$$\begin{aligned} \delta \int A_i dx^i &= \int \delta A_i dx^i + \int A_i \delta dx^i = \int A_{i,j} \delta x^j dx^i + \int A_i d\delta x^i \\ &= \int A_{i,j} \delta x^j dx^i + \int d(A_i \delta x^i) - \int dA_i \delta x^i = \int A_{i,j} \delta x^j dx^i + \int d(A_i \delta x^i) \\ &\quad - \int A_{i,j} dx^j \delta x^i = \int F_{ij} dx^j \delta x^i + \int d(A_i \delta x^i) \\ &= \int F_{ij} u^j \delta x^i ds + \int d(A_i \delta x^i). \end{aligned} \quad (3.1.36)$$

Thus the variation of (3.1.33) is, using the expression for the four-momentum $p^l = mc u^l$ (2.4.187),

$$\begin{aligned} \delta S &= mc \int \frac{D^{\{u_i\}}}{ds} \delta x^i ds - \frac{e}{c} \int F_{ij} u^j \delta x^i ds - mc \int d(u_i \delta x^i) - \frac{e}{c} \int d(A_i \delta x^i) \\ &= \int \left(\frac{D^{\{p_i\}}}{ds} - \frac{e}{c} \int F_{ij} u^j \right) \delta x^i ds - \int d \left(p_i \delta x^i + \frac{e}{c} A_i \delta x^i \right) \\ &= \int_1^2 \left(\frac{D^{\{p_i\}}}{ds} - \frac{e}{c} \int F_{ij} u^j \right) \delta x^i ds - \left(p_i + \frac{e}{c} A_i \right) \delta x^i \Big|_1^2, \end{aligned} \quad (3.1.37)$$

where the limits 1 and 2 denote the endpoints of the particle's world line. The principle of least action $\delta S = 0$ for arbitrary δx^i vanishing at the endpoints gives the Lorentz equation of motion of a particle of mass m and charge e in the electromagnetic field F_{ij} (2.7.110).

In a frame of reference which is locally geodesic and Galilean, the action (3.1.33) for a particle in the presence of the electromagnetic field gives

$$S = \int \left(-mcds + \frac{e}{c} \mathbf{A} \cdot d\mathbf{r} - e\phi dt \right) \quad (3.1.38)$$

or

$$S = \int \left(-\frac{mc^2}{\gamma} + \frac{e}{c} \mathbf{A} \cdot \mathbf{v} - e\phi \right) dt. \quad (3.1.39)$$

Thus the Lagrangian for this particle is

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \mathbf{A} \cdot \mathbf{v} - e\phi. \quad (3.1.40)$$

The Lagrange equations (3.1.22) for the Lagrangian (3.1.40) give

$$\begin{aligned} \frac{d}{dt} \left(\mathbf{p} + \frac{e}{c} \mathbf{A} \right) &= \frac{e}{c} \nabla (\mathbf{A} \cdot \mathbf{v}) - e \nabla \phi = \frac{e}{c} (\mathbf{v} \cdot \nabla) \mathbf{A} + \frac{e}{c} \mathbf{v} \times (\nabla \times \mathbf{A}) - e \nabla \phi \\ &= \frac{e}{c} \left(\frac{d\mathbf{A}}{dt} - \frac{\partial \mathbf{A}}{\partial t} \right) + \frac{e}{c} \mathbf{v} \times \mathbf{B} - e \nabla \phi, \end{aligned} \quad (3.1.41)$$

which is equivalent to (2.7.112). The Lagrangian for a system of particles is the sum of the Lagrangians (3.1.40) for each particle:

$$L = - \sum \left(mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \mathbf{A} \cdot \mathbf{v} - e\phi \right). \quad (3.1.42)$$

3.1.4 Conservation laws and integrals of motion

During the motion of a mechanical system with n degrees of freedom, the $2n$ quantities q_s and $\dot{q}_s$ vary with time according to the Lagrange equations. Since the general solution of the equations of motion contains $2n$ arbitrary constants, we can express these constants as functions of q_s and $\dot{q}_s$ that remain constant during the motion. Such functions are called *integrals of motion*. A system of particles which interact with one another but with no other bodies is said to be *closed*. For a closed system of particles, the Lagrangian does not depend explicitly on time, so the origin of time t_0 may be chosen arbitrarily as an additive constant. The remaining $2n - 1$ constants are functions of q_s and $\dot{q}_s$, so a closed mechanical system with n degrees of freedom has $2n - 1$ independent integrals of motion.

According to the Noether theorem, each continuous symmetry of a Lagrangian leads to some conservation law. For mechanical systems, the homogeneity and isotropy of space and time yields the constancy of certain quantities, which are said to be *conserved*. These quantities are additive, their values for a system composed of noninteracting parts are equal to the sums of their values for each part. Therefore the additivity of conserved integrals of motion relates the states of interacting bodies after the interaction to their states before the interaction, regardless of the nature of this interaction.

The variation of the action as a function of the coordinates (3.1.35) for a system of particles of masses m_a , using the geodesic equation of motion,

$$\delta S = - \sum_a [p_{ia} \delta x_a^i]_1^2, \quad (3.1.43)$$

where

$$p_{ia} = m_a c u_{ia} \quad (3.1.44)$$

is the four-momentum for the a th particle. The variation (3.1.43) is equivalent to

$$p_{ia} = - \frac{\partial S}{\partial x_a^i}, \quad (3.1.45)$$

so the four-momentum operator (1.6.75) (which is proportional to the differential generator of translation (1.6.66)) acting on the action S gives the four-momentum p_i (with the minus sign). If the spacetime is *homogeneous* then the mechanical properties of a closed system do not change under any translation of the system in space or time. Thus the variation δS (3.1.43) vanishes if $\delta x_a^i = \epsilon^i = \text{const}$:

$$\delta S = - \sum_a [p_{ia}]_1^2 \epsilon^i = 0 \quad (3.1.46)$$

or, using the arbitrariness of ϵ^i ,

$$\left[\sum_a p_{ia} \right]_1^2 = 0, \quad (3.1.47)$$

so the total four-momentum of a closed system is conserved:

$$\sum_a p_a^i = \text{const.} \quad (3.1.48)$$

This conservation, using the decomposition of p^i into the energy E and momentum $\mathbf{p}$:

$$p^i = \left(\frac{E}{c}, \mathbf{p} \right), \quad (3.1.49)$$

contains the conservation of the total energy of the system,

$$\sum_a E_a = \text{const}, \quad (3.1.50)$$

and the conservation of the total momentum of the system,

$$\sum_a \mathbf{p}_a = \text{const.} \quad (3.1.51)$$

For a system composed of one particle (3.1.43) becomes

$$\delta S = [\mathbf{p} \cdot \delta \mathbf{x} - E \delta t]_1^2, \quad (3.1.52)$$

Comparing (3.1.52) with (3.1.21) gives

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}, \quad (3.1.53)$$

$$E = \frac{\partial L}{\partial \mathbf{v}} \cdot \mathbf{v} - L = \mathbf{p} \cdot \mathbf{v} - L. \quad (3.1.54)$$

In generalized coordinates, we define the *generalized momenta* p_s conjugate to the coordinates q_s :

$$p_s = \frac{\partial L}{\partial \dot{q}_s}, \quad (3.1.55)$$

while the energy is

$$E = \frac{\partial L}{\partial \dot{q}_s} \dot{q}_s - L = p_s \dot{q}_s - L, \quad (3.1.56)$$

in agreement with (2.4.52). The variation of the action (3.1.21) in this notation is

$$\delta S = [p_s \delta q_s - E \delta t]_{t_1}^{t_2}, \quad (3.1.57)$$

which is equivalent to

$$p_s = \frac{\partial S}{\partial q_s}, \quad E = -\frac{\partial S}{\partial t}. \quad (3.1.58)$$

Substituting the Lagrangian (3.1.28) into (3.1.53) gives

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (3.1.59)$$

as in (2.4.210), while substituting it into (3.1.54) gives

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (3.1.60)$$

as in (2.4.209).

Under Lorentz transformations, the four-momentum p^i (3.1.49) transforms like the Lorentzian coordinate vector x^i , according to (1.6.97). The energy and momentum of a particle, which form such a four-vector, transform according to (1.6.95):

$$\begin{aligned} E &= \gamma(E_0 + \mathbf{V} \cdot \mathbf{p}_0), \\ \mathbf{p}_\perp &= \mathbf{p}_{\perp 0}, \quad \mathbf{p}_\parallel = \gamma\left(\mathbf{p}_{\parallel 0} + \frac{\mathbf{V}E_0}{c^2}\right), \end{aligned} \quad (3.1.61)$$

or equivalently

$$\begin{aligned} E_0 &= \gamma(E - \mathbf{V} \cdot \mathbf{p}), \\ \mathbf{p}_{\perp 0} &= \mathbf{p}_\perp, \quad \mathbf{p}_{\parallel 0} = \gamma\left(\mathbf{p}_\parallel - \frac{\mathbf{V}E}{c^2}\right), \end{aligned} \quad (3.1.62)$$

where $\mathbf{p}$ is the momentum of the particle in a frame of reference K , $\mathbf{p}_0$ is the momentum of the particle in a frame of reference K' , $\mathbf{V}$ is the velocity of the reference frame K' relative to K , and $\gamma = \frac{1}{\sqrt{1-V^2/c^2}}$. Accordingly, $E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$ is the kinetic energy of the particle in K , and $E_0 = \sqrt{\mathbf{p}_0^2 c^2 + m^2 c^4}$ is the kinetic energy of the particle in K' .

If the spacetime is *isotropic* then the mechanical properties of a closed system do not change under any rotation of the system in spacetime (spatial rotation or boost). Thus the variation δS vanishes if $\delta x_a^i = \epsilon^{ik} x_{ka}$, where $\epsilon^{ik} = -\epsilon^{ki} = \text{const}$ represents an infinitesimal Lorentz transformation:

$$\delta S = - \sum_a [p_a^i x_a^k]_1 \epsilon_{ik} = -\frac{1}{2} \sum_a [p_a^i x_a^k - p_a^k x_a^i]_1 \epsilon_{ik} = 0 \quad (3.1.63)$$

or, using the arbitrariness of ϵ_{ik} ,

$$\left[\sum_a M_a^{ik} \right]_1^2 = 0, \quad (3.1.64)$$

where

$$M_a^{ik} = x_a^i p_a^k - x_a^k p_a^i \quad (3.1.65)$$

is the tensor of angular momentum (2.4.58) for the a th particle. Therefore the total angular momentum of a closed system is conserved:

$$\sum_a M_a^{ik} = \text{const}. \quad (3.1.66)$$

We define the spatial *angular momentum vector*:

$$\mathbf{M} : M^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} M_{\beta\gamma} \quad (3.1.67)$$

or, due to (3.1.65),

$$\mathbf{M} = \mathbf{r} \times \mathbf{p}. \quad (3.1.68)$$

The isotropy of space thus implies the conservation of the total angular momentum vector.

The conservation of the total spatial vector $M^{0\alpha}$,

$$M^{0\alpha} = \sum_a M_a^{0\alpha} = \sum_a (ct\mathbf{p}_a - E_a \mathbf{r}_a) = \text{const}, \quad (3.1.69)$$

divided by the conservation of energy (3.1.50) gives

$$\mathbf{R} = \mathbf{V}t + \text{const}, \quad (3.1.70)$$

where

$$\mathbf{V} = \frac{c^2 \sum_a \mathbf{p}_a}{\sum_a E_a} \quad (3.1.71)$$

and

$$\mathbf{R} = \frac{\sum_a E_a \mathbf{r}_a}{\sum_a E_a}. \quad (3.1.72)$$

The relation (3.1.70) describes a uniform rectilinear motion (2.4.61) of the center of inertia of the system, that is, of the system as a whole. The constant (due to (3.1.51)) velocity of this motion is equal to $\mathbf{V}$ (2.4.62) and the coordinates of the center of inertia are given by the radius vector $\mathbf{R}$.

This is a generalization of the law of inertia for one particle (3.1.30) to a system of particles. so a free particle moves with a velocity which is constant in both magnitude and direction (uniform in a straight line). This is referred to as the *law of inertia*. The velocity of the system of particles as a whole (3.1.71) has the same form as the velocity for one particle (2.4.213). We can always choose a frame of reference in which the total momentum $\sum_a \mathbf{p}_a$ vanishes, so the system as a whole is at rest, $\mathbf{V} = 0$. If the particles interact then we must add to E_a the energy of the fields which carry the interaction. Since the components of the center of inertia (3.1.72) do not form the spatial components of any four-vector, they do not transform like the coordinates of a point under Lorentz transformations. Thus the location of the center of inertia relative to the particles depends on the frame of reference.

The angular momentum (3.1.68) contains the radius vector, so it depends on the choice of the point with respect to which it is defined. If the radius vectors of the a th particle in two different frames are $\mathbf{r}_a$ and $\mathbf{r}'_a$, and the origins of these frames are separated by a constant vector $\mathbf{a}$, then

$$\mathbf{r}_a = \mathbf{r}'_a + \mathbf{a}. \quad (3.1.73)$$

If $\mathbf{M}_a$ is the angular momentum of the particle in the unprimed frame and

$$\mathbf{M}'_a = \mathbf{r}'_a \times \mathbf{p}_a \quad (3.1.74)$$

is the angular momentum of this particle in the primed frame then

$$\mathbf{M} = \sum_a \mathbf{M}_a = \sum_a \mathbf{r}_a \times \mathbf{p}_a = \sum_a \mathbf{r}'_a \times \mathbf{p}_a + \mathbf{a} \times \sum_a \mathbf{p}_a = \mathbf{M}' + \mathbf{a} \times \mathbf{P}, \quad (3.1.75)$$

where $\mathbf{P}$ is the total momentum of the system. Thus, in the rest frame of reference, in which $\mathbf{P} = 0$, the angular momentum of the system does not depend on the choice of the origin of the Cartesian coordinates. If the system is closed, the conservation of its total angular momentum is independent of the choice of the origin because the total momentum of such system is also conserved.

The antisymmetric tensor of angular momentum (3.1.65) transforms under Lorentz transformations according to (1.6.100). If a system of particles is at rest as a whole in the frame K , in which its angular momentum is $\mathbf{M}$, and moves with velocity $\mathbf{V} = V\hat{z}$ in the frame K' , in which its angular momentum $\mathbf{M}'$ is defined with respect to the center of inertia $\mathbf{R}$ in K ($\mathbf{M}'$ in K' depends on the choice of the $\mathbf{r}$ because $\mathbf{P} \neq 0$), then K moves relative to K' with $\mathbf{V}$, so (1.6.100) gives

$$M'^{12} = M^{12}, \quad M'^{13} = \gamma \left(M^{13} + \frac{V}{c} M^{10} \right), \quad M'^{23} = \gamma \left(M^{23} + \frac{V}{c} M^{20} \right), \quad (3.1.76)$$

where $\gamma = \sqrt{1 - \frac{V^2}{c^2}}$. In K , $\sum_a \mathbf{p}_a = 0$ (system at rest) and $\sum_a E_a \mathbf{r}_a = 0$ (origin at the center of inertia), so (3.1.69) gives $M^{0\alpha} = 0$. Thus (3.1.76) reduces to

$$M'_x = \gamma M_x, \quad M'_y = \gamma M_y, \quad M'_z = M_z. \quad (3.1.77)$$

The Lagrangian for a charged particle in the presence of the electromagnetic field (3.1.40) gives, upon substitution into (3.1.53),

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} \mathbf{A}, \quad (3.1.78)$$

while substituting it into (3.1.54) gives

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\phi. \quad (3.1.79)$$

The relation (2.4.211) between the momentum and energy turns into

$$(E - e\phi)^2 = (\mathbf{p}c - e\mathbf{A})^2 + (mc^2)^2. \quad (3.1.80)$$

The momentum (3.1.78) and energy (3.1.79) of a charged particle are the sums of their values in the absence of the electromagnetic field, (3.1.59) and (3.1.60), called respectively the *kinematic momentum* and *kinematic energy*, and terms linear in the electromagnetic potential.

3.1.5 Nonrelativistic mechanics and Galileo principle of relativity

In the nonrelativistic limit, the Lagrangian for a free particle (3.1.28) becomes

$$L = -mc^2 + \frac{mv^2}{2}. \quad (3.1.81)$$

The first term in (3.1.81) is constant and thus it does not contribute to the Lagrange equations. The form of the second term can also be obtained from the invariance of the variation of the action under Galileo transformations (1.6.114), which is referred to as *Galileo's principle of relativity*. If an inertial frame K is moving with an infinitesimal velocity $\boldsymbol{\epsilon}$ relative to another inertial frame K' then the addition law for velocities (1.6.115) gives $\mathbf{v}' = \mathbf{v} + \boldsymbol{\epsilon}$. The Lagrangian in K' is related to the Lagrangian in K by

$$L(v'^2) = L(v^2 + 2\mathbf{v} \cdot \boldsymbol{\epsilon} + \epsilon^2) \approx L(v^2) + \frac{\partial L}{\partial v^2} 2\mathbf{v} \cdot \boldsymbol{\epsilon}. \quad (3.1.82)$$

Since the Lagrangian L' in K' can differ from the Lagrangian L in K only by the total time derivative of a function of the coordinates and time, the second term on the right-hand side of (3.1.82) must be linear in $\mathbf{v}$, so $\frac{\partial L}{\partial v^2}$ is independent of $\mathbf{v}$, and thus L is proportional to v^2 . The constant of proportionality characterizes the particle and we can denote it $\frac{m}{2}$, calling the scalar m the mass of the particle. If K is moving with a finite velocity $\mathbf{V}$ relative to K' then the addition law for velocities (1.6.115) gives $\mathbf{v}' = \mathbf{v} + \mathbf{V}$. The Lagrangian in K' is thus related to the Lagrangian in K by

$$L' = \frac{1}{2}mv'^2 = \frac{1}{2}m(\mathbf{v} + \mathbf{V})^2 = \frac{1}{2}mv^2 + m\mathbf{v} \cdot \mathbf{V} + \frac{1}{2}mV^2 = L + \frac{d}{dt} \left(m\mathbf{r} \cdot \mathbf{V} + \frac{1}{2}mV^2t \right), \quad (3.1.83)$$

in which the last term is a total time derivative and may be omitted. The integral $S = \frac{m}{2} \int_1^2 v^2 dt$ has a minimum if m is not negative, otherwise S would take arbitrarily large negative values for an arbitrarily rapid motion of the particle from point 1 to point 2. In the nonrelativistic limit, the Lagrangian (3.1.40) for a particle in the electromagnetic field is

$$L = -mc^2 + \frac{mv^2}{2} + \frac{e}{c} \mathbf{A} \cdot \mathbf{v} - e\phi. \quad (3.1.84)$$

We consider a closed system of particles. If the particles are nonrelativistic ($c \rightarrow \infty$) then the interactions between them propagate instantaneously. Otherwise the nonrelativistic addition law for velocities (1.6.115) would imply that the velocity of propagation of interaction and thus the interaction vanishes in some frame of reference, contradicting Galileo's principle of relativity. Thus the fields which carry these interactions depend on the positions of the particles at a given instant, so the interaction can be described by adding to the total Lagrangian of noninteracting particles $\sum_a \frac{1}{2}m_a v_a^2$ a certain function of the coordinates $-U(\mathbf{r}_a)$, called the *potential energy*:

$$L = \sum_a \frac{1}{2}m_a v_a^2 - U(\mathbf{r}_a) = T(\mathbf{v}_a) - U(\mathbf{r}_a). \quad (3.1.85)$$

The term T in (3.1.85) is referred to as the *kinetic energy*. The Lagrangian (3.1.85) is invariant under the time reversal $t \rightarrow -t$, so all motions which obey the laws of mechanics are reversible.

For arbitrary generalized coordinates q_s , the transformations

$$\mathbf{r}_a = \mathbf{r}_a(q_s), \quad \mathbf{v}_a = \frac{\partial \mathbf{r}_a}{\partial q_s} \dot{q}_s \quad (3.1.86)$$

turn the Lagrangian for a system of interacting nonrelativistic particles (3.1.85) into

$$L = \frac{1}{2} m_{rs}(q) \dot{q}_r \dot{q}_s - U(q) = T(q, \dot{q}) - U(q), \quad (3.1.87)$$

where

$$m_{rs} = \sum_a m_a \frac{\partial \mathbf{r}_a}{\partial q_r} \frac{\partial \mathbf{r}_a}{\partial q_s} \quad (3.1.88)$$

is the symmetric *mass tensor*, $m_{rs} = m_{sr}$. The Lagrangian L , due to the kinetic energy T , is a quadratic function of the velocities.

We consider a system A which is not closed and interacts with another system B executing a given motion; the system A moves in an *external field* due to the system B . If the system $A + B$ is closed then

$$L = T_A(q_A, \dot{q}_A) + T_B(q_B, \dot{q}_B) - U(q_A, q_B). \quad (3.1.89)$$

Substituting for q_B the given functions of time and omitting the term $T(q_B(t), \dot{q}_B(t))$, which depends on time only so it is the total time derivative of some function of time, gives

$$L = T_A(q_A, \dot{q}_A) - U(q_A, q_B(t)). \quad (3.1.90)$$

Thus the motion of a system in an external field is described by a Lagrangian in which the potential energy depends explicitly on time.

3.1.6 Momentum, force and Newton equations of motion

If the space is homogeneous then the mechanical properties of a closed system do not change under any translation of the system in space. In the Cartesian coordinates, a translation by a constant vector $\boldsymbol{\epsilon}$ results in $\mathbf{r} \rightarrow \mathbf{r} + \boldsymbol{\epsilon}$, so the corresponding change in L for a closed system is

$$\delta L = \sum_a \frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}_a = \boldsymbol{\epsilon} \cdot \sum_a \frac{\partial L}{\partial \mathbf{r}_a} = 0. \quad (3.1.91)$$

Since $\boldsymbol{\epsilon}$ is arbitrary, this condition is equivalent to

$$\sum_a \frac{\partial L}{\partial \mathbf{r}_a} = 0, \quad (3.1.92)$$

so the Lagrange equations (3.1.22) give

$$\mathbf{P} = \sum_a \mathbf{p}_a = \sum_a \frac{\partial L}{\partial \mathbf{v}_a} = \text{const}, \quad (3.1.93)$$

where $\mathbf{P}$ is the additive momentum of the system, equal to the sum of its values $\mathbf{p}_a$ for each particle, regardless of the nature of the interaction between them. The additivity of the momentum follows from the additivity of the Lagrangian. If a system is not closed but its Lagrangian does not depend explicitly on all the Cartesian coordinates, then the components of $\mathbf{P}$ which are parallel to the Cartesian coordinates which do not appear in L are conserved.

The Lagrange equations in the Cartesian coordinates (3.1.22) for the a th particle are equivalent to

$$\dot{\mathbf{p}}_a = \mathbf{f}_a, \quad (3.1.94)$$

where

$$\mathbf{f}_a = \frac{\partial L}{\partial \mathbf{r}_a} \quad (3.1.95)$$

is referred to as the *force* acting on the *ath* particle. This form of the equations of motion is called *Newton's equations* and constitutes *Newton's second law of dynamics*. The relation (3.1.94) is consistent with (2.4.73). The covariant form of (3.1.94) is

$$\frac{dp^i}{ds} = g^i, \quad (3.1.96)$$

where g^i is the force four-vector, parallel to the four-acceleration and thus perpendicular to the four-velocity, $g_i u^i = 0$. The components of g^i are

$$g^i = \left(\frac{\mathbf{f} \cdot \mathbf{v}\gamma}{c^2}, \frac{\mathbf{f}\gamma}{c} \right). \quad (3.1.97)$$

Combining (3.1.59) and (3.1.94) gives the equation of motion of a particle of mass m moving under the influence of the force $\mathbf{f}$:

$$\frac{d}{dt}(m\gamma\mathbf{v}) = \mathbf{f}. \quad (3.1.98)$$

As $v \rightarrow c$, the momentum tends to infinity, so accelerating a massive particle to the velocity of propagation of interaction c requires an infinite force and thus is impossible. In the nonrelativistic limit, the momentum (3.1.59) reduces to

$$\mathbf{p} = m\mathbf{v}. \quad (3.1.99)$$

The equation of motion (3.1.98) reduces to

$$m\mathbf{a} = \mathbf{f} = -\frac{\partial U}{\partial \mathbf{r}}, \quad (3.1.100)$$

which also results from substituting (3.1.85) into (3.1.22). Adding a constant to the potential energy does not affect Newton's equations (such a constant is a total time derivative of some function), so U is defined up to an additive constant, which usually is chosen such that $U \rightarrow 0$ as $\mathbf{r}_a \rightarrow \infty$. A field in which the same force $\mathbf{f}$ acts on a particle at any point is *uniform*. The potential energy in such field is

$$U = -\mathbf{f} \cdot \mathbf{r}. \quad (3.1.101)$$

In a weak gravitational field, the Lagrangian of a particle is

$$L = -mc^2 + \frac{mv^2}{2} - m\phi_g, \quad (3.1.102)$$

where ϕ_g is the gravitational potential (2.5.82), because the corresponding Lagrange equations (3.1.22) coincide with (2.5.84). The term $m\phi_g$ is thus the potential energy of the particle in a nonrelativistic gravitational field,

$$U_g = m\phi_g. \quad (3.1.103)$$

The action for (3.1.102) is

$$S = \int L dt = -mc \int \left(c - \frac{v^2}{2c} + \frac{\phi_g}{c} \right) dt, \quad (3.1.104)$$

which, compared with (3.1.25), gives

$$ds = \left(c - \frac{v^2}{2c} + \frac{\phi_g}{c} \right) dt. \quad (3.1.105)$$

Omitting terms vanishing in the limit $c \rightarrow \infty$ gives

$$ds^2 = (c^2 + 2\phi_g)dt^2 - d\mathbf{r}^2, \quad (3.1.106)$$

which is consistent with (2.5.82). In the presence of a weak gravitational field, the Lagrangian (3.1.84) turns into

$$L = -mc^2 + \frac{mv^2}{2} - m\phi_g + \frac{e}{c}\mathbf{A} \cdot \mathbf{v} - e\phi. \quad (3.1.107)$$

The relation (3.1.92) is equivalent to

$$\sum_a \mathbf{f}_a = 0, \quad (3.1.108)$$

so the sum of the forces in a closed system vanishes. For a system of two particles, $\mathbf{f}_1 + \mathbf{f}_2 = 0$, so the force on the first particle exerted by the second is equal in magnitude and opposite in direction to the force on the second particle exerted by the first (this statement is valid only for nonrelativistic particles, for which we can neglect fields propagating the interaction). This is the *action-reaction law* or *Newton's third law of dynamics*. For a system of particles, summing (3.1.94) over a gives

$$\dot{\mathbf{P}} = \sum_a \dot{\mathbf{p}}_a = \sum_a \mathbf{f}_a. \quad (3.1.109)$$

Since the forces of interaction between particles cancel out due to the action-reaction law, (3.1.109) becomes

$$\dot{\mathbf{P}} = \mathbf{F}, \quad (3.1.110)$$

where $\mathbf{F}$ is the sum of external forces acting on the particles. If no external forces act on a mechanical system then the system is closed, $\dot{\mathbf{P}} = 0$.

The Lagrangian L of a closed system does not depend explicitly on q_s , $\frac{\partial L}{\partial q_s} = 0$, so the Lagrange equations (3.1.15) with (3.1.55) give,

$$p_s = \text{const.} \quad (3.1.111)$$

If L depends on q_s then

$$F_s = \frac{\partial L}{\partial q_s} \quad (3.1.112)$$

are called *generalized forces*. The Lagrange equations (3.1.15) are in this notation

$$\dot{p}_s = F_s. \quad (3.1.113)$$

The form of the Lagrangian (3.1.87) gives the relation between generalized momenta and generalized velocities:

$$p_r = m_{rs}(q)\dot{q}_s. \quad (3.1.114)$$

Substituting (3.1.87) into the Lagrange equations (3.1.15) gives

$$m_{rs}\ddot{q}_r + \dot{m}_{rs}\dot{q}_r = \frac{1}{2}m_{rt,s}\dot{q}_r\dot{q}_t - \frac{\partial U}{\partial q_s}, \quad (3.1.115)$$

where

$$m_{rs,t} = \frac{\partial m_{rs}}{\partial q_t}, \quad (3.1.116)$$

or

$$m_{rs}\ddot{q}_r + m_{rs,t}\dot{q}_t\dot{q}_r - \frac{1}{2}m_{rt,s}\dot{q}_r\dot{q}_t = -\frac{\partial U}{\partial q_s}. \quad (3.1.117)$$

Defining the inverse of the mass tensor m_{rs}^{-1} ,

$$m_{rs}^{-1}m_{st} = \delta_{rt}, \quad (3.1.118)$$

and multiplying (3.1.117) by m_{us}^{-1} gives

$$\ddot{q}_u + [rt, u]\dot{q}_r\dot{q}_t = -m_{su}^{-1}\frac{\partial U}{\partial q_s}, \quad (3.1.119)$$

where the quantities $[rt, u]$ are analogous to the Christoffel symbols (1.4.26):

$$[rt, u] = \frac{1}{2}m_{us}^{-1}(m_{sr,t} + m_{st,r} - m_{rt,s}). \quad (3.1.120)$$

3.1.7 Energy

If the time is homogeneous then the mechanical properties of a closed system do not change under any translation of the system in time, so the Lagrangian of a closed system does not depend explicitly on time. The total time derivative of the Lagrangian $L(q, \dot{q})$ is thus, using the Lagrange equations (3.1.15),

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_s} \dot{q}_s + \frac{\partial L}{\partial \dot{q}_s} \ddot{q}_s = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} \dot{q}_s + \frac{\partial L}{\partial \dot{q}_s} \ddot{q}_s = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_s} \dot{q}_s \right) \quad (3.1.121)$$

or

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_s} \dot{q}_s - L \right) = \frac{dE}{dt} = 0. \quad (3.1.122)$$

Hence the energy of such system is conserved. The additivity of the energy follows from the additivity of the Lagrangian. Mechanical systems whose energy is conserved are said to be *conservative*. In the nonrelativistic limit, the energy of a particle of mass m moving with velocity $\mathbf{v}$ (3.1.60) reduces to

$$E = mc^2 + \frac{1}{2}mv^2, \quad (3.1.123)$$

where it is the sum of its rest energy mc^2 and its kinetic energy $\frac{1}{2}mv^2$. The energy is a positive quantity because $m > 0$ (if $m = 0$ then $E = Pc > 0$). In the nonrelativistic mechanics the energy can be of either sign, since adding a constant to the potential energy does not affect the Newton equations. The equivalence of mass and energy (2.4.212) is valid not only for a single particle, but also for a system of particles which is at rest as a whole, if m is the total mass of such system and E is its total energy. The energy of a composite body at rest contains, in addition to the rest energies of its constituent particles, their kinetic energies and the energies of their interactions. Thus $mc^2 \neq \sum_a m_a c^2$, so the mass, unlike energy, is not additive; the mass of a composite body is not equal to the sum of the masses of its parts:

$$m \neq \sum_a m_a. \quad (3.1.124)$$

Since the Lagrangian $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$ for a system of nonrelativistic particles is a quadratic function of $\dot{q}$, Euler's theorem on homogeneous functions gives

$$\frac{\partial L}{\partial \dot{q}_s} \dot{q}_s = \frac{\partial T}{\partial \dot{q}_s} \dot{q}_s = 2T. \quad (3.1.125)$$

Combining (3.1.56) and (3.1.125) gives

$$E = T(q, \dot{q}) + U(q). \quad (3.1.126)$$

In the Cartesian coordinates the energy of a system of nonrelativistic particles is the sum of the kinetic energy, which depends on the velocities of the particles, and the potential energy, which depends on their coordinates:

$$E = T(\mathbf{v}_a) + U(\mathbf{r}_a) = \sum_a \frac{1}{2} m_a v_a^2 + U(\mathbf{r}_a). \quad (3.1.127)$$

The change of kinetic energy in time is, due to (2.4.214) and (3.1.94),

$$\frac{dE}{dt} = \mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{v} \cdot \mathbf{F} = \frac{dW}{dt}, \quad (3.1.128)$$

where

$$dW = \mathbf{F} \cdot d\mathbf{r} \quad (3.1.129)$$

is an infinitesimal *work* done by a force $\mathbf{F}$ over a displacement $d\mathbf{r}$. Thus a work over a finite displacement is

$$W = \int \mathbf{F} \cdot d\mathbf{r}. \quad (3.1.130)$$

The quantity $P = \frac{dW}{dt}$ is called a *power*. If we regard a system of particles as a fluid and the force on this system is a surface force due to pressure (2.4.78) then (3.1.129) gives

$$dW = -p df \mathbf{n} \cdot d\mathbf{r} = -p dV. \quad (3.1.131)$$

Thus the pressure is given by

$$p = -\frac{\partial E}{\partial V}. \quad (3.1.132)$$

3.1.8 Center of mass

We consider a system of nonrelativistic particles. If a frame of reference K' moves with velocity $\mathbf{V}$ relative to a frame K then the velocities $\mathbf{v}'_a$ and $\mathbf{v}_a$ of the particles in these two frames are related by

$$\mathbf{v}_a = \mathbf{v}'_a + \mathbf{V}. \quad (3.1.133)$$

The momenta $\mathbf{P}'$ and $\mathbf{P}$ of the system are related by

$$\mathbf{P} = \sum_a m_a \mathbf{v}_a = \sum_a m_a \mathbf{v}'_a + \mathbf{V} \sum_a m_a = \mathbf{P}' + \mathbf{V} \sum_a m_a. \quad (3.1.134)$$

Thus we can choose a frame K' in which the system as a whole is at rest, $\mathbf{P}' = 0$. The velocity of this rest frame in K is

$$\mathbf{V} = \frac{\mathbf{P}}{\sum_a m_a} = \frac{\sum_a m_a \mathbf{v}_a}{\sum_a m_a}. \quad (3.1.135)$$

This velocity is the velocity of the system of particles as a whole. The relation (3.1.135) between the momentum $\mathbf{P}$ and velocity $\mathbf{V}$ of the system of particles is the same as that for one particle (3.1.99),

$$\mathbf{P} = M\mathbf{V}, \quad (3.1.136)$$

of mass

$$M = \sum_a m_a. \quad (3.1.137)$$

Thus the mass in nonrelativistic mechanics is additive. The relation (3.1.135) is the time derivative of

$$\mathbf{R} = \frac{\sum_a m_a \mathbf{r}_a}{M}. \quad (3.1.138)$$

The radius vector $\mathbf{R}$ of the system as a whole is the nonrelativistic limit of the center of inertia (3.1.72) of such system (since $E_a \approx m_a c^2$), and it is referred to as its *center of mass*. Since the components of the center of mass (3.1.138) transform like the coordinates of a point under Galileo transformations, the location of the center of mass relative to the particles does not depend on the frame of reference. The center of mass (3.1.138) contains the radius vectors of the particles, so it depends on the choice of the point with respect to which it is defined, like the angular momentum (3.1.68). If the origins of two frames are separated by a constant vector $\mathbf{a}$ then the radius vectors of the a th particle in these frames, $\mathbf{r}_a$ and $\mathbf{r}'_a$, satisfy (3.1.73). If $\mathbf{R}'$ is the center of mass in the primed frame,

$$\mathbf{R}' = \frac{\sum_a m_a \mathbf{r}'_a}{M}, \quad (3.1.139)$$

then

$$\mathbf{R}' = \mathbf{R} - \mathbf{a}. \quad (3.1.140)$$

If we choose $\mathbf{a} = \mathbf{R}$ then $\mathbf{R}'$ becomes zero, corresponding to setting the origin of the frame of reference at the center of mass of the system.

The energy of a mechanical system which is at rest as a whole is referred to as its *internal energy* ε . The energies E' and E of the system in the two frames K' and K are related by

$$\begin{aligned} E &= \frac{1}{2} \sum_a m_a v_a^2 + U = \frac{1}{2} \sum_a m_a (\mathbf{v}'_a + \mathbf{V})^2 + U = \frac{1}{2} \sum_a m_a v_a'^2 + U + \mathbf{V} \cdot \sum_a m_a \mathbf{v}'_a \\ &+ \frac{1}{2} M V^2 = E' + \mathbf{V} \cdot \mathbf{P}' + \frac{1}{2} M V^2. \end{aligned} \quad (3.1.141)$$

The corresponding Lagrangians are related by

$$L = L' + \mathbf{V} \cdot \mathbf{P}' + \frac{1}{2} M V^2, \quad (3.1.142)$$

so the corresponding actions are related by

$$S = S' + M \mathbf{V} \cdot \mathbf{R}' + \frac{1}{2} M V^2 t. \quad (3.1.143)$$

If the center of mass is at rest in K' then $\mathbf{P}' = 0$, so $E' = \varepsilon$, and (3.1.141) reduces to

$$E = \varepsilon + \frac{1}{2} M V^2. \quad (3.1.144)$$

Thus the energy of a system moving as a whole is the sum of its internal energy and the kinetic energy of its center of mass, which is called *König's theorem*.

3.1.9 Angular momentum and torque

If the space is isotropic then the mechanical properties of a closed system do not change under any rotation of the system in space. A Lorentz matrix (1.6.7) with $\epsilon_{0\alpha} = 0$, which represents an infinitesimal rotation, changes the spatial coordinates according to

$$\delta x_\alpha = -\epsilon_{\alpha\beta} x_\beta = -e_{\alpha\beta\gamma} x_\beta \delta\phi_\gamma \quad (3.1.145)$$

or

$$\delta \mathbf{r} = \delta \boldsymbol{\phi} \times \mathbf{r}, \quad (3.1.146)$$

where we denote the angle of this rotation (1.6.25) $\delta \boldsymbol{\phi}$. The corresponding change in the velocity relative to an inertial frame of reference is

$$\delta \mathbf{v} = \delta \boldsymbol{\phi} \times \mathbf{v}. \quad (3.1.147)$$

Thus the corresponding change in L for a closed system of particles is

$$\begin{aligned} \delta L &= \sum_a \left(\frac{\partial L}{\partial \mathbf{r}_a} \cdot \delta \mathbf{r}_a + \frac{\partial L}{\partial \mathbf{v}_a} \cdot \delta \mathbf{v}_a \right) = \sum_a (\dot{\mathbf{p}}_a \cdot \delta \boldsymbol{\phi} \times \mathbf{r}_a + \mathbf{p}_a \cdot \delta \boldsymbol{\phi} \times \mathbf{v}_a) \\ &= \delta \boldsymbol{\phi} \cdot \sum_a (\mathbf{r}_a \times \dot{\mathbf{p}}_a + \mathbf{v}_a \times \mathbf{p}_a) = \delta \boldsymbol{\phi} \cdot \frac{d}{dt} \sum_a \mathbf{r}_a \times \mathbf{p}_a \\ &= \delta \boldsymbol{\phi} \cdot \sum_a \dot{\mathbf{M}}_a = 0. \end{aligned} \quad (3.1.148)$$

Since $\delta \boldsymbol{\phi}$ is arbitrary, this condition gives the conservation of the total angular momentum of the system (3.1.68):

$$\sum_a \mathbf{M}_a = 0. \quad (3.1.149)$$

The additivity of the angular momentum follows from the additivity of the Lagrangian. There are no other additive integrals of motion; every closed system has 7 such integrals: energy, the 3 components of momentum and the 3 components of angular momentum. The component of the

angular momentum of a particle along any axis i coincides with its generalized momentum conjugate to the angle of rotation ϕ_i about this axis, regarded as a generalized coordinate q_s :

$$M_i = \frac{\partial L}{\partial \dot{\phi}_i}. \quad (3.1.150)$$

The Lagrange equations (3.1.15) give then

$$\dot{M}_i = \frac{\partial L}{\partial \phi_i}. \quad (3.1.151)$$

If a system is not closed but interacts with an external field which is symmetrical about some axis then the component of $\mathbf{M}$ along this axis relative to a point lying on the axis is conserved.

Differentiating the angular momentum for the a th particle with respect to time gives, using (3.1.59) and (3.1.94),

$$\dot{\mathbf{M}}_a = \frac{d}{dt}(\mathbf{r}_a \times \mathbf{p}_a) = \mathbf{v}_a \times \mathbf{p}_a + \mathbf{r}_a \times \mathbf{F}_a = \boldsymbol{\tau}_a, \quad (3.1.152)$$

where

$$\boldsymbol{\tau}_a = \mathbf{r}_a \times \mathbf{F}_a \quad (3.1.153)$$

is referred to as the *torque* acting on the a th particle. For a system of particles, summing (3.1.152) over a gives

$$\dot{\mathbf{M}} = \sum_a \dot{\mathbf{M}}_a = \sum_a \boldsymbol{\tau}_a. \quad (3.1.154)$$

Since the total angular momentum $\mathbf{M}$ for a closed system is conserved, the sum of internal (within the system) torques exerted on the particles vanishes. Thus (3.1.154) becomes

$$\dot{\mathbf{M}} = \mathbf{K}, \quad (3.1.155)$$

where $\mathbf{K}$ is the sum of external torques acting on the particles. If no external torques act on a mechanical system then $\dot{\mathbf{M}} = 0$. The relation (3.1.151) implies that the component of the torque acting on a particle along any axis i coincides with its generalized force related to the angle of rotation ϕ_i about this axis, regarded as a generalized coordinate q_s :

$$\tau_i = \frac{\partial L}{\partial \phi_i}. \quad (3.1.156)$$

Equation (3.1.155) is a Lagrange equation for the rotational coordinate ϕ .

If a frame of reference K' moves with velocity $\mathbf{V}$ relative to a frame K then the velocities $\mathbf{v}'_a$ and $\mathbf{v}_a$ of the particles in these two frames are related by (3.1.133). Thus the angular momenta $\mathbf{M}'$ and $\mathbf{M}$ of the system are related by

$$\mathbf{M} = \sum_a m_a \mathbf{r}_a \times \mathbf{v}_a = \sum_a m_a \mathbf{r}_a \times \mathbf{v}'_a + \sum_a m_a \mathbf{r}_a \times \mathbf{V} = \mathbf{M}' + M\mathbf{R} \times \mathbf{V}. \quad (3.1.157)$$

If the system as a whole is at rest in K' then $\mathbf{V}$ is the velocity of its center of mass and $M\mathbf{V}$ is its total momentum $\mathbf{P}$ (3.1.136) relative to K , so

$$\mathbf{M} = \mathbf{M}' + \mathbf{R} \times \mathbf{P}. \quad (3.1.158)$$

Thus the angular momentum of a moving system is the sum of its intrinsic angular momentum in a frame in which it is at rest and the (orbital) angular momentum of its motion as a whole.

The torque $\boldsymbol{\tau}$ contains the radius vector, so it depends on the choice of the point with respect to which it is defined, like the angular momentum. If the origins of two frames are separated by a constant vector $\mathbf{a}$ then the radius vectors of the a th particle in these frames, $\mathbf{r}_a$ and $\mathbf{r}'_a$, satisfy (3.1.73). If $\boldsymbol{\tau}_a$ is the torque acting on the particle in the unprimed frame and

$$\boldsymbol{\tau}'_a = \mathbf{r}'_a \times \mathbf{f}_a \quad (3.1.159)$$

is the torque acting on this particle in the primed frame then

$$\mathbf{K} = \sum_a \boldsymbol{\tau}_a = \sum_a \mathbf{r}_a \times \mathbf{f}_a = \sum_a \mathbf{r}'_a \times \mathbf{f}_a + \mathbf{a} \times \sum_a \mathbf{f}_a = \mathbf{K}' + \mathbf{a} \times \mathbf{F}. \quad (3.1.160)$$

Thus, if the total external force acting on the system of particles $\mathbf{F} = 0$ then the total torque acting on this system does not depend on the choice of the origin of the Cartesian coordinates. In this case, the system is said to be acted on by a *couple*. If $\mathbf{F}$ and $\mathbf{K}$ are perpendicular then we can choose the vector $\mathbf{a}$ for which $\mathbf{K}' = 0$, so $\mathbf{K} = \mathbf{a} \times \mathbf{F}$. In this case, the effect of all the forces acting on the system reduces to that of a single force $\mathbf{F}$. This choice is not unique: adding to $\mathbf{a}$ any vector parallel to $\mathbf{F}$ does not change $\mathbf{K}' = 0$.

3.1.10 Mechanical similarity

We consider a similarity transformation in which the coordinates $\mathbf{r}_i$ of all the particles (indexed by i) in a nonrelativistic mechanical system are multiplied by a constant factor a and the time by a constant factor b :

$$\mathbf{r}_i \rightarrow \mathbf{r}'_i = a\mathbf{r}_i, \quad t \rightarrow t' = bt. \quad (3.1.161)$$

This transformation changes the velocities $\mathbf{v}_i$ by a factor $\frac{a}{b}$, so the kinetic energy T changes by a factor $\frac{a^2}{b^2}$. If the potential energy of the system is a homogeneous function of the Cartesian coordinates,

$$U(a\mathbf{r}_i) = a^k U(\mathbf{r}_i), \quad (3.1.162)$$

where k is the degree of homogeneity of U , then it changes by a factor a^k . The equations of motion remain unchanged if the Lagrangian $T - U$ of the system changes by the same factor:

$$\frac{a^2}{b^2} = a^k, \quad (3.1.163)$$

which transforms a path of each particle into a geometrically similar path of different size. The relations between the ratios of mechanical quantities corresponding to the two paths and the ratio $\frac{t'}{t}$ of linear dimensions of these paths are

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{k}{2}}, \quad \frac{v'}{v} = \left(\frac{l'}{l}\right)^{\frac{k}{2}}, \quad \frac{p'}{p} = \left(\frac{l'}{l}\right)^{\frac{k}{2}}, \quad \frac{E'}{E} = \left(\frac{l'}{l}\right)^k, \quad \frac{M'}{M} = \left(\frac{l'}{l}\right)^{1+\frac{k}{2}}. \quad (3.1.164)$$

The kinetic energy T of a nonrelativistic system of particles is a quadratic function of the velocities, so Euler's theorem on homogeneous functions gives

$$2T = \sum_i \mathbf{v}_i \cdot \frac{\partial T}{\partial \mathbf{v}_i} = \sum_i \mathbf{v}_i \cdot \mathbf{p}_i = \frac{d}{dt} \left(\sum_i \mathbf{r}_i \cdot \mathbf{p}_i \right) - \sum_i \mathbf{r}_i \cdot \dot{\mathbf{p}}_i. \quad (3.1.165)$$

If the motion of the system is finite in space then averaging (3.1.165) over the time and using vanishing of the average of the time derivative of any bounded quantity gives

$$2\bar{T} = -\overline{\sum_i \mathbf{r}_i \cdot \dot{\mathbf{p}}_i} = -\overline{\sum_i \mathbf{r}_i \cdot \mathbf{f}_i} = \overline{\sum_i \mathbf{r}_i \cdot \frac{\partial U}{\partial \mathbf{r}_i}}. \quad (3.1.166)$$

If the potential energy of a system of particles is a homogeneous function of the coordinates (3.1.162) then Euler's theorem turns (3.1.166) into the *nonrelativistic virial theorem*:

$$2\bar{T} = k\bar{U}. \quad (3.1.167)$$

If the system is closed then its energy $E = T + U$ is constant, so (3.1.167) gives

$$\bar{T} = \frac{kE}{k+2}, \quad \bar{U} = \frac{2E}{k+2}. \quad (3.1.168)$$

The nonrelativistic limit of (2.4.208) gives

$$E = \sum_a m_a c^2 - \overline{\sum_a \frac{1}{2} m_a v^2} = \sum_a m_a c^2 - \bar{T}, \quad (3.1.169)$$

which is consistent with (3.1.168), if we omit the rest energies of the particles, for $k = -1$. Thus nonrelativistic charged particles interact with each other through a force which is proportional to the inverse of the distance between them.

3.1.11 Dissipation

If the mechanical energy E of a system is transformed into other, nonmechanical forms of energy, we refer to a process of such a transformation as the *dissipation of energy*. We define the *dissipative function*,

$$F = \frac{1}{2} \alpha_{st} \dot{q}_s \dot{q}_t, \quad (3.1.170)$$

which modifies the Lagrange equations (3.1.15) to include dissipation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} = \frac{\partial L}{\partial q_s} - \frac{\partial F}{\partial \dot{q}_s}. \quad (3.1.171)$$

Thus the generalized force $f_s = \frac{\partial L}{\partial q_s}$ is modified by an additional part,

$$f_s^{(fr)} = -\alpha_{st} \dot{q}_t, \quad (3.1.172)$$

called the *resistive force*. The change of energy during dissipation is

$$\frac{dE}{dt} = \frac{d}{dt} \left(\sum_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - L \right) = \sum_s \dot{q}_s \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} \right) = - \sum_s \dot{q}_s \frac{\partial F}{\partial \dot{q}_s} = -2F. \quad (3.1.173)$$

Therefore $F > 0$, so the quadratic form (3.1.170) is positive definite. If F were a linear function of $\dot{q}_s$ then the sign of $\frac{dE}{dt}$ could not be fixed negative. If F did not depend on $\dot{q}_s$ then $\frac{dE}{dt} = 0$. Thus (3.1.170) is the simplest function which can represent dissipation of energy in (3.1.171). For one particle, α_{st} is a 3×3 symmetric matrix. Since a particle is mechanically isotropic, α_{st} is diagonal,

$$\alpha_{st} = \alpha \delta_{st}, \quad (3.1.174)$$

where $\alpha > 0$ is a scalar. The resistive force (3.1.172) acting on a particle is then

$$\mathbf{f}^{(fr)} = -\alpha \mathbf{v}. \quad (3.1.175)$$

References: [2, 12].

3.2 Rigid bodies

3.2.1 Angular velocity

A system of particles in which the distances between the particles do not change is referred to as a *rigid body*. The interaction between the particles which imposes the constancy of their relative distances must propagate instantaneously. Thus the concept of a rigid body is valid only in non-relativistic mechanics. To describe the motion of a rigid body, we use two frames of reference: an inertial frame K and a moving frame K' rigidly fixed in the body. The origin of K' is usually taken at the center of mass of the body. The position of a rigid body with respect to K is determined by the position of K' : the 3 components of the radius vector $\mathbf{R}$ of the origin of K' in K and 3 angles representing the orientation of the axes of K' relative to the axes of K . Thus a rigid body is a mechanical system with 6 degrees of freedom. If the distances between the particles in a system

change under the action of external forces or torques, and they return to their original values when this action ends, then such a system is referred to as an *elastic body*.

An arbitrary displacement of a rigid body is the sum of an infinitesimal translation of the body as a whole $d\mathbf{R}$ and an infinitesimal rotation of the body about its center of mass. If $\mathbf{r}$ is the radius vector of the particle in K , $d\phi$ is an infinitesimal angle of rotation of K' with respect to K , and $\mathbf{r}'$ is the radius vector of the particle in K' , then (3.1.146) gives

$$d\mathbf{r} = d\mathbf{R} + d\phi \times \mathbf{r}'. \quad (3.2.1)$$

Dividing (3.2.1) by dt gives

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}', \quad (3.2.2)$$

where the vector

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} \quad (3.2.3)$$

is the velocity of the center of mass of the body, called the *translational velocity* of the body, and the pseudovector

$$\boldsymbol{\omega} = \frac{d\phi}{dt} \quad (3.2.4)$$

is the *angular velocity* of the rotation of the body. The term $\boldsymbol{\omega} \times \mathbf{r}$ is the *rotational velocity* of the body. Under rotations, boosts and translations, the angular velocity behaves like a spatial vector. The time derivative of the angular velocity is called the *angular acceleration*:

$$\boldsymbol{\epsilon} = \dot{\boldsymbol{\omega}}. \quad (3.2.5)$$

If we choose a different frame K'' rigidly fixed in the body, with the origin at a distance $\mathbf{a}$ from the origin of K' , then the radius vector $\mathbf{r}''$ of the particle in K'' satisfies

$$\mathbf{r}' = \mathbf{r}'' + \mathbf{a}. \quad (3.2.6)$$

Thus (3.2.2) becomes

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{a} + \boldsymbol{\omega} \times \mathbf{r}''. \quad (3.2.7)$$

If the translational velocity of the body in K'' is $\mathbf{V}'$ and its angular velocity is $\boldsymbol{\omega}'$ then (3.2.2) gives ($\mathbf{v}$ does not change because it is related to K)

$$\mathbf{v} = \mathbf{V}' + \boldsymbol{\omega}' \times \mathbf{r}''. \quad (3.2.8)$$

Comparing (3.2.8) with (3.2.7) gives

$$\mathbf{V}' = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{a}, \quad (3.2.9)$$

$$\boldsymbol{\omega}' = \boldsymbol{\omega}. \quad (3.2.10)$$

Thus the angular velocity of rotation of a rigid body at any instant is, unlike the translational velocity, independent of a system of coordinates chosen for a moving frame rigidly fixed in the body. Thus $\boldsymbol{\omega}$ is the angular velocity of the body. If $\mathbf{V}$ and $\boldsymbol{\omega}$ at any instant are perpendicular for some choice of the origin of K' then (3.2.9) implies that $\mathbf{V}'$ and $\boldsymbol{\omega}'$ are also perpendicular. In this case, (3.2.2) implies that the velocities $\mathbf{v}$ of all points of the body are perpendicular to $\boldsymbol{\omega}$, so we can choose the origin of K' whose velocity $\mathbf{V}' = 0$ at this instant. The corresponding motion is a pure rotation about an axis which passes through the origin of K' , called the *instantaneous axis of rotation*. If $\mathbf{V}$ and $\boldsymbol{\omega}$ are not perpendicular then we can choose the origin of K' for which $\mathbf{V}$ and $\boldsymbol{\omega}$ are parallel at a given instant.

3.2.2 Inertia tensor

The kinetic energy of a rigid body is given by substituting (3.2.2) (where we omit the prime) into (3.1.123) and summing over all the particles:

$$\begin{aligned}
T &= \sum_a \frac{1}{2} m_a v_a^2 = \sum_a \frac{1}{2} m_a (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_a)^2 \\
&= \sum_a \frac{1}{2} m_a V^2 + \sum_a m_a \mathbf{V} \cdot (\boldsymbol{\omega} \times \mathbf{r}_a) + \sum_a \frac{1}{2} m_a (\boldsymbol{\omega} \times \mathbf{r}_a)^2 \\
&= \frac{1}{2} M V^2 + \sum_a m_a \mathbf{r}_a \cdot (\mathbf{V} \times \boldsymbol{\omega}) + \sum_a \frac{1}{2} m_a (\omega^2 r_a^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_a)^2). \tag{3.2.11}
\end{aligned}$$

If the origin of the moving frame rigidly fixed to the body is at its center of mass then $\sum_a m_a \mathbf{r}_a = 0$ and (3.2.11) reduces to

$$T = \frac{1}{2} M V^2 + \sum_a \frac{1}{2} m_a (x_a^2 \delta_{\alpha\beta} - x_{\alpha a} x_{\beta a}) \omega_\alpha \omega_\beta = \frac{1}{2} M V^2 + \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta, \tag{3.2.12}$$

where the symmetric tensor

$$I_{\alpha\beta} = \sum_a m_a (x_a^2 \delta_{\alpha\beta} - x_{\alpha a} x_{\beta a}) \tag{3.2.13}$$

is referred to as the *inertia tensor*. Thus the kinetic energy of a rigid body is the sum of the kinetic energy of the translational motion of its center of mass, $\frac{1}{2} M V^2$, and the kinetic energy of the rotational motion of the body (*rotational kinetic energy*) about an axis passing through the center of mass,

$$T_{\text{rot}} = \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta. \tag{3.2.14}$$

The inertia tensor is additive, which follows from the additivity of the kinetic energy. If a rigid body is regarded as continuous then the mass of each particle must be replaced by the mass ρdV with density ρ contained in a volume element dV , and the summation over all particles by the integration over the volume of the body. The inertia tensor (3.2.13) becomes then

$$I_{\alpha\beta} = \int \rho (x^2 \delta_{\alpha\beta} - x_\alpha x_\beta) dV. \tag{3.2.15}$$

The inertia tensor (3.2.13) contains the radius vector, so it depends on the choice of the origin of a frame rigidly fixed in a rigid body. If the radius vectors of the a th particle composing the body in two different frames are $\mathbf{r}_a$ and $\mathbf{r}'_a$, and the origins of these frames are separated by a constant vector $\mathbf{a}$, then substituting (3.1.73) into (3.2.13) gives

$$\begin{aligned}
I'_{\alpha\beta} &= \sum_a m_a \left((\mathbf{r}_a - \mathbf{a})^2 \delta_{\alpha\beta} - (x_{\alpha a} - a_\alpha)(x_{\beta a} - a_\beta) \right) \\
&= \sum_a m_a \left((x_a^2 - 2\mathbf{x}_a \cdot \mathbf{a} + a^2) \delta_{\alpha\beta} - x_{\alpha a} x_{\beta a} + x_{\alpha a} a_\beta + x_{\beta a} a_\alpha - a_\alpha a_\beta \right) \\
&= \sum_a m_a (x_a^2 \delta_{\alpha\beta} - x_{\alpha a} x_{\beta a}) - M \left((2\mathbf{R} \cdot \mathbf{a} - a^2) \delta_{\alpha\beta} + (X_\alpha a_\beta + X_\beta a_\alpha - a_\alpha a_\beta) \right) \\
&= I_{\alpha\beta} - M \left((2\mathbf{R} \cdot \mathbf{a} - a^2) \delta_{\alpha\beta} + (X_\alpha a_\beta + X_\beta a_\alpha - a_\alpha a_\beta) \right), \tag{3.2.16}
\end{aligned}$$

where

$$I'_{\alpha\beta} = \sum_a m_a (x_a'^2 \delta_{\alpha\beta} - x'_{\alpha a} x'_{\beta a}) \tag{3.2.17}$$

and $\mathbf{R} = (X_\alpha)$ is the radius vector of the center of mass of the body (3.1.138). If the origin of the unprimed frame is at the center of mass of the body then $\mathbf{R} = 0$ and (3.2.16) reduces to

$$I'_{\alpha\beta} = I_{\alpha\beta} + M(a^2 \delta_{\alpha\beta} - a_\alpha a_\beta), \tag{3.2.18}$$

which is called the parallel-axis or *Steiner's theorem*.

The diagonal components of the inertia-tensor matrix are called the *moments of inertia* about the corresponding axes. The inertia tensor can be reduced to a diagonal form by an appropriate choice of the axes of a frame rigidly fixed in a rigid body, called the *principal axes of inertia*. The corresponding values of the diagonal components of $I_{\alpha\beta}$ are called the *principal moments of inertia*, I_1, I_2, I_3 . The rotational kinetic energy of the body (3.2.14) is then

$$T_{\text{rot}} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2), \quad (3.2.19)$$

where $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity along the principal axes of inertia. None of the three principal moments of inertia can be larger than the sum of the other two. If the body has a plane of symmetry then its center of mass and two of the principal axes of inertia must lie in that plane. If the body has an axis of symmetry then this axis is one of the principal axes of inertia and it passes through its center of mass.

The angular momentum of a rigid body, defined with respect to its center of mass, is given by substituting (3.2.2) (where we omit the prime) into (3.1.68) and summing over all the particles:

$$\mathbf{M} = \sum_a m_a \mathbf{r}_a \times (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_a) = \sum_a m_a \mathbf{r}_a \times (\boldsymbol{\omega} \times \mathbf{r}_a) = \sum_a m_a (r_a^2 \boldsymbol{\omega} - (\mathbf{r}_a \cdot \boldsymbol{\omega}) \mathbf{r}_a) \quad (3.2.20)$$

or

$$M_\alpha = I_{\alpha\beta} \omega_\beta. \quad (3.2.21)$$

If the axes of a frame rigidly fixed in the body coincide with the principal axes of inertia then

$$M_1 = I_1 \omega_1, \quad M_2 = I_2 \omega_2, \quad M_3 = I_3 \omega_3. \quad (3.2.22)$$

3.2.3 Eulerian angles

The orientation of the axes $x' = x_1, y' = x_2, z' = x_3$ of a frame K' moving with a rigid body relative to the axes x, y, z of an inertial K can be represented by 3 *Eulerian angles* θ, ϕ, ψ , as shown in Fig. 4. The origins of K and K' can be taken to coincide (at a point O) because the orientation of K' relative to K does not change under translations. The $x'y'$ plane intersects the xy plane along the *line of nodes* ON , in the direction of $\mathbf{z} \times \mathbf{z}'$, perpendicular to both $\mathbf{z}$ and $\mathbf{z}'$. The angle $\theta \in [0, \pi]$ is between the z and z' axes, $\phi \in [0, 2\pi]$ between ON and the x axis, and $\psi \in [0, 2\pi]$ between ON and the x' axis. The polar angle and azimuth of the direction z' with respect to K are θ and $\phi - \frac{\pi}{2}$. The polar angle and azimuth of the direction z with respect to K' are θ and $\frac{\pi}{2} - \psi$.

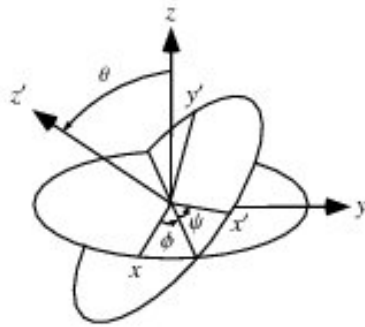


Figure 4: Eulerian angles.

The angular velocity $\dot{\theta}$ is along ON , so its nonvanishing components along the axes of K' are: $\dot{\theta}_1 = \dot{\theta} \cos\psi$, $\dot{\theta}_2 = -\dot{\theta} \sin\psi$. The angular velocity $\dot{\phi}$ is along the z -axis, so $\dot{\phi}_3 = \dot{\phi} \cos\theta$, $\dot{\phi}_1 =$

$\dot{\phi} \sin\theta \sin\psi$, $\dot{\phi}_2 = \dot{\phi} \sin\theta \cos\psi$. The angular velocity $\dot{\psi}$ is along the z' -axis, so its only nonvanishing component is $\dot{\psi}_3 = \dot{\psi}$. Thus the components of the angular velocity $\boldsymbol{\omega}$ along the axes of K' are

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\theta \sin\psi & 0 \\ -\sin\psi & \sin\theta \cos\psi & 0 \\ 0 & \cos\theta & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix}. \quad (3.2.23)$$

Substituting (3.2.23) into (3.2.21) gives the angular momentum of a rigid body in terms of the Eulerian angles and its derivatives. If the axes of K' are the principal axes of inertia of the body then (3.2.22) gives

$$M_1 = I_1(\cos\psi \dot{\theta} + \sin\theta \sin\psi \dot{\phi}), \quad M_2 = I_2(-\sin\psi \dot{\theta} + \sin\theta \cos\psi \dot{\phi}), \quad M_3 = I_3(\cos\theta \dot{\phi} + \dot{\psi}). \quad (3.2.24)$$

Substituting (3.2.23) into (3.2.14) gives the rotational kinetic energy of a rigid body in terms of the Eulerian angles and its derivatives. If the axes of K' are the principal axes of inertia of the body then (3.2.19) gives

$$T_{\text{rot}} = \frac{1}{2} \left(I_1(\cos\psi \dot{\theta} + \sin\theta \sin\psi \dot{\phi})^2 + I_2(-\sin\psi \dot{\theta} + \sin\theta \cos\psi \dot{\phi})^2 + I_3(\cos\theta \dot{\phi} + \dot{\psi})^2 \right). \quad (3.2.25)$$

A product of 3 rotations (which is a rotation) brings the frame K to K' : a rotation about the z -axis by an angle ϕ transforms the x -axis into a new x'' -axis (which coincides with the line of nodes), then a rotation about the x'' -axis by an angle θ transforms the z -axis into the z' -axis, and then a rotation about the z' -axis by an angle ψ transforms the x'' -axis into the x' -axis. The corresponding orthogonal matrix of this rotation is

$$R(\phi, \theta, \psi) = R_{z'}(\psi) R_{x'}(\theta) R_z(\phi), \quad (3.2.26)$$

where R_α are given by (1.6.33) and describe rotations with respect to moving axes. If a rotation R transforms an axis $\mathbf{n}$ to a new axis $\mathbf{n}'$ then a rotation R' about $\mathbf{n}'$ is related to the rotation R' about $\mathbf{n}$ by

$$R'_{\mathbf{n}'} = R R'_{\mathbf{n}} R^{-1}. \quad (3.2.27)$$

Thus we can rewrite (3.2.26) in terms of rotations with respect to the fixed axes of K :

$$\begin{aligned} R(\phi, \theta, \psi) &= \left(R_{x'}(\theta) R_z(\phi) \right) R_z(\psi) \left(R_{x'}(\theta) R_z(\phi) \right)^{-1} R_{x'}(\theta) R_z(\phi) \\ &= R_{x'}(\theta) R_z(\phi) R_z(\psi) = R_z(\phi) R_x(\theta) R_z^{-1}(\phi) R_z(\phi) R_z(\psi) \\ &= R_z(\phi) R_x(\theta) R_z(\psi). \end{aligned} \quad (3.2.28)$$

3.2.4 Newton and Euler equations

We consider a rigid body in an inertial frame of reference. The Lagrangian of a rigid body is given by

$$L = \frac{1}{2} M V^2 + \frac{1}{2} I_{ik} \omega_i \omega_k - U, \quad (3.2.29)$$

where the potential energy U is a function of the 6 degrees of freedom of the body. The Lagrange equations (3.1.15) for the 3 components of the radius vector $\mathbf{R}$ of the center of mass of the body are

$$\frac{d}{dt} \frac{\partial L}{\partial \mathbf{V}} - \frac{\partial L}{\partial \mathbf{R}} = 0. \quad (3.2.30)$$

The change of the potential energy under a translation of the body through a distance $\delta \mathbf{R}$ is

$$\delta U = \sum_a \frac{\partial U}{\partial \mathbf{r}_a} \delta \mathbf{r}_a = \sum_a \frac{\partial U}{\partial \mathbf{r}_a} \delta \mathbf{R} = - \sum_a \mathbf{f}_a \cdot \mathbf{R} = -\mathbf{F} \cdot \mathbf{R}, \quad (3.2.31)$$

so

$$\mathbf{F} = - \frac{\partial U}{\partial \mathbf{R}}. \quad (3.2.32)$$

Substituting (3.2.29) into (3.2.30) gives, using (3.1.136) and (3.2.32), the Newton equation of translational motion of a rigid body (3.1.110).

The Lagrange equations (3.1.15) for the 3 angles ϕ_α representing the orientation of the principal axes of inertia of the body relative to the axes of the inertial frame are

$$\frac{d}{dt} \frac{\partial L}{\partial \omega_\alpha} - \frac{\partial L}{\partial \phi_\alpha} = 0. \quad (3.2.33)$$

The change of the potential energy under a rotation of the body by an angle $\delta\phi$ is

$$\delta U = - \sum_a \mathbf{f}_a \cdot \mathbf{r}_a = - \sum_a \mathbf{f}_a \cdot (\delta\phi \times \mathbf{r}) = -\delta\phi \cdot \sum_a \mathbf{r}_a \times \mathbf{f}_a = -\delta\phi \cdot \sum_a \boldsymbol{\tau}_a = -\mathbf{K} \cdot \delta\phi, \quad (3.2.34)$$

so

$$\mathbf{K} = - \frac{\partial U}{\partial \phi}. \quad (3.2.35)$$

Substituting (3.2.29) into (3.2.33) and using (3.2.21) gives

$$\dot{M}_\alpha = - \frac{\partial U}{\partial \phi_\alpha} \quad (3.2.36)$$

or, using (3.2.35), the Newton equation of rotational motion of a rigid body (3.1.155).

If a rigid body interacts with a uniform field $\mathbf{E}$ then the force on each particle composing the body is $\mathbf{f} = e\mathbf{E}$, where e characterizes the properties of the particle with respect to the field. The corresponding total force $\mathbf{F} = \mathbf{E} \sum_a e_a$ and the total torque $\mathbf{K} = \sum_a e_a \mathbf{r}_a \times \mathbf{E}$. If $\sum_a e_a \neq 0$ then

$$\mathbf{K} = \mathbf{r}_0 \times \mathbf{F}, \quad (3.2.37)$$

where

$$\mathbf{r}_0 = \frac{\sum_a e_a \mathbf{r}_a}{\sum_a e_a}. \quad (3.2.38)$$

Thus the effect of a uniform field on a moving rigid body reduces to the action of a single force $\mathbf{F}$ applied at the point with the radius vector (3.2.38). In a uniform electric field, $\mathbf{E}$ is the electric field vector and e is the electric charge of the particle. In a uniform gravitational field, $\mathbf{E}$ is the gravitational acceleration due to gravity, e is the mass of the particle and $\mathbf{r}_0$ is the radius vector of the center of mass of the body.

We consider a frame of reference K' moving with a rigid body, whose axes are the principal axes of inertia, relative to an inertial frame K . If the change of any vector $\mathbf{A}$ in K' is $\frac{d'\mathbf{A}}{dt}$ then its change in K is, according to (3.2.2),

$$\frac{d\mathbf{A}}{dt} = \frac{d'\mathbf{A}}{dt} + \boldsymbol{\omega} \times \mathbf{A}. \quad (3.2.39)$$

Thus the Newton equations of motion of a rigid body (3.1.110) and (3.1.155) are equivalent to

$$\frac{d'\mathbf{P}}{dt} + \boldsymbol{\omega} \times \mathbf{P} = \mathbf{F}, \quad (3.2.40)$$

$$\frac{d'\mathbf{M}}{dt} + \boldsymbol{\omega} \times \mathbf{M} = \mathbf{K}. \quad (3.2.41)$$

Projecting (3.2.40) onto the axes of K' and using (3.1.136) gives

$$\begin{aligned} M(\dot{V}_1 + \omega_2 V_3 - \omega_3 V_2) &= F_1, \\ M(\dot{V}_2 + \omega_3 V_1 - \omega_1 V_3) &= F_2, \\ M(\dot{V}_3 + \omega_1 V_2 - \omega_2 V_1) &= F_3. \end{aligned} \quad (3.2.42)$$

Projecting (3.2.41) onto the axes of K' and using (3.2.22) (which is valid for the principal axes of inertia) gives *Euler's equations*:

$$\begin{aligned} I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 &= K_1, \\ I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 &= K_2, \\ I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 &= K_3. \end{aligned} \quad (3.2.43)$$

3.2.5 Noninertial frames of reference

In an inertial frame of reference K_0 , the Lagrangian for a nonrelativistic particle of mass m moving with velocity $\mathbf{v}_0$ in an external field represented by the potential energy U is

$$L_0 = \frac{1}{2}m\mathbf{v}_0^2 - U, \quad (3.2.44)$$

which gives the Newton equation of motion $m\frac{d\mathbf{v}_0}{dt} = -\frac{\partial U}{\partial \mathbf{r}}$. We consider a frame of reference K' , which moves relative to K_0 with translational velocity $\mathbf{V}(t)$. Substituting the relation $\mathbf{v}_0 = \mathbf{v}' + \mathbf{V}$, where $\mathbf{v}' = \frac{d\mathbf{r}'}{dt}$ is the velocity of the particle in K' , into (3.2.44) gives

$$L' = \frac{1}{2}m\mathbf{v}'^2 + m\mathbf{v}' \cdot \mathbf{V} + \frac{1}{2}m\mathbf{V}^2 - U = \frac{1}{2}m\mathbf{v}'^2 + \frac{d}{dt}(m\mathbf{r}' \cdot \mathbf{V}) - m\mathbf{r}' \cdot \frac{d\mathbf{V}}{dt} + \frac{d}{dt}\left(\frac{1}{2}m\mathbf{V}^2 t\right) - U, \quad (3.2.45)$$

which, after omitting total time derivatives, reduces to

$$L' = \frac{1}{2}m\mathbf{v}'^2 - m\mathbf{r}' \cdot \mathbf{A} - U, \quad (3.2.46)$$

where $\mathbf{A} = \frac{d\mathbf{V}}{dt}$ is the translational acceleration of K' . The Lagrange equations (3.1.22) in K' are $\frac{d}{dt}\frac{\partial L'}{\partial \mathbf{v}'} - \frac{\partial L'}{\partial \mathbf{r}'} = 0$, giving the Newton equation of motion:

$$m\frac{d\mathbf{v}'}{dt} = -\frac{\partial U}{\partial \mathbf{r}'} - m\mathbf{A}. \quad (3.2.47)$$

The effect of a translational motion of a frame of reference with acceleration $\mathbf{A}(t)$ is equivalent to the effect of a uniform field of force $-m\mathbf{A}$. This force acts on the particle in the direction opposite to the acceleration.

We consider a frame of reference K , which rotates relative to K' with angular velocity $\boldsymbol{\omega}(t)$ and whose origin coincides with the origin of K' , so $\mathbf{r} = \mathbf{r}'$. Substituting the relation $\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}$, where $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of the particle in K , into (3.2.46) gives

$$L = \frac{1}{2}m\mathbf{v}^2 + m\mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r})^2 - m\mathbf{r}' \cdot \mathbf{A} - U, \quad (3.2.48)$$

so

$$\frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + m\boldsymbol{\omega} \times \mathbf{r}, \quad \frac{\partial L}{\partial \mathbf{r}} = m\mathbf{v} \times \boldsymbol{\omega} + m(\boldsymbol{\omega} \times \mathbf{r}) \times \boldsymbol{\omega} - m\mathbf{A} - \frac{\partial U}{\partial \mathbf{r}}. \quad (3.2.49)$$

Thus the Lagrange equations (3.1.22) in K give the Newton equation of motion:

$$m\frac{d\mathbf{v}}{dt} = -\frac{\partial U}{\partial \mathbf{r}} - m\mathbf{A} - m\boldsymbol{\alpha} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \mathbf{v} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \quad (3.2.50)$$

where $\boldsymbol{\alpha}(t) = \dot{\boldsymbol{\omega}}$ is the angular acceleration of K . The effect of a rotational motion of a frame of reference with angular velocity $\boldsymbol{\omega}$ is equivalent to the effect of three forces: the force $m\boldsymbol{\alpha} \times \mathbf{r}$ arising from the nonuniformity of the rotation, the *Coriolis force*

$$\mathbf{f}_C = -2m\boldsymbol{\omega} \times \mathbf{v}, \quad (3.2.51)$$

and the *centrifugal force*

$$\mathbf{f}_c = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (3.2.52)$$

The Coriolis force depends on the velocity $\mathbf{v}$, but it is perpendicular to $\mathbf{v}$ so it does not produce any work. The centrifugal force acts in the direction opposite to $\mathbf{r}$ (away from the center of K) and its magnitude is $m\omega^2\rho$, where ρ is the distance of the particle from the axis of rotation.

3.2.6 Constraints and d'Alembert principle

A rigid body is in *equilibrium* if the total force and torque acting on it vanish:

$$\mathbf{F} = \sum \mathbf{f} = 0, \quad \mathbf{K} = \sum \mathbf{r} \times \mathbf{f} = 0. \quad (3.2.53)$$

According to (3.1.110) and (3.1.155), such body is at rest (*static*). The equations of equilibrium (3.2.53) do not depend on the choice of the origin with respect to which the torque is defined because of (3.1.160). For a system of rigid bodies, (3.2.53) must be satisfied for each body. The consistency of these equations with Newton's third law for each pair of bodies implies the appearance of additional forces exerted on each body by the bodies with which it is in contact and acting at the points of contact. These forces are referred to *reactions* and must be included in (3.2.53). They satisfy Newton's third law: the mutual reactions of two bodies in contact are equal in magnitude and opposite in direction. Solving the equations of equilibrium (3.2.53) gives the reactions. If the system of bodies is not in equilibrium then reactions must be included in (3.1.110) and (3.1.155). Solving these equations give the motion of the bodies and the reactions. In addition to reaction, dissipative forces of *friction* opposing the motion appear if two bodies in contact are in relative motion.

The motion of bodies in contact is a combination of *sliding* and *rolling*. Sliding is characterized by a reaction $\mathbf{R}$ perpendicular to the surfaces in contact and a friction $\mathbf{T}$ tangential to these surfaces and proportional in magnitude to the reaction,

$$T = \mu R. \quad (3.2.54)$$

The constant of proportionality μ is called the *coefficient of kinetic friction*. Two bodies in contact have a smaller number of degrees of freedom relative to these bodies in free motion. In sliding, the constraints impose relations between the coordinates of the bodies only; such constraints are called *holonomic*. The number of generalized coordinates n describing rigid bodies in sliding is $n = n_0 - k$, where n_0 is the number of degrees of freedom of the bodies in free motion and k is the number of (holonomic) constraints.

Rolling is characterized by no relative motion of the bodies at their instantaneous point of contact, with an arbitrarily oriented force of reaction and a friction torque opposing the rotation. In rolling, the velocities of the points in contact are thus equal. This condition is expressed by the *equations of constraint* on the generalized velocities $\dot{q}_s$:

$$\sum_s c_{js} \dot{q}_s = 0, \quad (3.2.55)$$

where the index $j = 1..k$ numbers these equations and the functions c_{js} depend only on the generalized coordinates q_s . If $\sum_s c_{js} \dot{q}_s$ is not the time derivative of some function of the coordinates then (3.2.55) cannot be integrated to relations between the coordinates of the bodies only; such constraints are called *nonholonomic*. For a rigid body rolling on another body at rest, (3.2.55) in the Cartesian coordinates gives

$$\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r} = 0, \quad (3.2.56)$$

where $\mathbf{r}$ is the radius vector of the instantaneous point of contact with respect to the center of mass of the rolling body. For sliding, the reaction is perpendicular to the surface of contact of the bodies. Since the motion in sliding is tangential to this surface, the work of forces of reaction along a *virtual displacement* (displacement at a fixed time) $\delta \mathbf{r}$, which is consistent with the constraints, vanishes:

$$\sum \mathbf{R} \cdot \delta \mathbf{r} = 0, \quad (3.2.57)$$

where the summation extends over all the bodies. This relation is equivalent to *d'Alembert's principle*:

$$\sum (\mathbf{f} - \dot{\mathbf{p}}) \cdot \delta \mathbf{r} = 0, \quad (3.2.58)$$

where $\mathbf{p}$ is the momentum of the body and $\mathbf{f}$ is the total force acting on it. The static ($\dot{\mathbf{p}} = 0$) case of d'Alembert's principle is called the *principle of virtual work*.

Multiplying (3.2.55) by δt gives

$$\sum_s c_{js} \delta q_s = 0, \quad (3.2.59)$$

so the variations δq_s are not independent. The dynamics of a mechanical system is given by the stationarity of the action $\delta S = 0$ with k additional constraints (3.2.59). Lagrange's method of finding conditional extrema gives

$$\delta S + \int \sum_{j,s} \lambda_j c_{js} \delta q_s dt = 0, \quad (3.2.60)$$

where λ_j are some functions of the coordinates, called *Lagrange multipliers*. The variations δq_s in (3.2.60) are now independent, so using (3.1.14) yields

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} + \sum_j \lambda_j c_{js} \right) \delta q_s dt = 0. \quad (3.2.61)$$

Since the variations δq_s are arbitrary, (3.2.61) gives the *Lagrange equations with constraints* for each degree of freedom $s = 1..n$:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \sum_j \lambda_j c_{js}. \quad (3.2.62)$$

These n second-order differential equations, together with k equations of constraint (3.2.55), are the equations of motion of the system for n coordinates q_s and k multipliers λ_j . The right-hand sides of (3.2.62) are the *generalized forces of reaction*:

$$R_s = \sum_j \lambda_j c_{js}. \quad (3.2.63)$$

The condition (3.2.59) is also valid for holonomic constraints. In this case, $\sum_s c_{js} \dot{q}_s$ in (3.2.55) is the time derivative of some function of the coordinates f_j , so (3.2.59) gives

$$\delta f_j = \sum_s c_{js} \delta q_s, \quad (3.2.64)$$

which then gives

$$c_{js} = \frac{\partial f_j}{\partial q_s}. \quad (3.2.65)$$

The Lagrange equations with constraints (3.2.62) become thus

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \sum_j \lambda_j \frac{\partial f_j}{\partial q_s}. \quad (3.2.66)$$

3.2.7 Maggi and Appell equations

References: [12].

3.3 Ideal fluids

3.3.1 Lagrange formulation

3.3.2 Euler formulation

References: [13, 14].

3.4 Hamiltonian mechanics

3.4.1 Legendre transformation, Hamilton and Routh equations

For a particle, the Hamiltonian is

$$H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}. \quad (3.4.1)$$

In the nonrelativistic limit, (3.4.1) reduces to

$$H = \frac{\mathbf{p}^2}{2m} + mc^2. \quad (3.4.2)$$

3.4.2 Poisson brackets

$$\{M_\alpha, M_\beta\} = e_{\alpha\beta\gamma} M_\gamma, \quad (3.4.3)$$

which resembles (1.6.37). This relation gives

$$\{M^2, M_\beta\} = \{M_\alpha M_\alpha, M_\beta\} = 0, \text{ which resembles (1.6.41)}. \quad (3.4.4)$$

The square of the angular momentum vector is therefore the Casimir invariant of the rotation group, that is, a quantity whose Poisson brackets with $\mathbf{M}$ vanish.

3.4.3 Maupertuis principle

3.4.4 Canonical transformations

3.4.5 Liouville theorem and distribution functions

3.4.6 Adiabatic motion

References: [12, 14].

3.5 Hamilton-Jacobi mechanics

3.5.1 Hamilton-Jacobi equation

$p_i = -\frac{\partial S}{\partial x^i}$ and $p_i p^i = m^2 c^2$ give the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} g^{ik} = m^2 c^2. \quad (3.5.1)$$

Substituting

$$S' = S + mc^2 t \quad (3.5.2)$$

into (3.5.1) in the Galilean frame of reference gives

$$\frac{1}{2m} (\nabla S')^2 - \frac{1}{2mc^2} \left(\frac{\partial S'}{\partial t} \right)^2 + \frac{\partial S'}{\partial t} = 0. \quad (3.5.3)$$

In the nonrelativistic limit, (3.5.3) reduces to

$$\frac{1}{2m} (\nabla S')^2 + \frac{\partial S'}{\partial t} = 0. \quad (3.5.4)$$

3.5.2 Canonical variables

References: [12, 14].

4 Applications

4.1 Mechanics

4.1.1 Orthonormal systems of coordinates

The position of a particle in the Cartesian coordinates is given by the radius vector

$$\mathbf{r} = (x, y, z) = x^\alpha \mathbf{e}_\alpha, \quad (4.1.1)$$

where the Cartesian unit vectors (*versors*) are constant vectors:

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1). \quad (4.1.2)$$

The velocity of the particle is

$$\mathbf{v} = v^\alpha \mathbf{e}_\alpha = \dot{x}^\alpha \mathbf{e}_\alpha, \quad (4.1.3)$$

so

$$v_\alpha = \dot{x}_\alpha. \quad (4.1.4)$$

The acceleration of the particle is

$$\mathbf{a} = a^\alpha \mathbf{e}_\alpha = \dot{v}^\alpha \mathbf{e}_\alpha, \quad (4.1.5)$$

so

$$a_\alpha = \ddot{x}_\alpha. \quad (4.1.6)$$

The Cartesian versors are *orthonormal* (unit and orthogonal):

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}. \quad (4.1.7)$$

We consider a transformation from the Cartesian coordinates x^α to generalized (curvilinear) coordinates q^β :

$$x^\alpha = x^\alpha(q^\beta). \quad (4.1.8)$$

The differential $d\mathbf{r}$ is thus

$$d\mathbf{r} = \mathbf{h}_\beta dq^\beta, \quad (4.1.9)$$

where

$$\mathbf{h}_\beta = \frac{\partial \mathbf{r}}{\partial q^\beta}. \quad (4.1.10)$$

The scalar product

$$\mathbf{h}_\alpha \cdot \mathbf{h}_\beta = \frac{\partial x^\gamma}{\partial q^\alpha} \frac{\partial x^\delta}{\partial q^\beta} \mathbf{e}_\gamma \cdot \mathbf{e}_\delta = \gamma_{\alpha\beta}, \quad (4.1.11)$$

where $\gamma_{\alpha\beta}$ is the spatial metric tensor (1.4.98). If $\gamma_{\alpha\beta}$ is diagonal then one can define a vector

$$\mathbf{n}_\alpha = \frac{\mathbf{h}_\alpha}{\sqrt{\gamma_{\alpha\alpha}}} \quad (4.1.12)$$

for each α . These vectors are orthonormal,

$$\mathbf{n}_\alpha \cdot \mathbf{n}_\beta = \delta_{\alpha\beta}, \quad (4.1.13)$$

and form a basis of an *orthonormal system of coordinates* in which the upper and lower indices α are identical.

The vectors $\mathbf{n}_\alpha$ generally are not constant. The time derivative $\dot{\mathbf{n}}_\alpha$ can be expressed as linear combinations of $\mathbf{n}_\beta$:

$$\dot{\mathbf{n}}_\alpha = \omega_{\alpha\beta} \mathbf{n}_\beta, \quad (4.1.14)$$

where the coefficients $\omega_{\alpha\beta}$ depend on the chosen coordinates q^α . Multiplying (4.1.14) by $\mathbf{n}_\gamma$ gives

$$\dot{\mathbf{n}}_\alpha \cdot \mathbf{n}_\gamma = \omega_{\alpha\beta} \mathbf{n}_\beta \cdot \mathbf{n}_\gamma = \omega_{\alpha\beta} \delta_{\beta\gamma} = \omega_{\alpha\gamma}, \quad (4.1.15)$$

so

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = (\mathbf{n}_\alpha \cdot \mathbf{n}_\beta)' = 0. \quad (4.1.16)$$

Thus the spatial tensor $\omega_{\alpha\beta}$ is antisymmetric. The velocity is given by

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial q^\alpha} \dot{q}^\alpha = \dot{q}^\alpha \mathbf{h}_\alpha = \sum_\alpha \sqrt{\gamma_{\alpha\alpha}} \dot{q}^\alpha \mathbf{n}_\alpha. \quad (4.1.17)$$

The acceleration is given by

$$\mathbf{a} = \sum_\alpha (\sqrt{\gamma_{\alpha\alpha}} \dot{q}^\alpha)' \mathbf{n}_\alpha + \sum_\alpha \sqrt{\gamma_{\alpha\alpha}} \dot{q}^\alpha \dot{\mathbf{n}}_\alpha = \sum_{\alpha,\beta} \left(\omega_{\alpha\beta} \sqrt{\gamma_{\alpha\alpha}} \dot{q}^\alpha + \delta_{\alpha\beta} (\sqrt{\gamma_{\alpha\alpha}} \dot{q}^\alpha)' \right) \mathbf{n}_\beta. \quad (4.1.18)$$

The passage from the Cartesian coordinates x, y, z to the *cylindrical coordinates* r, ϕ, z is given by

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z. \quad (4.1.19)$$

The radius vector is $\mathbf{r} = (r \cos \phi, r \sin \phi, z)$ and the metric tensor is

$$\gamma_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.1.20)$$

which gives

$$\mathbf{n}_r = (\cos \phi, \sin \phi, 0), \quad \mathbf{n}_\phi = (-\sin \phi, \cos \phi, 0), \quad \mathbf{n}_z = (0, 0, 1). \quad (4.1.21)$$

Thus $\dot{\mathbf{n}}_r = \dot{\phi} \mathbf{n}_\phi$, $\dot{\mathbf{n}}_\phi = -\dot{\phi} \mathbf{n}_r$ and $\dot{\mathbf{n}}_z = 0$, so

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & \dot{\phi} & 0 \\ -\dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.1.22)$$

The velocity is given by (4.1.17):

$$\mathbf{v} = \dot{r} \mathbf{n}_r + r \dot{\phi} \mathbf{n}_\phi + \dot{z} \mathbf{n}_z, \quad (4.1.23)$$

so

$$v_r = \dot{r}, \quad v_\phi = r \dot{\phi}, \quad v_z = \dot{z}. \quad (4.1.24)$$

The acceleration is given by

$$\mathbf{a} = (\dot{r} \mathbf{n}_r + r \dot{\phi} \mathbf{n}_\phi + \dot{z} \mathbf{n}_z)' = (\ddot{r} - r \dot{\phi}^2) \mathbf{n}_r + (r \ddot{\phi} + 2 \dot{r} \dot{\phi}) \mathbf{n}_\phi + \ddot{z} \mathbf{n}_z, \quad (4.1.25)$$

so

$$a_r = \ddot{r} - r \dot{\phi}^2, \quad a_\phi = r \ddot{\phi} + 2 \dot{r} \dot{\phi}, \quad a_z = \ddot{z}. \quad (4.1.26)$$

The gradient (1.4.124), divergence (1.4.126) and Laplacian (1.4.128) in the cylindrical coordinates are

$$\nabla \psi = \left(\frac{\partial \psi}{\partial r}, \frac{\partial \psi}{r \partial \phi}, \frac{\partial \psi}{\partial z} \right), \quad (4.1.27)$$

$$\operatorname{div} \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_\phi}{r \partial \phi} + \frac{\partial v_z}{\partial z}, \quad (4.1.28)$$

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{r^2 \partial \phi^2} + \frac{\partial^2}{\partial z^2}. \quad (4.1.29)$$

4.1.2 Uniformly accelerated motion

If the four-velocity u^i of a particle is constant then its motion is *uniform*. If the four-acceleration w^i of a particle is constant then its motion is *uniformly accelerated*. We consider $w^i = \text{const}$ and $\mathbf{v} \parallel \mathbf{a}$. If $\mathbf{v}$ is along the x direction, $\mathbf{v} = v\mathbf{e}_x$ and $\mathbf{a} = a\mathbf{e}_x$, then (1.6.119) gives

$$w^x = \frac{\gamma^2 a}{c^2} + \frac{\gamma^4 v^2 a}{c^4} = \frac{\gamma^4 a}{c^2}, \quad (4.1.30)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. We also have

$$w^x = \frac{du^x}{ds} = \frac{u^0}{c^2} \frac{d(\gamma v)}{dt} = \frac{\gamma}{c^2} \frac{d(\gamma v)}{dt}, \quad (4.1.31)$$

which gives

$$\frac{d}{dt}(\gamma v) = \gamma^3 a. \quad (4.1.32)$$

The relations (1.6.121) and (1.6.123) give the proper acceleration (1.6.123)

$$a_0^2 = \gamma^4 \left(a^2 + \frac{\gamma^2 v^2}{c^2} a^2 \right) = \gamma^6 a^2, \quad (4.1.33)$$

which brings (4.1.32) to

$$\frac{d}{dt}(\gamma v) = a_0. \quad (4.1.34)$$

If the time is chosen such that $v = 0$ at $t = 0$ then (4.1.34) integrates to

$$v = c \frac{a_0 t}{(c^2 + (a_0 t)^2)^{1/2}}. \quad (4.1.35)$$

As $t \rightarrow \infty$, $v \rightarrow c$. If the coordinates are chosen such that $x = \frac{c^2}{a_0}$, $y = z = 0$ at $t = 0$ then (4.1.35) integrates, using $v = \frac{dx}{dt}$, to

$$x^2 - (ct)^2 = \frac{c^4}{a_0^2}. \quad (4.1.36)$$

The motion is thus hyperbolic, with a parametric solution

$$ct = \frac{c^2}{a_0} \sinh \eta, \quad x = \frac{c^2}{a_0} \cosh \eta. \quad (4.1.37)$$

The proper time of the particle τ is given by

$$\tau = \int_0^t \frac{dt}{u^0} = \int_0^t (1 - v^2/c^2)^{1/2} dt = \frac{c}{a_0} \sinh^{-1} \frac{a_0 t}{c} = \frac{c\eta}{a_0}. \quad (4.1.38)$$

The world line (4.1.36) is represented by a hyperbola AC in Fig. 5. As $t \rightarrow \infty$, this curve asymptotically tends to the line $ct = x$, which is represented by a line OB . No signal emitted in the region on the left of this line can therefore reach a point moving along the world line (4.1.36). Such a point has an event horizon: no signals from world points which are behind the given point (in the instantaneous rest frame) at distances larger than $\frac{c^2}{a_0}$ at $t = 0$ ($|OA|$ in Fig. 5) can reach this point.

We consider the following transformation from the Cartesian coordinates ct, x, y, z to new coordinates (cT, X, y, z) :

$$ct = X \sinh \frac{cT}{\alpha}, \quad x = X \cosh \frac{cT}{\alpha}, \quad (4.1.39)$$

where $\alpha = \text{const}$. The inverse transformation is

$$cT = \alpha \tanh^{-1} \frac{ct}{x}, \quad X = \sqrt{x^2 - (ct)^2}. \quad (4.1.40)$$

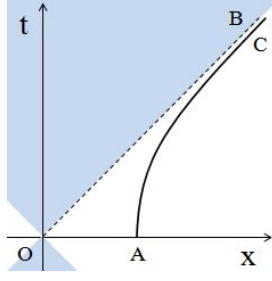


Figure 5: Event horizon in uniformly accelerated motion.

Substituting (4.1.39) into the Minkowski interval (1.4.85) gives

$$ds^2 = \frac{X^2}{\alpha^2} c^2 dT^2 - dX^2 - dy^2 - dz^2. \quad (4.1.41)$$

The interval (4.1.41) is called the *Rindler metric*. If a point is at rest in the Rindler metric,

$$X = \frac{c^2}{a_0}, \quad y = \text{const}, \quad z = \text{const}, \quad (4.1.42)$$

where $a_0 = \text{const}$, and we define

$$\eta = \frac{cT}{\alpha}, \quad (4.1.43)$$

then (4.1.39) gives (4.1.37). Thus a point at rest in the *Rindler coordinates* cT, X is moving with a constant proper acceleration a_0 in the Galilean coordinates.

In the nonrelativistic limit, $a_0 t \ll c$, (4.1.35) reduces to

$$v = a_0 t, \quad (4.1.44)$$

which integrates to

$$x = \frac{1}{2} a_0 t^2 + x_0. \quad (4.1.45)$$

Therefore a nonrelativistic uniformly accelerated motion is parabolic. As $t \rightarrow \infty$, $v \rightarrow \infty$, so the applicability of this limit is restricted to velocities much smaller than c . Formulae (4.1.44) and (4.1.45) are special cases of the nonrelativistic kinematics of a particle moving in a uniform field of force $\mathbf{f}$ with a constant acceleration $\mathbf{a} = \frac{\mathbf{f}}{m}$, which integrates to

$$\mathbf{v} = \mathbf{a}t + \mathbf{v}_0, \quad \mathbf{r} = \frac{1}{2} \mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{r}_0, \quad (4.1.46)$$

where $\mathbf{v}_0$ and $\mathbf{r}_0$ are the velocity and position of the particle at $t = 0$.

4.1.3 Uniformly rotating frame of reference

The Galilean frame of reference in the cylindrical coordinates is described, due to (4.1.20), by the interval

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2. \quad (4.1.47)$$

We consider a transformation to a frame of reference which rotates uniformly about the z -axis with angular velocity ω :

$$\phi = \phi' + \omega t. \quad (4.1.48)$$

The interval in this frame is

$$ds^2 = (c^2 - \omega^2 r^2) dt^2 - 2\omega r^2 d\phi' dt - dr^2 - r^2 d\phi'^2 - dz^2. \quad (4.1.49)$$

The rotating frame can be realized with physical bodies if $g_{00} > 0$ which gives

$$r < \frac{c}{\omega}, \quad (4.1.50)$$

representing the impossibility of exceeding the speed of propagation of interaction by physical bodies rotating with this frame. The square of an infinitesimal distance (1.4.97) corresponding to the interval (4.1.49) is

$$dl^2 = dr^2 + dz^2 + \frac{r^2 d\phi'^2}{1 - \frac{\omega^2 r^2}{c^2}}, \quad (4.1.51)$$

so the ratio of the circumference of a circle at $z = \text{const}$ to its radius is $2\pi/\sqrt{1 - \frac{\omega^2 r^2}{c^2}}$. The time difference between two synchronized infinitesimally separated points is, due to (1.4.101),

$$\Delta t = \frac{1}{c^2} \int \frac{\omega r^2 d\phi'}{1 - \frac{\omega^2 r^2}{c^2}}. \quad (4.1.52)$$

Thus the time difference along an infinitesimal closed curve connecting synchronized points is

$$\Delta t = \frac{1}{c^2} \oint \frac{\omega r^2 d\phi'}{1 - \frac{\omega^2 r^2}{c^2}}, \quad (4.1.53)$$

which, in the limit $\frac{\omega r}{c} \ll 1$, gives

$$\Delta t = \frac{\omega}{c^2} \oint r^2 d\phi' = \pm \frac{2\omega}{c^2} S, \quad (4.1.54)$$

where S is the surface area of the projection of the contour on the plane perpendicular to the z -axis and the sign $\pm$ depends on whether the synchronization takes place in the direction of the rotation or the opposite. If the length of the contour is L then the time of the propagation of a signal along the contour is thus

$$t = \frac{L}{c} \pm \frac{2\omega}{c^2} S, \quad (4.1.55)$$

so the speed of propagation of interaction in the rotating frame measured in the inertial frame is

$$c' = c \pm \frac{2\omega S}{L}. \quad (4.1.56)$$

The metric tensor corresponding to the interval (4.1.49) is

$$g_{ik} = \begin{pmatrix} 1 - \frac{\omega^2 r^2}{c^2} & 0 & -\frac{\omega r^2}{c} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\omega r^2}{c} & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g^{ik} = \begin{pmatrix} 1 & 0 & -\frac{\omega}{c} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\omega}{c} & 0 & \frac{\omega^2}{c^2} - \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (4.1.57)$$

Substituting the corresponding Christoffel symbols (1.4.26) into the metric geodesic equation (1.4.80) gives the equations of motion of a particle in a uniformly rotating frame of reference:

$$\frac{du^0}{ds} = 0, \quad \frac{du^z}{ds} = 0, \quad (4.1.58)$$

$$\frac{du^r}{ds} - r \left(\frac{\omega}{c} u^0 + u^\phi \right)^2 = 0, \quad (4.1.59)$$

$$\frac{du^\phi}{ds} + \frac{2}{r} u^r u^\phi + \frac{2\omega}{cr} u^0 u^r = 0. \quad (4.1.60)$$

Due to (4.1.58), we can take $s = ct$. Thus (4.1.59) and (4.1.60) give, using (4.1.26),

$$a^r = \frac{\partial v^r}{\partial t} - r v^{\phi 2} = \omega^2 r + 2\omega r v^\phi, \quad (4.1.61)$$

$$a^\phi = r \frac{\partial v^\phi}{\partial t} + 2v^r v^\phi = -2\omega v^r. \quad (4.1.62)$$

We consider a nonrelativistic particle in a frame of reference K which rotates uniformly relative to an inertial frame K_0 with angular velocity $\boldsymbol{\omega}$ and whose origin coincides with the origin of K_0 , so $\mathbf{r} = \mathbf{r}_0$. Substituting $\dot{\boldsymbol{\omega}} = 0$ and $\mathbf{A} = 0$ into the Lagrangian (3.2.48) gives

$$L = \frac{1}{2}m\mathbf{v}^2 + m\mathbf{v} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r})^2 - U, \quad (4.1.63)$$

so the equation of motion (3.2.50) is

$$m \frac{d\mathbf{v}}{dt} = -\frac{\partial U}{\partial \mathbf{r}'} - 2m\boldsymbol{\omega} \times \mathbf{v} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (4.1.64)$$

This nonrelativistic equation of motion in a uniformly rotating frame of reference is consistent with (4.1.61) and (4.1.62) for $U = 0$ (no external forces). Thus the relativistic Coriolis and centrifugal inertial forces have the same forms as their nonrelativistic expressions. The momentum $\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$ for the Lagrangian (4.1.63) is

$$\mathbf{p} = m\mathbf{v} + m\boldsymbol{\omega} \times \mathbf{r} = m\mathbf{v}_0 = \mathbf{p}_0, \quad (4.1.65)$$

where $\mathbf{v}_0$ is the velocity of the particle in K_0 . The momentum of the particle $\mathbf{p}$ in K coincides with the momentum $\mathbf{p}_0$ in K_0 . Thus the angular momentum of the particle $\mathbf{M} = \mathbf{r} \times \mathbf{p}$ in K coincides with the angular momentum $\mathbf{M}_0 = \mathbf{r}_0 \times \mathbf{p}_0$ in K_0 . The energy $E = \mathbf{p} \cdot \mathbf{v} - L$ for the Lagrangian (4.1.63) is

$$E = \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r})^2 + U, \quad (4.1.66)$$

where the term $-\frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r})^2$, which is independent of the velocity, is referred to as the *centrifugal potential energy*. Substituting $\mathbf{v}_0 = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}$ in (4.1.66) gives, using $E_0 = \frac{1}{2}m\mathbf{v}_0^2 + U$, the relation between the energies E and E_0 in the two frames:

$$E = \frac{1}{2}m\mathbf{v}_0^2 - m\mathbf{v}_0 \cdot (\boldsymbol{\omega} \times \mathbf{r}) + U = E_0 - m\boldsymbol{\omega} \cdot (\mathbf{r} \times \mathbf{v}_0) = E_0 - \mathbf{M} \cdot \boldsymbol{\omega}. \quad (4.1.67)$$

The relation $E = E_0 - \mathbf{M} \cdot \boldsymbol{\omega}$ is also valid for a system of particles due to the additivity of the Lagrangian.

4.1.4 Unidimensional nonrelativistic motion

In 1 dimension, the Lagrangian for a particle in an external field U is given by

$$L = \frac{1}{2}a(q)\dot{q}^2 - U(q), \quad (4.1.68)$$

where q is a generalized coordinate. In the Cartesian coordinates, the Lagrangian becomes

$$L = \frac{1}{2}m\dot{x}^2 - U(x). \quad (4.1.69)$$

The conservation of energy,

$$\frac{1}{2}m\dot{x}^2 + U(x) = E, \quad (4.1.70)$$

gives

$$t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x)}} + \text{const.} \quad (4.1.71)$$

The motion of the particle is possible in the regions where $U(x) < E$. The points where $U(x) = E$ are called *turning points*. At these points, the particle is at rest: $\dot{x} = 0$. If the region of motion is bounded by 2 turning points then such a motion is *finite*, that is, it takes place in a finite region of space. A finite motion is oscillatory with a period (the doubled time of the motion of the particle from one turning point to the other)

$$T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}, \quad (4.1.72)$$

where $x_1(E)$ and $x_2(E)$ are the coordinates of the turning points. The inverse function $x(U)$ is two-valued. If the region of motion is bounded by 1 or no turning points then such a motion is *infinite*.

If the potential energy $U(x)$ has a local minimum, whose coordinate can be taken at $x = 0$ and whose value can be set to 0, then (4.1.72) gives

$$T(E) = \sqrt{2m} \int_E^0 \frac{dx_1(U)}{dU} \frac{dU}{\sqrt{E-U}} + \sqrt{2m} \int_0^E \frac{dx_2(U)}{dU} \frac{dU}{\sqrt{E-U}}, \quad (4.1.73)$$

where $x_1(U)$ and $x_2(U)$ are the two branches of the function $x(U)$, as shown in Fig. (6). Integrating

$$\begin{aligned} \int_0^K \frac{T(E)dE}{\sqrt{K-E}} &= \sqrt{2m} \int_0^K \int_0^E \left(\frac{dx_2}{dU} - \frac{dx_1}{dU} \right) \frac{dU dE}{\sqrt{(K-E)(E-U)}} \\ &= \sqrt{2m} \int_0^K \left(\frac{dx_2}{dU} - \frac{dx_1}{dU} \right) dU \int_U^K \frac{dE}{\sqrt{(K-E)(E-U)}}. \end{aligned} \quad (4.1.74)$$

Using $\int_U^K dE/\sqrt{(K-E)(E-U)} = \pi$ and taking $K = U$ gives

$$x_2 - x_1 = \frac{1}{\pi\sqrt{2m}} \int_0^U \frac{T(E)dE}{\sqrt{U-E}}. \quad (4.1.75)$$

If the function $U(x)$ is symmetric with respect to its minimum then $x_2 = -x_1 = x$, yielding the function $x(U)$:

$$x(U) = \frac{1}{2\pi\sqrt{2m}} \int_0^U \frac{T(E)dE}{\sqrt{U-E}}. \quad (4.1.76)$$

Inverting this function gives the potential energy $U(x)$ as a functional of the period $T(E)$.

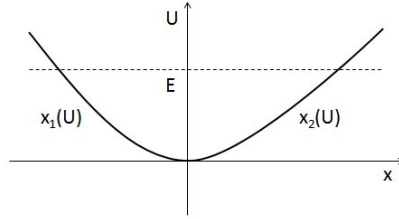


Figure 6: Potential energy in a finite motion.

4.1.5 Central field

We consider a nonrelativistic particle in an external field. If the potential energy U depends only on the distance r of the particle from the a fixed point, called the *center* of the field, then the field is referred to as *central*. The force acting on the particle is given by

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{r}} = -\frac{dU}{dr} \frac{\mathbf{r}}{r}. \quad (4.1.77)$$

Since the angular momentum $\mathbf{M}$ of the particle (or any system) in such a field relative to the center is conserved (cf. (3.1.151)),

$$\dot{\mathbf{M}} = \mathbf{r} \times \mathbf{F} = -\frac{1}{r} \frac{dU}{dr} \mathbf{r} \times \mathbf{r} = 0, \quad (4.1.78)$$

and $\mathbf{M}$ is perpendicular to $\mathbf{r}$, the radius vector $\mathbf{r}$ and the path of the particle lie in a plane perpendicular to the constant vector $\mathbf{M}$. The motion of the particle in a central field is therefore

two-dimensional. The Lagrangian of a particle in a central field $U(r)$ in the cylindrical coordinates (r, ϕ, z) is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r). \quad (4.1.79)$$

The relation $\frac{\partial L}{\partial \phi} = 0$ shows that ϕ is a cyclic coordinate. The Lagrange equations for ϕ , $\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$, give the conservation of the corresponding generalized momentum,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = M_z, \quad (4.1.80)$$

which coincides with the conserved angular momentum $\mathbf{M} = M_z \hat{z}$.

The area of an infinitesimal, triangular region bounded by the radius vectors $\mathbf{r}$, $\mathbf{r} + d\mathbf{r}$, and the element of the path $dr = r d\phi$ is

$$df = \frac{1}{2}r^2 d\phi, \quad (4.1.81)$$

as shown in Fig. (7). The absolute value of angular momentum $M = mr^2\dot{\phi}$ is therefore equal to

$$M = 2mf\dot{\phi}, \quad (4.1.82)$$

where $f\dot{\phi}$ is called the *sectorial velocity* of the particle. The conservation of angular momentum in a central field gives thus *Kepler's second law*: the radius vector of a particle in such a field sweeps out equal areas in equal times.

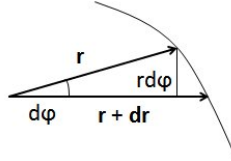


Figure 7: Radius vector.

The energy of the particle in a constant central field $U(r)$ is conserved,

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r). \quad (4.1.83)$$

Using (4.1.80) in (4.1.83) gives

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}, \quad (4.1.84)$$

where

$$U_{\text{eff}} = U(r) + \frac{M^2}{2mr^2} \quad (4.1.85)$$

is the *effective potential* and $M = M_z$. The term $\frac{M^2}{2mr^2}$ is the centrifugal potential energy (cf. (4.1.66)). The centrifugal energy diverges as r tends to 0. Therefore, the particle cannot reach the center of the field unless U tends to $-\infty$ as $r \rightarrow 0$ sufficiently enough so that U_{eff} tends there to a finite value. The motion of the particle is possible in the regions where $U_{\text{eff}} < E$. The points where $U_{\text{eff}} = E$ are the turning points for the r coordinate, $\dot{r} = 0$. At these points, however, the particle is not at rest: $\dot{\phi} \neq 0$. If the region of motion is bounded by 2 turning points for r , $r_1 \leq r \leq r_2$, then such a motion is finite. If the region of motion is bounded from below by 1 turning point for r , $r \geq r_1$, then such a motion is infinite.

Integrating (4.1.84) gives

$$t(r) = \int \frac{dr}{\sqrt{\frac{2}{m}(E - U) - \frac{M^2}{m^2 r^2}}} + \text{const}, \quad (4.1.86)$$

which upon inversion gives the motion of the particle, $r(t)$. Substituting $\dot{r} = \frac{dr}{d\phi}\dot{\phi}$ into (4.1.84) and integrating gives

$$\phi(r) = \int \frac{\frac{M}{r^2} dr}{\sqrt{2m(E - U) - \frac{M^2}{r^2}}} + \text{const}, \quad (4.1.87)$$

which upon inversion gives the path of the particle, $r(\phi)$. Since $\dot{\phi}$ does not change sign, $\phi(t)$ is a monotonic function. Substituting

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{M}{mr^2} \frac{dr}{d\phi} = -\frac{M}{m} \frac{d}{d\phi} \left(\frac{1}{r} \right) \quad (4.1.88)$$

into (4.1.84), and defining

$$u = \frac{1}{r}, \quad (4.1.89)$$

gives

$$E = \frac{M^2}{2m} \left[\left(\frac{du}{d\phi} \right)^2 + u^2 \right] + U(1/u). \quad (4.1.90)$$

Differentiating (4.1.90) with respect to ϕ and dividing by $\frac{M^2}{m} \frac{du}{d\phi}$ gives

$$\frac{d^2u}{d\phi^2} + u = -\frac{m}{M^2} \frac{d}{du} U(1/u). \quad (4.1.91)$$

The path of a particle in a central field is a closed curve if the change of ϕ during the period of the oscillation of the r coordinate,

$$\Delta\phi = 2 \int_{r_1}^{r_2} \frac{\frac{M}{r^2} dr}{\sqrt{2m(E - U) - \frac{M^2}{r^2}}}, \quad (4.1.92)$$

is a rational fraction of 2π . The only central potentials that lead to closed paths are $U \sim r^2$ ($\Delta\phi = \pi$) and $U \sim 1/r$ ($\Delta\phi = 2\pi$).

The concept of the potential energy is not valid in relativistic mechanics. For a charged particle in an electromagnetic field, however, the energy of the particle has an additive contribution from the field. We consider a relativistic particle of charge e in a central electrostatic field produced by a fixed charge e' . The motion of the particle takes place in a plane perpendicular to the constant angular momentum vector. The conserved energy of the particle is given by (3.1.80):

$$E = \sqrt{p^2 c^2 + m^2 c^4} + e\phi, \quad (4.1.93)$$

where $\phi = \frac{e'}{r}$ (4.2.7) and $\mathbf{A} = 0$. The momentum of the particle can be decomposed into the angular and radial parts:

$$p^2 = \frac{M^2}{r^2} + p_r^2. \quad (4.1.94)$$

The corresponding Hamilton-Jacobi equation is

$$-\frac{1}{c^2} \left(\frac{\partial S}{\partial t} + \frac{ee'}{r} \right)^2 + \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \phi} \right)^2 + m^2 c^2 = 0. \quad (4.1.95)$$

Putting $S = -Et + M\phi + S_r$ gives

$$S = -Et + M\phi + \int dr \sqrt{\frac{1}{c^2} \left(E - \frac{ee'}{r} \right)^2 - \frac{M^2}{r^2} - m^2 c^2}. \quad (4.1.96)$$

The motion of the particle is given by $\frac{\partial S}{\partial E} = \text{const}$, whereas its trajectory is given by $\frac{\partial S}{\partial M} = \text{const}$. If $ee' < 0$ then the motion is finite (and the path is not closed) for $E < mc^2$ and infinite for $E \geq mc^2$. If $ee' > 0$ then the motion is infinite.

The motion of two interacting particles regarded as a closed system is referred to as the *two-body problem*. For nonrelativistic particles, such a motion can be separated into the motion of the center of mass of the system and the motion of the particles relative to the center of mass. The Lagrangian of the system of two particles with masses m_1 and m_2 is given by

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(|\mathbf{r}_1 - \mathbf{r}_2|). \quad (4.1.97)$$

Defining the separation vector,

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad (4.1.98)$$

and using the center of mass of the system (3.1.138),

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m}, \quad (4.1.99)$$

where m is the total mass of the system (3.1.137),

$$m = m_1 + m_2, \quad (4.1.100)$$

gives

$$\mathbf{r}_1 = \frac{m_2}{m}\mathbf{r} + \mathbf{R}, \quad \mathbf{r}_2 = -\frac{m_1}{m}\mathbf{r} + \mathbf{R}. \quad (4.1.101)$$

The first terms on the right-hand sides in (4.1.101) are the coordinates of the two particles relative to the center of mass. Accordingly, the Lagrangian (4.1.97) becomes

$$L = \frac{1}{2}m\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(|\mathbf{r}|), \quad (4.1.102)$$

where

$$\mu = \frac{m_1m_2}{m_1 + m_2} \quad (4.1.103)$$

is referred to as the *reduced mass*. The Lagrangian (4.1.102) describes the free motion of a particle of mass m and the motion of a particle of mass μ in an external field $U(r)$. The motion of the two particles is then given by substituting $\mathbf{R}(t)$ and $\mathbf{r}(t)$ into (4.1.101). The generalized momenta corresponding to the coordinates $\mathbf{R}$ and $\mathbf{r}$ are

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{R}}} = m\dot{\mathbf{R}} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2, \quad \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = \mu\dot{\mathbf{r}}. \quad (4.1.104)$$

The quantity $\mathbf{P}$ is the total momentum of the system.

4.1.6 Kepler motion

We consider the motion of a particle with mass m in a nonrelativistic (Newtonian) gravitational field of a body with mass μ . Such a field is central, with the potential energy given by (3.1.103) and (4.3.59):

$$U(r) = -\frac{G\mu m}{r}. \quad (4.1.105)$$

The motion in such a field is referred to as a *Keplerian motion*. The effective potential is thus

$$U_{\text{eff}} = -\frac{\gamma}{r} + \frac{M^2}{2mr^2}, \quad (4.1.106)$$

where

$$\gamma = G\mu m > 0. \quad (4.1.107)$$

The particle therefore cannot reach the center of the field unless it moves radially, $M = 0$. The effective potential $U_{\text{eff}}(r)$ is shown in Fig. 8. Such a potential has a minimum at

$$U_{\text{min}} = -\frac{m\gamma^2}{2M^2}. \quad (4.1.108)$$

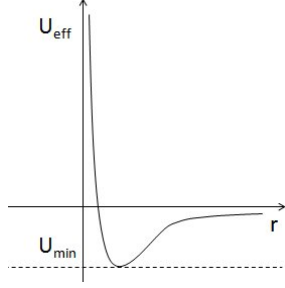


Figure 8: Effective potential in an attractive field $1/r$.

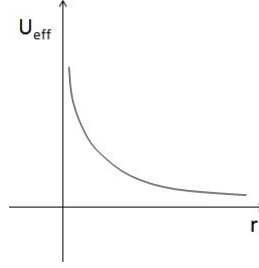


Figure 9: Effective potential in a repulsive field $1/r$.

The motion of the particle is finite for $E < 0$ and infinite for $E \leq 0$. If $\gamma < 0$ then the effective potential $U_{\text{eff}}(r)$ is shown in Fig. 9. The motion in such a field is infinite.

The Newton equation of motion (3.1.100) in a Newtonian field gives

$$m\ddot{\mathbf{r}} = -\frac{\gamma\mathbf{r}}{r^3}. \quad (4.1.109)$$

Since such a field is central, the angular momentum of the particle, $\mathbf{M} = m\mathbf{r} \times \dot{\mathbf{r}}$, is conserved (cf. (4.1.78)). The field is constant in time, so that the energy of the particle is conserved,

$$E = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{\gamma}{r}, \quad (4.1.110)$$

which can also be obtained by multiplying (4.1.109) by $\dot{\mathbf{r}}$ and integrating. We consider a quantity

$$\mathbf{L} = \frac{1}{m}\mathbf{p} \times \mathbf{M} - \frac{\gamma\mathbf{r}}{r}. \quad (4.1.111)$$

Differentiating $\mathbf{L}$ with respect to time gives, using (4.1.109),

$$\frac{d\mathbf{L}}{dt} = \ddot{\mathbf{r}} \times \mathbf{M} - \gamma \frac{d}{dt} \frac{\mathbf{r}}{\sqrt{\mathbf{r} \cdot \mathbf{r}}} = -\frac{\gamma}{r^3} \mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) + \gamma \left(\frac{(\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}}{r^3} - \frac{\dot{\mathbf{r}}}{r} \right) = 0. \quad (4.1.112)$$

The vector (4.1.111) is thus conserved and it is referred to as the *Laplace-Runge-Lenz vector*. Unlike the energy and angular momentum, this conserved quantity is not an additive constant of motion. The existence of such a one-valued function of $\mathbf{r}$ and $\mathbf{v}$ is related to a specific character of the motion in the $\frac{1}{r}$ potential, called degeneracy (cf. ??). The Laplace-Runge-Lenz vector is perpendicular to the angular momentum vector:

$$\mathbf{L} \cdot \mathbf{M} = \frac{1}{m}(\mathbf{p} \times \mathbf{M}) \cdot \mathbf{M} - \frac{\gamma\mathbf{r}}{r} \cdot (\mathbf{r} \times \mathbf{p}) = 0, \quad (4.1.113)$$

so that it lies in a plane of the motion and has 2 independent components. The number of the conserved quantities in the Keplerian motion is thus 6: 3 components of $\mathbf{M}$, 2 components of $\mathbf{L}$, and E , related to $2n$ arbitrary constants of the equations of motion for $n = 3$ degrees of freedom of a particle.

The square of the Laplace-Runge-Lenz vector is, using (4.1.110), equal to

$$\begin{aligned} L^2 &= \mathbf{L} \cdot \mathbf{L} = (\mathbf{v} \times \mathbf{M})^2 - \frac{2\gamma}{r} \mathbf{r} \cdot (\mathbf{v} \times \mathbf{M}) + \gamma^2 = v^2 M^2 - (\mathbf{v} \cdot \mathbf{M})^2 \\ &\quad - \frac{2\gamma}{r} \mathbf{M} \cdot (\mathbf{r} \times \mathbf{v}) + \gamma^2 = v^2 M^2 - \frac{2\gamma}{mr} M^2 + \gamma^2 = \frac{2EM^2}{m} + \gamma^2. \end{aligned} \quad (4.1.114)$$

We also have

$$\mathbf{L} \cdot \mathbf{r} = \frac{1}{m} \mathbf{r} \cdot (\mathbf{p} \times \mathbf{M}) - \gamma r = \frac{1}{m} \mathbf{M} \cdot (\mathbf{r} \times \mathbf{p}) - \gamma r = \frac{M^2}{m} - \gamma r. \quad (4.1.115)$$

Defining ϕ as the angle between $\mathbf{L}$ and the radius vector $\mathbf{r}$ gives $\mathbf{L} \cdot \mathbf{r} = Lr \cos\phi$, so that (4.1.115) turns into

$$r = \frac{p}{1 + e \cos\phi}, \quad (4.1.116)$$

where

$$p = \frac{M^2}{m\gamma}, \quad e = \frac{L}{\gamma}. \quad (4.1.117)$$

The relation $r(\phi)$ (4.1.116) is the equation of a conic-section curve with latus rectum $2p$ and eccentricity e , giving a mathematical formulation of *Kepler's first law*. It is a circle or radius p for $e = 0$, an ellipse for $0 < e < 1$, a parabola for $e = 1$, and a hyperbola for $e > 1$. Substituting (4.1.117) into (4.1.114) gives

$$E = \frac{\gamma(e^2 - 1)}{2p}, \quad e = \sqrt{1 + \frac{2EM^2}{m\gamma^2}}. \quad (4.1.118)$$

The path of the particle (4.1.116) can also be obtained from (4.1.87) or (4.1.91). The latter gives $\frac{d^2u}{d\phi^2} + u = \frac{m\gamma}{M^2} = \frac{1}{p}$.

The path of a particle in a Newtonian field for $e < 1$ ($E < 0$) is shown in Fig. 10. The origin of the field is in one of the two foci of the ellipse, F . The center of the ellipse is at point O . The semi-major axis of the ellipse is $|OP| = |FB| = a$, the semi-minor axis is $|OB| = b = a\sqrt{1 - e^2} = \frac{M}{\sqrt{-2mE}}$ (due to (4.1.118)), $|OF| = ae$, and $|FD| = p$. The Laplace-Runge-Lenz vector is directed along the vector FP . When $\phi = 0$, the particle is located at the point P which is closest to the center, called a *periastron*, for which $r = r_{\min} = \frac{p}{1+e}$. When $\phi = \pi$, the particle is located at the point A which is furthest from the center, called a *apoastron*, for which $r = r_{\max} = \frac{p}{1-e}$. The relation $r_{\min} + r_{\max} = 2a$ gives

$$p = a(1 - e^2), \quad (4.1.119)$$

so that $r_{\min} = a(1 - e)$, $r_{\max} = a(1 + e)$, and (4.1.118) yields

$$a = -\frac{\gamma}{2E}. \quad (4.1.120)$$

Accordingly, the equation of path (4.1.116) is

$$r = \frac{a(1 - e^2)}{1 + e \cos\phi}. \quad (4.1.121)$$

The periastron and apoastron are turning points: $r_{\min}$ and $r_{\max}$ are the two roots of the equation $U_{\text{eff}}(r) = E$. If $E = U_{\min}$ then $e = 0$.

Some properties of the Keplerian motion can be obtained using mechanical similarity. The potential energy in such a field is a homogeneous function of the Cartesian coordinates (3.1.162) with degree of homogeneity $k = -1$. The first relation in (3.1.164) gives thus

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{3/2}. \quad (4.1.122)$$

Accordingly, the period T of revolution of a particle along an elliptical orbit is related to the semi-major axis a of the ellipse by

$$T^2 \sim a^3, \quad (4.1.123)$$

which constitutes *Kepler's third law*. The value of T is given by the constancy of the sectorial velocity in (4.1.82), which constitutes Kepler's second law. This law gives $MT = 2\pi mab$, from which

$$T = 2\pi\sqrt{\frac{m}{\gamma}}a^{3/2}. \quad (4.1.124)$$

Substituting (4.1.118) and (4.1.120) into (4.1.86) gives

$$t = \sqrt{\frac{ma}{\gamma}} \int \frac{r dr}{\sqrt{a^2 e^2 - (r - a)^2}}. \quad (4.1.125)$$

Integrating (4.1.125) over the whole ellipse gives the period (4.1.124),

$$T = 2\sqrt{\frac{ma}{\gamma}} \int_{a(1-e)}^{a(1+e)} \frac{rdr}{\sqrt{a^2e^2 - (r-a)^2}}. \quad (4.1.126)$$

The expression (4.1.126) can also be derived using (4.1.88) and the equation of path (4.1.116). Substituting

$$r = a(1 - e \cos \xi), \quad (4.1.127)$$

where ξ is a parameter, into (4.1.125) and putting $t = 0$ when the particle is at the periastron gives *Kepler's equation*

$$\omega t = \xi - e \sin \xi, \quad (4.1.128)$$

where

$$\omega = \frac{2\pi}{T}. \quad (4.1.129)$$

Combining the equation of path (4.1.121) with (4.1.127) leads to

$$ae + r \cos \phi = a \cos \xi, \quad (4.1.130)$$

$$\tan \frac{\phi}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{\xi}{2}. \quad (4.1.131)$$

The relation (4.1.130) is represented in Fig. 11 which shows the *auxiliary circle* with radius a , corresponding to the Keplerian ellipse in Fig. 10. The angle ϕ between the major axis and the radius vector $\mathbf{r}$ is, in astronomy, denoted v and called the *true anomaly*. The angle ξ is between the major axis and a vector, which is directed from the center of the circle (and of the ellipse) to the point on the circle with the same horizontal coordinate as the particle. This angle is, in astronomy, denoted E and called the *eccentric anomaly*. The quantity ωt in (4.1.128) is, in astronomy, denoted M and called the *mean anomaly*. Kepler's equation (4.1.128) is then

$$M = E - e \sin E. \quad (4.1.132)$$

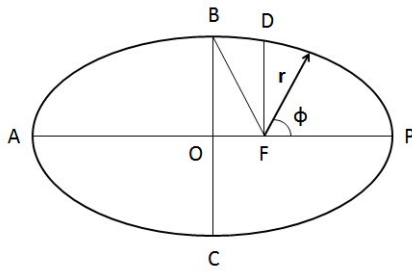


Figure 10: Keplerian motion on an ellipse.

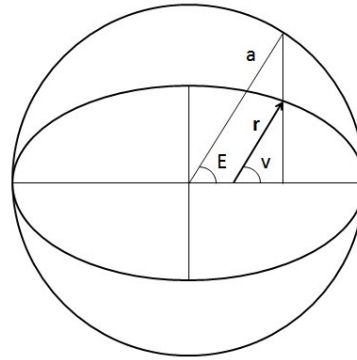


Figure 11: Auxiliary circle.

For $E = 0$, (4.1.118) indicates the motion on a parabola, $e = 1$. The only turning point is located at the periastron, $r_{\min} = \frac{p}{2}$, due to (4.1.116). Such a motion is given by (4.1.86):

$$t = \int \frac{rdr}{\sqrt{\frac{2\gamma}{m}r - \frac{M^2}{m^2}}}. \quad (4.1.133)$$

Substituting

$$r = \frac{1}{2}p(1 + \eta^2), \quad (4.1.134)$$

where η is a parameter, into (4.1.133) and putting $t = 0$ when the particle is at the periastron gives

$$\sqrt{\frac{\gamma}{mp^3}}t = \frac{1}{2}\left(\eta + \frac{\eta^3}{3}\right). \quad (4.1.135)$$

For $E > 0$, (4.1.118) indicates the motion on a hyperbola, $e > 1$, as shown in Fig. 12. The equation of path is

$$r = \frac{a(e^2 - 1)}{1 + e \cos \phi}, \quad (4.1.136)$$

where the semi-major axis of the hyperbola is given by

$$a = \frac{\gamma}{2E} = \frac{p}{e^2 - 1}. \quad (4.1.137)$$

The periastron is located at $|FP| = r_{\min} = a(e - 1)$, and $|FB| = p$. The parametric equations (4.1.127) and (4.1.128) are replaced by

$$r = a(e \cosh \xi - 1), \quad (4.1.138)$$

$$\omega t = e \sinh \xi - \xi. \quad (4.1.139)$$

Defining

$$\tan \frac{\phi}{2} = \sqrt{\frac{1+e}{e-1}} \tanh \frac{H}{2} \quad (4.1.140)$$

leads to Kepler's equation

$$M = e \sinh H - H. \quad (4.1.141)$$

If the field is repulsive, $\gamma < 0$, then the motion is shown in Fig. 13. The equation of path is

$$r = \frac{p}{e \cos \phi - 1}, \quad (4.1.142)$$

where

$$p = -\frac{M^2}{m\gamma}. \quad (4.1.143)$$

The periastron is located at $|FP| = r_{\min} = a(e + 1)$. The parametric equations (4.1.127) and (4.1.128) are replaced by

$$r = a(e \cosh \xi + 1), \quad (4.1.144)$$

$$\omega t = e \sinh \xi + \xi. \quad (4.1.145)$$

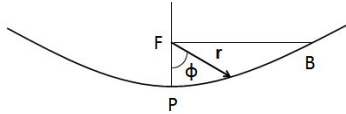


Figure 12: Keplerian motion on a hyperbola.

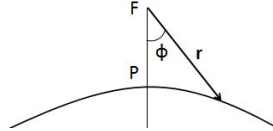


Figure 13: Motion on a hyperbola in a repulsive field.

We consider the motion of a particle in a perturbed Keplerian field, described by the Lagrangian (4.1.79) with

$$U(r) = -\frac{\gamma}{r} + \delta U(r), \quad (4.1.146)$$

where $\delta U(r)$ is a small correction to the Newtonian potential $-\frac{\gamma}{r}$. The change of ϕ during the period of the oscillation of the r coordinate (4.1.92) can be written as

$$\Delta\phi = -2\frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \sqrt{2m(E - U) - \frac{M^2}{r^2}} dr. \quad (4.1.147)$$

Substituting (4.1.146) into (4.1.147) and expanding in δU gives

$$\Delta\phi = 2\pi + \delta\phi, \quad (4.1.148)$$

where

$$\delta\phi = \frac{\partial}{\partial M} \int_{r_{\min}}^{r_{\max}} \frac{2m \delta U dr}{\sqrt{2m(E + \frac{\gamma}{r}) - \frac{M^2}{r^2}}} = \frac{\partial}{\partial M} \left(\frac{2m}{M} \int_0^\pi r^2 \delta U d\phi \right). \quad (4.1.149)$$

If $E \neq 0$ then we normalize the Laplace-Runge-Lenz vector by defining

$$\mathbf{D} = \frac{\mathbf{L}}{\sqrt{2m|E|}}. \quad (4.1.150)$$

The Poisson brackets of the components of such a vector and the components of the angular momentum vector are

$$[D_\alpha, D_\beta] = -\text{sgn}(E) e_{\alpha\beta\gamma} M_\gamma, \quad (4.1.151)$$

$$[D_\alpha, M_\beta] = e_{\alpha\beta\gamma} D_\gamma. \quad (4.1.152)$$

If $E < 0$ then the relations (3.4.3), (4.1.151) and (4.1.152) form the Lie algebra of the rotation group in the four-dimensional Euclidean space. The Casimir invariants C_i of such a group are the quantities whose Poisson brackets with $\mathbf{M}$ and $\mathbf{D}$ vanish:

$$\{C_i, M_\alpha\} = 0, \quad \{C_i, D_\alpha\} = 0. \quad (4.1.153)$$

These invariants are

$$C_1 = \mathbf{M} \cdot \mathbf{D} = 0, \quad C_2 = \mathbf{M}^2 + \mathbf{D}^2 = \frac{m\gamma^2}{2|E|}. \quad (4.1.154)$$

4.1.7 Decays and collisions of particles

We consider a free body with mass M at rest, decaying into two particles with masses m_1, m_2 and momenta $\mathbf{p}_{10}, \mathbf{p}_{20}$. The energies of the resulting particles are $E_{10} = \sqrt{\mathbf{p}_{10}^2 c^2 + m_1^2 c^4}$, $E_{20} = \sqrt{\mathbf{p}_{20}^2 c^2 + m_2^2 c^4}$. The conservation of momentum in this frame of reference gives $\mathbf{p}_{10} + \mathbf{p}_{20} = 0$, so that $E_{10}^2 - m_1^2 c^4 = E_{20}^2 - m_2^2 c^4$. Combining this relation with the conservation of energy, $E_{10} + E_{20} = Mc^2$, gives the values of E_{10} and E_{20} . The rest frame of the body is the center-of-inertia frame (*C-frame*) of the two resulting particles. If the internal state of particles does not change during the collision then such a collision is called *elastic*. The decay is possible if the *binding energy* $E_b = (M - m_1 - m_2)c^2$ is positive, otherwise the body is unstable with respect to this decay.

In a laboratory frame (*L-frame*), the body moves with velocity $\mathbf{V}$ and decays into the same two particles with momenta $\mathbf{p}_1, \mathbf{p}_2$ and energies $E_1 = \sqrt{\mathbf{p}_1^2 c^2 + m_1^2 c^4}$, $E_2 = \sqrt{\mathbf{p}_2^2 c^2 + m_2^2 c^4}$. We consider one of the emerging particles, omitting the subscripts 1,2. We take the plane in which the vectors $\mathbf{V}$, $\mathbf{p}$ and $\mathbf{p}_0$ lie as the xy -plane (the x -axis along the vector $\mathbf{V}$). We choose the L-frame as K and the C-frame as K' in (3.1.61) and (3.1.62), so that $\mathbf{V}$ is the velocity of the C-frame relative to the L-frame. If m is the mass of the particle, θ is the angle between $\mathbf{p}$ and $\mathbf{V}$, and θ_0 is the angle between $\mathbf{p}_0$ and $\mathbf{V}$, then (3.1.61) and (3.1.62) give

$$E_0 = \frac{E - (V/c)\sqrt{E^2 - m^2 c^4} \cos\theta}{\sqrt{1 - V^2/c^2}}, \quad E = \frac{E_0 + (V/c)\sqrt{E_0^2 - m^2 c^4} \cos\theta_0}{\sqrt{1 - V^2/c^2}}, \quad (4.1.155)$$

$$p \cos\theta = p_x = \frac{p_0 \cos\theta_0 + E_0 V/c^2}{\sqrt{1 - V^2/c^2}}, \quad p \sin\theta = p_y = p_0 \sin\theta_0. \quad (4.1.156)$$

The value of θ_0 depends on the law of interaction of the particles and their relative positions. If we regard the two resulting particles as a single composite system then the velocity of its motion as a whole is given by (3.1.71), $\frac{(\mathbf{p}_1 + \mathbf{p}_2)c^2}{E_1 + E_2} = \mathbf{V}$. The *invariant mass* of such a composite system,

$$m_{\text{inv}} = \frac{1}{c^2} \sqrt{\left(\sum_a E_a\right)^2 - \left(\sum_a \mathbf{p}_a\right)^2 c^2}, \quad (4.1.157)$$

is equal to $\frac{1}{c^2} \sqrt{(E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 c^2} = M$.

The relations (4.1.156) give $(p_x \sqrt{1 - V^2/c^2} - E_0 V/c^2)^2 + p_y^2 = p_0^2$, which is represented by a *kinematic ellipse* in Fig. 14. The major semiaxis of the ellipse is along the x -axis and it is equal to $\frac{p_0}{\sqrt{1 - V^2/c^2}}$. The minor semiaxis is along the y -axis and it is equal to p_0 . The vector $AO = \mathbf{V} \frac{E_0/c^2}{\sqrt{1 - V^2/c^2}}$ and the vector $AB = \mathbf{p}$. If $V < v_0 = \frac{p_0 c^2}{E_0}$ then A lies inside the ellipse. In this case, for each θ there is 1 value of E and thus 1 value of θ_0 . If $V > v_0$ then A lies outside the ellipse. In this case, for each θ there are 2 values of E and thus 2 values of θ_0 . Also, θ cannot exceed a value θ_{max} given by $\sin \theta_{\text{max}} = \frac{p_0 \sqrt{1 - V^2/c^2}}{mV}$. The angles of emergence of the two particles in the C-frame satisfy $\theta_{10} + \theta_{20} = \pi$. The separation angle in the L-frame, which is the angle between the directions of the particles after decay, is equal to $\Theta = \theta_1 + \theta_2$.

In the nonrelativistic limit, the relations (4.1.155) and (4.1.156) reduce to $\mathbf{v} = \mathbf{v}_0 + \mathbf{V}$, where $\mathbf{v}$ is the velocity of a decay particle in the L-frame, and $\mathbf{v}_0$ is the velocity of this particle in the C-frame. This relation gives $v^2 = v_0^2 + V^2 + 2v_0 V \cos \theta_0$ and $v_0^2 = v^2 + V^2 - 2vV \cos \theta$, from which $\tan \theta = \frac{v_0 \sin \theta_0}{v_0 \cos \theta_0 + V}$. Accordingly, the ellipse in Fig. 14 reduces to a circle of radius p_0 , where $AO = m\mathbf{V}$ and $AB = m\mathbf{v}$, so that $OB = m\mathbf{v}_0$. An equivalent circle of radius $v_0 = \frac{p_0}{m}$, where $AO = \mathbf{V}$, $AB = \mathbf{v}$ and $OB = \mathbf{v}_0$, is shown in Fig. 15. If $V > v_0$ then $\sin \theta_{\text{max}} = \frac{v_0}{V}$.

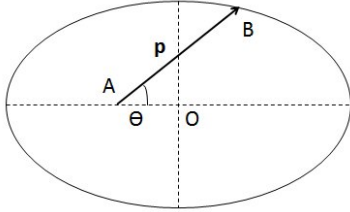


Figure 14: Kinematic ellipse of a decay.

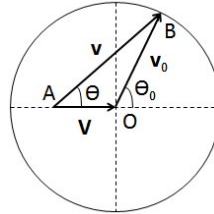


Figure 15: Nonrelativistic limit.

For a system of decaying particles that are randomly oriented in space, every decay particle of a given kind has the same energy in the C-frame and the angular distribution of their directions of motion is isotropic. The fraction of particles in a solid angle $d\Omega_0$ is thus equal to $dN = \frac{d\Omega_0}{4\pi}$, where $d\Omega_0 = 2\pi \sin \theta_0 d\theta_0$ and $\theta_0 \in [0, \pi]$. The angular distributions dN and the ranges of possible values of other quantities can be obtained from the relations between θ_0 and those quantities, such as (4.1.155). If a particle decays into 3 or more particles then the laws of conservation of energy and momentum allow more freedom in directions of motion and velocities of the emerging particles. The kinetic energy of each decay particle cannot exceed the maximum value which the particle has if the system of the other particles has the least possible mass. Such a mass is equal to the sum of the masses of those particles and describes the case where the particles move with the same velocity.

We consider two colliding particles with masses m_1, m_2 and four-momenta p_1^i, p_2^i . We consider an *elastic collision*, which is a collision during which the internal states of particles do not change. The conservation law of four-momentum is

$$p_1^i + p_2^i = p_1'^i + p_2'^i, \quad (4.1.158)$$

where the primed quantities refer to the particles after the collision. This law gives, using $p_{1i}p_1^i = p_{1i}'p_1'^i = m_1^2c^2$ and $p_{2i}p_2^i = p_{2i}'p_2'^i = m_2^2c^2$,

$$\begin{aligned} p_{1i}p_2^i - p_{1i}p_1'^i - p_{2i}p_1'^i + m_1^2c^2 &= 0, \\ p_{2i}p_1^i - p_{2i}p_2'^i - p_{1i}p_2'^i + m_2^2c^2 &= 0. \end{aligned} \quad (4.1.159)$$

Inelastic collisions, where the states of colliding particles change, are possible if the invariant mass (4.1.157) of the resulting particles does not exceed the invariant mass of the original particles.

The L-frame for collisions is defined as a frame of reference in which one of the colliding particles, for example m_2 , is at rest, so that $\mathbf{p}_2 = 0$ and $E_2 = m_2c^2$. The velocity of the motion of the system of such particles as a whole (3.1.71),

$$\mathbf{V} = \frac{\mathbf{p}_1c^2}{E_1 + m_2c^2}, \quad (4.1.160)$$

is the velocity of the C-frame relative to the L-frame. Therefore, before the collision, the particle with mass m_2 moves in the C-frame with velocity $-\mathbf{V}$, so that its momentum is

$$\mathbf{p}_{20} = -\frac{m_2\mathbf{V}}{1 - V^2/c^2}. \quad (4.1.161)$$

If θ_1 is the angle between $\mathbf{p}_1$ and $\mathbf{p}_1'$, and θ_2 is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2'$, then in the L-frame we obtain $p_{1i}p_2^i = E_1m_2$, $p_{1i}p_1'^i = E_1E_1'/c^2 - \mathbf{p}_1 \cdot \mathbf{p}_1' = E_1E_1'/c^2 - p_1p_1'\cos\theta_1$, $p_{2i}p_1^i = m_2E_1'$, $p_{2i}p_2'^i = m_2E_2'$ and $p_{1i}p_2'^i = E_1E_2'/c^2 - \mathbf{p}_1 \cdot \mathbf{p}_2' = E_1E_2'/c^2 - p_1p_2'\cos\theta_2$. These quantities are scalars so that they have the same values in all frames of reference. Substituting the above expressions into (4.1.159) gives the relations between the energies of the particles and the directions of their motion. The time component of (4.1.158) in the L-frame is the conservation of energy, $E_1 + m_2c^2 = E_1' + E_2'$. The energy in the L-frame transferred from the incident particle m_1 to the target particle m_2 is equal to $E_t = E_2' - m_2c^2$. If $m_1 > m_2$ then the scattering angle θ_1 has a maximum value given by $\sin\theta_{\max} = \frac{m_2}{m_1}$.

In the C-frame, the momenta of the two particles before the collision, $\mathbf{p}_{10} = \mathbf{p}_0$ and $\mathbf{p}_{20}$, satisfy $\mathbf{p}_{10} + \mathbf{p}_{20} = 0$. The conservation of momentum causes that the momenta of the particles rotate during the collision, remaining opposite in direction and equal in magnitude. The conservation of energy causes that the values of the momenta (and of the energies) do not change during the collision. If χ is the angle of such a rotation (*angle of scattering*), that is, the angle between $\mathbf{p}_{10}$ and $\mathbf{p}_{10}'$ or between $\mathbf{p}_{20}$ and $\mathbf{p}_{20}'$, then in the C-frame we obtain $p_{1i}p_2^i = E_{10}E_{20}/c^2 - \mathbf{p}_{10} \cdot \mathbf{p}_{20} = \sqrt{(p_0^2 + m_1^2c^2)(p_0^2 + m_2^2c^2)} + p_0^2$ and $p_{1i}p_1'^i = E_{10}E_{10}'/c^2 - \mathbf{p}_{10} \cdot \mathbf{p}_{10}' = E_{10}^2/c^2 - p_0^2\cos\chi = p_0^2(1 - \cos\chi) + m_1^2c^2$. Substituting the above expressions into (4.1.159) gives the relations between the energies of the particles and χ . A head-on collision corresponds to $\chi = \pi$.

An elastic collision of two particles with masses m_1, m_2 and momenta $\mathbf{p}_1, \mathbf{p}_2 = 0$ is equivalent (with respect to the resulting particles) to a decay of a particle with mass equal to $m_{\text{inv}} = \frac{1}{c^2}\sqrt{(E_1 + m_2c^2)^2 - \mathbf{p}_1^2c^2}$ (4.1.157), moving with velocity (4.1.160). Therefore, the vector $AB = \mathbf{p}_1'$ lies on an ellipse in Fig. 16, which is analogous to that in Fig. 14. The major semiaxis of this ellipse is equal to $\frac{p_0}{\sqrt{1-V^2/c^2}}$ and its minor semiaxis is equal to p_0 , where $p_0 = p_{10} = p_{20}' = p_{20} = \frac{m_2V}{\sqrt{1-V^2/c^2}}$

(due to (4.1.161)) and $V = \frac{p_1c^2}{E_1 + m_2c^2}$ (due to (4.1.160)). The length of the vector AC is equal to $V\frac{E_{10}/c^2}{\sqrt{1-V^2/c^2}} + \frac{p_{10}}{\sqrt{1-V^2/c^2}}$, where $E_{10} = \sqrt{p_0^2c^2 + m_1^2c^4}$, leading to $|AC| = p_1$. Thus we obtain $AC = \mathbf{p}_1$. The spatial components of (4.1.158) in the L-frame are the conservation of momentum, $\mathbf{p}_1 = \mathbf{p}_1' + \mathbf{p}_2'$, so that the vector $BC = \mathbf{p}_2'$. If $m_1 < m_2$ then A lies inside the ellipse. If $m_1 > m_2$ then A lies outside the ellipse.

In the nonrelativistic limit, the ellipse in Fig. 16 reduces to a circle of radius μv_1 in Fig. 17, where μ is the reduced mass (4.1.103), and the angle between the vectors OB and OC is χ . Thus we find

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2 \cos\chi}, \quad \theta_2 = \frac{\pi - \chi}{2}, \quad v_2' = \frac{2m_1v}{m_1 + m_2} \sin\frac{\chi}{2}. \quad (4.1.162)$$

This circle can also be constructed from the two-body problem. Differentiating (4.1.101) with respect to time gives

$$\mathbf{v}_1 = \frac{m_2 \mathbf{v}}{m_1 + m_2} + \mathbf{V}, \quad \mathbf{v}_2 = -\frac{m_1 \mathbf{v}}{m_1 + m_2} + \mathbf{V}, \quad (4.1.163)$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the particles before the collision, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity (the time derivative of the the separation vector (4.1.98)), and $\mathbf{V} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$ is the velocity of the center of mass of the particles (3.1.135) (the time derivative of $\mathbf{R}$ (4.1.99)). During the collision, the vector of relative velocity rotates without changing its magnitude, by an angle χ . If $\mathbf{n}$ is the unit vector along the direction of the velocity of the particle m_1 after the collision, then $\mathbf{v}$ in (4.1.163) turns into $v\mathbf{n}$, so that $\mathbf{v}'_1 = \frac{m_2 v \mathbf{n}}{m_1 + m_2} + \mathbf{V}$, $\mathbf{v}'_2 = -\frac{m_1 v \mathbf{n}}{m_1 + m_2} + \mathbf{V}$ or

$$\mathbf{p}'_1 = \mu v \mathbf{n} + \frac{m_1}{m_1 + m_2}(\mathbf{p}_1 + \mathbf{p}_2), \quad \mathbf{p}'_2 = -\mu v \mathbf{n} + \frac{m_2}{m_1 + m_2}(\mathbf{p}_1 + \mathbf{p}_2). \quad (4.1.164)$$

In the L-frame, $\mathbf{p}_2 = 0$, so that the formulae (4.1.164) correspond to Fig. 17 with $OB = \mu v \mathbf{n}$. A head-on collision corresponds to $\mathbf{n} = -\frac{\mathbf{v}}{v}$.

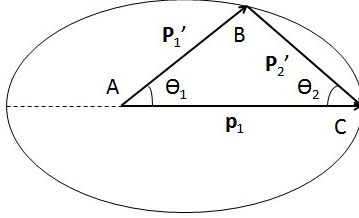


Figure 16: Kinematic ellipse of a collision.

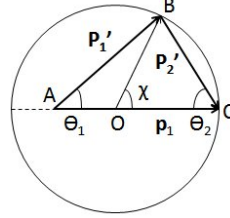


Figure 17: Nonrelativistic limit.

For two particles with masses m_1, m_2 , velocities $\mathbf{v}_1, \mathbf{v}_2$ and four-momenta p_1^i, p_2^i , we have from (3.1.59) and (3.1.60):

$$p_{1i} p_2^i = m_1 m_2 c^2 \frac{1 - \mathbf{v}_1 \cdot \mathbf{v}_2 / c^2}{\sqrt{(1 - v_1^2 / c^2)(1 - v_2^2 / c^2)}}. \quad (4.1.165)$$

In the rest frame of the particle m_2 , (4.1.165) becomes

$$p_{1i} p_2^i = \frac{m_1}{\sqrt{1 - v_{\text{rel}}^2 / c^2}} m_2 c^2, \quad (4.1.166)$$

where v_{rel} is the velocity of the particle m_1 in this frame,

$$v_{\text{rel}} = c \sqrt{1 - \frac{m_1^2 m_2^2 c^4}{(p_{1i} p_2^i)^2}}. \quad (4.1.167)$$

Because (4.1.167) is symmetric under interchanging m_1 and m_2 , v_{rel} is the *relative velocity* of the two particles. Substituting (4.1.165) into (4.1.167) gives

$$v_{\text{rel}} = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2 / c^2}}{1 - \mathbf{v}_1 \cdot \mathbf{v}_2 / c^2}. \quad (4.1.168)$$

In the nonrelativistic limit, (4.1.168) reduces to $v_{\text{rel}} = |\mathbf{v}_1 - \mathbf{v}_2|$, in agreement with $\mathbf{v}$ in (4.1.163). The relative velocity $d\mathbf{l}_{\mathbf{v}}$ between particles with velocities $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ gives thus the element of length in velocity space,

$$dl_{\mathbf{v}}^2 = \frac{d\mathbf{v}^2 - (\mathbf{v} \times d\mathbf{v})^2 / c^2}{(1 - v^2 / c^2)^2} = \frac{dv^2}{(1 - v^2 / c^2)^2} + \frac{v^2}{1 - v^2 / c^2} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.1.169)$$

where θ and ϕ are the polar angle and azimuth of the direction of $\mathbf{v}$. In terms of the rapidity (1.6.90), (4.1.169) is a line element in a three-dimensional Lobachevskii space (space of constant negative curvature),

$$dl_{\mathbf{v}}^2 = d\eta^2 + \sinh^2 \eta (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.1.170)$$

4.1.8 Scattering of particles

The trajectory of a particle with mass m moving in an external central field with potential $U(r)$ (in a plane) is deflected from a straight line, as shown in Fig. 18. The *impact parameter* ρ is the distance at which the particle would pass the center of the field if the field were absent. The *angle of deflection* χ is the angle between the asymptotes representing the trajectory of the particle far from the field. It is given by

$$\chi = |\pi - 2\phi_0|, \quad (4.1.171)$$

where ϕ_0 is the angle in the polar coordinates of the point of the trajectory which is closest to the center. If the motion is nonrelativistic then ϕ_0 results from (4.1.87):

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\frac{M}{r^2} dr}{\sqrt{2m(E - U) - \frac{M^2}{r^2}}}, \quad (4.1.172)$$

where $r_{\min}$ is the distance of this point from the center. Far from the field, the conserved energy and angular momentum of the particle are given by

$$E = \frac{1}{2}mv_{\infty}^2, \quad M = m\rho v_{\infty}, \quad (4.1.173)$$

where v_{∞} is the corresponding velocity of the particle. Accordingly, (4.1.172) becomes

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\frac{\rho}{r^2} dr}{\sqrt{1 - \rho^2/r^2 - \frac{2U}{mv_{\infty}^2}}}. \quad (4.1.174)$$

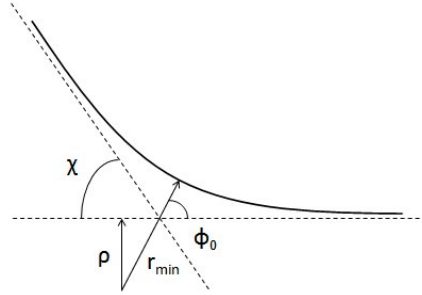


Figure 18: Deflection in a central field.

The deflection of a uniform (in cross section) beam of particles with the same velocities but with different impact parameters results in different angles of deflection. If dN is the number of particles scattered per unit time into the angles of deflection between χ and $\chi + d\chi$, and n is the number of particles passing per unit time through unit area of the cross section of the beam, then the *scattering cross section* is defined as

$$d\sigma = \frac{dN}{n}. \quad (4.1.175)$$

Particles which are scattered into the angles of deflection between χ and $\chi + d\chi$ have impact parameters between ρ and $\rho + d\rho$, so that $dN = n dS$, where $dS = 2\pi\rho d\rho$ is the area between the circles of radii ρ and $\rho + d\rho$. Therefore, we obtain

$$d\sigma = 2\pi\rho d\rho = 2\pi\rho \left| \frac{d\rho}{d\chi} \right| d\chi = \frac{\rho}{\sin\chi} \left| \frac{d\rho}{d\chi} \right| d\chi, \quad (4.1.176)$$

where $d\chi = 2\pi \sin\chi d\chi$ is the element of the solid angle between the cones with vertical angles χ and $\chi + d\chi$. Particles that reach the center of the field satisfy $E_{\infty} > U_{\text{eff,max}}$, where $U_{\text{eff,max}}$ is the

maximum value of the effective potential (4.1.85), which gives $\rho < \rho_{\max}$, where $\rho_{\max}$ is a solution of $U_{\text{eff},\max} = E_{\infty}$. The cross section for a particle to reach the center is thus given by $\sigma = \pi\rho_{\max}^2$.

Scattering of two particles with masses m_1, m_2 interacting through the potential energy $U(\mathbf{r})$, where $\mathbf{r}$ is the separation vector (4.1.98), is equivalent to scattering of a particle with reduced mass μ (4.1.103) by the field $U(r)$ whose origin is located at the center of mass of the two particles (4.1.99). Accordingly, scattering of a beam of particles by a central field is equivalent to scattering of two beams of particles. The cross section (4.1.176) is then a function of the angle of scattering χ in the C-frame. In the L-frame, in which one beam is at rest, the scattering cross section for incident particles is given by (4.1.176) upon substituting the relation $\chi(\theta_1)$ from (4.1.162), whereas the scattering cross section for particles initially at rest is given by (4.1.176) upon substituting the relation $\chi(\theta_2)$ from (4.1.162).

We consider two colliding beams of particles with particle number densities n_1, n_2 . In the rest frame of one beam (L-frame), the invariant number of collisions in a volume element dV and in time interval dt is equal to

$$d\nu = \sigma v_{\text{rel}} n_1 n_2 dV dt, \quad (4.1.177)$$

where $\sigma = \int d\sigma$ is the total scattering cross section given by (4.1.176) and v_{rel} is the relative velocity of the two beams. In the frame of reference in which the beams have velocities $\mathbf{v}_1, \mathbf{v}_2$, an invariant quantity which reduces in the L-frame to $n_1 n_2$, is equal to $\frac{n_1 n_2}{E_1 E_2} p_{1i} p_2^i$, due to $n = n_0 \gamma$ (2.5.103) and (3.1.60). The number of collisions in this frame is thus

$$d\nu = \sigma v_{\text{rel}} \frac{n_1 n_2}{E_1 E_2} p_{1i} p_2^i dV dt. \quad (4.1.178)$$

Substituting (4.1.165) and (4.1.168) into (4.1.178) gives the Pauli formula,

$$d\nu = \sigma \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2 / c^2} n_1 n_2 dV dt. \quad (4.1.179)$$

For a system of particles, the *distribution function* for the momenta of the particles, $f(\mathbf{p})$, is defined according to

$$dN = f(\mathbf{p}) d\mathbf{p}, \quad (4.1.180)$$

where dN is the number of particles whose components of momenta lie in the intervals $[p_x, p_x + dp_x]$, $[p_y, p_y + dp_y]$ and $[p_z, p_z + dp_z]$, and

$$d\mathbf{p} = dp_x dp_y dp_z \quad (4.1.181)$$

is a volume element in *momentum space*. An element of a hypersurface is a four-vector along the normal to the hypersurface. For the hypersurface $p_i p^i = m^2 c^2$, such a four-vector is parallel to p_i . The element (4.1.181), which is the 0th component of an element of this hypersurface, is thus proportional to $p_0 = \frac{E}{c}$. Accordingly, a quantity $\frac{d\mathbf{p}}{dE}$ is an invariant, which also results from

$$\frac{2}{c} \delta(p_i p^i - m^2 c^2) d^4 p = \frac{d\mathbf{p}}{\sqrt{\mathbf{p}^2 c^2 + m^2 c^4}} = \frac{d\mathbf{p}}{dE}, \quad (4.1.182)$$

where

$$d^4 p = dp^0 d\mathbf{p} \quad (4.1.183)$$

is an invariant. The invariance of dN in (4.1.180) yields the invariance of a quantity $f(\mathbf{p})E$:

$$f(\mathbf{p}')E' = f(\mathbf{p})E, \quad (4.1.184)$$

where the energies and momenta transform according to (3.1.61) or (3.1.62). In the spherical coordinates, $d\mathbf{p} = p^2 dp d\Omega$, where $d\Omega$ is the element of a solid angle around the direction of $\mathbf{p}$, so that $p dp = \frac{E dE}{c^2}$ leads to the invariance of $p dE d\Omega$.

The distribution function for the momenta and positions of particles, $f(\mathbf{p}, \mathbf{r})$, is defined according to

$$dN = f(\mathbf{p}, \mathbf{r}) d\mathbf{p} dV, \quad (4.1.185)$$

where dN is the number of particles whose components of momenta lie in a momentum-space volume element (4.1.181) and in a volume element dV . The product

$$dq = d\mathbf{p} dV \quad (4.1.186)$$

is an element of *phase space* (space of the momenta and positions of particles). Since $dV = dV_0 \sqrt{1 - v^2/c^2}$ (due to (1.6.107)) and $E = E_0/\sqrt{1 - v^2/c^2}$, where the subscripts 0 refer to the rest frame of a particle in the volume element dV , a quantity $E dV$ is an invariant. The invariance of $\frac{d\mathbf{p}}{dE}$ yields then the invariance of (4.1.186):

$$dq' = dq. \quad (4.1.187)$$

Accordingly, the invariance of dN in (4.1.185) yields the invariance of $f(\mathbf{p}, \mathbf{r})$:

$$f(\mathbf{p}', \mathbf{r}') = f(\mathbf{p}, \mathbf{r}). \quad (4.1.188)$$

4.1.9 Small oscillations

4.1.10 Motion of rigid bodies

We consider the motion of a free asymmetrical top with the principal moments of inertia $I_1 < I_2 < I_3$. The conserved energy E of the top is equal to (3.2.19). Substituting (3.2.22) into (3.2.19) gives

$$\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = 2E. \quad (4.1.189)$$

The terminus of the angular momentum vector $\mathbf{M}$ thus lies on an ellipsoid with semiaxes $2EI_1$, $2EI_2$ and $2EI_3$. The angular momentum $\mathbf{M}$ of the top is also conserved,

$$I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 = M_1^2 + M_2^2 + M_3^2 = M^2. \quad (4.1.190)$$

The terminus of $\mathbf{M}$ thus also lies on a sphere of radius M . When the vector $\mathbf{M}$ moves relative to the axes of inertia, its terminus moves along the line of intersection of the two surfaces. Such an intersection exists because $2EI_1 < M^2 < 2EI_3$. If M^2 is near $2EI_1$ then the sphere intersects the ellipsoid along 2 small, closed curves around the poles of the ellipsoid on the x_1 axis. As M^2 increases, the curves become larger. If $M^2 = 2EI_2$ then the sphere intersects the ellipsoid along 2 ellipses intersecting at the poles of the ellipsoid on the x_2 axis. As M^2 increases further, the curves become smaller and separate. If M^2 is near $2EI_3$ then the sphere intersects the ellipsoid along 2 closed curves around the poles of the ellipsoid on the x_3 axis. The curves of intersection are closed, so that the motion of $\mathbf{M}$ relative to the axes of inertia is periodic. Paths passing near the poles on the x_1 and x_3 axes lie entirely near these poles. However, paths passing near the poles on the x_2 axis go far from these poles. Therefore, the rotation of a top about the x_1 or x_3 axis is stable, while the rotation about the x_2 axis is unstable.

The relations (3.2.19) and (4.1.190) give ω_1 and ω_3 as functions of ω_2 : Substituting these components into Euler's equation (3.2.43) for $\omega_2(t)$ in the absence of external torques, $I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 = 0$, and defining

$$\begin{aligned} \tau &= \frac{\sqrt{(I_3 - I_2)(M^2 - 2EI_1)}}{\sqrt{I_1 I_2 I_3}} t, \\ w &= \frac{\sqrt{I_2(I_3 - I_2)}}{\sqrt{2EI_3 - M^2}} \omega_2, \\ k^2 &= \frac{(I_2 - I_1)(2EI_3 - M^2)}{(I_3 - I_2)(M^2 - 2EI_1)} < 1, \end{aligned} \quad (4.1.191)$$

gives

$$\frac{dw}{d\tau} = \sqrt{(1 - s^2)(1 - k^2 s^2)}. \quad (4.1.192)$$

Setting $t = 0$ when $\omega_2 = 0$ gives $\tau(w) = \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, which upon inverting gives $w(\tau)$ as a Jacobian elliptic function $w(\tau) = \text{sn}\tau$. The components of $\boldsymbol{\omega}$ are thus

$$\begin{aligned}\omega_1 &= \frac{\sqrt{2EI_3-M^2}}{\sqrt{I_1(I_3-I_1)}} \text{cn}\tau, \\ \omega_2 &= \frac{\sqrt{2EI_3-M^2}}{\sqrt{I_2(I_3-I_2)}} \text{sn}\tau, \\ \omega_3 &= \frac{\sqrt{M^2-2EI_1}}{\sqrt{I_3(I_3-I_1)}} \text{dn}\tau,\end{aligned}\tag{4.1.193}$$

where $\text{cn}\tau = \sqrt{1-\text{sn}^2\tau}$ and $\text{dn}\tau = \sqrt{1-k^2\text{sn}^2\tau}$. These functions are periodic in τ (with period $4\tau(w=1)$). Within a finite time, the vector $\boldsymbol{\omega}$ returns to the same position relative to the principal axes of inertia.

The polar angle and azimuth of the direction of the vector $\mathbf{M}$ (along the fixed z axis) with respect to the $x' = x_1, y' = x_2, z' = x_3$ axes fixed with the top are θ and $\frac{\pi}{2} - \psi$. The absolute motion of the top relative to the fixed x, y, z axes is therefore given by

$$\begin{aligned}M \sin\theta \sin\psi &= M_1 = I_1\omega_1, \\ M \sin\theta \cos\psi &= M_2 = I_2\omega_2, \\ M \cos\theta &= M_3 = I_3\omega_3,\end{aligned}\tag{4.1.194}$$

in which we used (3.2.22). These relations give θ and ψ as functions of $\boldsymbol{\omega}$ (4.1.193), $\cos\theta = \frac{I_3\omega_3}{M}$ and $\tan\psi = \frac{I_1\omega_1}{I_2\omega_2}$. Accordingly, $\theta(t)$ and $\psi(t)$ are periodic. The Eulerian angle ϕ does not appear in (4.1.194). The function $\phi(t)$ can be instead found by integrating

$$\dot{\phi} = \frac{M(I_1\omega_1^2 + I_2\omega_2^2)}{I_1^2\omega_1^2 + I_2^2\omega_2^2},\tag{4.1.195}$$

derived from (3.2.23). It turns out that $\phi(t)$ is not periodic. A free asymmetrical top thus never returns to the same position within a finite time.

We now consider the motion of a free symmetrical top with the principal moments of inertia $I_1 = I_2 < I_3$. The parameter k^2 in (4.1.191) vanishes, so that $\text{cn}\tau, \text{sn}\tau$ and $\text{dn}\tau$ reduce to $\cos\tau, \sin\tau$ and 1. Accordingly, the quantities (4.1.193) reduce to $\omega_1 = A \cos(\Omega t)$, $\omega_2 = A \sin(\Omega t)$ and $\omega_3 = \frac{\sqrt{M^2-2EI_1}}{\sqrt{I_3(I_3-I_1)}}$, where $A = \frac{\sqrt{2EI_3-M^2}}{\sqrt{I_1(I_3-I_1)}}$ and $\Omega = \frac{\sqrt{(I_3-I_1)(M^2-2EI_1)}}{I_1\sqrt{I_3}}$. Equivalently, Euler's equations (3.2.43) with $\mathbf{K} = 0$ give $\dot{\omega}_1 = -\Omega\omega_2$, $\dot{\omega}_2 = \Omega\omega_1$ and $\omega_3 = \text{const}$, where $\Omega = \frac{\omega_3(I_3-I_1)}{I_1}$, in agreement with (4.1.193). The vectors $\boldsymbol{\omega}$ and $\mathbf{M}$ therefore rotate uniformly with a constant angular velocity Ω about the axis of the top (the z axis). The relations (4.1.194) for the Eulerian angles give $\cos\theta = \frac{I_3\omega_3}{M} = \text{const}$ ($\dot{\theta} = 0$) and $\tan\psi = \cot(\Omega t)$, so that $\psi = \frac{\pi}{2} - \Omega t$ ($\dot{\psi} = -\Omega$). The relation (4.1.195) reduces to $\dot{\phi} = \frac{M}{I_1}$, which, using (3.2.24), gives $\dot{\psi} = M \cos\theta(\frac{1}{I_3} - \frac{1}{I_1})$. We also find $A = \dot{\phi} \sin\theta$, in agreement with (3.2.23).

The motion of a free symmetrical top can also be directly obtained from (3.2.23) and (4.1.194). Since the axes x' and y' can be chosen arbitrarily, because of the symmetry of the top, we can choose the x' -axis along the line of nodes, so that $\psi = 0$. Equations (3.2.23) therefore reduce to $\omega_1 = \dot{\theta}$, $\omega_2 = \dot{\phi} \sin\theta$ and $\omega_3 = \dot{\phi} \cos\theta + \psi$. The relations (4.1.194) reduce to $M_1 = 0$, $M_2 = M \sin\theta$ and $M_3 = M \cos\theta$. The relations (3.2.24) give thus $\dot{\theta} = 0$, $I_1\dot{\phi} = M$ and $I_3(\dot{\phi} \cos\theta + \psi) = M \cos\theta$, in agreement with the formulae obtained from Euler's equations.

Finally, we derive the motion of a free symmetrical top solely from (3.2.22). Since the axes x' and y' can be chosen arbitrarily, we can choose the y' -axis perpendicular to the plane containing the constant angular momentum vector $\mathbf{M}$ and the z' -axis. We have thus $M_2 = 0$ and then $\omega_2 = 0$, so that the vectors $\mathbf{M}$, $\boldsymbol{\omega}$ and the axis of the top (the z' -axis) are at every instant in the same plane, as shown in Fig. 19. The velocity of all points on the axis of the top, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, is therefore perpendicular to that plane. Accordingly, the axis of the top rotates uniformly about the vector $\mathbf{M}$, along the surface of a circular cone. Such a rotation is referred to as *precession*. If θ is the (constant) angle between the axis of the top and the vector $\mathbf{M}$ then the other components of the

angular momentum are $M_1 = M \sin\theta$ and $M_3 = M \cos\theta$. The angular velocity of the top about its axis is thus $\omega_3 = \frac{M \cos\theta}{I_3}$. We also have $\omega_1 = \frac{M \sin\theta}{I_1}$. Since $\mathbf{M}$, $\boldsymbol{\omega}$ and the z' -axis lie in the same plane, the vector $\boldsymbol{\omega}$ can be projected into the axis of the top and the direction of the angular momentum. The first component is equal to ω_3 , whereas the second one causes the motion of the axis of the top, so that it is equal to the angular velocity of precession ω_{pr} . The relation $\omega_1 = \omega_{\text{pr}} \sin\theta$ gives then $\omega_{\text{pr}} = \frac{M}{I_1}$. Therefore $\omega_{\text{pr}} = \dot{\phi}$, in agreement with $\dot{\phi}$ being the angular velocity of rotation about the z -axis.

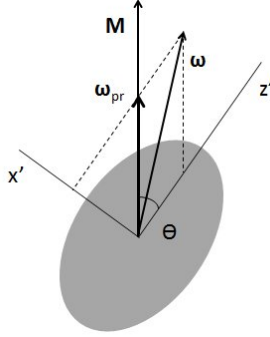


Figure 19: Symmetrical top.

4.1.11 Motion of ideal fluids

4.1.12 Motion in ideal fluids

4.2 Electromagnetism

4.2.1 Electrostatic field

A static (constant in time) electric field is called an *electrostatic field*. In such field, (2.7.38) reduces to

$$\text{curl } \mathbf{E} = 0, \quad (4.2.1)$$

which implies

$$\mathbf{E} = -\nabla\phi, \quad (4.2.2)$$

in agreement with (2.7.32). Substituting (4.2.2) into (2.7.89) gives the Poisson equation

$$\Delta\phi = -4\pi\rho. \quad (4.2.3)$$

The general solution of (4.2.3) is

$$\phi = \int \frac{\rho}{R} dV, \quad (4.2.4)$$

where $\mathbf{R}$ is the vector from the element of volume dV to the point at which ϕ is calculated. If the source of an electric field is a system of static point charges then (4.2.4) becomes

$$\phi = \sum_a \frac{e_a}{R_a}, \quad (4.2.5)$$

where $\mathbf{R}_a$ is the vector from a charge e_a to the point at which ϕ is calculated. Thus the electric field is

$$\mathbf{E} = \sum_a \frac{e_a \mathbf{R}_a}{R_a^3}, \quad (4.2.6)$$

in agreement with the principle of superposition for the electric field. For one charge e we have

$$\phi = \frac{e}{R}, \quad \mathbf{E} = \frac{e\mathbf{R}}{R^3}, \quad (4.2.7)$$

so the Lorentz force (2.7.112) exerted on a static charge e_a from a static charge e_b gives the *Cavendish-Coulomb law*:

$$\mathbf{f}_{ab} = \frac{e_a e_b \mathbf{R}_{ab}}{R_{ab}^3}, \quad (4.2.8)$$

where

$$\mathbf{R}_{ab} = \mathbf{R}_a - \mathbf{R}_b. \quad (4.2.9)$$

The Coulomb force (4.2.8) is repulsive for charges of the same sign and attractive for charges with opposite signs. According to Newton's third law, the force $\mathbf{f}_{ba}$ exerted on a static charge e_b from a static charge e_a is equal in magnitude and opposite in direction to $\mathbf{f}_{ab}$:

$$\mathbf{f}_{ba} = -\mathbf{f}_{ab}. \quad (4.2.10)$$

The energy of an electrostatic field is, due to (2.7.95),

$$U = \frac{1}{8\pi} \int E^2 dV = -\frac{1}{8\pi} \int \mathbf{E} \cdot \nabla \phi dV = -\frac{1}{8\pi} \int \nabla \cdot (\mathbf{E}\phi) dV + \frac{1}{8\pi} \int \phi \nabla \cdot \mathbf{E} dV. \quad (4.2.11)$$

The first term on the right of (4.2.11) can be transformed into a surface integral using the Gauß' theorem. This integral vanishes if the surface of integration is taken at infinity, so (2.7.89) and (4.2.11) give

$$U = \frac{1}{2} \int \rho \phi dV = \frac{1}{2} \sum_a e_a \phi_a, \quad (4.2.12)$$

where ϕ_a is the electric potential at the point occupied by a charge e_a . In calculating ϕ_a , we must not include the electric field produced by e_a because the corresponding contribution to the electrostatic energy, which is a *self-energy* (energy of an interaction of a point with itself), is infinite. Subtracting this self-energy, which process is a *renormalization* of the energy, from U gives

$$U = \frac{1}{2} \sum_a e_a \phi'_a, \quad (4.2.13)$$

where

$$\phi'_a = \sum_{b \neq a} \frac{e_b}{R_{ab}}. \quad (4.2.14)$$

For two static charges e_1 and e_2 separated by a distance r , the energy of their interaction is thus $U = \frac{e_1 e_2}{r}$, in agreement with $-\nabla U = \mathbf{f}$, where $\mathbf{f}$ is given by (4.2.8).

We consider a system of static point charges e_a located at $\mathbf{r}_a$ and take the origin of the Cartesian system of coordinates anywhere within the system. The electric potential at a point with the radius vector $\mathbf{R}$ is thus

$$\phi = \sum_a \frac{e_a}{|\mathbf{R} - \mathbf{r}_a|}. \quad (4.2.15)$$

If $R \gg r_a$ then the electric potential (4.2.15) can be expanded in a Taylor series with respect to the small quantity $\frac{\mathbf{r}_a}{R}$:

$$\phi \approx \phi_0 + \phi_1 = \frac{1}{R} \sum_a e_a - \nabla \frac{1}{R} \cdot \sum_a e_a \mathbf{r}_a = \frac{1}{R} \sum_a e_a - \nabla \frac{1}{R} \cdot \mathbf{d}, \quad (4.2.16)$$

where the spatial vector

$$\mathbf{d} = \sum_a e_a \mathbf{r}_a \quad (4.2.17)$$

is referred to as the *electric dipole moment* of the system. It has 3 independent components. If we shift the origin by a constant vector $\mathbf{a}$ then

$$\mathbf{r}'_a = \mathbf{r}_a + \mathbf{a}, \quad (4.2.18)$$

so

$$\mathbf{d}' = \mathbf{d} + \mathbf{a} \sum_a e_a. \quad (4.2.19)$$

Therefore if the total electric charge of the system is zero then its dipole moment does not depend on the choice of the origin. In this case, (4.2.16) becomes

$$\phi = \frac{\mathbf{d} \cdot \mathbf{R}}{R^3}, \quad (4.2.20)$$

which gives

$$\mathbf{E} = \frac{3(\mathbf{d} \cdot \mathbf{n})\mathbf{n} - \mathbf{d}}{R^3}, \quad (4.2.21)$$

where

$$\mathbf{n} = \frac{\mathbf{R}}{R} \quad (4.2.22)$$

is the unit vector in the direction of $\mathbf{R}$. If the charge-to-mass ratio $\frac{e_a}{m_a} = \frac{e}{m}$ is equal for all the charges in the system then the dipole moment of the system is proportional to the radius vector of its center of mass $\mathbf{X}$:

$$\mathbf{d} = \frac{e}{m} \mathbf{X}. \quad (4.2.23)$$

The next term in a Taylor expansion of the electric potential (4.2.15) is (omitting the subscripts a)

$$\phi_2 = \frac{1}{2} \sum e x_\alpha x_\beta \frac{\partial^2}{\partial X_\alpha \partial X_\beta} \frac{1}{R} = \frac{1}{6} D_{\alpha\beta} \frac{\partial^2}{\partial X_\alpha \partial X_\beta} \frac{1}{R} = \frac{1}{2R^3} D_{\alpha\beta} n_\alpha n_\beta, \quad (4.2.24)$$

where x_α are the components of $\mathbf{r}$, X_α are the components of $\mathbf{R}$, and the symmetric spatial tensor

$$D_{\alpha\beta} = \sum e(3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) = D_{\beta\alpha} \quad (4.2.25)$$

is the *electric quadrupole moment* of the system. This tensor is traceless,

$$D_{\alpha\alpha} = 0, \quad (4.2.26)$$

so it has 5 independent components. Similarly, the l -th term ϕ_l in a Taylor expansion of the electric potential (4.2.15) depends on the 2^l -pole moment of a system of charges which is a totally symmetric tensor of rank l with $2l + 1$ independent components. The expansion of $\frac{1}{|\mathbf{R} - \mathbf{r}|}$ can be written as

$$\frac{1}{|\mathbf{R} - \mathbf{r}|} = \sum_{l=0}^{\infty} \frac{r^l}{R^{l+1}} P_l(\cos \chi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{r^l}{R^{l+1}} Y_{lm}^*(\theta, \phi) Y_{lm}(\vartheta, \varphi), \quad (4.2.27)$$

where χ is the angle between $\mathbf{R}$ and $\mathbf{r}$, θ and ϕ are the polar angle and azimuth of the direction along $\mathbf{R}$, and ϑ and φ are the polar angle and azimuth of the direction along $\mathbf{r}$. Therefore ϕ_l is given by

$$\phi_l = \frac{1}{R^{l+1}} \sum_{m=-l}^l \sqrt{\frac{4\pi}{2l+1}} Q_{lm} Y_{lm}^*(\theta, \phi), \quad (4.2.28)$$

where

$$Q_{lm} = \sqrt{\frac{4\pi}{2l+1}} \sum_a e_a r_a^l Y_{lm}(\vartheta_a, \varphi_a). \quad (4.2.29)$$

We consider a system of static charges in an external field with potential ϕ , whose spatial variations are small within the system, $\frac{R|\nabla\phi|}{\phi} \ll 1$. The electrostatic energy (4.2.12) of such a system is

$$U = \sum_a e_a \phi(\mathbf{r}_a) \approx U_0 + U_1 + U_2, \quad (4.2.30)$$

where

$$U_0 = \phi_0 \sum_a e_a, \quad (4.2.31)$$

$$U_1 = \nabla\phi_0 \cdot \sum_a e_a \mathbf{r}_a = -\mathbf{d} \cdot \mathbf{E}_0, \quad (4.2.32)$$

$$U_2 = \frac{1}{2} \sum e_a x_\alpha x_\beta \frac{\partial^2 \phi_0}{\partial x_\alpha \partial x_\beta} = \frac{1}{6} D_{\alpha\beta} \frac{\partial^2 \phi_0}{\partial x_\alpha \partial x_\beta}, \quad (4.2.33)$$

where the subscripts 0 refer to the field at the origin. For two static dipoles $\mathbf{d}_1$ and $\mathbf{d}_2$ separated by a vector $\mathbf{R}$, the energy of their interaction is given by (4.2.21) and (4.2.32):

$$U = \frac{(\mathbf{d}_1 \cdot \mathbf{d}_2) - 3(\mathbf{d}_1 \cdot \mathbf{n})(\mathbf{d}_2 \cdot \mathbf{n})}{R^3}. \quad (4.2.34)$$

The total force on a system in an electrostatic field in the dipole approximation is

$$\mathbf{F} = \mathbf{E}_0 \sum_a e_a + (\mathbf{d} \cdot \nabla) \mathbf{E}_0. \quad (4.2.35)$$

The total torque on such a system is

$$\mathbf{K} = \sum_a e_a \mathbf{r}_a \times \mathbf{E}_0 = \mathbf{d} \times \mathbf{E}_0. \quad (4.2.36)$$

For a nonrelativistic scattering in a Coulomb field, $U(r) = -\frac{\gamma}{r}$, where $\gamma = \text{const}$, (4.1.171) and (4.1.174) give the impact parameter as a function of the scattering angle,

$$\rho^2 = \frac{\gamma^2}{m^2 v_\infty^4} \cot^2 \frac{\chi}{2}. \quad (4.2.37)$$

Substituting (4.2.37) into (4.1.176) gives the *Rutherford formula* for the scattering cross section in such a field:

$$d\sigma = \left(\frac{\gamma}{2mv_\infty^2} \right)^2 \frac{d\Omega}{\sin^4 \frac{\chi}{2}}. \quad (4.2.38)$$

4.2.2 Magnetostatic field

A static magnetic field is called a *magnetostatic field*. We consider a system of charges which move with finite velocities within a finite region of space. Such a motion is *stationary*; in this case we can average the magnetic field over the time and regard it as magnetostatic. Thus (2.7.33) gives

$$\text{div } \bar{\mathbf{B}} = 0, \quad (4.2.39)$$

which implies

$$\bar{\mathbf{B}} = \nabla \times \bar{\mathbf{A}}, \quad (4.2.40)$$

as in (2.7.33). After averaging, (2.7.90) simplifies to

$$\text{curl } \bar{\mathbf{B}} = \frac{4\pi}{c} \bar{\mathbf{j}}. \quad (4.2.41)$$

Substituting (4.2.40) into (4.2.41) gives

$$\nabla(\nabla \cdot \bar{\mathbf{A}}) - \Delta \bar{\mathbf{A}} = \frac{4\pi}{c} \bar{\mathbf{j}}. \quad (4.2.42)$$

Gauge invariance of the electromagnetic potential allows to impose one condition on A_i . If we impose the *Coulomb gauge*,

$$\nabla \cdot \mathbf{A} = 0, \quad (4.2.43)$$

then (4.2.42) reduces to the Poisson equation

$$\Delta \bar{\mathbf{A}} = -\frac{4\pi}{c} \bar{\mathbf{j}}. \quad (4.2.44)$$

The general solution of (4.2.44) is

$$\bar{\mathbf{A}} = \frac{1}{c} \int \frac{\bar{\mathbf{j}}}{R} dV, \quad (4.2.45)$$

where $\mathbf{R}$ is the vector from the element of volume dV to the point at which $\bar{\mathbf{A}}$ is calculated. Thus the magnetic field is given by the *Biot-Savart-Laplace law*:

$$\bar{\mathbf{B}} = \frac{1}{c} \int \frac{\bar{\mathbf{j}} \times \mathbf{R}}{R^3} dV. \quad (4.2.46)$$

If the source of a magnetic field is a system of stationary point charges then (4.2.45) becomes

$$\bar{\mathbf{A}} = \frac{1}{c} \sum_a \frac{\bar{e}_a \mathbf{v}_a}{R_a} dV, \quad (4.2.47)$$

where $\mathbf{R}_a$ is the vector from a charge e_a to the point at which $\bar{\mathbf{A}}$ is calculated.

We consider a system of stationary point charges e_a located $\mathbf{r}_a$ and take the origin of the Cartesian system of coordinates anywhere within the system. The magnetic potential at a point with the radius vector $\mathbf{R}$ is thus

$$\bar{\mathbf{A}} = \frac{1}{c} \sum_a \frac{\bar{e}_a \mathbf{v}_a}{|\mathbf{R} - \mathbf{r}_a|}. \quad (4.2.48)$$

If $R \gg r_a$ then the magnetic potential (4.2.48) can be expanded in a Taylor series with respect to the small quantity $\frac{r_a}{R}$ (omitting the subscripts a):

$$\begin{aligned} \bar{\mathbf{A}} &\approx \frac{1}{cR} \sum \bar{e} \mathbf{v} - \frac{1}{c} \sum \overline{e \mathbf{v} \left(\mathbf{r} \cdot \nabla \frac{1}{R} \right)} = \frac{1}{cR} \frac{d}{dt} \sum e \mathbf{r} + \frac{1}{cR^3} \sum \overline{e \mathbf{v} (\mathbf{r} \cdot \mathbf{R})} \\ &= \frac{1}{cR^3} \left(\frac{1}{2} \frac{d}{dt} \sum e \mathbf{r} (\mathbf{r} \cdot \mathbf{R}) + \frac{1}{2} \sum \left(\mathbf{v} (\mathbf{r} \cdot \mathbf{R}) - \mathbf{r} (\mathbf{v} \cdot \mathbf{R}) \right) \right) = \frac{1}{2cR^3} \sum e (\mathbf{r} \times \mathbf{v}) \times \mathbf{R} \\ &= \frac{\bar{\mathbf{m}} \times \mathbf{R}}{R^3}, \end{aligned} \quad (4.2.49)$$

where the spatial pseudovector

$$\bar{\mathbf{m}} = \frac{1}{2c} \sum_a e_a \mathbf{r}_a \times \mathbf{v}_a \quad (4.2.50)$$

is referred to as the *magnetic moment* of the system. Thus the magnetic field is

$$\bar{\mathbf{B}} = \frac{3(\bar{\mathbf{m}} \cdot \mathbf{n})\mathbf{n} - \bar{\mathbf{m}}}{R^3}. \quad (4.2.51)$$

The total force on a system of charges in a magnetostatic field vanishes:

$$\bar{\mathbf{F}} = \sum \frac{e}{c} \overline{\mathbf{v} \times \mathbf{B}} = \frac{d}{dt} \sum \frac{e}{c} \overline{\mathbf{r} \times \mathbf{B}} = 0. \quad (4.2.52)$$

The total torque on such a system is, analogously to (4.2.49),

$$\bar{\mathbf{K}} = \sum \frac{e}{c} \overline{\mathbf{r} \times (\mathbf{v} \times \mathbf{B})} = \sum \frac{e}{c} \left(\overline{\mathbf{v} (\mathbf{r} \cdot \mathbf{B})} - \frac{1}{2} \mathbf{B} \frac{d}{dt} r^2 \right) = \bar{\mathbf{m}} \times \mathbf{B}. \quad (4.2.53)$$

The expression for the torque on a magnetic dipole (4.2.53) is similar to that for the torque on an electric dipole (4.2.36). If the charges are nonrelativistic, $v_a \ll c$, and their charge-to-mass ratio $\frac{e_a}{m_a} = \frac{e}{m}$ is equal for all the charges in the system then the magnetic moment of the system is proportional to its angular momentum:

$$\mathbf{m} = \frac{e}{2mc} \mathbf{M}. \quad (4.2.54)$$

A general system with a total mass m and electric charge e has

$$\mathbf{m} = \frac{ge}{2mc} \mathbf{M}, \quad (4.2.55)$$

where the constant of proportionality g is called the *gyromagnetic ratio*. Thus a system of nonrelativistic particles with the same charge-to-mass ratio has $g = 1$. Combining (4.2.53) and (4.2.54) gives the *Larmor precession*,

$$\dot{\mathbf{M}} = \boldsymbol{\omega} \times \mathbf{M}, \quad (4.2.56)$$

with angular velocity

$$\boldsymbol{\omega} = -\frac{e}{2mc} \mathbf{B}. \quad (4.2.57)$$

If a magnetostatic field is uniform (in space) then

$$\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} \quad (4.2.58)$$

is the solution of (2.7.33). Substituting (4.2.58) into the term $\sum \frac{e}{c} \mathbf{A} \cdot \mathbf{v}$ in the Lagrangian (3.1.42) turns this term into $\mathbf{m} \cdot \mathbf{B}$. Thus the magnetostatic (potential) energy of such a system is

$$U = -\mathbf{m} \cdot \mathbf{B}. \quad (4.2.59)$$

The expression for the energy of a magnetic dipole (4.2.59) is similar to that for the energy of an electric dipole (4.2.32).

4.2.3 Motion in constant uniform electromagnetic field

The equation of motion (2.7.110) of a charged particle in an electromagnetic field integrates to

$$u^i(\tau) = (e^{\frac{e}{mc} \int_0^\tau F(\tau') d\tau'})^i_j u^j(0), \quad (4.2.60)$$

where the exponent of a matrix A is defined as a Taylor expansion:

$$(e^A)^i_j = \delta_j^i + A^i_j + \frac{1}{2} A^i_k A^k_j + \frac{1}{6} A^i_k A^k_l A^l_j + \dots \quad (4.2.61)$$

If the field is constant and uniform then (4.2.60) becomes

$$u^i(\tau) = (e^{\frac{e\tau}{mc} F})^i_j u^j(0). \quad (4.2.62)$$

Equations (2.7.112) and (2.7.114) integrate to

$$\begin{aligned} \mathbf{p} - \mathbf{p}_0 &= e(\mathbf{E}t + \mathbf{r} \times \mathbf{B}), \\ E - E_0 &= e\mathbf{E} \cdot \mathbf{r}. \end{aligned} \quad (4.2.63)$$

The mixed components of the electromagnetic field tensor are

$$F^i_j = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \quad (4.2.64)$$

which gives

$$(F^2)^i_j = F^i_k F^k_j = \begin{pmatrix} \mathbf{E}^2 & -\mathbf{E} \times \mathbf{B} \\ \mathbf{E} \times \mathbf{B} & E_\alpha E_\beta + B_\alpha B_\beta - \mathbf{B}^2 \delta_{\alpha\beta} \end{pmatrix}. \quad (4.2.65)$$

If $\mathbf{E} \cdot \mathbf{B} = 0$ then $(F^3)^i_j = F^i_k F^k_l F^l_j = (\mathbf{E}^2 - \mathbf{B}^2) F^i_j$. This case includes a constant uniform electric field and a constant uniform magnetic field.

We consider the motion of a particle with mass m and charge e in a constant uniform electric field $\mathbf{E}$, which we choose to be along the z axis. The particle moves in such a field in a plane, which we choose as the xz -plane. If the initial velocity of the particle is $\mathbf{v}(0) = v(0)\hat{e}_x$ then its initial four-velocity is

$$u^0(0) = \gamma(0), \quad u^x(0) = \frac{\gamma(0)v(0)}{c}, \quad \gamma(0) = \frac{1}{\sqrt{1 - \frac{v^2(0)}{c^2}}}. \quad (4.2.66)$$

The equation of motion (4.2.62) gives

$$u^0 = \gamma(0) \cosh(w\tau), \quad u^x = u^x(0), \quad u^y = 0, \quad u^z = \gamma(0) \sinh(w\tau), \quad (4.2.67)$$

where

$$w = \frac{e|\mathbf{E}|}{mc}. \quad (4.2.68)$$

If $t(0) = x(0) = y(0) = z(0) = 0$ then the trajectory of the particle is a catenary curve:

$$t = \frac{\gamma(0)}{w} \sinh(w\tau), \quad x = \gamma(0)v(0)\tau, \quad y = \text{const}, \quad z = \frac{\gamma(0)c}{w} (\cosh(w\tau) - 1). \quad (4.2.69)$$

In the nonrelativistic limit, $\gamma(0) \approx 1$ and $w\tau \ll 1$, this curve reduces to a parabola:

$$t = \tau, \quad x = v(0)t, \quad y = \text{const}, \quad z = \frac{1}{2} c w t^2 = \frac{1}{2} a_0 t^2, \quad (4.2.70)$$

where

$$a_0 = \frac{e|\mathbf{E}|}{m}. \quad (4.2.71)$$

If $\mathbf{v}(0) = 0$ then the motion of a particle in a constant uniform electric field $\mathbf{E}$ coincides with the relativistic motion (4.1.37) with constant proper acceleration a_0 (the electric field does not change under boosts along the direction of the field). This motion can also be obtained by integrating (2.7.112) and using (2.4.213).

We consider the motion of a particle with mass m and charge e in a constant uniform magnetic field $\mathbf{B}$, which we choose to be along the z axis. If the initial velocity of the particle is $\mathbf{v}(0) = (v_x(0), 0, v_z(0))$ then its initial four-velocity is

$$u^0(0) = \gamma(0), \quad u^x(0) = \frac{\gamma(0)v_x(0)}{c}, \quad u^z(0) = \frac{\gamma(0)v_z(0)}{c}, \quad \gamma(0) = \frac{1}{\sqrt{1 - \frac{v_x^2(0) + v_z^2(0)}{c^2}}}. \quad (4.2.72)$$

The equation of motion (4.2.62) gives

$$u^0 = \gamma(0), \quad u^x = u^x(0) \cos(\omega\tau), \quad u^y = -u^x(0) \sin(\omega\tau), \quad u^z = u^z(0), \quad (4.2.73)$$

where

$$\omega = \frac{e|\mathbf{B}|}{mc}. \quad (4.2.74)$$

The first equation in (4.2.73) represents the constancy of the energy in the magnetic field. If $t(0) = x(0) = z(0) = 0$ and $y(0) = \frac{u^x(0)}{\omega}$ then the trajectory of the particle is a helix:

$$t = \gamma(0)\tau, \quad x = \frac{\sin(\omega\tau)}{\omega} u^x(0), \quad y = \frac{\cos(\omega\tau)}{\omega} u^x(0), \quad z = u^z(0)\tau. \quad (4.2.75)$$

The axis of this helix is parallel to the direction of the field and its radius is

$$r = \frac{u^x(0)}{\omega}. \quad (4.2.76)$$

This motion can also be obtained by integrating (2.7.112) and using (2.4.213) and the constancy of the energy in the magnetic field.

We consider the motion of a particle with mass m and charge e in a superposition of a constant uniform electric field and a constant uniform magnetic field if both fields are mutually perpendicular and equal in magnitude. Thus $(F^3)^i_j = 0$, so if the particle is at rest at $\tau = 0$ then the equation of motion (4.2.62) gives

$$u^0 = 1 + \frac{1}{2}(\omega\tau)^2, \quad \mathbf{u} = \frac{\mathbf{E}}{E}\omega\tau + \frac{\mathbf{E} \times \mathbf{B}}{2E^2}(\omega\tau)^2. \quad (4.2.77)$$

As $\tau \rightarrow \infty$, the direction of the velocity of the particle tends to that of $\mathbf{E} \times \mathbf{B}$.

4.2.4 Electromagnetic waves

In the absence of electric charges, the Maxwell equations (2.7.37), (2.7.38), (2.7.89) and (2.7.90) reduce to

$$\nabla \cdot \mathbf{B} = 0, \quad (4.2.78)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \quad (4.2.79)$$

$$\nabla \cdot \mathbf{E} = 0, \quad (4.2.80)$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\dot{\mathbf{E}}. \quad (4.2.81)$$

Gauge invariance of the electromagnetic potential allows to impose on it one condition. If we impose

$$\phi = 0 \quad (4.2.82)$$

then (2.7.32) simplifies to

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}}. \quad (4.2.83)$$

Thus (4.2.81) gives

$$\nabla \times (\nabla \times \mathbf{A}) = -\Delta \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\frac{1}{c^2}\ddot{\mathbf{A}}. \quad (4.2.84)$$

This equation is invariant under a transformation

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla f(\mathbf{r}), \quad (4.2.85)$$

where $f(\mathbf{r})$ is any function of the spatial coordinates. Therefore we can impose another constraint on $\mathbf{A}$. If we choose the Coulomb gauge (4.2.43) then (4.2.84) reduces to the *d'Alembert wave equation*:

$$\square \mathbf{A} = \frac{1}{c^2}\ddot{\mathbf{A}} - \Delta \mathbf{A} = 0. \quad (4.2.86)$$

Its solutions are called *electromagnetic waves*. They have 2 degrees of freedom because the 2 conditions (4.2.82) and (4.2.43) constrain the 4 components of A_i . The fields $\mathbf{E}$ and $\mathbf{B}$ also satisfy the wave equation (4.2.86). The second Maxwell-Minkowski equation in the absence of charges (2.7.75) becomes

$$F^{ik}_{,k} = 0. \quad (4.2.87)$$

Substituting (2.7.24) into (4.2.87) gives

$$\partial^i \partial_i A_j - \partial_j (\partial^i A_i) = 0. \quad (4.2.88)$$

If we impose on the electromagnetic potential, due to gauge invariance, the *Lorentz gauge*

$$\partial^i A_i = 0 \quad (4.2.89)$$

then (4.2.88) simplifies to

$$\partial^i \partial_i A_j = 0, \quad (4.2.90)$$

which is identical with (4.2.86). This equation is invariant under a transformation

$$A_i \rightarrow A_i + \partial_i f, \quad (4.2.91)$$

where f is any function satisfying the wave equation $\partial^i \partial_i f = 0$. This invariance allows imposing on A_i another condition, such as $A_0 = 0$ (4.2.82).

A solution of (4.2.86) which depends on time and only one spatial coordinate describes a *plane wave*. If we choose the Cartesian frame of reference such that this coordinate is x then (4.2.86) gives

$$\frac{\partial^2 A_i(t, x)}{c^2 \partial t^2} = \frac{\partial^2 A_i(t, x)}{\partial^2 x}, \quad (4.2.92)$$

whose solution is

$$A_i = f_i^{(1)}(t - x/c) + f_i^{(2)}(t + x/c), \quad (4.2.93)$$

where $f_i^{(1)}$ and $f_i^{(2)}$ are vector functions of their arguments. If $f^{(2)} = 0$ then the field has the same values for events (t, x) which satisfy the relation

$$x = ct + \text{const.} \quad (4.2.94)$$

Thus this field propagates along the x -axis with velocity c , representing a *plane wave* moving in the positive direction along this axis. Therefore electromagnetic waves move with the velocity of propagation of interaction. If $f^{(1)} = 0$ then the field represents a plane wave moving in the negative direction along this axis. If we impose the Coulomb gauge then $\frac{\partial A_x}{\partial x} = 0$, which gives, due to (4.2.92), $-cE_x = \frac{\partial A_x}{\partial t} = \text{const.}$ A constant electric field does not represent a wave, so

$$A_x = 0. \quad (4.2.95)$$

The electric and magnetic fields corresponding to a plane wave propagating along the $+x$ -axis are

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial(t - x/c)}, \quad (4.2.96)$$

$$\mathbf{B} = \nabla(t - x/c) \times \frac{\partial \mathbf{A}}{\partial(t - x/c)} = \mathbf{n} \times \mathbf{E}, \quad (4.2.97)$$

where $\mathbf{n} = \nabla x$ is the unit vector in the direction of propagation of the wave. A plane wave is *transverse* (perpendicular to the wave vector):

$$\mathbf{E} \perp \mathbf{n}, \quad \mathbf{B} \perp \mathbf{n}, \quad (4.2.98)$$

which gives

$$E^2 = B^2, \quad (4.2.99)$$

$$W = \frac{E^2}{4\pi} = -\sigma_{xx}, \quad (4.2.100)$$

$$\mathbf{S} = \frac{c}{4\pi} E^2 \mathbf{n} = cW \mathbf{n}. \quad (4.2.101)$$

The relation between the energy density W and the momentum density $\frac{\mathbf{S}}{c^2}$ is the same as for a particle moving with the velocity of propagation of interaction c . Under a Lorentz transformation with $\mathbf{V} = c\beta\hat{z}$, W transforms like the 00 component of the tensor (2.4.71), according to (1.6.100):

$$W = \frac{1}{1 - \frac{V^2}{c^2}} \left(W' + 2 \frac{V}{c^2} S'_z - \frac{V^2}{c^2} \sigma'_{zz} \right). \quad (4.2.102)$$

If α' is the angle between the z' -axis (which coincides with the z -axis) and $\mathbf{n}$ then $S'_z = cW' \cos \alpha'$ and $\sigma'_{zz} = -W' \cos^2 \alpha'$, so

$$W = \frac{(1 + \frac{V}{c} \cos \alpha')^2}{1 - \frac{V^2}{c^2}} W'. \quad (4.2.103)$$

A plane wave reflecting from a plane whose normal vector is $\mathbf{N}$ exerts on the plane the force $f_\alpha = -(\sigma_{\alpha\beta} + \sigma'_{\alpha\beta})N_\beta$, where $\sigma_{\alpha\beta}$ and $\sigma'_{\alpha\beta}$ are the stress tensors of the wave before and after the reflection. This force is thus $\mathbf{f} = W\mathbf{n}(\mathbf{N} \cdot \mathbf{n}) + W'\mathbf{n}'(\mathbf{N} \cdot \mathbf{n}')$.

A solution of (4.2.86) which is a periodic function of time can be represented as a superposition of *monochromatic waves*. A monochromatic plane wave along the x -axis is a sinusoidal solution of (4.2.86),

$$\mathbf{A} = \text{Re}(\mathbf{A}_0 e^{-i\omega(t-x/c)}), \quad (4.2.104)$$

where ω is the *angular frequency* of the wave. If this wave propagates in the direction of $\mathbf{r}$ then

$$\mathbf{A} = \text{Re}(\mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}), \quad (4.2.105)$$

where

$$\mathbf{k} = \frac{\omega}{c} \mathbf{n} \quad (4.2.106)$$

is the *wave vector*. A monochromatic plane wave satisfies, due to the wave equation,

$$\Delta \mathbf{A} + \frac{\omega^2}{c^2} \mathbf{A} = 0. \quad (4.2.107)$$

For linear operations on $\mathbf{A}$ we can omit the real-part operator Re . Thus $\mathbf{A} = \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$, from which we obtain

$$\mathbf{E} = ik\mathbf{A}, \quad \mathbf{B} = i\mathbf{k} \times \mathbf{A}, \quad (4.2.108)$$

where

$$k = |\mathbf{k}|. \quad (4.2.109)$$

The electric field is a real vector,

$$\mathbf{E} = \text{Re}(\mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}), \quad (4.2.110)$$

where $\mathbf{E}_0$ is a complex vector. Its square can be written as

$$\mathbf{E}_0^2 = |\mathbf{E}_0|^2 e^{-2i\alpha}, \quad (4.2.111)$$

where α is the *phase* of the wave, so

$$\mathbf{E}_0 = \mathbf{b} e^{-i\alpha}, \quad \mathbf{b}^2 = |\mathbf{E}_0|^2, \quad (4.2.112)$$

where $\mathbf{b}$ is a complex vector whose square is a real scalar. Writing

$$\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2, \quad (4.2.113)$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are real vectors, gives then

$$\mathbf{b}_1 \cdot \mathbf{b}_2 = 0, \quad (4.2.114)$$

so these vectors are perpendicular to each other. Equation (4.2.110) gives

$$\mathbf{E} = \text{Re}(\mathbf{b} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t - \alpha)}) \quad (4.2.115)$$

or, if $\mathbf{b}_1$ is along the y -axis and $\mathbf{b}_2$ along the z -axis,

$$E_y = b_1 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \alpha), \quad E_z = b_2 \sin(\omega t - \mathbf{k} \cdot \mathbf{r} + \alpha), \quad (4.2.116)$$

which describes an *elliptically polarized* wave:

$$\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1. \quad (4.2.117)$$

If $\mathbf{b}_1 = \mathbf{b}_2$ then a wave is *circularly polarized*. If $\mathbf{b}_1 = 0$ or $\mathbf{b}_2 = 0$ then a wave is *plane polarized* or *linearly polarized*. Similar expressions can be obtained for the vector $\mathbf{B}$.

The frequency and wave vector form a four-vector:

$$k^i = \left(\frac{\omega}{c}, \mathbf{k} \right), \quad (4.2.118)$$

which is a null vector,

$$k^i k_i = 0. \quad (4.2.119)$$

The potential (4.2.105) is then

$$\mathbf{A} = \text{Re}(\mathbf{A}_0 e^{-ik_i x^i}). \quad (4.2.120)$$

The energy-momentum tensor (2.7.94) for a monochromatic plane wave is

$$T^{ij} = \frac{Wc^2}{\omega^2} k^i k^j. \quad (4.2.121)$$

If a source at rest in a frame K emits a monochromatic wave with frequency ω then its frequency ω' in a frame K' in which the source moves with velocity v along the x -axis is given by the corresponding Lorentz transformation of the vector k^i :

$$k^0 = \frac{k'^0 - \frac{v}{c} k'^1}{\sqrt{1 - v^2/c^2}}. \quad (4.2.122)$$

If α' is the angle between $\mathbf{v}$ and $\mathbf{k}'$ then $k'^1 = k' \cos \alpha'$, so

$$\omega' = \omega \frac{\sqrt{1 - v^2/c^2}}{1 - \frac{v}{c} \cos \alpha'}, \quad (4.2.123)$$

as in formula (1.6.113) for the Doppler effect.

A wave can be represented as a superposition of monochromatic waves with different frequencies, which is referred to as *spectral resolution*. A periodic field $f(t)$ can be represented as a discrete superposition of monochromatic waves with frequencies which are integral multiples of the *fundamental frequency* ω_0 (Fourier series):

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{-in\omega_0 t}. \quad (4.2.124)$$

The inverse relation is

$$f_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{in\omega_0 t} dt, \quad (4.2.125)$$

where

$$T = \frac{2\pi}{\omega_0} \quad (4.2.126)$$

is the *period* of the field. Since f is real, $f_{-n} = f_n^*$. The time average of f^2 satisfies, using $\bar{f} = 0$,

$$\overline{f^2} = 2 \sum_{n=1}^{\infty} |f_n|^2. \quad (4.2.127)$$

A nonperiodic field $f(t)$ which vanishes for $t \rightarrow \pm\infty$ can also be represented as a superposition of monochromatic waves, but the distribution of the frequencies is continuous (Fourier transform):

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{\omega} e^{-i\omega t} d\omega. \quad (4.2.128)$$

The inverse relation is

$$f_\omega = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \quad (4.2.129)$$

Since f is real, $f_{-\omega} = f_\omega^*$. The time integral of f^2 satisfies

$$\int_{-\infty}^{\infty} f^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} |f_\omega|^2 d\omega. \quad (4.2.130)$$

We consider an electromagnetic field in the absence of charges enclosed inside a finite volume taken as a parallelepiped with sides a, b, c . We can expand the field $\mathbf{A}$ inside this parallelepiped into plane waves through a Fourier series:

$$\mathbf{A}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (4.2.131)$$

where the summation extends over all values of $\mathbf{k} = (k_x, k_y, k_z)$ with

$$k_x = \frac{2\pi n_x}{a}, \quad k_y = \frac{2\pi n_y}{b}, \quad k_z = \frac{2\pi n_z}{c}, \quad (4.2.132)$$

where n_x, n_y, n_z are integers. The field $\mathbf{A}$ is real, so

$$\mathbf{A}_{-\mathbf{k}} = \mathbf{A}_{\mathbf{k}}^*. \quad (4.2.133)$$

The Coulomb gauge implies

$$\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}} = 0, \quad (4.2.134)$$

while the wave equation gives

$$\ddot{\mathbf{A}}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 \mathbf{A}_{\mathbf{k}} = 0, \quad (4.2.135)$$

where

$$\omega_{\mathbf{k}} = ck. \quad (4.2.136)$$

The vectors $\mathbf{A}_{\mathbf{k}}$ are thus harmonic functions of time. We can choose these functions to represent running waves:

$$\mathbf{A} = \sum_{\mathbf{k}} (\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} + \mathbf{a}_{\mathbf{k}}^* e^{-i\mathbf{k} \cdot \mathbf{r}}), \quad (4.2.137)$$

or

$$\mathbf{A}_{\mathbf{k}} = \mathbf{a}_{\mathbf{k}} + \mathbf{a}_{-\mathbf{k}}^*. \quad (4.2.138)$$

Thus $\mathbf{a}_{\mathbf{k}} \sim e^{i\omega_{\mathbf{k}} t}$ and $\mathbf{a}_{\mathbf{k}}^* \sim e^{-i\omega_{\mathbf{k}} t}$, so

$$\dot{\mathbf{A}}_{\mathbf{k}} = -ick(\mathbf{a}_{\mathbf{k}} - \mathbf{a}_{-\mathbf{k}}^*). \quad (4.2.139)$$

The electric and magnetic fields are given by

$$\mathbf{E} = -\frac{1}{c} \sum_{\mathbf{k}} \dot{\mathbf{A}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (4.2.140)$$

$$\mathbf{B} = i \sum_{\mathbf{k}} (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (4.2.141)$$

so the total energy and momentum of the electromagnetic field inside the parallelepiped are

$$E = \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{B}^2) dV = \sum_{\mathbf{k}} E_{\mathbf{k}}, \quad (4.2.142)$$

$$\mathbf{p} = \frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B} dV = \sum_{\mathbf{k}} \frac{\mathbf{k}}{k} \frac{E_{\mathbf{k}}}{c}, \quad (4.2.143)$$

where

$$E_{\mathbf{k}} = \frac{k^2 V}{2\pi} \mathbf{a}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}^* \quad (4.2.144)$$

and $V = abc$ is the volume of the parallelepiped.

Defining

$$\mathbf{Q}_{\mathbf{k}} = \sqrt{\frac{V}{4\pi c^2}} (\mathbf{a}_{\mathbf{k}} + \mathbf{a}_{\mathbf{k}}^*), \quad (4.2.145)$$

$$\mathbf{P}_{\mathbf{k}} = \dot{\mathbf{Q}}_{\mathbf{k}} = -i\omega_{\mathbf{k}} \sqrt{\frac{V}{4\pi c^2}} (\mathbf{a}_{\mathbf{k}} - \mathbf{a}_{\mathbf{k}}^*), \quad (4.2.146)$$

allows to write the energy of the field (4.2.142) as a Hamiltonian:

$$H = \sum_{\mathbf{k}} \frac{1}{2} (\mathbf{P}_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2 \mathbf{Q}_{\mathbf{k}}^2). \quad (4.2.147)$$

The Hamilton equations for this Hamiltonian are

$$\dot{\mathbf{Q}}_{\mathbf{k}} = \frac{\partial H}{\partial \mathbf{P}_{\mathbf{k}}} \rightarrow \mathbf{P}_{\mathbf{k}} = \dot{\mathbf{Q}}_{\mathbf{k}}, \quad (4.2.148)$$

$$\dot{\mathbf{P}}_{\mathbf{k}} = -\frac{\partial H}{\partial \mathbf{Q}_{\mathbf{k}}} \rightarrow \ddot{\mathbf{Q}}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 \mathbf{Q}_{\mathbf{k}} = 0, \quad (4.2.149)$$

in accordance with (4.2.135) and (4.2.145). The vectors $\mathbf{Q}_{\mathbf{k}}$ and $\mathbf{P}_{\mathbf{k}}$ are perpendicular to the wave vector $\mathbf{k}$, so they have 2 independent components $\alpha = 1, 2$ which determine the polarization of the wave. Thus we can put in (4.2.147)

$$\mathbf{Q}_{\mathbf{k}}^2 = \sum_{\alpha} \mathbf{Q}_{\mathbf{k}\alpha}^2, \quad \mathbf{P}_{\mathbf{k}}^2 = \sum_{\alpha} \mathbf{P}_{\mathbf{k}\alpha}^2. \quad (4.2.150)$$

The Hamiltonian (4.2.147) is thus the sum of the Hamiltonians corresponding to one-dimensional harmonic oscillators:

$$H = \sum_{\mathbf{k}\alpha} H_{\mathbf{k}\alpha}, \quad H_{\mathbf{k}\alpha} = \frac{1}{2} (\mathbf{P}_{\mathbf{k}\alpha}^2 + \omega_{\mathbf{k}}^2 \mathbf{Q}_{\mathbf{k}\alpha}^2). \quad (4.2.151)$$

The field of charges can also be expanded into plane waves,

$$f(\mathbf{r}) = \int_{-\infty}^{\infty} f_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}, \quad f_{\mathbf{k}} = \int_{-\infty}^{\infty} f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} dV. \quad (4.2.152)$$

We consider a charge q in uniform rectilinear motion with velocity $\mathbf{v}$, so that $\rho = q\delta(\mathbf{r} - \mathbf{v}t)$. The scalar potential satisfies then $\square\phi = 4\pi q\delta(\mathbf{r} - \mathbf{v}t)$, so that its Fourier transform satisfies $(\square\phi)_{\mathbf{k}} = 4\pi q e^{-i\mathbf{k}\cdot\mathbf{v}t}$. The relation $(\square\phi)_{\mathbf{k}} = \frac{\ddot{\phi}_{\mathbf{k}}}{c^2} + \mathbf{k}^2 \phi_{\mathbf{k}}$ gives then

$$\phi_{\mathbf{k}} = 4\pi q \frac{e^{-i\mathbf{k}\cdot\mathbf{v}t}}{\mathbf{k}^2 - (\mathbf{k} \cdot \mathbf{v}/c)^2}. \quad (4.2.153)$$

The vector potential is given by $\mathbf{A}_{\mathbf{k}} = \frac{\mathbf{v}}{c} \phi_{\mathbf{k}}$. The dependence of the Fourier-transformed fields on the Fourier-transformed potentials results from (2.7.32) and (2.7.33) by substituting ∇ by $i\mathbf{k}$:

$$\mathbf{E}_{\mathbf{k}} = -i\mathbf{k}\phi_{\mathbf{k}} - \frac{\dot{\mathbf{A}}_{\mathbf{k}}}{c}, \quad (4.2.154)$$

$$\mathbf{B}_{\mathbf{k}} = i\mathbf{k} \times \mathbf{A}_{\mathbf{k}}. \quad (4.2.155)$$

For a static charge, $\phi_{\mathbf{k}} = \frac{4\pi q}{\mathbf{k}^2}$ and $\mathbf{E}_{\mathbf{k}} = -\frac{4\pi i q}{\mathbf{k}^2} \mathbf{k}$, so that its electric field is resolved into waves directed along the wave vector (*longitudinal waves*).

4.2.5 Retarded potentials

In the presence of electric charges, the Maxwell-Minkowski equation (2.7.75) and the Lorentz gauge (4.2.89) give

$$\square A^i = \frac{1}{c^2} \frac{\partial^2 A^i}{\partial t^2} - \triangle A^i = \frac{4\pi}{c} j^i. \quad (4.2.156)$$

The time component of (4.2.156) is

$$\square \phi = 4\pi \rho, \quad (4.2.157)$$

whose physical solution vanishing at infinity is

$$\phi(t, \mathbf{r}) = \int \frac{\rho(t - R/c, \mathbf{r}')}{R} dV', \quad (4.2.158)$$

where

$$R = |\mathbf{r} - \mathbf{r}'| \quad (4.2.159)$$

is the distance from the source at $\mathbf{r}'$ to the field point $\mathbf{r}$. The spatial components of (4.2.156) are

$$\square \mathbf{A} = \frac{4\pi}{c} \mathbf{j}, \quad (4.2.160)$$

whose physical solution vanishing at infinity is

$$\mathbf{A}(t, \mathbf{r}) = \int \frac{\mathbf{j}(t - R/c, \mathbf{r}')}{cR} dV'. \quad (4.2.161)$$

The potentials (4.2.158) and (4.2.161) are referred to as the *retarded potentials*. If we replace the minus sign in (4.2.158) and (4.2.161) with the plus sign then the corresponding potentials, which also are solutions of (4.2.157) and (4.2.160), are called the *advanced potentials*. Only the retarded potentials are physical, in accordance with the finite velocity of propagation of interaction c and the principle of causality. Their values at t depend on the distribution of electric charges at the earlier, *retarded time*,

$$t_R = t - R/c, \quad (4.2.162)$$

which is the time at which a signal emitted from the source located at $\mathbf{r}'$ reaches the field point located at $\mathbf{r}$ at time t . Adding to the right-hand sides of (4.2.158) and (4.2.161) the general solutions ϕ_0 and $\mathbf{A}_0$ of the corresponding homogeneous equations $\frac{\partial^2 \phi}{c^2 \partial t^2} - \triangle \phi = 0$ and (4.2.86) does not alter (4.2.156). The quantities ϕ_0 and $\mathbf{A}_0$ correspond to the external field acting on the system. For a static field, (4.2.157) and (4.2.160) reduce to (4.2.3) and (4.2.44). In the absence of charges, (4.2.157) and (4.2.160) reduce to the wave equations $\partial_t \partial^i \phi = 0$ and (4.2.86).

We consider an electric charge e moving along a path given by a radius vector $\mathbf{r}_0(t)$. The vector from the charge to the field point located at $\mathbf{r}$ is thus $\mathbf{R}(t) = \mathbf{r} - \mathbf{r}_0$. The electromagnetic potential $A^i(t, \mathbf{r})$ depends on the motion of the charge at the retarded time t' satisfying

$$t' + \frac{R(t')}{c} = t. \quad (4.2.163)$$

In the frame of reference in which the charge is at rest at time t' , $\mathbf{v}_0(t') = \dot{\mathbf{r}}_0(t') = 0$, the potentials (at time t) are

$$\phi = \frac{e}{R(t')}, \quad \mathbf{A} = 0. \quad (4.2.164)$$

Therefore, in the frame of reference in which the charge is moving at time t' and its four-velocity is u^i , the electromagnetic potential is

$$A^i = \frac{eu^i}{u^k R_k}, \quad (4.2.165)$$

where R^i is the four-vector with components

$$R^i = (c(t - t'), \mathbf{R}). \quad (4.2.166)$$

This vector is a null vector, $R^i R_i = 0$. The temporal and spatial components of (4.2.165) are referred to as the *Lienard-Wiechert potentials*:

$$\phi = \frac{e}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}, \quad (4.2.167)$$

$$\mathbf{A} = \frac{e\mathbf{v}}{c(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})}. \quad (4.2.168)$$

The left-hand sides of (4.2.167) and (4.2.168) are evaluated at t , while their right-hand sides must be evaluated at t' . The Lienard-Wiechert potentials can also be obtained by substituting $\rho(t, \mathbf{r}) = e\delta(\mathbf{r} - \mathbf{r}_0(t))$ and $\mathbf{j}(t, \mathbf{r}) = e\mathbf{v}\delta(\mathbf{r} - \mathbf{r}_0(t))$ into the retarded potentials (4.2.158) and (4.2.161).

Differentiating $R(t') = c(t - t')$ with respect to t and using $\frac{\partial \mathbf{R}(t')}{\partial t'} = -\mathbf{v}(t')$ gives $\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'} \frac{\partial t'}{\partial t} = -\frac{\mathbf{R} \cdot \mathbf{v}}{R} \frac{\partial t'}{\partial t} = c(1 - \frac{\partial t'}{\partial t})$, so

$$\frac{\partial t'}{\partial t} = \frac{R}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}. \quad (4.2.169)$$

Differentiating $R(t') = c(t - t')$ with respect to $\mathbf{r}$ gives $\frac{\partial t'}{\partial \mathbf{r}} = -\frac{1}{c} \frac{\partial R(t')}{\partial \mathbf{r}} = -\frac{1}{c} (\frac{\partial R}{\partial t'} \frac{\partial t'}{\partial \mathbf{r}} + \frac{\mathbf{R}}{R})$, so

$$\frac{\partial t'}{\partial \mathbf{r}} = -\frac{\mathbf{R}}{c(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})}. \quad (4.2.170)$$

Substituting these formulae into (2.7.32) and (2.7.33), written as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{c \partial t'} \frac{\partial t'}{\partial t} - \nabla \phi - \frac{\partial \phi}{\partial t'} \frac{\partial t'}{\partial \mathbf{r}}, \quad (4.2.171)$$

$$\mathbf{B} = \nabla \times \mathbf{A} + \frac{\partial t'}{\partial \mathbf{r}} \times \frac{\partial \mathbf{A}}{\partial t'}, \quad (4.2.172)$$

gives the electric and magnetic field produced by a moving charge:

$$\mathbf{E} = \frac{e(1 - \frac{v^2}{c^2})(\mathbf{R} - \frac{\mathbf{v}}{c}R)}{(R - \frac{\mathbf{R} \cdot \mathbf{v}}{c})^3} + \frac{e\mathbf{R} \times ((\mathbf{R} - \frac{\mathbf{v}}{c}R) \times \dot{\mathbf{v}})}{c^2(R - \frac{\mathbf{R} \cdot \mathbf{v}}{c})^3}, \quad (4.2.173)$$

$$\mathbf{B} = \frac{1}{R} \mathbf{R} \times \mathbf{E}, \quad (4.2.174)$$

where

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t'}. \quad (4.2.175)$$

The left-hand side of (4.2.173) is evaluated at t , while its right-hand side must be evaluated at t' . For large R , the first term on the right of (4.2.173) varies like $\frac{1}{R^2}$, while the second term varies like $\frac{1}{R}$. The first term on the right of (4.2.173) does not depend on the acceleration $\dot{\mathbf{v}}$, so it corresponds to the electric field produced by a uniformly moving charge. If the velocity $\mathbf{v}$ is constant then

$$\mathbf{R}(t') - \frac{\mathbf{v}}{c}R(t') = \mathbf{R}(t') - \mathbf{v}(t - t') = \mathbf{R}(t) \quad (4.2.176)$$

and

$$R(t') - \frac{\mathbf{R}(t') \cdot \mathbf{v}}{c} = \sqrt{R^2(t) - \frac{1}{c^2}(\mathbf{v} \times \mathbf{R}(t))^2} = R(t) \sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta(t)}, \quad (4.2.177)$$

where $\theta(t)$ is the angle between $\mathbf{R}(t)$ and $\mathbf{v}$. Thus (4.2.173) reduces to

$$\mathbf{E} = \frac{e\mathbf{R}(t)}{R^3(t)} \frac{1 - \frac{v^2}{c^2}}{(1 - \frac{v^2}{c^2} \sin^2 \theta(t))^{3/2}}, \quad (4.2.178)$$

while (4.2.174) is equivalent to

$$\mathbf{B} = \frac{1}{c} \mathbf{v} \times \mathbf{E}. \quad (4.2.179)$$

These fields can also be obtained by applying a Lorentz transformation from the frame in which the charge is at rest and where $\phi = \frac{e}{R}$ and $\mathbf{A} = 0$ to the frame in which the charge moves with velocity $\mathbf{v}$.

Spectral resolution (4.2.128) of ρ and ϕ into $\rho_\omega e^{-i\omega t}$ and $\phi_\omega e^{-i\omega t}$ turns the wave equation (4.2.157) into $\Delta\phi_\omega + k^2\phi_\omega = -4\phi\rho_\omega$, and the formula for the retarded potential (4.2.158) into

$$\phi_\omega = \int \rho_\omega \frac{e^{ikR}}{R} dV. \quad (4.2.180)$$

Analogously, (4.2.161) gives

$$\mathbf{A}_\omega = \int \mathbf{j}_\omega \frac{e^{ikR}}{cR} dV. \quad (4.2.181)$$

The inverse relation (4.2.129) applied to ρ_ω and substituted into (4.2.180) gives $\phi_\omega = \int \frac{\rho}{R} e^{i(\omega t + kR)} dV dt$. Substituting to this relation the charge density of an electric charge e moving along a path given by a radius vector $\mathbf{r}_0(t)$, $\rho(t, \mathbf{r}) = e\delta(\mathbf{r} - \mathbf{r}_0(t))$, gives the spectral representation of the scalar Lienard-Wiechert potential:

$$\phi_\omega = e \int \frac{1}{R} e^{i\omega(t+R/c)} dt. \quad (4.2.182)$$

Analogously, the vector potential gives

$$\mathbf{A}_\omega = e \int \frac{\dot{\mathbf{r}}_0}{cR} e^{i\omega(t+R/c)} dt. \quad (4.2.183)$$

For a discrete spectral resolution of the field, (4.2.182) becomes

$$\phi_n = \frac{e}{T} \int_0^T \frac{1}{R} e^{in\omega_0(t+R/c)} dt. \quad (4.2.184)$$

4.2.6 Electromagnetic field in second approximation

The Lagrangian for a particle of mass m and charge e moving with velocity $\mathbf{v}$ in a field represented by the potentials ϕ and $\mathbf{A}$ is given by (3.1.40). Because the velocity of propagation of interaction is finite, the field must be considered as a system with degrees of freedom independent of those of the particles. If the velocities of the particles in a system of charges are small compared to c then the degrees of freedom of the field can be represented in terms of the coordinates and velocities of the particles. The Lagrangian for the system can then be described in terms of the coordinates and velocities of the particles only up to terms of second order in $\frac{v}{c}$, because moving charges radiate electromagnetic waves (which have their own degrees of freedom) at the third order. Expanding the retarded scalar potential in series of powers of R/c and using the conservation of the total charge yields

$$\phi = \int \frac{\rho}{R} dV - \frac{\partial}{\partial t} \int \rho dV + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int R\rho dV = \int \frac{\rho}{R} dV + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int R\rho dV, \quad (4.2.185)$$

where all the quantities are evaluated at the same time t . Expanding the retarded vector potential in series of powers of R/c yields

$$\mathbf{A} = \frac{1}{c} \int \frac{\rho\mathbf{v}}{R} dV. \quad (4.2.186)$$

The retarded potentials due to one charge are

$$\phi = \frac{e}{R} + \frac{e}{2c^2} \frac{\partial^2 R}{\partial t^2}, \quad (4.2.187)$$

$$\mathbf{A} = \frac{e\mathbf{v}}{cR}. \quad (4.2.188)$$

Applying a gauge transformation (2.7.12) with

$$\alpha = -\frac{e}{2c} \frac{\partial R}{\partial t} \quad (4.2.189)$$

brings the potentials (4.2.187) and (4.2.188) to

$$\phi = \frac{e}{R}, \quad (4.2.190)$$

$$\mathbf{A} = \frac{e\mathbf{v}}{cR} + \frac{e}{2c} \nabla \frac{\partial R}{\partial t} = \frac{e}{2cR} (\mathbf{v} + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}). \quad (4.2.191)$$

The Lagrangian for a charge e_a is given by (3.1.40) in which $m = m_a$, $e = e_a$, and ϕ and $\mathbf{A}$ are the potentials of the field produced by all the other charges at the position of e_a , given by the sums of (4.2.190) and (4.2.191). This Lagrangian is thus, omitting the constant term $-m_a c^2$,

$$L_a = \frac{m_a v_a^2}{2} + \frac{m_a v_a^4}{8} - e_a \sum_{b \neq a} \frac{e_b}{R_{ab}} + \frac{e_a}{2c^2} \sum_{b \neq a} \frac{e_b}{R_{ab}} (\mathbf{v}_a \cdot \mathbf{v}_b + (\mathbf{v}_a \cdot \mathbf{n}_{ab})(\mathbf{v}_b \cdot \mathbf{n}_{ab})), \quad (4.2.192)$$

where

$$\mathbf{R}_{ab} = \mathbf{r}_a - \mathbf{r}_b, \quad \mathbf{n}_{ab} = \frac{\mathbf{R}_{ab}}{R_{ab}}. \quad (4.2.193)$$

Consequently, the Lagrangian for a system of charges is

$$L = \sum_a \frac{m_a v_a^2}{2} + \sum_a \frac{m_a v_a^4}{8c^2} - \sum_{a,b \neq a} \frac{e_a e_b}{2R_{ab}} + \sum_{a,b \neq a} \frac{e_a e_b}{4c^2 R_{ab}} (\mathbf{v}_a \cdot \mathbf{v}_b + (\mathbf{v}_a \cdot \mathbf{n}_{ab})(\mathbf{v}_b \cdot \mathbf{n}_{ab})), \quad (4.2.194)$$

and is called the *Darwin Lagrangian*.

The corresponding Darwin Hamiltonian is obtained from (4.2.194) by substituting $\mathbf{v}_a = \frac{\mathbf{p}_a}{m_a}$ and changing the signs of all the terms except the nonrelativistic kinetic term $\sum_a \frac{m_a v_a^2}{2}$ (these other terms are regarded as the potential energy):

$$H = \sum_a \frac{p_a^2}{2m_a} - \sum_a \frac{p_a^4}{8c^2 m_a^3} + \sum_{a,b \neq a} \frac{e_a e_b}{2R_{ab}} - \sum_{a,b \neq a} \frac{e_a e_b}{4c^2 m_a m_b R_{ab}} (\mathbf{p}_a \cdot \mathbf{p}_b + (\mathbf{p}_a \cdot \mathbf{n}_{ab})(\mathbf{p}_b \cdot \mathbf{n}_{ab})). \quad (4.2.195)$$

The center of inertia of the system $\mathbf{R}$ in this approximation is given by (3.1.72) in which we must add to E_a the energy of the field:

$$\mathbf{R} = \frac{\sum_a E_a \mathbf{r}_a + \int W \mathbf{r} dV}{\sum_a E_a + \int W dV}. \quad (4.2.196)$$

In the numerator, we have

$$\begin{aligned} \int W \mathbf{r} dV &= \frac{1}{8\pi} \int \mathbf{E}^2 \mathbf{r} dV = \frac{1}{8\pi} \int (\nabla \phi)^2 \mathbf{r} dV = \frac{1}{8\pi} \int \mathbf{r} \left(\nabla \frac{\phi^2}{2} \cdot d\mathbf{f} \right) \\ &\quad - \frac{1}{8\pi} \int \nabla \frac{\phi^2}{2} dV - \frac{1}{8\pi} \int \phi \Delta \phi \mathbf{r} dV. \end{aligned} \quad (4.2.197)$$

At infinity, the first two terms on the right of (4.2.197) vanish, which gives

$$\int W \mathbf{r} dV = -\frac{1}{8\pi} \int \phi \Delta \phi \mathbf{r} dV = \frac{1}{2} \int \rho \phi \mathbf{r} dV = \frac{1}{2} \sum_a e_a \phi_a \mathbf{r}_a. \quad (4.2.198)$$

Therefore the radius vector of the center of inertia is

$$\mathbf{R} = \frac{1}{E} \sum_a \mathbf{r}_a \left(m_a c^2 + \frac{p_a^2}{2m_a} + e_a \sum_{b \neq a} \frac{e_b}{2R_{ab}} \right), \quad (4.2.199)$$

where the energy of the system is

$$E = \sum_a \left(m_a c^2 + \frac{p_a^2}{2m_a} \right) + \sum_{a,b \neq a} \frac{e_a e_b}{2R_{ab}}. \quad (4.2.200)$$

4.2.7 Electromagnetic radiation

We consider a system of charges ρdV located $\mathbf{r}$ and take the origin of the Cartesian system of coordinates anywhere within the system. If $\mathbf{R}$ is the radius vector of a point at which we measure the field and this point is located at a distance which is large compared to the size of the system, $R \gg r$, then $|\mathbf{R} - \mathbf{r}| \approx R - \mathbf{r} \cdot \mathbf{n}$, where $\mathbf{n} = \frac{\mathbf{R}}{R}$. The retarded potentials in this approximation are

$$\phi_t = \frac{1}{R} \int \rho_{t-R/c+\mathbf{r} \cdot \mathbf{n}/c} dV, \quad (4.2.201)$$

$$\mathbf{A}_t = \frac{1}{cR} \int \mathbf{j}_{t-R/c+\mathbf{r} \cdot \mathbf{n}/c} dV. \quad (4.2.202)$$

The vector Lienard-Wiechert potential (4.2.168) from a single charge is

$$\mathbf{A}_t = \frac{e\mathbf{v}_{t'}}{cR(1 - \frac{\mathbf{v}_{t'} \cdot \mathbf{n}}{c})}, \quad (4.2.203)$$

where the retarded time t' satisfies

$$t' + \frac{R}{c} - \frac{1}{c} \mathbf{r}_{t'} \cdot \mathbf{n} = t. \quad (4.2.204)$$

If the field point is also located at a distance which is large compared to the wavelength of the radiated electromagnetic waves (in a *wave zone*), $R \gg \lambda$, then these waves can be approximated as plane waves. The magnetic field of this wave is given by (4.2.97):

$$\mathbf{B} = \frac{1}{c} \dot{\mathbf{A}} \times \mathbf{n}, \quad (4.2.205)$$

while the electric field is

$$\mathbf{E} = \mathbf{B} \times \mathbf{n} = \frac{1}{c} (\dot{\mathbf{A}} \times \mathbf{n}) \times \mathbf{n}. \quad (4.2.206)$$

The energy of the radiation per unit time and per unit solid angle do is called the *intensity* of the radiation. It is equal to the flux of the Poynting vector through a surface element $R^2 do$ of a sphere of radius R :

$$dI = \mathbf{S} \cdot d\mathbf{f}, \quad (4.2.207)$$

which gives, due to (4.2.101),

$$dI = \frac{c}{4\pi} \mathbf{B}^2 R^2 do. \quad (4.2.208)$$

Since, for large R , (4.2.173) varies like $\frac{1}{R}$, the intensity (4.2.208) is independent of R . Continuous spectral resolution of the vector potential,

$$\mathbf{A}_\omega = \frac{e^{ikR}}{cR} \int \mathbf{j}_\omega e^{-i\mathbf{k} \cdot \mathbf{r}} dV, \quad (4.2.209)$$

gives, using (4.2.130),

$$\mathbf{B} = i\mathbf{k} \times \mathbf{A}_\omega, \quad (4.2.210)$$

$$\mathbf{E} = \frac{ic}{\omega} \mathbf{k} \times (\mathbf{A}_\omega \times \mathbf{k}), \quad (4.2.211)$$

$$dI_{\mathbf{n}\omega} = \frac{c}{2\pi} |\mathbf{B}_\omega|^2 R^2 do \frac{d\omega}{2\pi}. \quad (4.2.212)$$

Analogously, discrete spectral resolution gives

$$dI_{\mathbf{n}n} = \frac{c}{2\pi} |\mathbf{B}_n|^2 R^2 do. \quad (4.2.213)$$

If a radiating system of charges is nonrelativistic then the wavelength of the radiated electromagnetic waves λ is large compared to the size of the system a because $\lambda \sim ac/v$. Expanding the retarded vector potential (4.2.202) in a Taylor series with respect to the small quantity $\mathbf{r} \cdot \mathbf{n}/R$ gives

$$\mathbf{A}_t = \frac{1}{cR} \int \mathbf{j}_{t-R/c} dV + \frac{1}{c^2 R} \frac{\partial}{\partial(t-R/c)} \int (\mathbf{r} \cdot \mathbf{n}) \mathbf{j}_{t-R/c} dV, \quad (4.2.214)$$

which for the system of charges becomes

$$\mathbf{A} = \frac{1}{cR} \sum e \mathbf{v} + \frac{1}{c^2 R} \frac{\partial}{\partial t} \sum e \mathbf{v} (\mathbf{r} \cdot \mathbf{n}). \quad (4.2.215)$$

The quantity $\mathbf{r} \cdot \mathbf{n}/c$ is on the order of a/c , where Using $\mathbf{v}(\mathbf{r} \cdot \mathbf{n} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{r}(\mathbf{r} \cdot \mathbf{n})) + \frac{1}{2} (\mathbf{r} \times \mathbf{v}) \times \mathbf{n}$ and the definitions of the electric and magnetic dipole momenta turns (4.2.215) into

$$\mathbf{A} = \frac{1}{cR} \dot{\mathbf{d}} + \frac{1}{2c^2 R} \frac{\partial^2}{\partial t^2} \sum e \mathbf{r} (\mathbf{r} \cdot \mathbf{n}) + \frac{1}{cR} \dot{\mathbf{m}} \times \mathbf{n}. \quad (4.2.216)$$

Since adding to $\mathbf{A}$ any vector proportional to $\mathbf{n}$ does not alter the fields (4.2.205) and (4.2.206), we can write (4.2.216) as

$$\mathbf{A} = \frac{1}{cR} \dot{\mathbf{d}} + \frac{1}{6c^2 R} \frac{\partial^2}{\partial t^2} \sum e (3\mathbf{r}(\mathbf{r} \cdot \mathbf{n}) - \mathbf{n}r^2) + \frac{1}{cR} \dot{\mathbf{m}} \times \mathbf{n} = \frac{1}{cR} \dot{\mathbf{d}} + \frac{1}{6c^2 R} \ddot{\mathbf{D}} + \frac{1}{cR} \dot{\mathbf{m}} \times \mathbf{n}, \quad (4.2.217)$$

where the components of the electric quadrupole vector $\mathbf{D}$ are

$$D_\alpha = D_{\alpha\beta} n_\beta. \quad (4.2.218)$$

Thus the magnetic field is given by

$$\mathbf{B} = \frac{1}{c^2 R} \left(\ddot{\mathbf{d}} \times \mathbf{n} + \frac{1}{6c} \dot{\mathbf{D}} \times \mathbf{n} + (\ddot{\mathbf{m}} \times \mathbf{n}) \times \mathbf{n} \right). \quad (4.2.219)$$

Substituting (4.2.219) into (4.2.208) and integrating over the solid angle gives, using $\int n_\alpha n_\beta d\Omega = \frac{4\pi}{3} \delta_{\alpha\beta}$ and $\int n_\alpha n_\beta n_\gamma n_\delta d\Omega = \frac{4\pi}{15} (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma})$,

$$I = \frac{2}{3c^3} \ddot{\mathbf{d}}^2 + \frac{1}{180c^5} \dot{\mathbf{D}}_{\alpha\beta}^2 + \frac{2}{3c^3} \ddot{\mathbf{m}}^2. \quad (4.2.220)$$

The first term on the right of (4.2.220) represents the *electric dipole radiation*, the second one the *electric quadrupole radiation*, and the third one the *magnetic dipole radiation*. These terms must be evaluated at the time $t - R/c$. For the electric dipole radiation, (4.2.212) and (4.2.213) integrated over the solid angle give

$$dI_\omega = \frac{4\omega^4}{3c^3} |\mathbf{d}_\omega|^2 \frac{d\omega}{2\pi}, \quad (4.2.221)$$

$$I_n = \frac{4\omega_0^4 n^4}{3c^3} |\mathbf{d}_n|^2, \quad (4.2.222)$$

where $\mathbf{d}_\omega$ and $\mathbf{d}_n$ are the spectral representations of $\mathbf{d}$. The recoil force acting on a closed radiating system is

$$F_\alpha = \int \sigma_{\alpha\beta} n_\beta R^2 d\Omega, \quad (4.2.223)$$

which, using the Maxwell stress tensor (2.7.97) and the radiation fields for the vector potential (4.2.217), gives

$$F_\alpha = -\frac{1}{c^4} \left(\frac{1}{15c} \dot{\mathbf{D}}_{\alpha\beta} \ddot{\mathbf{d}}_\beta + \frac{2}{3} (\ddot{\mathbf{d}} \times \ddot{\mathbf{m}})_\alpha \right). \quad (4.2.224)$$

If all the charges in a closed system move with velocities $v \ll c$ and have the same charge-to-mass ratio then they do not produce the electric dipole radiation because the dipole moment of the system

is proportional to the conserved radius vector of its center of mass (4.2.23). They also do not produce the magnetic dipole radiation because the magnetic moment of such a system is proportional to its conserved angular momentum (4.2.54).

If the field point is located at a distance which is not large compared to the wavelength of the radiated electromagnetic waves then these waves cannot be approximated as plane waves. In the electric-dipole approximation, (4.2.217) gives $\mathbf{A} = \frac{\dot{\mathbf{d}}}{cR}$, so the magnetic field is

$$\mathbf{B} = \frac{1}{c} \nabla \times \frac{\dot{\mathbf{d}}}{R}. \quad (4.2.225)$$

After spectral resolution, in which $\mathbf{d} \sim \mathbf{d}_\omega e^{-i\omega(t-R/c)}$ and $\mathbf{B} \sim \mathbf{B}_\omega e^{-i\omega t}$, (4.2.225) gives

$$\mathbf{B}_\omega = -ik \nabla \times \left(\mathbf{d}_\omega \frac{e^{ikR}}{R} \right). \quad (4.2.226)$$

To calculate the scalar potential, we use the Lorentz gauge, obtaining

$$\phi = -\nabla \cdot \frac{\mathbf{d}}{R}, \quad (4.2.227)$$

so the electric field is

$$\mathbf{E} = \nabla \left(\nabla \cdot \frac{\mathbf{d}}{R} \right) - \frac{\ddot{\mathbf{d}}}{c^2 R} = \nabla \times \left(\nabla \times \frac{\mathbf{d}}{R} \right). \quad (4.2.228)$$

The angular momentum radiated away by electromagnetic waves is

$$\dot{M}_\alpha = \int e_{\alpha\beta\gamma} x_\beta \sigma_{\gamma\delta} n_\delta R^2 d\omega, \quad (4.2.229)$$

which gives, using the Maxwell stress tensor (2.7.97),

$$\dot{\mathbf{M}} = \frac{2}{3c^3} \dot{\mathbf{d}} \times \ddot{\mathbf{d}}. \quad (4.2.230)$$

In the next approximation, the potentials due to the electric quadrupole moment are

$$\phi = \frac{1}{6} \frac{\partial^2}{\partial X_\alpha \partial X_\beta} \frac{D_{\alpha\beta}}{R}, \quad A_\alpha = -\frac{1}{6c} \frac{\partial}{\partial X_\beta} \frac{\dot{D}_{\alpha\beta}}{R}, \quad (4.2.231)$$

and the potentials due to the magnetic dipole moment are

$$\phi = 0, \quad \mathbf{A} = \nabla \times \frac{\mathbf{m}}{R}. \quad (4.2.232)$$

If the intensity of the radiation in the rest frame K' of the radiating source is

$$dI' = f(\cos\theta', \phi') d(\cos\theta') d\phi', \quad (4.2.233)$$

then in the frame K moving relative to K' with velocity $\mathbf{V}$ we have $dE' = \frac{dE - \mathbf{V} \cdot d\mathbf{P}}{\sqrt{1-V^2/c^2}} = dE \frac{1-V\cos\theta/c}{\sqrt{1-V^2/c^2}}$, $\phi = \phi'$ and $dt = dt' / \sqrt{1-V^2/c^2}$. Thus $dI = (dE/dt) d(\cos\theta) d\phi$ gives, using (1.6.112),

$$dI = \frac{(1-V^2/c^2)^2}{(1-V\cos\theta/c)^3} f\left(\frac{\cos\theta - V/c}{1-V\cos\theta/c}, \phi\right) d(\cos\theta) d\phi. \quad (4.2.234)$$

For the dipole radiation, $f = \text{const} \cdot \sin^2\theta'$, (4.2.234) gives

$$dI = \text{const} \frac{(1-V^2/c^2)^3}{(1-V\cos\theta/c)^5} \sin^2\theta d(\cos\theta) d\phi. \quad (4.2.235)$$

4.2.8 Radiation reaction

Expanding the retarded scalar potential in series of powers of R/c up to the cubic term yields

$$\phi = \int \frac{\rho}{R} dV + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int R \rho dV - \frac{1}{6c^3} \frac{\partial^3}{\partial t^3} \int R^2 \rho dV, \quad (4.2.236)$$

while expanding the retarded vector potential yields

$$\mathbf{A} = \frac{1}{c} \int \frac{\rho \mathbf{v}}{R} dV - \frac{1}{c^2} \frac{\partial}{\partial t} \int \rho \mathbf{v} dV. \quad (4.2.237)$$

The last terms on the right-hand sides of (4.2.236) and (4.2.237) correspond to the radiation of electromagnetic waves. Applying a gauge transformation (2.7.12) with

$$\alpha = \frac{1}{6c^2} \frac{\partial^2}{\partial t^2} \int R^2 \rho dV \quad (4.2.238)$$

brings these terms to

$$\phi_{rad} = 0, \quad \mathbf{A}_{rad} = -\frac{2}{3c^3} \sum e \dot{\mathbf{v}}, \quad (4.2.239)$$

so the corresponding electric field is

$$\mathbf{E}_{rad} = \frac{2}{3c^3} \ddot{\mathbf{d}}. \quad (4.2.240)$$

The force acting on a charge e in this field is then

$$\mathbf{f} = \frac{2e}{3c^3} \ddot{\mathbf{d}} \quad (4.2.241)$$

The sum of the forces (4.2.241) acting on charges in a closed, electrically neutral system vanishes, so it does not appear in the total recoil force (4.2.224). If the charge is point-like then expanding $\mathbf{A}$ to terms of higher power of R/c does not change (4.2.241) because these terms vanish as $R \rightarrow 0$. The energy radiated per unit time equals to the total power:

$$\sum \mathbf{f} \cdot \mathbf{v} = \frac{2}{3c^3} \ddot{\mathbf{d}} \cdot \sum e \mathbf{v} = \frac{2}{3c^3} \ddot{\mathbf{d}} \cdot \dot{\mathbf{d}} = \frac{2}{3c^3} \frac{d}{dt} (\dot{\mathbf{d}} \cdot \dot{\mathbf{d}}) - \frac{2}{3c^3} \ddot{\mathbf{d}}^2, \quad (4.2.242)$$

so the average loss of the energy (*radiation damping*) through the electric dipole radiation is

$$\overline{\frac{dE}{dt}} = \overline{\sum \mathbf{f} \cdot \mathbf{v}} = -\frac{2}{3c^3} \overline{\ddot{\mathbf{d}}^2}, \quad (4.2.243)$$

in agreement with the first term on the right of (4.2.220). The angular momentum radiated per unit time equals to the total torque:

$$\sum \mathbf{r} \times \mathbf{f} = \frac{2}{3c^3} \sum e \mathbf{r} \times \dot{\mathbf{d}} = \frac{2}{3c^3} \mathbf{d} \times \dot{\mathbf{d}} = \frac{2}{3c^3} \frac{d}{dt} (\mathbf{d} \times \dot{\mathbf{d}}) - \frac{2}{3c^3} \dot{\mathbf{d}} \times \ddot{\mathbf{d}}, \quad (4.2.244)$$

so the average loss of the angular momentum through the electric dipole radiation is

$$\overline{\frac{d\mathbf{M}}{dt}} = \overline{\sum \mathbf{r} \times \mathbf{f}} = -\frac{2}{3c^3} \overline{\dot{\mathbf{d}} \times \ddot{\mathbf{d}}}, \quad (4.2.245)$$

in agreement with (4.2.230).

For a nonrelativistic particle with mass m and electric charge e , its equation of motion must also include the recoil force (4.2.241) due to the interaction with its own electromagnetic field, which is called the *Abraham-Lorentz force*:

$$\mathbf{f} = \frac{2e^2}{3c^3} \ddot{\mathbf{v}}. \quad (4.2.246)$$

This force is a recoil force on an accelerating charged particle caused by emitting an electromagnetic radiation, and it is also called the *radiation-reaction force*. It is proportional to the particle's jerk. Thus the equation of motion is

$$m\dot{\mathbf{v}} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B} + \frac{2e^2}{3c^3}\ddot{\mathbf{v}}, \quad (4.2.247)$$

or

$$m\dot{\mathbf{v}} = \mathbf{F}_{ext} + mt_0\ddot{\mathbf{v}}, \quad (4.2.248)$$

where

$$t_0 = \frac{2e^2}{3mc^3} \quad (4.2.249)$$

is on the order of the time of propagation of interaction through a distance equal to the *classical radius* of the particle,

$$r_0 = \frac{e^2}{mc^2}. \quad (4.2.250)$$

The classical radius of a particle is on the order of the radius of a uniformly charged sphere whose electrostatic energy and rest energy are on the same order. Integrating (4.2.247) once gives

$$m\dot{\mathbf{v}} = \frac{1}{t_0} \int_t^\infty e^{-\frac{t'-t}{t_0}} \mathbf{F}_{ext}(t') dt', \quad (4.2.251)$$

which means that future values of the external force unphysically (violating causality) affect the acceleration of the particle in the present, indicating limits of the *classical* (based on the principle of least action) description of electrodynamics. The weight factor $e^{-\frac{t'-t}{t_0}}$ falls off rapidly for times greater than t_0 in the future.

To eliminate these noncausal pre-acceleration effects, caused by the third-order character of (4.2.247), we can apply to this equation a reduction of order due to Landau and Lifshitz. Differentiating (4.2.247) with respect to the time, in the frame of reference in which the particle is instantaneously at rest, gives

$$\ddot{\mathbf{v}} = \frac{e}{m} \dot{\mathbf{E}} + \frac{e}{mc} \dot{\mathbf{v}} \times \mathbf{B}. \quad (4.2.252)$$

Substituting (4.2.247) with $\mathbf{v} = 0$ and without (4.2.246) into (4.2.252) gives

$$\ddot{\mathbf{v}} = \frac{e}{m} \dot{\mathbf{E}} + \frac{e^2}{m^2 c} \mathbf{E} \times \mathbf{B}, \quad (4.2.253)$$

so the Abraham-Lorentz force (4.2.246) becomes in this approximation

$$\mathbf{f} = \frac{2e^3}{3mc^3} \dot{\mathbf{E}} + \frac{2e^4}{3m^2 c^4} \mathbf{E} \times \mathbf{B}. \quad (4.2.254)$$

Thus (4.2.247) turns into a second-order equation, which is free of noncausal solutions,

$$m\dot{\mathbf{v}} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B} + \frac{2e^3}{3mc^3} \dot{\mathbf{E}} + \frac{2e^4}{3m^2 c^4} \mathbf{E} \times \mathbf{B}. \quad (4.2.255)$$

The above method of reduction of order is valid if the force $\mathbf{f} \ll e\mathbf{E}$. If the frequency of the electromagnetic waves radiated by the particle is ω then $\dot{\mathbf{E}} \sim \omega\mathbf{E}$. The first term in (4.2.254) is much smaller than $e\mathbf{E}$ if the wavelength of these waves is much larger than the classical radius of the particle:

$$\lambda = \frac{c}{\omega} \gg r_0. \quad (4.2.256)$$

The second term in (4.2.254) is much smaller than $e\mathbf{E}$ if the magnetic field is not too strong:

$$|\mathbf{B}| \ll \frac{m^2 c^4}{e^3}. \quad (4.2.257)$$

The above conditions limit the classical description of the interaction of charged particles with electromagnetic waves and thus the classical theory of electromagnetic fields.

In the frame of reference in which the particle is instantaneously at rest, the radiated energy and momentum are

$$dE = \frac{2e^2}{3c^3} a^2 dt, \quad d\mathbf{p} = 0. \quad (4.2.258)$$

Thus the relativistic expression for the radiated momentum four-vector in any frame of reference is, using (1.6.122),

$$dp^i = -\frac{2e^2}{3c} \frac{du^k}{ds} \frac{du_k}{ds} dx^i. \quad (4.2.259)$$

Substituting here (2.7.110) in the locally Galilean system of coordinates gives

$$dp^i = -\frac{2e^4}{3m^2 c^5} F_{kl} u^l F^{km} u_m dx^i. \quad (4.2.260)$$

The time components of (4.2.259) and (4.2.260) gives the intensity of the radiation:

$$I = \frac{2e^2}{3c^3} \frac{a^2 - (\mathbf{v} \times \mathbf{a})^2/c^2}{(1 - v^2/c^2)^3} = \frac{2e^4}{3m^2 c^3} \frac{(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})^2 - (\frac{\mathbf{v}}{c} \cdot \mathbf{E})^2}{1 - v^2/c^2}. \quad (4.2.261)$$

In the case of a uniform motion on a circle in a constant uniform magnetic field, $\mathbf{E} = 0$ and $\mathbf{B} \perp \mathbf{v}$, leading to the *synchrotron radiation*:

$$I = \frac{2e^4}{3m^2 c^5} \frac{B^2 v^2}{1 - v^2/c^2}. \quad (4.2.262)$$

Substituting the Lienard-Wiechert electric (4.2.173) and magnetic (4.2.174) fields in the wave zone (for large R),

$$\mathbf{E} = \frac{e}{c^2 R} \frac{\mathbf{n} \times \left((\mathbf{n} - \frac{\mathbf{v}}{c}) \times \mathbf{a} \right)}{(1 - \frac{\mathbf{n} \cdot \mathbf{v}}{c})^3}, \quad \mathbf{B} = \mathbf{n} \times \mathbf{E}, \quad (4.2.263)$$

into (4.2.208) gives

$$dI = \frac{e^2}{4\pi c^3} \left(\frac{2(\mathbf{n} \cdot \mathbf{a})(\mathbf{v} \cdot \mathbf{a})}{c(1 - \mathbf{n} \cdot \mathbf{v}/c)^5} + \frac{a^2}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^4} - \frac{(1 - v^2/c^2)(\mathbf{n} \cdot \mathbf{a})^2}{(1 - \mathbf{n} \cdot \mathbf{v}/c)^6} \right) d\Omega, \quad (4.2.264)$$

whose right-hand side must be evaluated at the retarded time t' . Integrating (4.2.264) over the solid angle gives (4.2.261), while the angular distribution of the total radiated energy is

$$dE_{\mathbf{n}} = \int dI dt = \int dI (1 - \mathbf{n} \cdot \mathbf{v}/c) dt'. \quad (4.2.265)$$

The relativistic equation of motion of a charged particle in an external electromagnetic field in the locally Galilean system of coordinates, including radiation damping, is

$$mc \frac{du^i}{ds} = \frac{e}{c} F^{ik} u_k + g^i, \quad (4.2.266)$$

where the relativistic Abraham-Lorentz force four-vector g^i must be orthogonal to u^i and its spatial components must reduce in the nonrelativistic limit to (4.2.246). These requirements yield

$$g^i = \frac{2e^2}{3c} \left(\frac{d^2 u^i}{ds^2} - u^i u^k \frac{d^2 u_k}{ds^2} \right). \quad (4.2.267)$$

Substituting here (4.2.266) without g^i gives the approximated expression for the relativistic Abraham-Lorentz force:

$$g^i = \frac{2e^3}{3mc^3} \frac{\partial F^{ik}}{\partial x^l} u_k u^l - \frac{2e^4}{3m^2 c^5} F^{il} F_{kl} u^k + \frac{2e^4}{3m^2 c^5} F_{kl} u^l F^{km} u_m u^i. \quad (4.2.268)$$

In the nonrelativistic limit, the spatial components of (4.2.268) reduce to (4.2.255).

4.2.9 Scattering of electromagnetic waves

We consider a plane electromagnetic wave impinging on a system of charges. The resulting motion of the charges produces an electromagnetic radiation. This process can be regarded as a scattering of the incident wave on the system of charges. The intensity of the radiation dI is proportional to the energy flux of the incident wave $S = |\mathbf{S}|$. Thus the scattering of an electromagnetic wave on the system is characterized by their ratio, called the *scattering cross section* for waves:

$$d\sigma = \frac{dI}{S}. \quad (4.2.269)$$

For a free nonrelativistic charge at rest, we can neglect the magnetic field in the Lorentz force, so the equation of motion of the charge is $m\ddot{\mathbf{r}} = e\mathbf{E}$ and thus $\ddot{\mathbf{d}} = \frac{e^2}{m}\mathbf{E}$, from which (4.2.208) and (4.2.219) give

$$dI = \frac{e^4}{4\pi m^2 c^3} (\mathbf{E} \times \mathbf{n})^2 d\omega. \quad (4.2.270)$$

This expression also results from (4.2.264) for $\mathbf{v} = 0$. For a linearly polarized wave, $\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$. If the wave is weak then the charge executes a small motion, so $\mathbf{r} \approx 0$. Since the Poynting vector is $\mathbf{S} = \frac{c}{4\pi} \mathbf{E}^2$, (4.2.269) gives

$$d\sigma = r_0^2 \sin^2 \theta d\omega, \quad (4.2.271)$$

where r_0 is the classical radius of the charge (4.2.250) and θ is the angle between $\mathbf{n}$ and $\mathbf{E}$. Integrating (4.2.271) over the solid angle gives the *Thomson formula*:

$$\sigma = \frac{8\pi}{3} r_0^2. \quad (4.2.272)$$

The time-averaged force (4.2.254) exerted on a scattered charge is

$$\mathbf{f} = \frac{2e^4}{3m^2 c^4} \overline{\mathbf{E}^2 \mathbf{n}} = \sigma \bar{W} \mathbf{n}. \quad (4.2.273)$$

In the (locally) rest frame of the charge, its acceleration is thus $a_0 = \sigma \bar{W}_0 / m$. In any frame, using (4.1.34) and (4.2.103), this equation gives the speed of the charge v :

$$\frac{d}{dt} \frac{v}{\sqrt{1 - v^2/c^2}} = \frac{\sigma \bar{W}}{m} \frac{1 - v/c}{1 + v/c}. \quad (4.2.274)$$

In the rest frame of the charge, the frequency of the scattering wave is equal to the frequency of the incident wave, $\omega' = \omega$. In any frame, in which the charge moves with velocity $\mathbf{V}$, this condition is $k'_i u'^i = k_i u^i$ or

$$\omega' \left(1 - \frac{V}{c} \cos \theta' \right) = \omega \left(1 - \frac{V}{c} \cos \theta \right). \quad (4.2.275)$$

For an elliptically polarized wave (with $\mathbf{r} \approx 0$), $\mathbf{E} = \mathbf{E}_1 \cos \omega t + \mathbf{E}_2 \sin \omega t$, which gives for a free charge at rest:

$$d\sigma = r_0^2 \frac{(\mathbf{E}_1 \times \mathbf{n})^2 + (\mathbf{E}_2 \times \mathbf{n})^2}{\mathbf{E}_1^2 + \mathbf{E}_2^2} d\omega. \quad (4.2.276)$$

For a linearly polarized wave impinging on a free electric dipole which is a rotator with moment of inertia I and frequency ω_0 , $\ddot{\mathbf{d}} = \omega_0 \times \mathbf{d}$, the equation of motion of the dipole is $\mathbf{K} = I\ddot{\omega}_0 = \mathbf{d} \times \mathbf{E}$. If $|\omega_0| \ll \omega$ then $\ddot{\mathbf{d}} = I^{-1}(\mathbf{d} \times \mathbf{E}) \times \mathbf{d}$, which leads to $d\sigma$ upon using (4.2.208) and (4.2.219). If an incident wave is unpolarized then averaging (4.2.271) over all the directions perpendicular to $\mathbf{n}$ and using $\overline{E_\alpha E_\beta} = \frac{1}{2} \mathbf{E}^2 (\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2})$ gives

$$d\sigma = \frac{1}{2} r_0^2 (1 + \cos^2 \vartheta) d\omega, \quad (4.2.277)$$

where ϑ is the angle between $\mathbf{n}$ and $\mathbf{k}$.

4.2.10 Light

The wave vector

$$k^i = \frac{dx^i}{d\lambda}, \quad (4.2.278)$$

where λ is the affine parameter. The wave vector satisfies

$$\frac{dk^i}{d\lambda} + \Gamma_{jl}^i k^j k^l = 0 \quad (4.2.279)$$

and

$$k^i k_i = 0. \quad (4.2.280)$$

It also satisfies

$$k_i = \frac{\partial\psi}{\partial x^i}, \quad (4.2.281)$$

where ψ is the eikonal. Thus (4.2.280) gives

$$g^{ik} \frac{\partial\psi}{\partial x^i} \frac{\partial\psi}{\partial x^k} = 0. \quad (4.2.282)$$

The Fermat principle is

$$\delta \int k_\alpha dx^\alpha = 0. \quad (4.2.283)$$

4.3 Gravity

4.3.1 Constant gravitational field

A gravitational field is *constant* if the metric and torsion tensors do not depend on the time coordinate x^0 . In this case, the time coordinate is called the *world time*. The constancy of the metric tensor $g_{ik} = g_{ik}(x^\alpha)$ is invariant under transformations

$$x^0 \rightarrow x^0 + f(x^\alpha), \quad (4.3.1)$$

where f is an arbitrary function of the spatial coordinates. The three-dimensional scalar g_{00} , three-dimensional tensor $\gamma_{\alpha\beta}$ (1.4.98) and three-dimensional curl of g_α (1.4.100),

$$f_{\alpha\beta} = g_{\beta,\alpha} - g_{\alpha,\beta}, \quad (4.3.2)$$

are invariant under (4.3.1). The spatial metric tensor $\gamma_{\alpha\beta}$ is used to lower contravariant spatial indices. The tensor $\gamma^{\alpha\beta}$, which is inverse to $\gamma_{\alpha\beta}$, is used to raise covariant spatial indices. If the source of a field is at rest, both directions of time, t and $-t$, are equivalent. Such a field is referred to as *static* and the corresponding metric tensor has

$$g_{0\alpha} = 0. \quad (4.3.3)$$

If $g_{0\alpha} \neq 0$ then t and $-t$ are not equivalent and the field is referred to as *stationary*. The proper time is given by

$$\tau = \frac{1}{c} \sqrt{g_{00}} x^0. \quad (4.3.4)$$

For a weak field,

$$\tau = \frac{x^0}{c} \left(1 + \frac{\phi}{c^2} \right). \quad (4.3.5)$$

In a stationary gravitational field, the interval is

$$ds^2 = g_{00}(dx^0 - g_\alpha dx^\alpha)^2 - dl^2, \quad (4.3.6)$$

where dl^2 is given by (1.4.97). The three-dimensional velocity of a particle,

$$v^\alpha = \frac{dx^\alpha}{d\tau}, \quad (4.3.7)$$

is defined in terms of the synchronized proper time (corresponding to the difference in x^0 between two synchronized infinitesimally separated points (1.4.101)):

$$d\tau = \frac{1}{c}\sqrt{g_{00}}(dx^0 - \delta x^0) = \frac{1}{c}\sqrt{g_{00}}(dx^0 - g_{\alpha}dx^\alpha). \quad (4.3.8)$$

Thus the interval (4.3.6) becomes

$$ds^2 = g_{00}(dx^0 - g_{\alpha}dx^\alpha)^2 \left(1 - \frac{v^2}{c^2}\right), \quad (4.3.9)$$

where

$$v^2 = \gamma_{\alpha\beta}v^\alpha v^\beta. \quad (4.3.10)$$

Using (4.3.6) in the definition of the four-velocity gives

$$u^\alpha = \frac{v^\alpha}{c\sqrt{1 - v^2/c^2}}, \quad u^0 = \frac{1}{\sqrt{g_{00}}\sqrt{1 - v^2/c^2}} + g_{\alpha}u^\alpha, \quad (4.3.11)$$

from which we find

$$u_0 = \frac{\sqrt{g_{00}}}{\sqrt{1 - v^2/c^2}}. \quad (4.3.12)$$

Thus a particle in a stationary gravitational field has a constant energy:

$$E_0 = -c \frac{\partial S}{\partial x^0} = mc^2 u_0 = \frac{mc^2 \sqrt{g_{00}}}{\sqrt{1 - v^2/c^2}} = \text{const}, \quad (4.3.13)$$

where S is the principal function. Since a static field is the special case of a stationary field, (4.3.13) is also the energy of a particle in a static gravitational field. In a weak field, (4.3.13) reduces to

$$E_0 = mc^2 + \frac{mv^2}{2} + m\phi. \quad (4.3.14)$$

The force acting on a particle in a constant gravitational field is equal to

$$f^\alpha = \frac{\nabla p^\alpha}{dt} = c \left(1 - \frac{v^2}{c^2}\right) \frac{\nabla p^\alpha}{ds} = c \left(1 - \frac{v^2}{c^2}\right) u^\beta p^\alpha{}_{;\beta}, \quad (4.3.15)$$

where ∇ is the three-dimensional Riemannian covariant differential and colon denotes the three-dimensional Riemannian covariant derivative, both defined with respect to the three-dimensional Christoffel symbols (1.4.26) constructed from the spatial metric tensor $\gamma_{\alpha\beta}$ (1.4.98) instead of g_{ik} ,

$$\{\delta_{\alpha\beta}\} = \frac{1}{2}\gamma^{\delta\zeta}(\gamma_{\zeta\alpha,\beta} + \gamma_{\zeta\beta,\alpha} - \gamma_{\alpha\beta,\zeta}). \quad (4.3.16)$$

Using the metric geodesic equation (1.4.80) valid for a particle in the gravitational field turns (4.3.15) into

$$f^\alpha = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \left(-(\ln\sqrt{g_{00}})_\alpha + \sqrt{g_{00}}(g_{\beta,\alpha} - g_{\alpha,\beta}) \frac{v^\beta}{c} \right). \quad (4.3.17)$$

In the spatial-vector notation, this equation is

$$\mathbf{f} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \left(-\nabla \ln\sqrt{g_{00}} + \sqrt{g_{00}} \frac{\mathbf{v}}{c} \times (\nabla \times \mathbf{g}) \right). \quad (4.3.18)$$

If a particle is at rest, $\mathbf{v} = 0$, then the force (4.3.18) has a potential:

$$\mathbf{f} = -\nabla(mc^2 \ln \sqrt{g_{00}}). \quad (4.3.19)$$

If $v \ll c$ then the second term on the right of (4.3.18) has the form of the Coriolis force (3.2.51) with the angular velocity

$$\boldsymbol{\omega} = -\frac{c}{2} \sqrt{g_{00}} \nabla \times \mathbf{g}. \quad (4.3.20)$$

For a ray of light in a stationary gravitational field, its frequency, measured in terms of the world time, is constant:

$$\omega_0 = -c \frac{\partial \psi}{\partial x^0} = \text{const}, \quad (4.3.21)$$

where ψ is the eikonal. If the frequency is measured in terms of the proper time then

$$\omega = -\frac{\partial \psi}{\partial \tau} = -\frac{\partial \psi}{\partial x^0} \frac{c}{\sqrt{g_{00}}} = \frac{\omega_0}{\sqrt{g_{00}}}. \quad (4.3.22)$$

For a weak field, (4.3.22) reduces to

$$\omega = \omega_0 \left(1 - \frac{\phi}{c^2}\right). \quad (4.3.23)$$

This decrease of the frequency of light in a gravitational field is referred to as the *redshift*.

The null condition for the wave vector, $k_i k^i = 0$, can be written, analogously to (4.3.6), as

$$g_{00}(k^0 - g_\alpha k^\alpha)^2 - \gamma_{\alpha\beta} k^\alpha k^\beta = 0. \quad (4.3.24)$$

The constant frequency satisfies

$$\frac{\omega_0}{c} = k_0 = g_{00}(k^0 - g_\alpha k^\alpha), \quad (4.3.25)$$

so (4.3.24) gives

$$k^\alpha = \frac{\omega_0}{c\sqrt{g_{00}}} \frac{dx^\alpha}{dl}. \quad (4.3.26)$$

The Fermat principle (4.2.283) reads

$$\delta \int \left(\frac{dl}{\sqrt{g_{00}}} + g_\alpha dx^\alpha \right) = 0. \quad (4.3.27)$$

In a constant gravitational field, g_α appears in the Einstein field equations only in the tensor $f_{\alpha\beta}$ which is invariant under (4.3.1). The terms containing g_α or $g_{(\alpha;\beta)}$, which are not invariant under (4.3.1), vanish. The component R_0^0 of the Riemannian Ricci tensor R_{ik} can be written, using (1.4.32) and (1.4.35), as

$$\begin{aligned} R_0^0 &= g^{i0} R_{i0} = g^{i0} (\{ \overset{j}{i} \}_0, \overset{j}{j} - \{ \overset{j}{i} \}_j, \overset{j}{j} + \{ \overset{k}{i} \}_0 \{ \overset{j}{k} \}_j - \{ \overset{k}{i} \}_j \{ \overset{j}{k} \}_0) = \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{i0} \{ \overset{j}{i} \}_0), \overset{j}{j} \\ &- \frac{1}{\sqrt{-g}} (\sqrt{-g}), \overset{j}{j} g^{i0} \{ \overset{j}{i} \}_0 - g^{i0}, \overset{j}{j} \{ \overset{j}{i} \}_0 + g^{i0} (\{ \overset{k}{i} \}_0 \{ \overset{j}{k} \}_j - \{ \overset{k}{i} \}_j \{ \overset{j}{k} \}_0) \\ &= \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{i0} \{ \overset{j}{i} \}_0), \overset{j}{j} - \{ \overset{k}{k} \}_j g^{i0} \{ \overset{j}{i} \}_0 - (g^{i0}, \overset{j}{j} - \{ \overset{i}{k} \}_j g^{k0} - \{ \overset{0}{k} \}_j g^{ik}) \{ \overset{j}{i} \}_0 \\ &+ g^{i0} (\{ \overset{k}{i} \}_0 \{ \overset{j}{k} \}_j - \{ \overset{k}{i} \}_j \{ \overset{j}{k} \}_0) = \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{i0} \{ \overset{\alpha}{i} \}_0), \alpha + g^{ik} \{ \overset{0}{k} \}_j \{ \overset{j}{i} \}_0 \\ &= \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{i0} \{ \overset{\alpha}{i} \}_0), \alpha + \frac{1}{2} g^{ik} \{ \overset{0}{k} \}_j g^{jl} (g_{li,0} + g_{l0,i} - g_{i0,l}) \\ &= \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{i0} \{ \overset{\alpha}{i} \}_0), \alpha + g^{k(i} g^{l)j} \{ \overset{0}{k} \}_j g_{0[l,i]} = \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{i0} \{ \overset{\alpha}{i} \}_0), \alpha. \end{aligned} \quad (4.3.28)$$

Therefore, we have

$$\int R_0^0 \sqrt{-\mathbf{g}} dV = \oint \sqrt{-\mathbf{g}} g^{i0} \{i_0^\alpha\} df_\alpha. \quad (4.3.29)$$

We define

$$h = g_{00}. \quad (4.3.30)$$

If the metric tensor satisfies $g_{0\alpha} = 0$ then $g^{0\alpha} = 0$ and $g^{00} = 1/g_{00}$. In this case, R_0^0 in (4.3.28) reduces, using (1.4.106), (1.4.107) and (1.4.128), to the *Levi-Civita identity*:

$$\begin{aligned} R_0^0 &= \frac{1}{\sqrt{h\mathfrak{s}}} (\sqrt{h\mathfrak{s}} g^{00} \{0_0^\alpha\})_{,\alpha} = \frac{1}{2\sqrt{h\mathfrak{s}}} \left(\sqrt{h\mathfrak{s}} h^{-1} g^{\alpha\beta} (2g_{\beta 0,0} - g_{00,\beta}) \right)_{,\alpha} \\ &= \frac{1}{\sqrt{h\mathfrak{s}}} (\sqrt{\mathfrak{s}} \gamma^{\alpha\beta} \sqrt{h}_{,\beta})_{,\alpha} = \frac{1}{\sqrt{h}} \Delta \sqrt{h}. \end{aligned} \quad (4.3.31)$$

If the torsion tensor vanishes then the Einstein equations for a constant field are

$$\begin{aligned} \frac{1}{h} R_{00} &= \frac{1}{\sqrt{h}} (\sqrt{h})_{;\alpha}{}^\alpha + \frac{h}{4} f_{\alpha\beta} f^{\alpha\beta} = \kappa \left(\frac{\epsilon + p}{1 - v^2/c^2} - \frac{\epsilon - p}{2} \right), \\ \frac{1}{\sqrt{h}} R_0^\alpha &= -\frac{\sqrt{h}}{2} f^{\alpha\beta}{}_{;\beta} - \frac{3}{2} f^{\alpha\beta} (\sqrt{h})_{;\beta} = \kappa \frac{\epsilon + p}{1 - v^2/c^2} \frac{v^\alpha}{c}, \\ R^{\alpha\beta} &= P^{\alpha\beta} + \frac{h}{2} f^{\alpha\gamma} f^\beta{}_\gamma - \frac{1}{\sqrt{h}} (\sqrt{h})_{;\alpha\beta} = \kappa \left(\frac{\epsilon + p}{1 - v^2/c^2} \frac{v^\alpha v^\beta}{c^2} \right. \\ &\quad \left. + \frac{\epsilon - p}{2} \gamma^{\alpha\beta} \right), \end{aligned} \quad (4.3.32)$$

where $P_{\alpha\beta}$ is the three-dimensional Ricci tensor constructed from the spatial metric tensor $\gamma_{\alpha\beta}$ instead of g_{ik} .

4.3.2 Synchronous system of reference

A system of coordinates in which

$$g_{00} = 1, \quad g_{0\alpha} = 0 \quad (4.3.33)$$

is referred to as a *synchronous system of reference*. In such a system, we also have

$$g_\alpha = 0, \quad \gamma_{\alpha\beta} = -g_{\alpha\beta}, \quad \mathbf{g} = -\mathfrak{s}, \quad g^{00} = 1, \quad g^{0\alpha} = 0, \quad (4.3.34)$$

and the interval reduces to

$$ds^2 = c^2 dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta. \quad (4.3.35)$$

In a synchronous system of reference, the difference in x^0 between two synchronized infinitesimally separated points (1.4.101) vanishes. For a given metric g_{ik} , there exists an infinite number of coordinate transformations leading to synchronous systems of reference. An infinitesimal transformation from one synchronous system to another is given by

$$ct \rightarrow ct + \phi(x^\alpha), \quad x^\alpha \rightarrow x^\alpha + \xi^\alpha(x^i), \quad (4.3.36)$$

where ϕ and ξ are infinitesimal functions. Such a transformation does not change $g_{00} = 1$, whereas the condition $g_{0\alpha} = 0$ yields the relation between ξ and ϕ :

$$\xi^\alpha = c\phi_{,\beta} \int \gamma^{\alpha\beta} dt + f^\alpha(x^\beta), \quad (4.3.37)$$

where f^α are arbitrary infinitesimal functions. The corresponding change in $\gamma_{\alpha\beta}$ is given by

$$\gamma_{\alpha\beta} \rightarrow \gamma_{\alpha\beta} - \xi_{\alpha;\beta} - \xi_{\beta;\alpha} - \phi \kappa_{\alpha\beta}. \quad (4.3.38)$$

In a synchronous system of reference, time lines $u^0 = 1$, $u^\alpha = 0$ are geodesic, that is, they identically satisfy the metric geodesic equation (1.4.80). They are also normal to the hypersurface $t = \text{const}$: $n_i = (ct)_{,i}$ gives the normalized normal vector $n_0 = 1$, $n_\alpha = 0$, which then gives $n_0 = 1$, $n^\alpha = 0$, leading to

$$n^i \propto u^i. \quad (4.3.39)$$

A synchronous system of reference for a particle in a gravitational field can be constructed from the Hamilton-Jacobi equation (3.5.1) for the principal function S . A complete integral of this equation depends on four parameters, out of which one is an additive function of the other three: $S = f(x^i, \xi^\alpha) + A(\xi^\alpha)$. The world trajectory, which is a metric geodesic, is given by

$$\frac{\partial S}{\partial x^\alpha} = 0. \quad (4.3.40)$$

Choosing

$$x^{i'} = \left(-\frac{S}{mc}, \xi^\alpha \right) \quad (4.3.41)$$

as the new coordinates gives, using (3.5.1),

$$g^{0'0'} = g^{ik} \frac{\partial x^{0'}}{\partial x^i} \frac{\partial x^{0'}}{\partial x^k} = g^{ik} \frac{\partial(-S/mc)}{\partial x^i} \frac{\partial(-S/mc)}{\partial x^k} = 1. \quad (4.3.42)$$

The four-velocity of the particle in the new coordinates is, using $mcu_i = p_i = -\frac{\partial S}{\partial x^i}$, (4.3.40) and (4.3.41),

$$u_{i'} = -\frac{\partial(S/mc)}{\partial x^{i'}} = (1, \mathbf{0}). \quad (4.3.43)$$

The new four-velocity $u^{i'}$ satisfies the metric geodesic equation (1.4.80) in the new coordinates. The hypersurface of constant time in the new coordinates is $-S/mc = \text{const}$. The corresponding normal vector is, using (4.3.40),

$$n_{i'} = \frac{\partial(-S/mc)}{\partial x^{i'}} = (-1, \mathbf{0}), \quad (4.3.44)$$

which is proportional to $u_{i'}$. Thus the primed coordinate system of reference (4.3.41) for this particle is synchronous:

$$g_{0'\alpha'} = 0. \quad (4.3.45)$$

We define

$$\kappa_{\alpha\beta} = \gamma_{\alpha\beta,0}, \quad (4.3.46)$$

which gives

$$\kappa_\alpha^\alpha = \kappa_{\alpha\beta} \gamma^{\alpha\beta} = (\ln s)_{,0}. \quad (4.3.47)$$

The components of the Riemann tensor are

$$\begin{aligned} R_{0\alpha 0\beta} &= \frac{1}{2} \kappa_{\alpha\beta,0} - \frac{1}{4} \kappa_{\alpha\gamma} \kappa_{\beta}^\gamma, \\ R_{0\alpha\beta\gamma} &= \frac{1}{2} (\kappa_{\alpha\gamma;\beta} - \kappa_{\alpha\beta;\gamma}), \\ R_{\alpha\beta\gamma\delta} &= -P_{\alpha\beta\gamma\delta} + \frac{1}{4} (\kappa_{\alpha\delta} \kappa_{\beta\gamma} - \kappa_{\alpha\gamma} \kappa_{\beta\delta}), \end{aligned} \quad (4.3.48)$$

where colon denotes the three-dimensional Riemannian covariant derivative with respect to the three-dimensional Christoffel symbols (4.3.16) and $P_{\alpha\beta\gamma\delta}$ is the three-dimensional Riemann tensor constructed from the spatial metric tensor $\gamma_{\alpha\beta}$ instead of g_{ik} . The components of the corresponding Ricci tensor are

$$\begin{aligned} R_0^0 &= -\frac{1}{2} \kappa_{\alpha,0}^\alpha - \frac{1}{4} \kappa_\alpha^\beta \kappa_\beta^\alpha, \\ R_\alpha^0 &= \frac{1}{2} (\kappa_{\alpha;\beta}^\beta - \kappa_{\beta;\alpha}^\beta), \\ R_\alpha^\beta &= -P_\alpha^\beta - \frac{1}{2\sqrt{s}} (\sqrt{s} \kappa_\alpha^\beta)_{,0}, \end{aligned} \quad (4.3.49)$$

where $P_{\alpha\beta}$ is the three-dimensional Riemannian Ricci tensor constructed from the spatial metric tensor $\gamma_{\alpha\beta}$ instead of g_{ik} . For a perfect fluid, the Einstein equations for R_0^0 give

$$R_0^0 = \kappa \left(T_0^0 - \frac{1}{2} T \right) = \kappa \left(\frac{\epsilon + 3p}{2} + \frac{\epsilon + p}{1 - v^2/c^2} \frac{v^2}{c^2} \right). \quad (4.3.50)$$

If $\kappa_{\alpha\beta} = 0$ then the first equation in (4.3.49) and the Einstein equations require vanishing of the energy-momentum tensor T_{ik} . In this case, the third equation in (4.3.49) leads to $P_{\alpha\beta} = 0$, which then gives $P_{\alpha\beta\gamma\delta} = 0$ because of (1.4.74). However, the condition $P_{\alpha\beta\gamma\delta} = 0$ means that the space is Euclidean: the gravitational field vanishes. Therefore a synchronous system of reference in the presence of a gravitational field is nonstationary.

For a perfect fluid, if $p \neq 0$ then the motion is nongeodesic (cf. (2.5.93)). Therefore in a synchronous system of reference, the worldlines of particles are not time lines, so matter is not at rest. In such a coordinate system one cannot construct a comoving frame of reference. If $p = 0$ then a synchronous and comoving frame of reference yields $u_0 = 1$, $u_\alpha = 0$, which then gives

$$u_{\alpha;\beta} - u_{\beta;\alpha} = 0. \quad (4.3.51)$$

This equation is equivalent to

$$\text{curl} \mathbf{v} = 0. \quad (4.3.52)$$

Thus a synchronous and comoving frame of reference can be constructed only if the motion of dust ($p = 0$) is irrotational.

For a perfect fluid, the first equation in (4.3.49) and (4.3.50) give

$$\frac{1}{2} \kappa_{\alpha,0}^\alpha + \frac{1}{4} \kappa_\alpha^\beta \kappa_\beta^\alpha \leq 0. \quad (4.3.53)$$

The algebraic inequality

$$\kappa_\beta^\alpha \kappa_\alpha^\beta \geq \frac{1}{3} (\kappa_\alpha^\alpha)^2 \quad (4.3.54)$$

leads then to the *Landau inequality*,

$$\left((\kappa_\alpha^\alpha)^{-1} \right)_{,0} \geq \frac{1}{6}. \quad (4.3.55)$$

This inequality, together with (4.3.47), means that the determinant of the spatial metric tensor, $\mathfrak{s}$, becomes zero in a finite time. The determinant of the metric tensor, $\mathfrak{g}$, becomes zero in a finite time as well. Thus physical synchronous systems of reference cannot be constructed for all values of the time coordinate t .

4.3.3 Nonrelativistic gravity

The Poisson equation for a nonrelativistic gravitational field (2.5.88) has the general solution

$$\phi = -G \int \frac{\mu}{R} dV, \quad (4.3.56)$$

where $R = |\mathbf{R}|$ and $\mathbf{R}$ is the vector from the element of volume dV to the point at which ϕ is calculated. If the source of a gravitational field is a system of static point masses then (4.3.56) becomes

$$\phi = -G \sum_a \frac{m_a}{R_a}, \quad (4.3.57)$$

where $\mathbf{R}_a$ is the vector from a mass m_a to the point at which ϕ is calculated. Thus the gravitational acceleration in (2.5.84) is

$$\mathbf{g} = -G \sum_a \frac{m_a \mathbf{R}_a}{R_a^3}, \quad (4.3.58)$$

representing the principle of superposition for the gravitational field. For one mass m we have

$$\phi = -\frac{Gm}{R}, \quad \mathbf{g} = -\frac{Gm\mathbf{R}}{R^3}, \quad (4.3.59)$$

so the force $\mathbf{f} = m\mathbf{g}$ exerted on a nonrelativistic mass m_a from a nonrelativistic mass m_b gives *Newton's law of universal gravitation*:

$$\mathbf{f}_{ab} = -\frac{Gm_a m_b \mathbf{R}_{ab}}{R_{ab}^3}, \quad (4.3.60)$$

where

$$\mathbf{R}_{ab} = \mathbf{R}_a - \mathbf{R}_b. \quad (4.3.61)$$

The gravitational potential (4.3.59) is negative. Thus the force of gravitation (4.3.60) is attractive. According to Newton's third law, the force $\mathbf{f}_{ba}$ exerted on a nonrelativistic mass m_b from a nonrelativistic mass m_a is equal in magnitude and opposite in direction to $\mathbf{f}_{ab}$:

$$\mathbf{f}_{ba} = -\mathbf{f}_{ab}. \quad (4.3.62)$$

The action for the gravitational field corresponding to G (2.5.85) is

$$S_g = -\frac{1}{8\pi G} \int \int (\nabla\phi)^2 dV dt, \quad (4.3.63)$$

which gives the total action for the nonrelativistic gravitational field and matter:

$$S = \int \int \left(\frac{\mu v^2}{2} - \mu\phi - \frac{1}{8\pi G} (\nabla\phi)^2 \right) dV dt. \quad (4.3.64)$$

Varying the action (4.3.64) with respect to the gravitational potential ϕ gives, for $\delta\phi$ vanishing at the surface embracing the volume of integration,

$$\delta S = \int \int \left(-\mu\delta\phi + \frac{1}{4\pi G} \Delta\phi\delta\phi \right) dV dt. \quad (4.3.65)$$

The stationarity of this action, $\delta S = 0$, under $\delta\phi$ gives the Poisson equation (2.5.85).

The integrand in (4.3.64) is the nonrelativistic Lagrangian density for the gravitational field and matter. The corresponding Hamiltonian density is given by changing the signs of the terms in (4.3.64) containing ϕ (corresponding to the potential energy in nonrelativistic mechanics). Thus the energy for the gravitational field and matter is

$$\mathbf{E} = \int \left(\frac{\mu v^2}{2} + \mu\phi + \frac{1}{8\pi G} (\nabla\phi)^2 \right) dV. \quad (4.3.66)$$

Using $\int (\nabla\phi)^2 dV = -\int \phi \Delta\phi dV + \oint \phi \nabla\phi \cdot d\mathbf{S}$ and that the surface integral vanishes at infinity (since ϕ depends on R according to (4.3.59)), gives, together with (2.5.85),

$$\mathbf{E} = \int \left(\frac{\mu v^2}{2} - \frac{1}{8\pi G} (\nabla\phi)^2 \right) dV. \quad (4.3.67)$$

Therefore the energy of the nonrelativistic gravitational field is given by

$$U = -\frac{1}{8\pi G} \int (\nabla\phi)^2 dV = -\frac{1}{8\pi G} \int (\mathbf{g})^2 dV. \quad (4.3.68)$$

This energy is not equal to the $(-\mathbf{g})_{t_{00}}$ component of the Landau-Lifshitz energy-momentum complex, where t^{ik} is the Landau-Lifshitz energy-momentum pseudotensor (2.5.30); $(-\mathbf{g})_{T_{ik}}$ also contributes to U . Analogously to (4.2.11), we have

$$U = \frac{1}{8\pi G} \int \mathbf{g} \cdot \nabla\phi dV = \frac{1}{8\pi G} \int \nabla \cdot (\mathbf{g}\phi) dV - \frac{1}{8\pi G} \int \phi \nabla \cdot \mathbf{g} dV. \quad (4.3.69)$$

The first term on the right of (4.3.69) can be transformed into a surface integral using the Gauß' theorem. This integral vanishes if the surface of integration is taken at infinity, so (2.5.85) and (4.3.69) give

$$U = \frac{1}{2} \int \mu \phi dV = \frac{1}{2} \sum_a m_a \phi_a, \quad (4.3.70)$$

where ϕ_a is the gravitational potential at the point occupied by a mass m_a . In calculating ϕ_a , we must not include the gravitational field produced by m_a because the corresponding contribution to the gravitational energy (a self-energy), is infinite. Subtracting this self-energy from U (renormalizing the masses of the particles) gives

$$U = \frac{1}{2} \sum_a m_a \phi'_a, \quad (4.3.71)$$

where

$$\phi'_a = -G \sum_{b \neq a} \frac{m_b}{R_{ab}}. \quad (4.3.72)$$

We consider a system of static point masses m_a located at $\mathbf{r}_a$ and take the origin of the Cartesian system of coordinates anywhere within the system. The gravitational potential at a point with the radius vector $\mathbf{R}$ is thus

$$\phi = -G \sum_a \frac{m_a}{|\mathbf{R} - \mathbf{r}_a|}. \quad (4.3.73)$$

If $R \gg r_a$ then the gravitational potential (4.3.73) can be expanded in a Taylor series with respect to the small quantity $\frac{\mathbf{r}_a}{R}$:

$$\begin{aligned} \phi &\approx \phi_0 + \phi_1 + \phi_2 = -\frac{G}{R} \sum_a m_a + G \sum_a m_a \mathbf{r}_a \cdot \nabla \frac{1}{R} - \frac{G}{2} \sum_a m_a x_\alpha x_\beta \frac{\partial^2}{\partial X_\alpha \partial X_\beta} \frac{1}{R} \Big|_a \\ &= -\frac{GM}{R} + GM \boldsymbol{\rho} \cdot \nabla \frac{1}{R} - \frac{G}{6} D_{\alpha\beta} \frac{\partial^2}{\partial X_\alpha \partial X_\beta} \frac{1}{R} \Big|_a = -\frac{GM}{R} - \frac{GM \boldsymbol{\rho} \cdot \mathbf{n}}{R^2} \\ &\quad - \frac{G}{2R^3} D_{\alpha\beta} n_\alpha n_\beta, \end{aligned} \quad (4.3.74)$$

where M is the total mass of the system (3.1.137), $\boldsymbol{\rho}$ denotes the radius vector of its center of mass (3.1.138), x_α are the components of $\mathbf{r}$, X_α are the components of $\mathbf{R}$, $\mathbf{n}$ is the unit vector in the direction of $\mathbf{R}$, and the symmetric spatial tensor

$$D_{\alpha\beta} = \sum_a m_a (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta})|_a = D_{\beta\alpha} \quad (4.3.75)$$

is the tensor of *mass quadrupole moment* of the system. This tensor can be written as

$$D_{\alpha\beta} = \int \mu (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) dV, \quad (4.3.76)$$

where μ is the mass density, $M = \int \mu dV$. The tensor of mass quadrupole moment is traceless,

$$D_{\alpha\alpha} = 0, \quad (4.3.77)$$

so it has 5 independent components. It is related to the inertia tensor (3.2.13) by

$$D_{\alpha\beta} = I_{\gamma\gamma} \delta_{\alpha\beta} - 3I_{\alpha\beta}. \quad (4.3.78)$$

If we choose the origin of the frame of reference at the center of mass of the system then $\boldsymbol{\rho} = 0$ and (4.3.74) reduces to

$$\phi = -\frac{GM}{R} - \frac{G}{2R^3} D_{\alpha\beta} n_\alpha n_\beta. \quad (4.3.79)$$

This reduction is possible because the masses of the particles are of the same (positive) sign. The electric charges of the particles can have different signs. In the analogous expression for the electric potential (4.2.16), we can eliminate the electric-dipole term ϕ_1 , which corresponds to the center-of-mass term ϕ_1 in (4.3.74), only if the total electric charge of the system is different from zero.

4.3.4 Schwarzschild metric

A *centrally* or *spherically symmetric* gravitational field is the field in which the generators of rotations (1.6.58) are also generators of isometries (transformations that do not change the metric tensor). The generators (1.6.58) in the Cartesian coordinates are

$$J_x = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \quad J_y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad J_z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}. \quad (4.3.80)$$

In the spherical coordinates (r, θ, ϕ) , they are

$$\begin{aligned} J_x &= -\sin\phi \frac{\partial}{\partial\theta} - \cot\theta \cos\phi \frac{\partial}{\partial\phi}, \\ J_y &= \cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi}, \\ J_z &= \frac{\partial}{\partial\phi}. \end{aligned} \quad (4.3.81)$$

We can write them as

$$J_\alpha = \xi_\alpha^i \partial_i, \quad (4.3.82)$$

where

$$\begin{aligned} \xi_x^0 &= \xi_y^0 = \xi_z^0 = 0, \quad \xi_x^r = \xi_y^r = \xi_z^r = 0, \\ \xi_x^\theta &= -\sin\phi, \quad \xi_x^\phi = -\cot\theta \cos\phi, \\ \xi_y^\theta &= \cos\phi, \quad \xi_y^\phi = -\cot\theta \sin\phi, \\ \xi_z^\theta &= 0, \quad \xi_z^\phi = 1. \end{aligned} \quad (4.3.83)$$

They are generators of isometries if ξ_α^i are Killing vectors. For the Levi-Civita connection, (1.4.42) gives

$$g_{ik} \xi_{\alpha,j}^k + g_{jk} \xi_{\alpha,i}^k + g_{ij,k} \xi_\alpha^k = 0. \quad (4.3.84)$$

For $\alpha = z$, (4.3.84) gives

$$g_{ij,\phi} = 0, \quad (4.3.85)$$

so the metric tensor g_{ij} is independent of ϕ . For $\alpha = x, y$, we consider different components g_{ij} . If $i, j \in \{0, r\}$ then (4.3.84) gives

$$\begin{aligned} \alpha = x : & -g_{ij,\theta} \sin\phi - g_{ij,\phi} \cot\theta \cos\phi = 0, \\ \alpha = y : & g_{ij,\theta} \cos\phi - g_{ij,\phi} \cot\theta \sin\phi = 0, \end{aligned} \quad (4.3.86)$$

which, using (4.3.85), yields

$$g_{ij,\theta} = 0. \quad (4.3.87)$$

Thus

$$g_{00} = g_{00}(t, r), \quad g_{0r} = g_{0r}(t, r), \quad g_{rr} = g_{rr}(t, r). \quad (4.3.88)$$

If $i \in \{0, r\}$ and $j = \theta$ then (4.3.84) gives

$$\begin{aligned} \alpha = x : & g_{i\phi} \frac{\cos\phi}{\sin^2\theta} - g_{i\theta,\theta} \sin\phi - g_{i\theta,\phi} \cot\theta \cos\phi = 0, \\ \alpha = y : & g_{i\phi} \frac{\sin\phi}{\sin^2\theta} + g_{i\theta,\theta} \cos\phi - g_{i\theta,\phi} \cot\theta \sin\phi = 0, \end{aligned} \quad (4.3.89)$$

which, using (4.3.85), yields

$$g_{i\phi} = 0, \quad g_{i\theta,\theta} = 0. \quad (4.3.90)$$

If $i \in \{0, r\}$ and $j = \phi$ then (4.3.84) gives

$$\begin{aligned}\alpha = x : & -g_{i\theta}\cos\phi + g_{i\phi}\cot\theta\sin\phi - g_{i\phi,\theta}\sin\phi - g_{i\phi,\phi}\cot\theta\cos\phi = 0, \\ \alpha = y : & -g_{i\theta}\sin\phi - g_{i\phi}\cot\theta\cos\phi + g_{i\phi,\theta}\cos\phi - g_{i\phi,\phi}\cot\theta\sin\phi = 0,\end{aligned}\quad (4.3.91)$$

which, using (4.3.85) and (4.3.90), yields

$$g_{i\theta} = 0. \quad (4.3.92)$$

Thus

$$g_{0\theta} = g_{0\phi} = g_{r\theta} = g_{r\phi} = 0. \quad (4.3.93)$$

If $i = \theta$ and $j = \theta$ then (4.3.84) gives

$$\begin{aligned}\alpha = x : & 2g_{\theta\phi}\frac{\cos\phi}{\sin^2\theta}\cos\phi - g_{\theta\theta,\theta}\sin\phi = 0, \\ \alpha = y : & 2g_{\theta\phi}\frac{\sin\phi}{\sin^2\theta}\cos\phi + g_{\theta\theta,\theta}\cos\phi = 0,\end{aligned}\quad (4.3.94)$$

which yields

$$g_{\theta\phi} = 0, \quad g_{\theta\theta,\theta} = 0. \quad (4.3.95)$$

If $i = \theta$ and $j = \phi$ then (4.3.84) gives

$$\begin{aligned}\alpha = x : & -g_{\theta\theta}\cos\phi + g_{\theta\phi}\cot\theta\sin\phi + g_{\phi\phi}\frac{\cos\phi}{\sin^2\theta} - g_{\theta\phi,\theta}\sin\phi = 0, \\ \alpha = y : & -g_{\theta\theta}\sin\phi - g_{\theta\phi}\cot\theta\cos\phi + g_{\phi\phi}\frac{\sin\phi}{\sin^2\theta} + g_{\theta\phi,\theta}\cos\phi = 0,\end{aligned}\quad (4.3.96)$$

which, using (4.3.95), yields

$$g_{\phi\phi} = \sin^2\theta \, g_{\theta\theta}. \quad (4.3.97)$$

If $i = \phi$ and $j = \phi$ then (4.3.84) gives

$$\begin{aligned}\alpha = x : & 2g_{\phi\phi}\cot\theta\sin\phi - 2g_{\theta\phi}\cos\phi - g_{\phi\phi,\theta}\sin\phi = 0, \\ \alpha = y : & -2g_{\phi\phi}\cot\theta\cos\phi - 2g_{\theta\phi}\sin\phi + g_{\phi\phi,\theta}\cos\phi = 0,\end{aligned}\quad (4.3.98)$$

which, using (4.3.95) and (4.3.97), leads to the identity. Thus

$$g_{\theta\theta} = g_{\theta\theta}(t, r), \quad g_{\theta\phi} = 0, \quad g_{\phi\phi} = \sin^2\theta \, g_{\theta\theta}. \quad (4.3.99)$$

Combining (4.3.88), (4.3.93) and (4.3.99) gives the general form of the metric tensor in a spherically symmetric gravitational field ($r = 1, \theta = 2, \phi = 3$):

$$g_{ij} = \begin{pmatrix} g_{00}(t, r) & g_{01}(t, r) & 0 & 0 \\ g_{01}(t, r) & g_{11}(t, r) & 0 & 0 \\ 0 & 0 & g_{22}(t, r) & 0 \\ 0 & 0 & 0 & \sin^2\theta \, g_{22}(t, r) \end{pmatrix} \quad (4.3.100)$$

or

$$ds^2 = g_{00}(t, r)c^2dt^2 + 2g_{01}(t, r)c dt \, dr + g_{11}(t, r)dr^2 + g_{22}(t, r)(d\theta^2 + \sin^2\theta \, d\phi^2). \quad (4.3.101)$$

This form can also be obtained from general considerations. We can apply a coordinate transformation from t, r, θ, ϕ to $t'(t, r), r'(t, r), \theta, \phi$ such that

$$g_{0'1'} = 0, \quad g_{22} = g_{22}(r'). \quad (4.3.102)$$

Accordingly, the interval (4.3.101) becomes (omitting the primes)

$$ds^2 = g_{00}(t, r)c^2dt^2 + g_{11}(t, r)dr^2 + g_{22}(r)(d\theta^2 + \sin^2\theta \, d\phi^2). \quad (4.3.103)$$

In the locally Galilean frame of reference, (4.3.103) should coincide with the Minkowski metric tensor (1.4.84) in the spherical coordinates,

$$ds^2 = c^2 dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (4.3.104)$$

which implies that $g_{00}(t, r) > 0$, $g_{11} < 0$, $g_{22} < 0$. Thus we can write

$$g_{00}(t, r) = e^{\nu(t, r)}, \quad g_{11} = -e^{\lambda(t, r)}, \quad g_{22} = -f^2(r), \quad (4.3.105)$$

where ν , λ and f are real functions. We can still apply coordinate transformations $t \rightarrow t'(t)$ and $r \rightarrow r'(r)$ without changing the form of the interval (4.3.104). Specifically, we can use $r \rightarrow r'(r)$ to define the radial coordinate r such that

$$f(r) = r, \quad (4.3.106)$$

for which a circle composed of points with $r = R$ has the circumference $2\pi R$ and the sphere composed of points with $r = R$ has the surface area $4\pi R^2$. The corresponding radius, defined as the distance from the origin to a point where $r = R$, is $\int_0^R e^{\lambda/2} dr$, which is equal to R only if $\lambda = 0$. Thus the interval for a spherically symmetric gravitational field is given by

$$ds^2 = e^{\nu(t, r)} c^2 dt^2 - e^{\lambda(t, r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (4.3.107)$$

where we can still use $t \rightarrow t'(t)$ to redefine the time coordinate t . The gravitational field given by the interval (4.3.107) is symmetric about the origin $r = 0$.

The Christoffel symbols for the interval (4.3.107) that do not vanish are

$$\begin{aligned} \{^0_{00}\} &= \frac{\dot{\nu}}{2}, \quad \{^0_{01}\} = \frac{\nu'}{2}, \quad \{^0_{11}\} = \frac{\dot{\lambda}}{2} e^{\lambda-\nu}, \\ \{^1_{00}\} &= \frac{\nu'}{2} e^{\nu-\lambda}, \quad \{^1_{01}\} = \frac{\dot{\lambda}}{2}, \quad \{^1_{11}\} = \frac{\lambda'}{2}, \quad \{^2_{22}\} = -r e^{-\lambda}, \quad \{^3_{33}\} = -r \sin^2\theta e^{-\lambda}, \\ \{^2_{12}\} &= \{^3_{13}\} = \frac{1}{r}, \quad \{^2_{33}\} = -\sin\theta \cos\theta, \quad \{^3_{23}\} = \cot\theta, \end{aligned} \quad (4.3.108)$$

where dot denotes a derivative with respect to ct and prime denotes a derivative with respect to r . The corresponding components of the Einstein tensor that do not vanish identically are

$$\begin{aligned} G^0_0 &= \frac{1}{r^2} - e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right), \\ G^1_1 &= \frac{1}{r^2} - e^{-\lambda} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right), \\ G^2_2 = G^3_3 &= -\frac{1}{2} e^{-\lambda} \left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu' \lambda'}{2} + \frac{\nu' - \lambda'}{r} \right) + \frac{1}{2} e^{-\nu} \left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\nu} \dot{\lambda}}{2} \right), \\ G^1_0 &= -e^{-\lambda} \frac{\dot{\lambda}}{r}. \end{aligned} \quad (4.3.109)$$

In vacuum, the Einstein equations (as well as the Einstein-Cartan equations) give $G^i_j = 0$. The last equation in (4.3.109) gives

$$\dot{\lambda} = 0 \rightarrow \lambda = \lambda(r). \quad (4.3.110)$$

Adding the first and second equation in (4.3.109) gives

$$(\nu + \lambda)' = 0 \rightarrow \nu + \lambda = f(t). \quad (4.3.111)$$

However, we can use $t \rightarrow t'(t)$ to redefine the time coordinate t such that $f(t) = 0$. Thus

$$\nu = -\lambda, \quad (4.3.112)$$

which also leads to

$$\dot{\nu} = 0 \rightarrow \nu = \nu(r). \quad (4.3.113)$$

Therefore the metric tensor of a spherically symmetric gravitational field does not depend on time. The existence of the three Killing vectors associated with the generators of rotations (1.6.58) implies the existence of the fourth Killing vector associated with time translation:

$$\xi_0^i = \delta_0^i. \quad (4.3.114)$$

A spherically symmetric gravitational field in vacuum is static, which constitutes the *Birkhoff theorem*.

We consider a spherically symmetric gravitational field in vacuum around a massive sphere. Such a field vanishes at infinity. Integrating the first equation in (4.3.109), with the condition $e^\lambda \rightarrow 1$ at $r \rightarrow \infty$, gives

$$e^{-\lambda} = e^\nu = 1 + \frac{C}{r}, \quad (4.3.115)$$

where C is a constant. Comparing (4.3.115) with (2.5.82) and (4.3.59) yields

$$C = -r_g, \quad (4.3.116)$$

where

$$r_g = \frac{2Gm}{c^2} \quad (4.3.117)$$

is called the gravitational or *Schwarzschild radius* corresponding to the mass m . The corresponding interval,

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (4.3.118)$$

describes the *Schwarzschild metric* for a spherically symmetric gravitational field. For a spherically symmetric cavity inside the sphere, the regularity of the metric at the center, $r = 0$, requires $C = 0$. Thus the gravitational field there vanishes. For a spherically symmetric gravitational field inside a massive sphere of radius R , the regularity of the metric at the center yields $\lambda = 0$ at $r = 0$. Integrating the first equation in (4.3.109) then gives

$$\lambda = -\ln\left(1 - \frac{\kappa}{r} \int_0^r T_0^0 r^2 dr\right), \quad (4.3.119)$$

yielding $e^\lambda \geq 1$. The first and second equation in (4.3.109) give also $\nu' + \lambda' \geq 0$, yielding $\nu + \lambda \leq 0$. Outside the sphere, (4.3.119) gives

$$\lambda = -\ln\left(1 - \frac{\kappa}{r} \int_0^R T_0^0 r^2 dr\right) = -\ln\left(1 - \frac{\kappa}{r} \int_0^R \tilde{\epsilon} r^2 dr\right), \quad (4.3.120)$$

where $\tilde{\epsilon}$ is the effective energy density (2.5.115) of a spin fluid. Without spin, $\tilde{\epsilon} = \epsilon$ is the ordinary energy density of an ideal fluid. Comparing (4.3.120) with (4.3.115) and (4.3.117) gives

$$m = \frac{4\pi}{c^2} \int_0^R \tilde{\epsilon} r^2 dr < \frac{4\pi}{c^2} \int_0^R \tilde{\epsilon} e^{\lambda/2} r^2 dr = \frac{1}{c^2} \int \tilde{\epsilon} dV. \quad (4.3.121)$$

This inequality indicates the *gravitational mass defect*.

The spatial interval (1.4.97) for the Schwarzschild metric is given by

$$dl^2 = \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.3.122)$$

The proper time τ is related to the coordinate time t by $d\tau = \sqrt{g_{00}}dt$, yielding $d\tau \leq dt$. Thus the time is slower in the presence of a gravitational field, in accordance with the time dilation in noninertial frames of reference and the principle of equivalence. The distance between two points with the same angular components of the spherical coordinates, located at r_1 and r_2 , is bigger than

the difference between their radial components: $\int_{r_1}^{r_2} \frac{dr}{\sqrt{1-r_g/r}} \geq r_2 - r_1$. For a weak gravitational field, the metric (4.3.118) can be approximated as

$$ds^2 = ds_0^2 - \frac{r_g}{r}(c^2 dt^2 + dr^2), \quad (4.3.123)$$

where ds_0^2 is the interval corresponding to the Minkowski tensor of a flat spacetime (1.4.84). For a plane passing through the origin of the Schwarzschild field, we can take $\theta = \pi/2$. The spatial interval (4.3.122) on this plane is

$$dl^2 = \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \quad (4.3.124)$$

The form of the surface of rotation on which the geometry is the same as on this plane is

$$dl^2 = dr^2 + dz^2 + r^2 d\phi^2 = dr^2(1 + z'^2) + r^2 d\phi^2, \quad (4.3.125)$$

where $z' = \frac{dz}{dr}$ and $1 + z'^2 = (1 - r_g/r)^{-1}$, yielding *Flamm's paraboloid*:

$$z = 2\sqrt{r_g(r - r_g)}. \quad (4.3.126)$$

The surface of rotation (4.3.126) represents embedding of the non-Euclidean, two-dimensional equatorial plane of the Schwarzschild geometry into the three-dimensional Euclidean space. Such a space has no physical meaning and merely serves to visualize Flamm's paraboloid, as shown in Fig. 20.

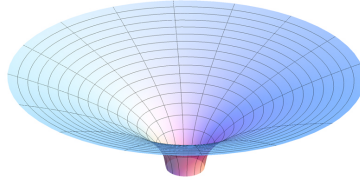


Figure 20: Flamm paraboloid.

The nonzero components of the curvature tensor which for the Schwarzschild field is equal to the Riemann tensor (1.4.44) are

$$R^{01}_{01} = R^{23}_{23} = -\frac{r_g}{r^3}, \quad R^{02}_{02} = R^{03}_{03} = R^{12}_{12} = R^{13}_{13} = \frac{r_g}{2r^3}. \quad (4.3.127)$$

The *Kretschmann scalar*,

$$I = R_{ijkl}R^{ijkl}, \quad (4.3.128)$$

is singular at the origin $r = 0$:

$$I = \frac{12r_g^2}{r^6}. \quad (4.3.129)$$

Thus the Schwarzschild field has a spacetime (curvature) singularity at the origin, indicating its unphysical character. The nonzero components of the spatial Ricci tensor (1.4.74) (the Ricci tensor constructed from the spatial metric tensor (1.4.98)) are

$$P_1^1 = -\frac{r_g}{r^3}, \quad P_2^2 = P_3^3 = \frac{r_g}{2r^3}. \quad (4.3.130)$$

Thus the spatial Ricci scalar $P = 0$. For a plane perpendicular to the radius, the Gauß curvature (1.4.76) is

$$K = \frac{P_{2323}}{\gamma_{22}\gamma_{33}} = -P_1^1 > 0. \quad (4.3.131)$$

Thus the sum of the angles Δ of a small triangle on such a plane embracing its intersection with the above radius is $\Delta > \pi$. For a plane passing through the origin, $K < 0$. Thus the sum of the angles of a small triangle on such a plane is $\Delta < \pi$, except for triangles embracing the origin, for which $\Delta > \pi$.

The choice (4.3.106) of the radial coordinate r is not unique. Using the Birkhoff theorem, we can write the interval for a spherically symmetric gravitational field as

$$ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - e^{\mu(r)} r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.3.132)$$

and use the freedom of applying a transformation $r \rightarrow r'(r)$ to impose a constraint

$$f(\nu, \lambda, \mu) = 0. \quad (4.3.133)$$

The Einstein equations in vacuum for the metric (4.3.132) give the *de Sitter formulae*:

$$e^\nu = 1 - \frac{r_g}{R}, \quad e^\lambda = \frac{R'^2}{1 - \frac{r_g}{R}}, \quad e^\mu = \frac{R^2}{r^2}, \quad (4.3.134)$$

where $R = R(r)$ is a function fixed by (4.3.133) and $R' = \frac{dR}{dr}$. The *Schwarzschild-Hilbert* constraint $\mu = 0$ gives $R = r$ (4.3.106) and the Schwarzschild metric (4.3.118). *Droste's* constraint $\lambda = 0$ yields

$$r = R \left(1 - \frac{r_g}{R} \right)^{1/2} + \frac{r_g}{2} \ln \frac{\sqrt{R} + \sqrt{R - r_g}}{\sqrt{R} - \sqrt{R - r_g}} + \text{const.} \quad (4.3.135)$$

Schwarzschild's constraint $\lambda + 2\mu + \nu = 0$ gives

$$R = (r^3 + r_g^3)^{1/3}, \quad (4.3.136)$$

which is equivalent to (4.3.118) with the origin at $r = -r_g$. *Weyl's* constraint $\lambda - \mu = 0$ gives

$$R = r \left(1 + \frac{r_g}{4r} \right)^2, \quad (4.3.137)$$

which yields the metric describing a spherically symmetric gravitational field in the *isotropic spherical coordinates*:

$$ds^2 = \frac{(1 - r_g/(4r))^2}{(1 + r_g/(4r))^2} c^2 dt^2 - (1 + r_g/(4r))^4 (dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)). \quad (4.3.138)$$

The *Einstein-Rosen* constraint $\lambda - \mu + \nu + \ln 4 = 0$ yields

$$R = r_g + \frac{r^2}{4r_g}, \quad (4.3.139)$$

which is equivalent to (4.3.138).

A hypersurface of constant radius, $f(r) = r - \text{const} = 0$, has the normalized normal vector (2.5.138) given in the spherical coordinates (ct, r, θ, ϕ) by

$$n_i = (0, 1, 0, 0). \quad (4.3.140)$$

In a Schwarzschild field,

$$n_i n^i = \frac{r_g}{r} - 1. \quad (4.3.141)$$

Thus the Schwarzschild sphere $r = r_g$ is an event horizon.

4.3.5 Motion in Schwarzschild field

define black hole and white hole

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad (4.3.142)$$

tortoise radial coordinate

$$dr_* = \frac{dr}{1 - \frac{r_g}{r}} \quad (4.3.143)$$

$$r_* = r + r_g \ln\left(\frac{r}{r_g} - 1\right) \quad (4.3.144)$$

free-fall velocity

$$\frac{V}{c} = \sqrt{\frac{r_g}{r}} \quad (4.3.145)$$

proper coordinates

$$\begin{aligned} dT^2 &= \left(1 - \frac{r_g}{r}\right) dt^2, \\ dR^2 &= \left(1 - \frac{r_g}{r}\right)^{-1} dr^2. \end{aligned} \quad (4.3.146)$$

We consider a massive particle moving radially in the gravitational field described by the metric (??). For brevity, we use

$$h = g_{00} = \frac{(1 - r_g/(4r))^2}{(1 + r_g/(4r))^2}, \quad f = -g_{rr} = (1 + r_g/(4r))^4. \quad (4.3.147)$$

The motion of the particle is given by the radial geodesic equations. If the particle is at rest at $r = r_0$, then these equations are

$$\frac{dt}{d\tau} = u^0 = \sqrt{h_0}/h, \quad (4.3.148)$$

$$\frac{dr}{cd\tau} = u^r = \epsilon(h_0 h^{-1} f^{-1} - f^{-1})^{1/2}, \quad (4.3.149)$$

where τ is the proper time of the particle, $h_0 = h|_{r=r_0}$, and $\epsilon = -1 (+1)$ for an infalling (outgoing) motion. We consider a particle falling into a black hole, $\epsilon = -1$. As $r \rightarrow r_g/4$, h goes to zero and $f \rightarrow 16$, so both u^0 and u^r become infinite. Even if the initial motion were not purely radial, the components u^0 , u^r would still become infinite at $r = r_g/4$, with u^θ , u^ϕ remaining finite. Therefore, each motion of a massive particle becomes effectively radial at the surface $r = r_g/4$.

A distant observer situated in a nearly Galilean spacetime measures the velocity of the infalling particle as

$$v_d = \frac{dr}{dt} = c \frac{u^r}{u^0} = -c \frac{(h_0 f^{-1} h - f^{-1} h^2)^{1/2}}{\sqrt{h_0}}. \quad (4.3.150)$$

As $r \rightarrow r_g/4$, v_d goes to zero. Writing $r = r_g/4 + \xi$, where $0 < \xi \ll r_g$, gives $v_d \approx -c\xi/(2r_g)$ and thus $r - r_g/4 \sim \exp(-ct/(2r_g))$, so the particle reaches the surface of a black hole $r = r_g/4$ after an infinite time t . This surface is an event horizon for a distant observer, as it is for the standard Schwarzschild metric [?]. The proper time $\Delta\tau$ of the particle for moving radially from $r = r_0$ to $r = r_g/4$ is finite, which can be shown by considering $r_0 = r_g/4 + \xi$:

$$c\Delta\tau = \int_{r_g/4}^{r_g/4+\xi} dr/u^r \approx r_g. \quad (4.3.151)$$

After reaching the surface $r = r_g/4$, the particle continues moving; its radial coordinate r decreases to $r_1 = r_g^2/(16r_0) \approx r_g/4 - \xi$ (at which $u^r = 0$) in a proper time $\Delta\tau = r_g/c$. The radial motion of

a massive particle (in terms of the proper time) in the spacetime (??) for $r \leq r_g/4$ is the image (in the sense of the method of image charges for spheres in electrostatics) of the particle's motion for $r \geq r_g/4$. Applying the transformation (??) to Eqs. (4.3.148) and (4.3.149) in the region $r \leq r_g/4$ gives

$$\frac{dt}{d\tau} = \sqrt{h_0}/h, \quad (4.3.152)$$

$$\frac{dr'}{cd\tau} = -\epsilon(h_0 h^{-1} f^{-1} - f^{-1})^{1/2}, \quad (4.3.153)$$

where $h = (1 - r_g/(4r'))^2/(1 + r_g/(4r'))^2$ and $f = (1 + r_g/(4r'))^4$. An infalling radial geodesic motion inside a black hole appears, in terms of the new radial coordinate r' , as an outgoing motion from a white hole (the time reversal of a black hole).

The local velocity of the particle v_l , measured in terms of the proper time, as determined by static clocks synchronized along the trajectory of the particle, is related to u^0 by $u^0 = (h(1 - v_l^2/c^2))^{-1/2}$ [2]. As the particle moves from $r = r_0$ to $r = r_g/4$, v_l increases from zero to c , and as the particle moves from $r = r_g/4$ to $r = r_1$, v_l decreases to zero. In a Schwarzschild field, v_l exceeds c inside a black hole, which does not violate Einstein's theory of relativity because the interior of a Schwarzschild black hole is not static and neither can be clocks synchronized along the trajectory of the particle.

4.3.6 Other coordinates for Schwarzschild field

The Schwarzschild metric (4.3.118) has a coordinate singularity at the surface $r = r_g$, where $g_{00} = 0$ and g_{11} is infinite. The components of the curvature tensor (4.3.127) and the Kretschmann scalar (4.3.129) are finite at this surface, though. Therefore an appropriate coordinate transformation, that is singular at $r = r_g$, can remove this apparent singularity. Defining new coordinates $\tau_{\pm}, R$ such that

$$\begin{aligned} \pm cd\tau_{\pm} &= cdt + f(r)dr_*, \\ dR &= cdt + \frac{dr_*}{f(r)}, \end{aligned} \quad (4.3.154)$$

where dr_* is the tortoise coordinate (4.3.143), turns (4.3.118) into

$$ds^2 = \frac{1 - r_g/r}{1 - f^2} (c^2 d\tau_{\pm}^2 - f^2 dR^2) - r^2 d\Omega^2. \quad (4.3.155)$$

We also have

$$R \mp c\tau_{\pm} = \int \frac{1 - f^2}{f} dr_*. \quad (4.3.156)$$

The physical case of gravitational attraction (black hole) corresponds to plus signs in (4.3.154). The metric (4.3.155) is regular at $r = r_g$ if $f(r_g) = 1$. Taking

$$f(r) = \sqrt{\frac{r_g}{r}} \quad (4.3.157)$$

turns (4.3.155) into the *Lemaître metric*:

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r(\tau, R)} dR^2 - r^2(\tau, R) d\Omega^2, \quad (4.3.158)$$

where $\tau = \tau_+$ and $r(\tau, R)$ is given by integrating (4.3.156):

$$r = \left(\frac{3}{2}(R - c\tau) \right)^{2/3} r_g^{1/3}. \quad (4.3.159)$$

The Lemaître metric (4.3.158) is regular at the gravitational radius. It is singular at $r = 0$, where $c\tau = R$. The Lemaître metric is synchronous, $g_{00} = 1$ and $g_{0\alpha} = 0$. Thus the bodies at rest in the

Lemaître coordinates (τ, R) are freely falling in the gravitational field. The time coordinate τ is the proper time. Radial null geodesics, $ds = d\Omega = 0$, are given by

$$\frac{cd\tau}{dR} = \pm \sqrt{\frac{r_g}{r}}, \quad (4.3.160)$$

where plus sign corresponds to a radially outgoing signal and minus sign to a radially infalling signal. Since all physical world lines must lie within the local light cones, no signal can escape from inside the Schwarzschild sphere, where $\frac{dr}{d\tau} < 0$, as shown in Fig. 21. Thus signals emitted there radially inwards and outwards both reach the origin. The radially falling bodies reach the gravitational radius and then the origin within finite proper time. The unphysical case of gravitational repulsion (white hole) corresponds to minus signs in (4.3.154). In this case, no signal can enter inside the Schwarzschild sphere. In Fig. 21, this case corresponds to $\tau = -\tau_-$.

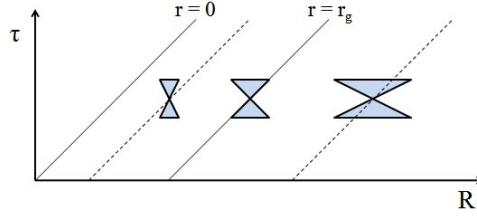


Figure 21: Lemaître coordinates.

We can define new time coordinates v, u according to

$$\begin{aligned} v &= ct + r_*, \\ u &= ct - r_*. \end{aligned} \quad (4.3.161)$$

The v coordinate is the ingoing null coordinate or *advanced Eddington-Finkelstein time* and u is the outgoing null coordinate or *retarded Eddington-Finkelstein time*, analogously to the advanced and retarded time (4.2.162). Replacing t by v turns (4.3.118) into the *ingoing Eddington-Finkelstein metric*:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dv^2 - 2dv dr - r^2 d\Omega^2. \quad (4.3.162)$$

The metric (4.3.162) is regular at $r = r_g$ and describes a black hole. Radial null geodesics are given by

$$dv = 0, \quad dv = 2dr_*, \quad (4.3.163)$$

where the first case corresponds to a radially infalling signal and the second one to a radially outgoing signal. Since all physical world lines must lie within the local light cones, no signal can escape from inside the Schwarzschild sphere, where $\frac{dr}{dv} < 0$, as shown in Fig. 22. Replacing t by u turns (4.3.118) into the *outgoing Eddington-Finkelstein metric*:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) du^2 + 2du dr - r^2 d\Omega^2. \quad (4.3.164)$$

The metric (4.3.164) is regular at $r = r_g$ and describes a white hole. Radial null geodesics are given by

$$du = 0, \quad du = -2dr_*, \quad (4.3.165)$$

where the first case corresponds to a radially outgoing signal and the second one to a radially infalling signal. No signal can enter inside the Schwarzschild sphere. This case is described by Fig. 22, in which v is replaced by $-u$.

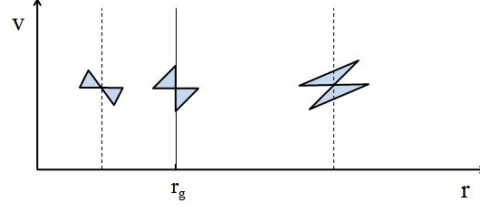


Figure 22: Eddington-Finkelstein coordinates.

We can define new time coordinates $\tilde{v}, \tilde{u}$ according to

$$\begin{aligned} d\tilde{v} &= cdt + \sqrt{\frac{r_g}{r}} dr_*, \\ d\tilde{u} &= cdt - \sqrt{\frac{r_g}{r}} dr_*. \end{aligned} \quad (4.3.166)$$

The $\tilde{v}$ coordinate is the *advanced Painlevé-Gullstrand time* and $\tilde{u}$ is the *retarded Painlevé-Gullstrand time*. Integrating (4.3.166) gives

$$\begin{aligned} \tilde{v} &= ct + 2\sqrt{r_g r} + r_g \ln \frac{\sqrt{r} - \sqrt{r_g}}{\sqrt{r} + \sqrt{r_g}}, \\ \tilde{u} &= ct - 2\sqrt{r_g r} - r_g \ln \frac{\sqrt{r} - \sqrt{r_g}}{\sqrt{r} + \sqrt{r_g}} \end{aligned} \quad (4.3.167)$$

Replacing t by $\tilde{v}$ turns (4.3.118) into the *ingoing Painlevé-Gullstrand metric*:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) d\tilde{v}^2 - 2\sqrt{\frac{r_g}{r}} d\tilde{v} dr - dr^2 - r^2 d\Omega^2. \quad (4.3.168)$$

The metric (4.3.168) is regular at $r = r_g$ and describes a black hole. Radial null geodesics are given by

$$d\tilde{v} = \left(\sqrt{\frac{r_g}{r}} \pm 1\right) dr_*, \quad (4.3.169)$$

where plus sign corresponds to a radially outgoing signal and minus sign to a radially infalling signal. Replacing t by $\tilde{u}$ turns (4.3.118) into the *outgoing Painlevé-Gullstrand metric*:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) d\tilde{u}^2 + 2\sqrt{\frac{r_g}{r}} d\tilde{u} dr - dr^2 - r^2 d\Omega^2. \quad (4.3.170)$$

The metric (4.3.170) is regular at $r = r_g$ and describes a white hole. Radial null geodesics are given by

$$d\tilde{u} = \left(-\sqrt{\frac{r_g}{r}} \pm 1\right) dr_*, \quad (4.3.171)$$

where plus sign corresponds to a radially outgoing signal and minus sign to a radially infalling signal.

The Painlevé-Gullstrand coordinates can also be obtained through a special Lorentz transformation (1.6.93) with the free-fall velocity V (4.3.145) in the R -direction applied (locally) to the stationary proper time T (4.3.146):

$$\begin{aligned} d\tilde{v} &= \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \left(cdT + \frac{V}{c} dR\right), \\ d\tilde{u} &= \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \left(cdT - \frac{V}{c} dR\right). \end{aligned} \quad (4.3.172)$$

Thus the Painlevé-Gullstrand time is the proper time measured by clocks that fall freely in the spherically symmetric gravitational field. This result agrees with the metric geodesic equations (1.4.81) corresponding to (4.3.168) and (4.3.170), for which, respectively, $\frac{d\tilde{v}}{ds} = 1$ and $\frac{d\tilde{u}}{ds} = 1$.

Replacing both t and r by the null coordinates v and u (4.3.161) turns (4.3.118) into

$$ds^2 = \left(1 - \frac{r_g}{r}\right) du dv - r^2 d\Omega^2. \quad (4.3.173)$$

Defining the *Kruskal coordinates*,

$$\begin{aligned} U &= e^{-u/(2r_g)}, \quad V = e^{v/(2r_g)}, \\ T + R &= V, \quad T - R = -U, \end{aligned} \quad (4.3.174)$$

turns (4.3.173) into the *Kruskal-Szekeres metric*:

$$ds^2 = \frac{4r_g^3}{r} e^{-r/r_g} (dT^2 - dR^2) - r^2 d\Omega^2, \quad (4.3.175)$$

where $r = r(T, R)$ is given by a transcendental equation,

$$e^{r/r_g} \left(\frac{r}{r_g} - 1 \right) = R^2 - T^2. \quad (4.3.176)$$

The metric (4.3.175) is regular at $r = r_g$ and singular at $r = 0$. It is also singular ($\det g_{ij} = 0$) for $r_g = 0$. T is the time coordinate and R is the radial coordinate. Region I in Fig. 23 ($V > 0, U > 0$) represents the exterior of a black hole ($r > r_g$), where the transformation from (t, r) to (T, R) is given by

$$\begin{aligned} T &= \left(\frac{r}{r_g} - 1 \right)^{1/2} e^{r/(2r_g)} \sinh \frac{ct}{2r_g}, \\ R &= \left(\frac{r}{r_g} - 1 \right)^{1/2} e^{r/(2r_g)} \cosh \frac{ct}{2r_g}. \end{aligned} \quad (4.3.177)$$

Region II in Fig. 23 ($V > 0, U < 0$) represents the interior of a black hole ($r < r_g$), where the transformation from (t, r) to (T, R) is given by

$$\begin{aligned} T &= \left(1 - \frac{r}{r_g} \right)^{1/2} e^{r/(2r_g)} \cosh \frac{ct}{2r_g}, \\ R &= \left(1 - \frac{r}{r_g} \right)^{1/2} e^{r/(2r_g)} \sinh \frac{ct}{2r_g}. \end{aligned} \quad (4.3.178)$$

The lines $r = \text{const}$ are given by $UV = \text{const}$ and the lines $t = \text{const}$ by $V/U = \text{const}$. The curvature singularity $r = 0$ is given by $UV = -1$ and the event horizon by $UV = 0$. Radial null geodesics, which are the boundaries of light cones, are given by $dU dV = 0$: $V = \text{const}$ describes an ingoing null geodesic and $U = \text{const}$ describes an outgoing null geodesic.

A Schwarzschild black hole is represented by $V > 0$. However, the coordinates (V, U) can be extended to $V < 0$. Region III in Fig. 23 ($V < 0, U < 0$) represents the exterior of a white hole ($r > r_g$), where the transformation from (t, r) to (T, R) is given by

$$\begin{aligned} T &= -\left(\frac{r}{r_g} - 1 \right)^{1/2} e^{r/(2r_g)} \sinh \frac{ct}{2r_g}, \\ R &= \left(\frac{r}{r_g} - 1 \right)^{1/2} e^{r/(2r_g)} \cosh \frac{ct}{2r_g}. \end{aligned} \quad (4.3.179)$$

Region III is mathematically equivalent to the time-reversed region I. Region IV in Fig. 23 ($V < 0, U > 0$) represents the interior of a white hole ($r < r_g$), where the transformation from (t, r) to (T, R) is given by

$$\begin{aligned} T &= \left(1 - \frac{r}{r_g}\right)^{1/2} e^{r/(2r_g)} \cosh \frac{ct}{2r_g}, \\ R &= -\left(1 - \frac{r}{r_g}\right)^{1/2} e^{r/(2r_g)} \sinh \frac{ct}{2r_g}. \end{aligned} \quad (4.3.180)$$

Region IV is mathematically equivalent to the time-reversed region II. These 4 regions represent a *maximally extended Schwarzschild solution*. The Schwarzschild metric (4.3.118) is covered only by region I or region III. The ingoing Eddington-Finkelstein and Painlevé-Gullstrand metrics are covered by regions I and II. The outgoing Eddington-Finkelstein and Painlevé-Gullstrand metrics are covered by regions III and IV.

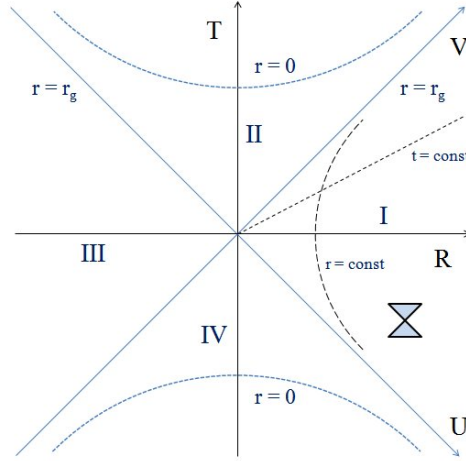


Figure 23: Kruskal coordinates.

4.3.7 Weyl isotropic metric

The interval of the static, spherically symmetric gravitational field in vacuum, expressed in isotropic spherical coordinates (ct, ρ, θ, ϕ) , is given by the *Weyl isotropic metric* (4.3.138):

$$ds^2 = \frac{(1 - r_g/(4\rho))^2}{(1 + r_g/(4\rho))^2} c^2 dt^2 - (1 + r_g/(4\rho))^4 (d\rho^2 + \rho^2 d\Omega^2), \quad (4.3.181)$$

where $0 \leq \rho < \infty$ is the radial coordinate in Weyl's constraint. This metric does not change its form under the radial coordinate transformation

$$\rho \rightarrow \rho' = \frac{r_g^2}{16\rho}, \quad (4.3.182)$$

and is Galilean for $\rho \rightarrow \infty$. Therefore it is also Galilean for $\rho \rightarrow 0$, describing an *Einstein-Rosen bridge* or wormhole: two Schwarzschild solutions of the Einstein field equations (a black hole and white hole) connected at the common event horizon $r = r_g/4$. The metric tensor at the horizon is singular, $\det g_{ik} = 0$. The nonzero components of the Riemann curvature tensor for this metric are given by

$$\begin{aligned} R^{02}_{02} &= R^{03}_{03} = R^{12}_{12} = R^{13}_{13} = \frac{r_g}{2\rho^3(1 + r_g/(4\rho))^6}, \\ R^{01}_{01} &= R^{23}_{23} = -\frac{r_g}{\rho^3(1 + r_g/(4\rho))^6}, \end{aligned} \quad (4.3.183)$$

so the Kretschmann scalar (4.3.128) is finite everywhere,

$$I = 12r_g^2 \rho^{-6} (1 + r_g/(4\rho))^{-12}, \quad (4.3.184)$$

going to zero as $\rho \rightarrow \infty$ and $\rho \rightarrow 0$. For $\rho \gg r_g/4$, the interval (4.3.181) reduces to

$$ds^2 = \left(1 - \frac{r_g}{\rho}\right) c^2 dt^2 - \left(1 + \frac{r_g}{\rho}\right) (dx^2 + dy^2 + dz^2). \quad (4.3.185)$$

The Weyl metric for $\rho > r_g/4$ describes the exterior sheet of a Schwarzschild black hole. The transformation of the radial coordinate,

$$\rho \rightarrow r = \rho(1 + r_g/(4\rho))^2, \quad (4.3.186)$$

brings the interval (4.3.181) into the standard Schwarzschild form (4.3.118). The spacetime given by the metric (4.3.181) for $\rho < r_g/4$ is regarded by observers at $\rho > r_g/4$ as the interior of a black hole. Because of the invariance of the metric (4.3.181) under the transformation (4.3.182), this interior is an image of the other exterior sheet. Since the motion through the event horizon of an Einstein-Rosen bridge is unidirectional, this interior is equivalent to the exterior sheet of a Schwarzschild white hole. The spacetime of an Einstein-Rosen bridge is regular everywhere. The surface of rotation representing embedding of the non-Euclidean, two-dimensional equatorial plane of the Einstein-Rosen geometry into the three-dimensional Euclidean space is the combination of two Flamm's paraboloids, as shown in Fig. 24.

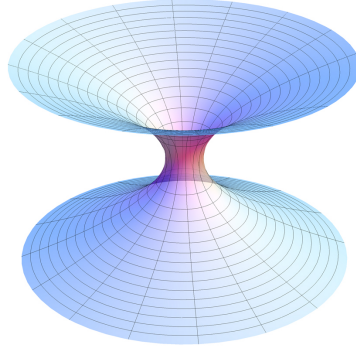


Figure 24: Einstein-Rosen bridge.

We define a new radial coordinate ξ such that

$$r = r_g + \frac{\xi^2}{4r_g}. \quad (4.3.187)$$

The Schwarzschild metric (4.3.118) becomes

$$ds^2 = \frac{\xi^2}{4r_g^2 + \xi^2} c^2 dt^2 - \frac{4r_g^2 + \xi^2}{4r_g^2} d\xi^2 - \left(r_g + \frac{\xi^2}{4r_g}\right)^2 d\Omega^2, \quad (4.3.188)$$

which is equivalent to the Weyl metric (4.3.181), describing an Einstein-Rosen bridge. Near the horizon, where $\xi \approx 0$, (4.3.188) reduces to

$$ds^2 = \frac{\xi^2}{4r_g^2} c^2 dt^2 - d\xi^2 - r_g^2 d\Omega^2. \quad (4.3.189)$$

Introducing new coordinates $\tilde{t}, \tilde{r}$ such that

$$c\tilde{t} = \xi \sinh \frac{ct}{2r_g}, \quad \tilde{r} = \xi \cosh \frac{ct}{2r_g}, \quad (4.3.190)$$

turns (4.3.189) into

$$ds^2 = c^2 d\tilde{t}^2 - d\tilde{r}^2 - r_g^2 d\Omega^2. \quad (4.3.191)$$

Thus the $(\tilde{t}, \tilde{r})$ subspace of the spacetime represented by the interval (4.3.191) is flat. Accordingly, the (t, ρ) subspace of the spacetime represented by the interval (4.3.181) near the event horizon is nearly flat. The interval (4.3.189) is analogous to the Rindler metric (4.1.41) and the coordinates ct, ξ in (4.3.189) are analogous to cT, X (4.1.40). If $\xi = c^2/a_0$, where $a_0 = \text{const}$, then (4.3.190) gives a hyperbola $\tilde{r}^2 - (c\tilde{t})^2 = c^4/a_0^2$, which is analogous to (4.1.36). Therefore (4.3.187) gives the proper acceleration of a massive particle near the event horizon:

$$a_0 \approx \frac{1}{2r_g \sqrt{1 - r_g/r}}. \quad (4.3.192)$$

4.3.8 Interior Schwarzschild solution

We consider a spherically symmetric gravitational field in spacetime filled with an ideal fluid. The general form of the metric in such a field is given by the interval (4.3.101). We can apply a coordinate transformation from t, r, θ, ϕ to $\tau(t, r), R(t, r), \theta, \phi$ such that

$$g_{01} = 0, \quad u^1 = \frac{dR}{d\tau} = 0. \quad (4.3.193)$$

The second condition means that no radial motion takes place in this frame of reference. Since the spherical symmetry also requires $u^2 = u^3 = 0$, this frame of reference is a comoving frame. Accordingly, the interval (4.3.101) becomes

$$ds^2 = e^{\nu(\tau, R)} c^2 d\tau^2 - e^{\lambda(\tau, R)} dR^2 - e^{\mu(\tau, R)} R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.3.194)$$

where ν, λ and μ are real functions. We can still apply coordinate transformations $\tau \rightarrow \tau'(\tau)$ and $R \rightarrow R'(R)$ without changing the form of the interval (4.3.194). The components of the Einstein tensor corresponding to (4.3.194) that do not vanish identically are

$$\begin{aligned} G_0^0 &= -e^{-\lambda} \left(\mu'' - \frac{2}{R^2} + \frac{3}{4} \left(\mu' + \frac{2}{R} \right)^2 - \frac{1}{2} \left(\mu' + \frac{2}{R} \right) \lambda' \right) + \frac{1}{2} e^{-\nu} \left(\dot{\lambda} \dot{\mu} + \frac{\dot{\mu}^2}{2} \right) + \frac{e^{-\mu}}{R^2}, \\ G_1^1 &= -\frac{1}{2} e^{-\lambda} \left(\frac{1}{2} \left(\mu' + \frac{2}{R} \right)^2 + \left(\mu' + \frac{2}{R} \right) \nu' \right) + e^{-\nu} \left(\ddot{\mu} - \frac{1}{2} \dot{\mu} \dot{\nu} + \frac{3}{4} \dot{\mu}^2 \right) + \frac{e^{-\mu}}{R^2}, \\ G_2^2 &= G_3^3 = -\frac{1}{4} e^{-\lambda} \left(2\nu'' + \nu'^2 + 2\mu'' - \frac{4}{R^2} + \left(\mu' + \frac{2}{R} \right)^2 - \nu' \lambda' \right. \\ &\quad \left. + \left(\mu' + \frac{2}{R} \right) (\nu' - \lambda') \right) + \frac{1}{4} e^{-\nu} (2\ddot{\lambda} + \dot{\lambda}^2 + 2\ddot{\mu} + \dot{\mu}^2 - \dot{\lambda} \dot{\nu} - \dot{\mu} \dot{\nu} + \dot{\lambda} \dot{\mu}), \\ G_0^1 &= \frac{1}{2} e^{-\lambda} \left(2\dot{\mu}' + \left(\mu' + \frac{2}{R} \right) (\dot{\mu} - \dot{\lambda}) - \nu' \dot{\mu} \right), \end{aligned} \quad (4.3.195)$$

where dot denotes differentiation with respect to τ and prime denotes differentiation with respect to R . The Einstein equations in this frame of reference are

$$G_0^0 = \kappa \epsilon, \quad G_1^1 = G_2^2 = G_3^3 = -\kappa p, \quad G_0^1 = 0. \quad (4.3.196)$$

The corresponding conservation law $T^{ik}_{;k} = 0$ gives

$$\dot{\lambda} + 2\dot{\mu} = -\frac{2\dot{\epsilon}}{\epsilon + p}, \quad \nu' = -\frac{2p'}{\epsilon + p}, \quad (4.3.197)$$

where the constants of integration depend on the allowed transformations $\tau \rightarrow \tau'(\tau)$ and $R \rightarrow R'(R)$. For a spherically symmetric gravitational field in spacetime filled with a spin fluid, $\epsilon \rightarrow \tilde{\epsilon}$ and $p \rightarrow \tilde{p}$.

If a spherically symmetric sphere of radius a is in hydrostatic equilibrium then the metric does not depend on the time coordinate and the interval (4.3.194) reduces to

$$ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.3.198)$$

where we used the freedom of applying a coordinate transformation $r \rightarrow r'(r)$ to set $\mu(r) = 0$, as in (4.3.107). The Einstein equations (4.3.195) reduce to (cf. (4.3.109))

$$\begin{aligned} \frac{1}{r^2} - e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) &= \kappa \epsilon, \\ \frac{1}{r^2} - e^{-\lambda} \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) &= -\kappa p. \end{aligned} \quad (4.3.199)$$

Integrating the first equation in (4.3.199) gives

$$e^\lambda = \left(1 - \frac{\kappa}{r} \int_0^r \epsilon r^2 dr \right)^{-1} = \left(1 - \frac{r_g(r)}{r} \right)^{-1}, \quad (4.3.200)$$

where

$$r_g(r) = \kappa \int_0^r \epsilon(r) r^2 dr \quad (4.3.201)$$

is the gravitational radius of the sphere of radius r centered at the origin (cf. (4.3.117), (4.3.119) and (4.3.121)). The second equation in (4.3.199) and the second equation in (4.3.197) yield

$$p' = -\frac{1}{2}(\epsilon + p) \left(e^\lambda \left(\frac{1}{r} + \kappa p r \right) - \frac{1}{r} \right) = -\frac{1}{2r^2}(\epsilon + p) e^\lambda (r_g(r) + \kappa p r^3), \quad (4.3.202)$$

which, upon substituting (4.3.200), gives the *Tolman-Oppenheimer-Volkoff equation*:

$$\frac{dp}{dr} = -\frac{1}{2r^2} \frac{(\epsilon + p)(r_g(r) + \kappa p r^3)}{1 - \frac{r_g(r)}{r}}. \quad (4.3.203)$$

In the nonrelativistic limit, (4.3.203) reduces to

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}, \quad (4.3.204)$$

where ρ is the mass density and

$$m(r) = 4\pi \int_0^r \rho(r) r^2 dr \quad (4.3.205)$$

is the mass of the sphere of radius r centered at the origin. The hydrostatic condition (4.3.204) can also be derived for a spherically symmetric system from (2.5.100) for $\mathbf{v} = 0$, using (4.3.56).

If the energy density inside the sphere is homogeneous, $\epsilon = \text{const}$, then (4.3.200) gives

$$e^{\lambda(r)} = \left(1 - \frac{1}{3} \kappa \epsilon r^2 \right)^{-1}, \quad (4.3.206)$$

which, with (4.3.202), leads to

$$p' = -\frac{1}{2}(\epsilon + p) \left(1 - \frac{1}{3} \kappa \epsilon r^2 \right)^{-1} \left(\frac{\epsilon}{3} + p \right) \kappa r. \quad (4.3.207)$$

Integrating this equation with the condition that the pressure vanishes at the boundary of the sphere, $p(a) = 0$, gives

$$p(r) = \epsilon \frac{h(r) - h(a)}{3h(a) - h(r)}, \quad (4.3.208)$$

where

$$h(r) = \left(1 - \frac{1}{3}\kappa\epsilon r^2\right)^{1/2}. \quad (4.3.209)$$

The pressure (4.3.208) is a decreasing function of r . The condition that $p(0)$ be finite gives

$$a > \frac{9}{8}r_g. \quad (4.3.210)$$

Physically realistic equations of state obey $p \leq \epsilon/3$. Thus we have $p(0) \leq \epsilon/3$, which yields

$$a \geq \frac{9}{5}r_g. \quad (4.3.211)$$

The second equation in (4.3.197) integrates to

$$e^{\nu(r)} = e^{\nu(a)} \left(1 + \frac{p(r)}{\epsilon}\right)^{-2}. \quad (4.3.212)$$

The continuity of the metric at the boundary of the sphere, $r = a$, requires (cf. (4.3.112))

$$e^{\nu(a)} = e^{-\lambda(a)}. \quad (4.3.213)$$

Thus the interval for the *interior Schwarzschild metric* for a constant energy density is given by

$$ds^2 = \frac{9}{4} \left(h(a) - \frac{1}{3}h(r) \right)^2 c^2 dt^2 - h^{-2}(r) dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (4.3.214)$$

4.3.9 Tolman solution

4.3.10 Kerr metric

The spacetime geometry in the region surrounding a mass m rotating with angular momentum L is given by the *Kerr metric*:

$$\begin{aligned} ds^2 = & \left(1 - \frac{r_g r}{\rho^2}\right) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + a^2 \sin^2\theta \frac{r_g r}{\rho^2}\right) \sin^2\theta d\phi^2 \\ & + 2a \sin^2\theta \frac{r_g r}{\rho^2} c dt d\phi, \end{aligned} \quad (4.3.215)$$

where

$$a = \frac{L}{mc}, \quad (4.3.216)$$

$$\rho^2 = r^2 + a^2 \cos^2\theta, \quad (4.3.217)$$

$$\Delta = r^2 - r_g r + a^2. \quad (4.3.218)$$

A coordinate transformation,

$$\begin{aligned} x + iy &= (r + ia) \sin\theta e^{i\phi}, \\ z &= r \cos\theta, \end{aligned} \quad (4.3.219)$$

brings the coordinates (r, θ, ϕ) , referred to as the *Boyer-Lindquist coordinates*, to the *Kerr-Schild coordinates* (x, y, z) , in which the Kerr metric becomes

$$g_{ij} = \eta_{ij} + f k_i k_j, \quad (4.3.220)$$

where

$$\begin{aligned} k_0 &= 1, \quad k_x = \frac{rx + ay}{r^2 + a^2}, \quad k_y = \frac{ry - ax}{r^2 + a^2}, \quad k_z = \frac{z}{r}, \\ f &= \frac{r_g r^3}{r^4 + a^2 z^2}. \end{aligned} \quad (4.3.221)$$

The vector $\mathbf{k}$ is a unit vector. The relation between the Boyer-Lindquist and Kerr-Schild coordinates is the same as the relation between the oblate spheroidal and Cartesian coordinates. The Kerr metric is the stationary ($\partial g_{ij}/\partial t = 0$), axisymmetric ($\partial g_{ij}/\partial \phi = 0$), asymptotically flat ($g_{ij} \rightarrow \eta_{ij}$ as $r \rightarrow \infty$), vacuum ($T_{ij} = 0$) solution of the Einstein equations. !

4.3.11 Motion in Kerr field

4.3.12 Reissner-Nordström metric

4.3.13 Kerr-Newman metric

The spacetime geometry in the region surrounding a mass m of charge q rotating with angular momentum L is given in the Boyer-Lindquist coordinates (r, θ, ϕ) (4.3.219) by the *Kerr-Newman metric*:

$$ds^2 = \left(1 - \frac{r_g r - r_q^2}{\rho^2}\right) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + a^2 \sin^2 \theta \frac{r_g r - r_q^2}{\rho^2}\right) \times \sin^2 \theta d\phi^2 + 2a \sin^2 \theta \frac{r_g r - r_q^2}{\rho^2} c dt d\phi, \quad (4.3.222)$$

where a and ρ^2 are given by (4.3.216) and (4.3.217), and

$$\Delta = r^2 - r_g r + a^2 + r_q^2, \quad (4.3.223)$$

where

$$r_q^2 = \frac{Gq^2}{c^4}. \quad (4.3.224)$$

The electromagnetic potential in the Kerr-Newman field is given by

$$A_i dx^i = \frac{qr c dt}{\rho^2} - \frac{qra \sin^2 \theta d\phi}{\rho^2}. \quad (4.3.225)$$

This metric can be written in the Kerr-Schild form (4.3.220), where f is generalized to

$$f = \frac{r^2}{r^4 + a^2 z^2} (r_g r - r_q^2). \quad (4.3.226)$$

The electromagnetic potential in these coordinates is given by

$$A_i = \frac{qr^3}{r^4 + a^2 z^2} k_i. \quad (4.3.227)$$

The Kerr-Newman metric is the stationary ($\partial g_{ij}/\partial t = 0$), axisymmetric ($\partial g_{ij}/\partial \phi = 0$), asymptotically flat ($g_{ij} \rightarrow \eta_{ij}$ as $r \rightarrow \infty$) solution of the Einstein-Maxwell equations without sources ($j^i = 0$).

At large distances $r \gg a$, the electric potential in (4.3.225) reduces to the Coulomb potential $\phi = \frac{q}{r}$ and the magnetic potential in (4.3.225) reduces to (4.2.49) for the magnetic moment

$$\mathbf{m} = qa\hat{z}. \quad (4.3.228)$$

Thus we have

$$\mathbf{m} = \frac{q}{mc} \mathbf{M}, \quad (4.3.229)$$

so the gyromagnetic ratio (4.2.55) for the Kerr-Newman field is $g = 2$ which is twice larger than that for a nonrelativistic particle or a system of identical nonrelativistic particles.

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4.3.14 Weak gravitational field

The component $h_{03} = \frac{2GM}{rc^2} \sin^2 \theta$ satisfies the field equation $P_{03} = 0$ linearized in h_{03} . Thus, for a slowly rotating body, $h_{03} = \frac{2GM}{rc^2} \sin^2 \theta$ is valid even if the gravitational field is strong.

4.3.15 Gravitational waves

4.3.16 Kottler-de Sitter metric

In the presence of the cosmological constant Λ , the Lagrangian density (2.5.21) gives the field equations

$$P_{ik} - \frac{1}{2} P g_{ik} = \kappa T_{ik} + \Lambda g_{ik}. \quad (4.3.230)$$

For a spherically symmetric gravitational field around a massive sphere, these equations give the *Kottler metric*:

$$ds^2 = \left(1 - \frac{r_g}{r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{r_g}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.3.231)$$

which generalizes the Schwarzschild metric (4.3.118). If $r_g = 0$, the Kottler metric (4.3.231) reduces to the *de Sitter metric*:

$$ds^2 = \left(1 - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.3.232)$$

For the de Sitter metric (4.3.232), the curvature is given by

$$R_{ijkl} = -\frac{\Lambda}{3} (g_{ik} g_{jl} - g_{il} g_{jk}). \quad (4.3.233)$$

4.3.17 Friedmann-Lemaître-Robertson-Walker metric

4.4 Spinors

4.4.1 Free spinors

4.4.2 Dirac equation in central electric field

4.4.3 Schrödinger equation in central electric field

4.4.4 Dirac equation in uniform magnetic field

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