

A HISTORICAL OVERVIEW OF CONNECTIONS IN GEOMETRY

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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DEDICATION

To Pandora. This thesis was accomplished for you and in spite of you.

“Mathematics is very akin to Art; a mathematical theory not only must be rigorous, but it must also satisfy our mind in quest of simplicity, of harmony, of beauty...”
—Charles Ehresmann

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ABSTRACT

This thesis is an attempt to untangle/clarify the modern theory of connections in Geometry. Towards this end a historical approach was taken and original as well as secondary sources were used. An overview of the most important historical developments is given as well as a modern look at how the various definitions of connection are related.

PREFACE

In this thesis I hope to clear up some of the confusions surrounding a connection in Differential Geometry; or, at least some of the things that confused me when I was trying to figure out just what exactly a connection is. In particular, 1) How is a covariant derivative operator related to a connection? 2) How is parallel transport related to the previous two notions? 3) What was the first definition of a connection? 4) What is a connection in the most general sense? I hope that I answer all of these questions satisfactorily in the following pages. I have also provided a chart of the hierarchy of connections via the use of Lie subgroups.

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CHAPTER 1

THE BACK STORY

The motivation for the idea of “connection” comes from two separate developments in the history of Mathematics; namely, those of Non-Euclidean Geometry and Differential Calculus. The concept of “connection” is crucial to both.

1.1 “Anti”- Euclidean Space

Ever since Euclid conceived it, the parallel postulate has been subject to much controversy. For a long time it was believed to be provable by the other four postulates. In fact several attempts at proving the parallel postulate from the other four were actually accepted as proofs, but eventually a mistake was found. By 1817 Carl Friedrich Gauss had become convinced that the fifth postulate was independent of the other four postulates and that a consistent geometry, different from the Euclidean one, was possible. At that time he referred to such a geometry as “anti-Euclidean”. In 1824 the term “non-Euclidean Geometry” appeared for the first time in a letter Gauss sent to Franz Adolf Taurinus who was one of the first to publish results about non-Euclidean Geometry.[6] At the time Gauss was working on this, thinking was dominated by the philosophy of Immanuel Kant who had stated that Euclidean geometry is *the inevitable necessity of thought*. It is said that Gauss disliked controversy and thus kept his “anti-Euclidean” work to himself. Despite Gauss’ reticence, the idea of a non-Euclidean geometry gained ground over the next 3 decades thanks especially to the publications of János Bolyai and Nikolai Lobachevsky. However, it would be a long while yet before this new Geometry was unconditionally accepted by pure Geometers. In [11] K.Reich suggests that the missing concept of parallelism may have been partially to blame for the delay, as it has always been a central theme for the foundations of Geometry.

When Gauss developed the theory of surfaces during the years 1825-1827 he used the fundamental property that “there exist parametrizations of any surface by coordinate

systems which produce vectors in the tangent plane at that point.” Gauss focused mainly on the influence of curvature on the geometry of the surface using geodesic curves on surfaces embedded in 3-dimensional Euclidean space. In doing so, he established geometric properties *extrinsically* and then made them *intrinsic* by proving that the properties really do not depend on the embedding. His was the first systematic use of geodesics to make a general Geometry, yet it did not include a notion of parallelism.[6]

The generalization to higher dimensions of what Gauss did for surfaces is due to Bernhard Riemann and is what makes up the content of his lecture *Über die Hypothesen welche der Geometrie zu Grunde liegen* (“On the hypotheses that lie at the foundation of Geometry”). In Part I of Riemann’s lecture, the concept of an “ n -fold extended quantity” (now called a manifold of dimension n) is introduced. Although the details are fuzzy (there are no equations in this particular work of Riemann’s as it was intended for non-mathematicians), it is clear that Riemann’s “ n -dimensional spaces” are characterized by the fact that they are locally like n -dimensional Euclidean space, which Riemann referred to as *flat*. [6] In Part II Riemann considers what are now called Riemannian metrics and curvature. He uses, just as Gauss did, the shortest path to build a geodesic coordinate system.

Although Dedekind in his biography of Riemann called the lecture “a masterpiece of exposition”, it is generally believed that Gauss was the only attendee who really appreciated what Riemann had done. Not surprising that Gauss should be so impressed as he was Riemann’s advisor at the time and even chose the topic. Regardless of its reception at the time, no one today would deny that the ideas contained within are some of the most important in the entire history of Differential Geometry. Riemann, like Gauss, did not include a concept of parallelism; in fact, the word parallel was not even mentioned.[11]

1.2 The Differential Calculus

The idea of a derivative was considered by several European mathematicians in the early 17th century, but Isaac Newton and Gottfried Leibniz must share the credit (even if they do it grudgingly) for their independent developments of The Calculus (classically

referred to as “infinitesimal calculus”) which consisted of Differential Calculus and Integral Calculus. When defining the derivative, Isaac Newton considered variables as changing with respect to time; Leibniz thought of variables x and y as ranging over sequences of infinitely close values. Both Newton and Leibniz provided a formula for finding the second derivative. In Euclidean space, the second derivative of a dependent variable y with respect to an independent variable x is represented (in Leibniz’s notation, still used today) as $\frac{d^2y}{dx^2}$ and generalizes to higher dimensions via partial derivatives as $\frac{\partial^2 f}{\partial x^2}$ where $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

With the rise of non-Euclidean geometry there also arose a problem with the Differential Calculus. To calculate the second derivative it is necessary to compare tangent vectors at distinct points of curves in space. If the space is Euclidean, vectors can be easily compared because as long as the magnitude and direction of a vector stays the same, it can be moved about freely. However, in the case of curves restricted to surfaces this no longer works. Without a specified method of how it was to be done, vectors were restricted to their base points. When Riemann developed his n -dimensional surfaces (now known as differential manifolds), there still was no prescription for moving vectors based at one point of a space to another point and thus no satisfactory solution to the problem of how to define second derivatives (i.e., acceleration) on curved surfaces.

CHAPTER 2

FIRST ENCOUNTERS

The concept of connection was implemented at least as early as 1869 but it was almost a half century later before the mathematical community recognized a connection for what it actually was.

2.1 Γ

In 1869 Elwin Bruno Christoffel published *Über die Transformation der homogenen Differentialausdrücke zweiten Grades* in which he introduced the famous symbols Γ_{ij}^k (Christoffel's notation was $\{^k_{ji}\}$). In his paper, Christoffel is interested in “conditions that differential expressions of degree 2 must verify to be able to transform one into another by a change of variables.” Toward this end Christoffel considers two expressions of the same Riemannian metric (g) and begins by assuming the relation

$$\sum_{i,j=1}^n g_{ij}(x^k) dx^i dx^j = \sum_{\alpha,\beta=1}^n g_{\alpha\beta}(y^\gamma) dy^\alpha dy^\beta$$

where the coordinates (x^k) and (y^γ) and their differentials are related by the transformation f (i.e., $y = f(x)$). In order to isolate a second partial derivative of the transformation f , Christoffel introduces the coefficients noted today

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^n g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^l} \right).$$

Thus, he obtains the fundamental equation

$$\frac{\partial^2 x^\gamma}{\partial x^i \partial x^l} = \sum_{j=1}^n \Gamma_{li}^j \frac{\partial f^\gamma}{\partial x^j} - \sum_{\beta\alpha=1}^n \Gamma_{\beta\alpha}^\gamma \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\alpha}{\partial x^l}, \quad (*)$$

which describes the way in which the Γ_{ij}^k transform under change of coordinates.[6]

Christoffel's point of view is strictly that of a mathematician interested in “invariant theory”. The goal of his paper was to solve a problem of “classification” and he certainly

did not investigate any relationships with any particular curves in space.[6] What Christoffel did not realize was the fact that the symbols he had discovered themselves determine a *connection*. Specifically, they are the “coefficients” of the connection. In fact, a generalized form of equation (*) is used in [13] to define a (*classical*) *connection* as “an assignment of n^3 numbers to each coordinate system, such that equation (*) holds” (i.e., *connection coefficients* in more general settings).

2.2 ∇

It was Gregorio Ricci-Curbastro who noticed that the Christoffel symbols obtained from the Riemannian metric could be used to create a “coordinate free” differential calculus. This extension of the Differential Calculus allowed a modification of partial differentiation to spaces with curvature (i.e. non-Euclidean) by means of a covariant derivative. As stated in [6], Ricci defined the covariant derivative of a vector field X ($X = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}$ in local coordinates) to be:

A field of endomorphisms of TM whose components $X^i_{;j}$ are calculated from the components (X^i) of X , by the formula¹

$$X^i_{;j} = \frac{\partial X^i}{\partial x^j} + \sum_{k=1}^n \Gamma^k_{ij} X^k.$$

In Euclidean space, the sum is zero and hence the covariant derivative is just an ordinary partial derivative.

Ricci takes great care that an equation derived in one coordinate system is valid in any other. This coordinate independence is the reason for the name “Absolute” which was given to this extension of the Differential Calculus. In his 1893 publication Ricci explains: “For the sake of brevity I will denote by the name the ‘absolute differential calculus’ the totality of methods which I named in other occasions covariant and contravariant derivatives, because they can be applied to any fundamental form independently of the choice of the independent variables and require instead that the latter are completely general and arbitrary.”[2]

¹In modern notation the covariant derivative $X^i_{;j}$ is written as $(\nabla_j X)^i$ or $(\nabla_{\partial_j} X)^i$ or $(\nabla_{\frac{\partial}{\partial x^j}} X)^i$ where ∇ is referred to as a *covariant derivative operator*.

Ricci developed his calculus throughout the years 1884-1900 but despite the significance of what Ricci had discovered, the Absolute Differential Calculus remained relatively unknown. In 1899 Ricci, with the help of his student Tullio Levi-Civita, wrote a final exposition on the Absolute Differential Calculus. It appeared in the *Mathematische Annalen* in 1901. The goal of this final publication was to “put the [ADC] within the capacity of everybody who needed it”. However, as stated by Levi-Civita himself, “for many more years it was used almost exclusively by its inventor and a few of his students”. The essential shift in appreciation for the Absolute Differential Calculus resulted from Einstein’s theory of General Relativity. As Levi-Civita stated, this physical theory was “the great trial, foretold by Beltrami, where Ricci’s calculus revealed itself to be not only useful but truly indispensable”. However, even at this point, the new calculus was purely formal without geometric interpretation.[2]

At this point a connection is essentially right under the nose of the mathematical community; in the form of a covariant derivative, which stayed purely an analytic algorithm until 1917.

2.3 $\mathbb{P}$

Tullio Levi-Civita introduced the concept of parallelism on a Riemannian manifold in his paper *Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura riemanniana* published in 1917. There were two other independent discoveries of “parallelism” by G. Hessenberg and J.A. Schouten. However, Levi-Civita had priority of publication and so today it is referred to as “Levi-Civita’s parallelism”. [11] Levi-Civita began his famous work with the goal of reducing the formal apparatus commonly used in order to introduce the Riemann symbols relative to the curvature of a certain 4-dimensional manifold and he did make some progress toward this goal. However, it was the geometric interpretation of the Riemann curvature and the notion of parallel directions in an arbitrary Riemannian manifold discovered along the way which makes up the majority of the paper and for which it is so well known today.[7]

Levi-Civita's original idea began with the observation that any Riemannian manifold V_n could be embedded in a Euclidean space of higher dimension S_N . Although manifolds weren't fully understood until at least sometime in the 1950s, it is indeed what Levi-Civita had in mind. He then used parallel displacement in the embedding space to move a tangent vector α at P to the nearby point P' . The resulting vector is not necessarily in the tangent space at P' ; however, by orthogonal projection onto the tangent space at P' Levi-Civita obtains a vector α' which is the tangent vector at P' parallel to the tangent vector α at P . This definition is not intrinsic but Levi-Civita saw that if P' is restricted to being a point infinitesimally close to P , then the definition can be made intrinsic and the parallel vector is determined by the metric.[6]

Indeed, let x_i ($i = 1, \dots, n$) be the general coordinates of V_n , dx_i the increments corresponding to the passage from P to P' , $\xi^{(i)}$ the parameters related to any direction (α) going out of P , and $\xi^{(i)} + d\xi^{(i)}$ those related to an infinitely close direction (α') going out of P' . Then, according to Levi-Civita the parallelism condition is expressed by the n equations

$$d\xi^{(i)} + \sum_{j,l=1}^n \left\{ \begin{matrix} j l \\ i \end{matrix} \right\} dx_j \xi^{(l)} = 0$$

where $\left\{ \begin{matrix} j l \\ i \end{matrix} \right\}$ is the Christoffel symbol. These equations are invariant for any transformation of coordinates and contain only quantities given in the manifold V_n . 'Once the law is known according to which one passes from a point to a point infinitely close to it, one is able immediately to accomplish the displacement of parallel directions along any arbitrary curve C' (Levi-Civita, 1917,p.3).

Assuming that $x_i = x_i(s)$ are the parametric equations of the curve, then the functions $\xi^{(i)}(s)$ are determined by the equations

$$\frac{d\xi^{(i)}}{ds} + \sum_{j,l=1}^n \left\{ \begin{matrix} j l \\ i \end{matrix} \right\} \frac{dx_j}{ds} \xi^{(l)} = 0.$$

As a consequence of this definition it follows that the parallel displacement from a point P to another point P' of V_n depends upon the choice of curve joining them

along which the parallel displacement is made. In addition, Levi-Civita remarked that the directions of the tangents along the same geodesic are all mutually parallel; this is a generalization to arbitrary manifolds of a basic property of straight lines in Euclidean spaces. He also observed that the parallel displacement along any arbitrary path of two different directions preserves the angle formed by them.[2]

Levi-Civita is frequently given credit for the discovery and naming of a “connection”. What he actually discovered was a *parallelism structure* $\mathbb{P}$. The significance of Levi-Civita’s 1917 paper was in the geometric information that it gleaned about the covariant derivative. In the special case of a manifold isometrically embedded into a higher dimensional Euclidean space, the covariant derivative can be viewed as the orthonormal projection of the Euclidean derivative along a tangent vector onto the manifold’s tangent space. Once Levi-Civita pointed out the parallelism that was inherent in the covariant derivative operator, it was not long before the theory of connections was truly born.

2.4 A First Definition

The term “connection” first appears in Hermann Weyl’s 1918 text *Reine Infinitesimal Geometrie*. In section 3 he describes an *affine connection* as “that which determines into which vector at P' a vector at an infinitesimally close point P will transform under parallel displacement from P to P' .” A condition which Weyl requires of parallel displacement in this instance is that *the transfer of the totality of vectors from P to the infinitely close point P' by means of parallel displacement produces an affine transformation of the vectors at P to the vectors at P'* . The transformation is an affine one in the sense that it preserves collinearity and ratios of distances but not necessarily angles or lengths. Weyl goes on to define *components of the affine connection* to be quantities $\Gamma_{rs}^i = \Gamma_{sr}^i$. In section 4 Weyl defines a *metric connection* to be an affine connection on a metric manifold (by which he must mean a Riemannian manifold since the associated quadratic form is now assumed to be positive definite; i.e. a Riemannian metric) where the Γ_{rs}^i are Christoffel symbols.[14]

Levi-Civita's parallel displacement via a covariant derivative operation certainly fits Weyl's definition of "connection" and is why today Levi-Civita's system of parallel transport $\mathbb{P}$ is called the *Levi-Civita connection*, as is the covariant derivative operator ∇ associated to it.

At this point the reader should recognize that the concept of connection has indeed fixed two important problems. A connection, in the form of a covariant derivative, allows for a well defined second derivative on a curved surface in any pseudo-Riemannian space (and thus, Riemannian and Euclidean spaces as well). A connection, in the form of a system of parallel transport, allows for a notion of parallelism in the same.

CHAPTER 3

GENERALIZATION

Directly after the introduction of the theory of connections in 1917, a generalization for spaces other than Riemannian, and situations other than those specific to the *Levi-Civita connection*, was sought. Weyl was the first to make a successful generalization. His “affine connection” successfully generalized the “Levi-Civita connection” for spaces other than Riemannian. Weyl’s was only a small step in the process of generalization. During the next couple of decades E. Cartan took center stage in the development of connection theory as he sought to bring F. Klein’s view of Geometry to bear upon Differential Geometry. By 1950 this view was subsumed to an even wider frame; that of connections in principal fiber bundles.

3.1 Lie groups

Into the burgeoning field of Differential Geometry came the algebraic concept of a group. A Lie group is essentially a group which is also a differentiable manifold so that the group operations are compatible with the smooth structure. Such groups are named for Sophus Lie who laid the foundations with his work on continuous transformation groups. At the time Lie was working on this, his friend and colleague Felix Klein was working on what is today called the *Erlangen Program* (because the work took place in Erlangen, Germany). In 1872 Klein published his influential research program under the title *Vergleichende Betrachtungen über neuere geometrische Forschungen*. Klein’s Erlangen Program was innovative in several ways, one of which was the idea that group theory was the most useful way of organizing geometrical knowledge. By this time, several non-Euclidean geometries had already emerged, but their relationships with one another had not been adequately clarified. Klein’s program showed that a geometry could be defined by the properties of a space that are invariant under a given group of transformations. However, Klein’s view of Geometry

applied to whole spaces. “Localizing it, as one might say, and attaining clarity about what was part of the definition and what had to be added on (or might even be incompatible) was by no means an obvious move to make.”[6]

By the time Einstein developed the Theory of General Relativity, Élie Cartan was already an expert in the theory of *infinitesimal Lie groups* (i.e., *Lie algebras*) which he had been working on at least since 1892. During that year Lie came to Paris at the invitation of Darboux. The main purpose of his visit was to meet Cartan and the two had many discussions during Lie’s six month stay. In one article Cartan writes that Lie was interested “with a great good will in the research of young French mathematicians” and that “Posterity will see in him only the genius who created the theory of transformation groups, and we French shall never be able to forget the ties, which bind us to him and which make his memory dear to us.”[1]

Once he had come into contact with Einstein’s theory, Cartan immediately began working on a general framework which would link Differential Geometry to an “infinitesimalized generalization of Klein’s Erlangen program.”[12] Toward this end Cartan worked with what he called “generalized spaces” (*espace généralisé*). Classically a generalized space was “a space of tangent spaces (which may or may not be the spaces of tangent vectors) such that two infinitely near tangent spaces are related by an infinitesimal transformation of a given Lie group” i.e., a space with a (possibly nonlinear) connection.[4] At about this same time Weyl had begun his own research into the representation theory of Lie groups. Weyl used the Absolute Differential Calculus of Ricci, whereas Cartan used his own calculus of differential forms.[12]

The successful generalization of Klein’s Erlangen program to infinitesimal geometry provided an impressive conceptual frame for studying different types of geometries such as Riemannian, Lorentzian, Weylian, affine, conformal, projective, and others. Thanks to Cartan, all of these geometries were not only characterized by connections and curvature but also allowed for the new phenomenon of torsion.

Despite the progress being made in the theory of connections, there was still a lack of unification. The ambiguous definition of Cartan’s “generalized spaces” did not help to clarify the situation. With the introduction of fiber bundles we are finally able to express the concepts related to “connection” in a more satisfactory way and obtain the most general definition.

3.2 Fiber Bundles

It was Cartan’s student Charles Ehresmann who finally untangled and successfully classified all of the generalized and specific connections which had emerged in the first half of the 20th century. He began this endeavor in an effort to understand Cartan’s connections from a global point of view (as well as being influenced by Lie and Ernest Vessiot). Toward this end Ehresmann introduced the concept of *fiber bundles* (independently of Whitney and Steenrod). Ehresmann published his first notes on the subject during the period 1941-1944 in which he defines *locally trivial principal bundles* and their *associated fiber bundles* (also locally trivial¹). In his 1943 paper *Sur les espaces fibrés associés à une variété différentiable* a manifold is defined by means of an atlas of local charts for the first time.[8]

A main characteristic of *principal bundles* is that they are fiber bundles along with a Lie group G which acts freely on the fibers. This group essentially tells what kind of transformations are allowed between the model space F and the fibers. The group G is referred to as the structure group of the bundle. Thus, a *principal bundle* is often referred to as a *principal G -bundle*. An important problem which Ehresmann worked on was reduction of the structure group G of a *principal G -bundle* to a subgroup H of G . Ehresmann proved that the reduction of a structure group G is equivalent to the existence of a section in an associated bundle with standard fiber G/H . [8] Furthermore, given a *principal G -bundle* (P, M, G) and a subgroup H of G , it can happen that $(P, M, G) \cong (P, M, H)^2$. This reduction

¹Topological fiber bundles over general spaces may or may not be locally trivial. However, (C^∞) fiber bundles are assumed to be locally trivial since the base space is assumed to be paracompact and paracompactness implies local triviality.

²To obtain this equivalence it is necessary that the G -valued cocycle on M takes values only in H .

of the structure group is such an important concept because the smaller the subgroup, the more restricted are the kinds of transformations which are allowed between the model space and the fibers (i.e., the smaller the group structure, the more you know!).

3.3 A Minimal Definition

In *Les connexions infinitésimales dans un espace fibré différentiable*, published in 1951, Charles Ehresmann defines an *infinitesimal connection* (known today as an *Ehresmann connection*) on a locally trivial fiber bundle to be a distribution of n -contact elements (where n is the dimension of the base space) which is transverse to the fibers and satisfies the path lifting property. In the case of a principal G -bundle, Ehresmann adds the condition that “the distribution must be invariant under the right action of G ” and goes on to show that “the connection associates an infinitesimal displacement to any vector tangent to the base.” [8]

Ehresmann’s concept can be described in modern terms as follows: Let E be a differentiable (C^∞) fiber bundle (E, π, M, F) . An Ehresmann connection on E is a subbundle $\mathcal{H}$ of TE , which is complementary to $\ker(\pi_*)$. In other words, for each $v \in E$, $T_v E = \ker(\pi_*) \oplus \mathcal{H}_v$. The bundle $\mathcal{H}$ is called the horizontal bundle and the bundle $\ker(\pi_*)$ is called the vertical bundle. The path lifting property says that given any smooth path γ on M joining $x_0 = \gamma(t_0)$ to $x_1 = \gamma(t_1)$, and any $v_0 \in E_{\gamma(t_0)}$, there exists a horizontal smooth path $\bar{\gamma}$ joining $v_0 \in \pi^{-1}(x_0)$ to $v_1 \in \pi^{-1}(x_1)$ which projects on γ by π (i.e., $\pi(v(t)) = \gamma(t)$ for all t).

To see a little more of the situation, consider an open covering $\mathcal{U}$ of M . For any open set U in $\mathcal{U}$, there exists a diffeomorphism (given by a bundle chart) between $\pi^{-1}U$ and $U \times F$. Thus, each element in $\pi^{-1}U$ can be identified with a pair $(p, \mathbf{v})$; p tells where you are (horizontally) relative to the base M and $\mathbf{v}$ tells where you are (vertically) in the fiber over p . Let x^i be local coordinates in U and let y^j be coordinates along the fibers in $\pi^{-1}U$. Then, for $p \in M|_U$ we have $p = (x^1, x^2, \dots, x^n)$ and for $\mathbf{v} \in E|_{\pi^{-1}U}$ we have $\mathbf{v} = (y^1, y^2, \dots, y^n)$. Now, for some other open set $U' \in \mathcal{U}$ we have (via another bundle chart) coordinates $x^{i'}$ in U' and $y^{j'}$ along the fibers in $\pi^{-1}U'$. So, $p \in M|_{U'}$ implies $p = (x^{1'}, x^{2'}, \dots, x^{n'})$ and $\mathbf{v} \in E|_{\pi^{-1}U'}$

implies $\mathbf{v} = (y^{1'}, y^{2'}, \dots, y^{n'})$. If $U \cap U' \neq \emptyset$ say, $U \cap U' = T$, then on $\pi^{-1}T$ we have $p = x^{i'}(x^i)$ and $\mathbf{v} = y^{j'}(x^i, y^j)$.

The choice of a chart induces vector fields $\frac{\partial}{\partial x^i}$ on M and $\frac{\partial}{\partial y^j}$ on E with the transformation laws:

$$\frac{\partial f(x', y')}{\partial x^{i'}} = \frac{\partial f(x, y)}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} + \frac{\partial f(x, y)}{\partial y^j} \frac{\partial y^j}{\partial x^{i'}}; \quad \frac{\partial f(x', y')}{\partial y^{j'}} = \frac{\partial f(x, y)}{\partial y^j} \frac{\partial y^j}{\partial y^{j'}}$$

Therefore,

$$\frac{\partial}{\partial x^{i'}} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} + \frac{\partial y^j}{\partial x^{i'}} \frac{\partial}{\partial y^j}; \quad \frac{\partial}{\partial y^{j'}} = \frac{\partial y^j}{\partial y^{j'}} \frac{\partial}{\partial y^j}$$

and the inverse:

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial}{\partial x^{i'}} + \frac{\partial y^{j'}}{\partial x^i} \frac{\partial}{\partial y^{j'}}; \quad \frac{\partial}{\partial y^j} = \frac{\partial y^{j'}}{\partial y^j} \frac{\partial}{\partial y^{j'}}$$

Vector fields $\frac{\partial}{\partial y^j}$ span what is called the vertical subspace in TE. This subspace, consisting of vectors tangent to the fibers, is independent of the choice of chart. There are no *a priori* distinguished “horizontal” subspaces. The subspaces determined by vector fields $\frac{\partial}{\partial x^i}$ change when the charts change. A preferred distribution of horizontal subspaces is given by an *Ehresmann connection*. Once such a connection is given, vector fields $\frac{\partial}{\partial x^i}$ tangent to M can be lifted to horizontal vector fields $\bar{\frac{\partial}{\partial x^i}}$ tangent to E.

In light of Ehresmann’s definition, a connection can be seen as a purely geometric object. All the concepts attached to that of a connection can be interpreted in this context and we can make a top down classification of all connections via restrictions on the group G .

CHAPTER 4

MODERNITY

Thanks to Charles Ehresmann there is now a consensus on what a connection really is (a distribution of horizontal subspaces), as well as a consistent terminology.

4.1 Equivalence

At this point the reader might wonder (as I did) how in the world this abstract concept of a particular distribution on a fiber bundle is related to the concept of covariant differentiation or parallel transport which have also been (and still sometimes are) called connections. The simple answer is that a system of parallel transport and a covariant derivative operator each result in a distribution of horizontal subspaces. In fact, on any differentiable fiber bundle there is a bijective correspondence between a connection $\mathcal{H}$ and a parallelism structure $\mathbb{P}$. On any vector bundle there is a bijective correspondence between a covariant derivative operator ∇^1 and a connection $\mathcal{H}$. For proofs that all three are equivalent geometric concepts see [10].

4.2 Terminology

A connection is said to live “on” a fiber bundle or “in” the tangent bundle to a fiber bundle. So, a connection on E is a connection in TE . However, connections were originally studied only in the tangent bundle case and were referred to as on a manifold. This terminology stuck even after the generalization of connections on manifolds to connections on fiber bundles, which is why today when the literature refers to a connection on M , one assumes this to mean that the connection is in TTM rather than a more general TE which makes sense anyway because if a bundle is not specified, the only one that is naturally present over M is TM .

¹In 1950 Jean-Louis Koszul unified the various notions of covariant differentiation in general vector bundles which arose in the 1940’s and also showed a bijective correspondence between ∇ and $\mathbb{P}$

4.3 Conclusions

Before I studied the history of connections I did not understand (nor quite believe) why Differential Geometry is sometimes described as being “the study of a connection on a principal bundle”. The trueness of the statement is now quite obvious. After all, Differential Geometry takes place on a differential manifold (which is naturally part of a principal bundle, TM) and a connection is the indispensable structure which allows for both the “differential” aspect and the “geometry” aspect.

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APPENDICES

APPENDIX A

Note on affine connections

When Weyl generalized the Levi-Civita connection to obtain linear parallel transport (with zero torsion) in a vector bundle, he called it an “affine” connection. Shortly afterward, Cartan dropped the symmetry requirement on the Christoffel symbols to obtain arbitrary linear parallel transport (with the possibility of torsion) in a vector bundle which he also referred to as an “affine” connection, with the comment that “The expression ‘affine connection’ is borrowed from H. Weyl, although it will be used here in a more general context.”

Cartan’s generalization reflects the role played by the group of affine transformations which can be seen as an extension of the linear group to include translations. In essence, admitting parallel translation as a legitimate motion changes the automorphism group of $\mathbb{R}^n$ from GL_n to A_n . This led to the modern definition of *affine connection* as a non-linear connection with structure group $GL_n \leq G \leq A_n$.

Today the term “affine” is used mostly in the sense of Weyl to refer to linear parallel transport (without torsion). However, close attention should be paid to the context whenever the term *affine* is used since aside from the three formulations given above, there are at least two more (lesser used) definitions of *affine connection* (see [10]).

APPENDIX B

Note on Cartan connections

In this thesis, I have not delved too deeply into *Cartan connections*. Cartan connections are a specialized type of principal connections which are quite rigidly tied to the underlying geometry of the base manifold by what by what Cartan refers to as a “soldering” of the total space to the base space. All of the technical ways in which Cartan connections are related to other types of connections can be found by reading Cartan’s work. But, be forewarned, as Weyl himself, in reviewing one of Cartan’s books, stated “Cartan is undoubtedly the greatest living master in Differential Geometry...I must admit that I found the book, like most of Cartan’s papers, hard reading...”[3] a sentiment that I found over and over again in reference to Cartan’s work. Thus, I would recommend the following sources for information on *Cartan connections*.

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APPENDIX C

Chart of connections

