

# PS2

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## 1. Operations on tensors

a.

Let  $S$  and  $T$  be  $(p, q)$  tensors. Show that  $S + T$  is also a  $(p, q)$  tensor.

b.

Let  $S$  be a  $(p, q)$  tensor and  $T$  be an  $(r, s)$  tensor. Show that  $S \otimes T$  is a  $(p+r, q+s)$  tensor.

c.

Let  $T$  be a  $(p, q)$  tensor. Show that  $T^{\rho\mu_1\cdots\mu_{p-1}}_{\rho\nu_1\cdots\nu_{q-1}}$  is a  $(p-1, q-1)$  tensor.

c.

Let  $T$  be a  $(p, 0)$  tensor. Show that  $T_{(\mu_1\cdots\mu_p)}$  is a symmetric tensor and  $T_{[\mu_1\cdots\mu_p]}$  is an antisymmetric tensor, i.e. show that the symmetry/antisymmetry is preserved by coordinate transformations.

d.

Let  $T$  be a  $(p, q)$  tensor. Show that  $\partial_\rho T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}$  is a  $(p, q+1)$  tensor when restricted to inertial frames, i.e. when the coordinate transformations are Poincare transformations.

## 2. Levi-Civita tensor

The Levi-Civita symbol  $\tilde{\epsilon}_{\mu_1\cdots\mu_n}$  is totally antisymmetric and defined such that  $\tilde{\epsilon}_{01\cdots(n-1)} = 1$  in all coordinate systems. Although the Levi-Civita symbol is invariant under Lorentz transformations, it doesn't transform as a tensor under general coordinate transformations. On the other hand, the following object is a tensor in all coordinates systems:

$$\epsilon_{\mu_1\cdots\mu_n} = \sqrt{-g}\tilde{\epsilon}_{\mu_1\cdots\mu_n}$$

where  $g$  is the determinant of the metric  $g_{\mu\nu}$ . Verify that

$$\epsilon_{\mu'_1 \dots \mu'_n} = \frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\mu'_1}} \dots \frac{\partial x^{\mu_n}}{\partial \tilde{x}^{\mu'_n}} \epsilon_{\mu_1 \dots \mu_n}$$

where  $\epsilon_{\mu_1 \dots \mu_n}$  is the Levi-Civita tensor in the  $x$  coordinate system and  $\epsilon_{\mu'_1 \dots \mu'_n}$  is the Levi-Civita tensor in the  $x'$  coordinate system. Show that the volume element

$$\epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_n} \equiv \sqrt{-g} dx^0 \wedge dx^1 \wedge \dots \wedge dx^{n-1}$$

is invariant under coordinate transformations.

### 3. Maxwell's equations in inertial coordinates

Show that in an inertial coordinate system, Maxwell's equations can be written as follows:

$$\partial_\alpha F^{\alpha\beta} = J^\beta, \quad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

(in appropriate units) where  $J$  is the current 4-vector. The Maxwell stress-energy tensor is defined as

$$T^{\mu\nu} = F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \eta^{\mu\nu}$$

where  $F^\nu{}_\alpha = \eta_{\mu\alpha} F^{\nu\mu}$ ,  $F_{\alpha\beta} = \eta_{\nu\alpha} \eta_{\mu\beta} F^{\nu\mu}$ . Assuming that  $J$  vanishes, show that

$$\partial_\mu T^{\mu\nu} = 0.$$

### 4. Energy-momentum tensor of a perfect fluid

The energy momentum tensor of a perfect fluid in Minkowski space is given by

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu - p \eta^{\mu\nu}$$

where  $\rho$  is the rest-frame energy density,  $U^\mu$  is the 4-velocity of the fluid, and  $p$  is the rest-frame pressure. The motion of the fluid is described by the conservation equation

$$\partial_\mu T^{\mu\nu} = 0$$

By projecting this equation onto  $U_\nu$  ( $U_\nu \partial_\mu T^{\mu\nu} = 0$ ), show that

$$\partial_\mu (\rho U^\mu) + p \partial_\mu U^\mu = 0. \quad (1)$$

By projecting onto the 3d space perpendicular to  $U_\nu$  using the tensor

$$P_{\mu\nu} = U_\mu U_\nu - g_{\mu\nu},$$

show that

$$(\rho + p) \frac{dU^\mu}{d\tau} = -P^{\mu\nu} \partial_\nu p \quad (2)$$

where  $\tau$  is the proper time of the fluid. Approximating the velocity of the fluid by  $U^\mu \sim (1, \vec{w})$  show that in the non-relativistic limit, eqs. 1 and 2 reduce to

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{w}) = 0, \quad \rho \partial_t \vec{w} + \rho (\vec{w} \cdot \vec{\nabla}) \vec{w} = -\vec{\nabla} p.$$