

## Lecture 9

# Covariant electrodynamics

# 1. Lorentz Group

Consider **Lorentz transformations** – pseudo-orthogonal transformations in 4-dimensional vector space (Minkowski space)

4-vectors:  $x = (ct, \mathbf{r})^T \equiv (t, \mathbf{x})^T, \quad q = (\omega/c, \mathbf{q})^T \quad (1)$

Notation:

**T = transposition**  $(x_1, x_2, x_3, x_4)^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

with **contravariant components**

(upper indices)

$$\begin{aligned} x^0 &= ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z; \\ q^0 &= \omega/c, \quad q^1 = q_x, \quad q^2 = q_y, \quad q^3 = q_z \end{aligned} \quad (2)$$

Note: **covariant components** (lower indices) are defined by

$$\begin{aligned} x_0 &= ct, \quad x_1 = -x, \quad x_2 = -y, \quad x_3 = -z \\ q_0 &= \omega/c, \quad q_1 = -q_x, \quad q_2 = -q_y, \quad q_3 = -q_z \end{aligned} \quad (3)$$

The transformation between covariant and contravariant components is

$$x_\mu = \sum_{\nu=0}^3 g_{\mu\nu} x^\nu =: g_{\mu\nu} x^\nu \quad \mu = 0, 1, 2, 3 \quad (4)$$

# 1. Lorentz Group

---

**Pseudometric tensor:**

$$(g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (5)$$

**For any space-time vector we get:**

$$\sum_{\mu,\nu=0}^3 g_{\mu\nu} x^\mu x^\nu = \sum_{\mu=0}^3 x^\mu x_\mu = \sum_{\mu=0}^3 x'^\mu x'_\mu. \quad (6)$$

where the vector  $x'$  is the result of a **Lorentz transformation**  $\Lambda^\mu_\nu$ :

$$x'^\mu = \sum_{\nu} \Lambda^\mu_\nu x^\nu; \quad \mu, \nu = 0, 1, 2, 3 \quad (7)$$

such that

$$\sum_{\mu=0}^3 x^\mu x_\mu = c^2 t^2 - \mathbf{r}^2 = \text{const.} \quad (8)$$

• **Pseudo-orthogonality relation:**

$$\sum_{\mu,\nu} g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \quad (9)$$

which implies that the **pseudometric tensor is Lorentz invariant** !

# 1. Lorentz Group

---

In matrix form:

$$\Lambda^T g \Lambda = g \quad (10)$$

where  $g^2 = 1_4$ , and  $1_4$  is the 4x4 unitary matrix  $\rightarrow (g \Lambda^T g) \Lambda = 1_4$  (11)

The **inverse Lorentz transformation** reads:

$$\Lambda^{-1} = g \Lambda^T g. \quad (12)$$

From (9), (6) we obtain:

$$\sum_{\mu} x'^{\mu} x'_{\mu} = \sum_{\mu\nu} g_{\mu\nu} x'^{\mu} x'^{\nu} = \sum_{\mu,\nu,\rho,\sigma} g_{\mu\nu} \Lambda^{\mu}_{\rho} x^{\rho} \Lambda^{\nu}_{\sigma} x^{\sigma} = \sum_{\rho\sigma} g_{\rho\sigma} x^{\rho} x^{\sigma} = \sum_{\rho} x_{\rho} x^{\rho}. \quad (13)$$

For the **transformation in  $x^1$  direction with velocity  $\beta = v/c$**   
the transformation matrix is given by

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (14)$$

with determinant

$$\det \Lambda = \gamma^2 - \gamma^2 \beta^2 = \gamma^2 (1 - \beta^2) = 1 \quad (15)$$

$$\gamma^2 = 1/(1-\beta^2)$$

# 1. Lorentz Group

---

## Properties of Lorentz transformations:

### 1) Two Lorentz transformations

$$x'^{\mu} = \sum_{\nu} \Lambda^{\mu}_{\nu} x^{\nu}, \quad x''^{\rho} = \sum_{\mu} \Lambda'^{\rho}_{\mu} x'^{\mu}, \quad (16)$$

applied successively

$$x''^{\rho} = \sum_{\mu, \nu} \Lambda'^{\rho}_{\mu} \Lambda^{\mu}_{\nu} x^{\nu} = \sum_{\nu} \Lambda''^{\rho}_{\nu} x^{\nu} \quad (17)$$

appear as a single Lorentz transformation. In matrix form:

$$\Lambda''^T g \Lambda'' = (\Lambda' \Lambda)^T g (\Lambda' \Lambda) = \Lambda'^T \underbrace{\Lambda'^T g \Lambda'}_g \Lambda = \Lambda'^T g \Lambda = g, \quad (18)$$

$$\Lambda'^T g \Lambda' = \Lambda^T g \Lambda = g \quad (19)$$

2) The neutral element is the 4x4 unitary matrix  $1_4$  for a Lorentz boost with  $v=0$

3) For each  $\Lambda$  exists the inverse transformation:

$$\det (\Lambda^T g \Lambda) = \det \Lambda^T \det g \det \Lambda = (\det \Lambda)^2 \det g = \det g = -1, \quad (20)$$

$$(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1 \neq 0. \quad (21)$$

4) The matrix multiplication is associative  $\rightarrow$  the Lorentz transformation is

**associative also**, ( i.e. the order in which the operations are performed does not matter as long as the sequence of the operands is not changed.)

## 2. Lorentz Group: scalars, vectors, tensors

---

1) **Lorentz-Scalars** transform as:

$$\Psi \xrightarrow{\Lambda} \Psi' = \Psi \quad (22)$$

**Lorentz transformation**

$\Psi$ : e.g. electric charge

2) **Lorentz-Vectors** are defined by the transformation properties as:

$$A^\mu \rightarrow A'^\mu = \sum_\nu \Lambda^\mu{}_\nu A^\nu \quad (23)$$

→ Transformation of covariant components:

$$\begin{aligned} A'_\mu &= \sum_\nu g_{\mu\nu} A'^\nu = \sum_{\nu,\rho} g_{\mu\nu} \Lambda^\nu{}_\rho A^\rho \\ &= \sum_{\nu,\rho,\sigma} g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma} A_\sigma = \sum_\sigma (g\Lambda g)_\mu{}^\sigma A_\sigma \\ &= \sum_\sigma (\Lambda^{-1})^\sigma{}_\mu A_\sigma, \end{aligned} \quad (24)$$

## 2. Lorentz Group: vectors, tensors

---

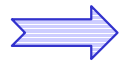
### 3) Lorentz-Tensor

Transformation of the covariant components of a Lorentz tensor is defined as:

$$T'^{\mu\nu} = \sum_{\rho,\sigma} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma} \quad (25)$$

→ Transformation of contravariant-covariant components:

$$T_\mu{}^\nu = \sum_\rho g_{\mu\rho} T^{\rho\nu}, \quad T^{\mu\nu} = \sum_{\rho,\sigma} g^{\mu\rho} g^{\nu\sigma} T_{\rho\sigma}. \quad (26)$$



$$T'^\mu{}_\nu = \sum_{\rho,\sigma} (\Lambda^{-1})^\mu{}_\rho \Lambda^\nu{}_\sigma T^\rho{}_\sigma. \quad (27)$$

### 4) Higher tensor products (Kronecker products)

e.g. tensor of 3rd order:  $C_\mu{}^\nu{}_\rho = A_\mu T^\nu{}_\rho$  (28)

$A_\mu$  is a vector-component

$T^\nu{}_\rho$  is a tensor-component

## 2. Lorentz Group: vectors, tensors

---

### 5) Covariant trace

Consider e.g. the trace of a **tensor of 2nd order**:  $\Psi = \sum_{\mu} T_{\mu}^{\mu}$  (29)

After Lorentz transformation it has to transform as a scalar:

$$\Psi' = \sum_{\mu} T_{\mu}'^{\mu} = \sum_{\mu, \nu, \rho} (\Lambda^{-1})^{\nu}_{\mu} \Lambda^{\mu}_{\rho} T_{\nu}^{\rho} = \sum_{\nu, \rho} \delta^{\nu}_{\rho} T_{\nu}^{\rho} = \sum_{\nu} T_{\nu}^{\nu} = \Psi \quad (30)$$

Cf. Eq.(27)

### 6) Fundamental Levi-Civita-Pseudo-tensor (or permutation tensor)

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu, \nu, \rho, \sigma) \text{ is an even permutation of } (0, 1, 2, 3) \\ -1 & \text{if } (\mu, \nu, \rho, \sigma) \text{ is an odd permutation of } (0, 1, 2, 3) \\ 0 & \text{in all other cases} \end{cases} \quad (31)$$

Pseudo-tensor,

since  $\epsilon^{\alpha\beta\gamma\delta} = \sum_{\mu, \nu, \rho, \sigma} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} \epsilon_{\mu\nu\rho\sigma} = \det g \epsilon_{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta}$  (32)

After Lorentz transformation we get:

$$\epsilon'^{\mu\nu\rho\sigma} = \sum_{\alpha, \beta, \gamma, \delta} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} \Lambda^{\rho}_{\gamma} \Lambda^{\sigma}_{\delta} \epsilon^{\alpha\beta\gamma\delta} = \det \Lambda \epsilon^{\mu\nu\rho\sigma} \quad (33)$$

## 2. Lorentz Group: vectors, tensors

---

### ■ Covariant differentiation:

$$\partial'_\mu \Psi' =: \frac{\partial \Psi'}{\partial x'^\mu} = \sum_\nu \frac{\partial \Psi}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} = \sum_\nu (\Lambda^{-1})^\nu{}_\mu \frac{\partial \Psi}{\partial x^\nu} \quad (34)$$

Since  $\Psi'$  is a Lorentz scalar and

$$x^\mu = \sum_\rho (\Lambda^{-1})^\mu{}_\rho x'^\rho \quad (35)$$

➤ the 4-derivative is a covariant **4-vector**:  $\partial_\mu \psi = \frac{\partial \psi}{\partial x^\mu} \quad (36)$

**Note:** the total derivative is a **scalar**:  $d\psi = \sum_\mu dx^\mu \frac{\partial \psi}{\partial x^\mu} = \sum_\mu dx^\mu \partial_\mu \psi \quad (37)$

➤ The 4-divergence of a Lorentz vector is a **Lorentz scalar**:

$$\sum_\nu \partial'_\nu A'^\nu = \sum_{\mu,\nu,\sigma} (\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\sigma \partial_\mu A^\sigma = \sum_{\mu,\rho} \delta^\mu{}_\rho \partial_\mu A^\rho = \sum_\mu \partial_\mu A^\mu \quad (38)$$

## 2. Lorentz Group: vectors, tensors

---

If one introduces the **4-vector**  $A_\mu = \partial_\mu \Psi = \frac{\partial \Psi}{\partial x^\mu},$  (39)

it follows from (38):  $\sum_{\mu,\nu} g^{\mu\nu} \partial_\mu A_\nu = \sum_{\mu,\nu} g^{\mu\nu} \partial_\mu \partial_\nu \Psi = \sum_{\mu,\nu} g^{\mu\nu} \partial'_\mu \partial'_\nu \Psi'$  (40)

Thus the **D'Alembert-Operator**

$$\sum_{\mu,\nu} g^{\mu\nu} \partial_\mu \partial_\nu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2, \quad \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (41)$$

is also **invariant under Lorentz transformations**, i.e.

$$\sum_{\nu\sigma} g^{\nu\sigma} \frac{\partial^2}{\partial x^\nu \partial x^\sigma} A_\mu = \sum_{\nu\sigma} g^{\nu\sigma} \partial_\nu \partial_\sigma A_\mu = \sum_\nu \partial_\nu \partial^\nu A_\mu \quad (42)$$

**In general: The scalar product of two Lorentz vectors is a Lorentz scalar:**

$$\sum_\mu A'^\mu B'_\mu = \sum_\mu \sum_{\sigma\rho} \Lambda^\mu{}_\rho A^\rho (\Lambda^{-1})^\sigma{}_\mu B_\sigma = \sum_{\rho,\sigma} A^\rho \delta^\sigma_\rho B_\sigma = \sum_\rho A^\rho B_\rho. \quad (43)$$

### 3. Vector current

The continuity equation:

$$\vec{\nabla} \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0. \quad (44)$$

New notation:  $j^0 = c\rho; \quad j^1 = j_x; \quad j^2 = j_y; \quad j^3 = j_z$  (45)

$\rho$  is the **charge density** and  $j_1, j_2, j_3$  are the **3-components of the current**  
In covariant notation we get:

$$\sum_{\mu} \partial_{\mu} j^{\mu} = \sum_{\mu} \frac{\partial}{\partial x^{\mu}} j^{\mu} = 0. \quad (46)$$

• In a system with a charge distribution **at rest** we have:

$$j'^0 = c\rho_0, \quad j'^1 = j'^2 = j'^3 = 0. \quad (47)$$

As a covariant component of a 4-vector,  $j_{\mu}$  must be also invariant under **Lorentz transformations** with velocity  $\beta = v/c$  in  **$x_1$ -direction**:

$$x^0 = \gamma(\beta x'^1 + x'^0); \quad x^1 = \gamma(x'^1 + \beta x'^0); \quad x^2 = x'^2; \quad x^3 = x'^3, \quad (48)$$

and transforms as

with

$$\begin{aligned} j^0 &= c\gamma\rho_0; & j^1 &= \gamma\rho_0 v; & j^2 &= 0; & j^3 &= 0. \\ \rho &= \gamma\rho_0. \end{aligned} \quad (49)$$

### 3. Vector current

---

The Lorentz transformation of the **volume** from frame  $\Sigma$  to  $\Sigma'$  reads:

$$dV = \frac{dV_0}{\gamma} \quad (50)$$

One obtains in any frame the **charge invariance**:

$$Q = \int_V dV \rho = \int \frac{dV_0}{\gamma} \gamma \rho_0 = \int dV_0 \rho_0 \quad (51)$$

## 4. The four-potential

---

- Consider the **vector potential**  $\mathbf{A}$  and **scalar potential**  $\Phi$

**The Lorentz gauge reads:**

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0 \quad (52)$$

**The wave equation:**

$$\sum_{\mu} \partial_{\mu} \partial^{\mu} \mathbf{A} = \mu_0 \mathbf{j}; \quad \sum_{\mu} \partial_{\mu} \partial^{\mu} \Phi = -\frac{\rho}{\epsilon_0} \quad (53)$$

$\epsilon_0$  is the electric constant,  $\mu_0$  is the magnetic constant.

One can also write in **covariant notation**:

$$(A^{\mu}) = \left( \frac{1}{c} \Phi, \mathbf{A} \right)^T \quad (54)$$

using that  $\epsilon_0 \mu_0 = c^{-2}$  (55)

Then Eq. (53) reads: 
$$\sum_{\nu} \partial_{\nu} \partial^{\nu} A^{\mu} = \mu_0 j^{\mu} \quad (56)$$

The **Lorentz covariance** of Eq. (52) can be also written in the form:

$$\sum_{\mu} \partial_{\mu} A^{\mu} = \sum_{\mu} \frac{\partial}{\partial x^{\mu}} A^{\mu} = 0 \quad (57)$$

# 4. The four-potential

## Consequences:

- Lorentz invariance of the continuity equation and gauge condition:

If in the frame  $\Sigma$

$$\sum_{\mu} \frac{\partial}{\partial x^{\mu}} j^{\mu} = 0; \quad \sum_{\mu} \frac{\partial}{\partial x^{\mu}} A^{\mu} = 0 \quad (57)$$

then in the frame  $\Sigma'$  we also have

$$\sum_{\mu} \frac{\partial}{\partial x'^{\mu}} j'^{\mu} = 0; \quad \sum_{\mu} \frac{\partial}{\partial x'^{\mu}} A'^{\mu} = 0. \quad (58)$$

Since eq. (53) is covariant, the same result will be in  $\Sigma$  and  $\Sigma'$

$$\sum_{\nu} \partial'_{\nu} \partial'^{\nu} A'_{\mu} = \mu_0 j'_{\mu}, \quad (59)$$

thus

$$\sum_{\nu, \rho} \Lambda^{\mu}_{\nu} \partial_{\rho} \partial^{\rho} A^{\nu} = \sum_{\rho} \partial_{\rho} \partial^{\rho} A'^{\mu} = \sum_{\rho} \partial'_{\rho} \partial'^{\rho} A'^{\mu} = \mu_0 \sum_{\nu} \Lambda^{\mu}_{\nu} j^{\nu} = \mu_0 j'^{\mu}. \quad (60)$$

This is the covariance of classical electrodynamics!

## 5. Plane-waves

---

**Consider plane-waves in vacuum in the inertial system  $\Sigma$ :**

$$A^\mu = A^{(0)\mu} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] = A^{(0)\mu} \exp\left(-i \sum_{\lambda} k_{\lambda} x^{\lambda}\right) \quad (61)$$

**with**  $k^0 = \frac{\omega}{c}; \quad k^1 = k_x; \quad k^2 = k_y; \quad k^3 = k_z.$  (62)

**Due to the covariance of the homogenous wave equation**

$$\sum_{\nu} \partial_{\nu} \partial^{\nu} A^{\mu} = 0 \quad (63)$$

**a transformation to another system  $\Sigma'$  gives:**

$$A'^{\mu}(x'^{\rho}) = \sum_{\nu} \Lambda^{\mu}_{\nu} A^{\nu}(x^{\lambda}) = \left( \sum_{\nu} \Lambda^{\mu}_{\nu} A^{(0)\nu} \right) \exp\left(-i \sum_{\lambda} k_{\lambda} x^{\lambda}\right) = A'^{(0)\mu} \exp\left(-i \sum_{\rho} k'_{\rho} x'^{\rho}\right) \quad (64)$$

**Accordingly, the phase of the plane wave should be also invariant:**

$$\sum_{\lambda} k_{\lambda} x^{\lambda} = \sum_{\lambda} k'_{\lambda} x'^{\lambda} \quad (65)$$

**as in case of a point-like excitation, where the wave fronts are spherical surfaces moving with the velocity of light  $c$ .**

## 5. Plane-waves

---

Since the phase in Eq. (64) is an invariant scalar product, the  $k_\mu$  must be the covariant components of a 4-vector, which transform as:

$$\begin{aligned} k'_x &= \gamma \left( k_x - \frac{v}{c^2} \omega \right); & k'_y &= k_y; & k'_z &= k_z; \\ \omega' &= \gamma(\omega - vk_x). \end{aligned} \quad (66)$$

Using the dispersion relation:

$$\frac{\omega}{k} = c = \frac{\omega'}{k'}, \quad (67)$$

and denoting the angles  $\phi$  and  $\phi'$  with the direction of  $\mathbf{k}$  and  $\mathbf{k}'$  with respect to the direction of  $\mathbf{v}$  (i.e. the  $x$ -direction in case of (28)) we obtain:

$$\omega' = \gamma\omega(1 - \beta \cos \phi) \quad (68)$$

$$\cos \phi' = \frac{k}{k'} \gamma (\cos \phi - \beta) = \frac{\cos \phi - \beta}{1 - \beta \cos \phi}. \quad (69)$$

## 5. Plane-wave

---

Equation (68) describes the **Doppler-effect**, which apart from a **longitudinal effect**,

$$\omega' = \omega \frac{1 \mp \beta}{\sqrt{1 - \beta^2}} = \omega \left[ 1 \mp \beta + \mathcal{O}(\beta^2) \right] \quad (70)$$

for  $\beta \ll 1$  and  $\phi = 0, \pi$ ,

also implies a **transversal effect**

$$\omega' = \gamma\omega = \frac{\omega}{\sqrt{1 - \beta^2}} = \omega \left[ 1 + \frac{\beta^2}{2} + \mathcal{O}(\beta^4) \right] \quad (71)$$

for  $\phi = \pm\pi/2$

which is a typical relativistic phenomenon. This effect was shown experimentally in 1938 within the investigation of radiation from moving H-atoms. Well known is the longitudinal effect in the radiation of distant galaxies (**red shift**) which demonstrates that these objects are moving away from us.

Furthermore, both phenomena describe the **relativistic aberration of light**, i.e. the apparent change of position of light sources due to the motion of the earth around the sun relative to stars.

## 6. Transformation of the fields **E** and **B**

---

Once knowing the fields **A** and **Φ**, one can compute the **fields E and B** via

$$\mathbf{B} = \nabla \times \mathbf{A}; \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad (72)$$

Let's rewrite Eq. (72) in covariant form with coordinates  $x_\mu$  and the components of the 4-potential  $A_\mu$ . For example we obtain:

$$E_x = -\partial_1(cA^0) - c\partial_0A^1 = c(\partial_0A_1 - \partial_1A_0), \quad B_1 = \partial_2A^3 - \partial_3A^2 = -(\partial_2A_3 - \partial_3A_2). \quad (73)$$

Eq. (73) suggests to introduce the **antisymmetric field-strength tensor of second rank**

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = -F_{\nu\mu} \quad (74)$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & -B_z & B_y \\ -\frac{1}{c}E_y & B_z & 0 & -B_x \\ -\frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} \quad (75)$$

## 6. Transformation of the fields E and B

---

The contravariant components are

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & -B_z & B_y \\ \frac{1}{c}E_y & B_z & 0 & -B_x \\ \frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} \quad (76)$$

Now we know the transformation properties of the fields E and B since the contravariant components transform as (25)

$$F'^{\mu\nu} = \sum_{\rho,\sigma} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma} \quad (77)$$

For the special Lorentz boost (14) we obtain:

$$\begin{aligned} E'_x &= E_x; & B'_x &= B_x; & E'_y &= \gamma(E_y - vB_z); & B'_y &= \gamma(B_y + \frac{v}{c^2}E_z); \\ E'_z &= \gamma(E_z + vB_y); & B'_z &= \gamma(B_z - \frac{v}{c^2}E_y). \end{aligned} \quad (78)$$

---

$$(14): \quad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

transformation in  $x^1$  direction with velocity  $\beta = v/c$

## 6. Transformation of the fields E and B

---

For a Lorentz-Boost with **velocity  $\mathbf{v}$**  in arbitrary direction holds that the **parallel components (in direction of  $\mathbf{v}$ ) are conserved:**

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}; \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad (79)$$

while the **transverse components** transform as:

$$\mathbf{E}'_{\perp} = \gamma [\mathbf{E}_{\perp} + (\mathbf{v} \times \mathbf{B})]; \quad \mathbf{B}'_{\perp} = \gamma \left[ \mathbf{B}_{\perp} - \frac{1}{c^2} (\mathbf{v} \times \mathbf{E}) \right] \quad (80)$$

**The inversion**

$$\mathbf{E}_{\perp} = \gamma [\mathbf{E}'_{\perp} - (\mathbf{v} \times \mathbf{B}')]; \quad \mathbf{B}_{\perp} = \gamma \left[ \mathbf{B}'_{\perp} + \frac{1}{c^2} (\mathbf{v} \times \mathbf{E}') \right] \quad (81)$$

is obtained – in analogy to the coordinate transformation - by replacing  $\mathbf{v} \rightarrow -\mathbf{v}$ . Equations (80) and (81) demonstrate the connection between the electromagnetic fields E and B in different frames.

# 7. Maxwell equations

---

Now we can rewrite the Maxwell equations for the electromagnetic field in covariant form. We focus on the case of the vacuum and recall **the Maxwell equations in conventional notation**:

$$\text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{div } \mathbf{B} = 0, \quad (82)$$

$$\text{div } \mathbf{E} = \frac{1}{\epsilon_0} \rho, \quad \text{rot } \mathbf{B} = \mu_0 \left( \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \quad (83)$$

The equations (82) are the **homogeneous Maxwell equations**. They can be fulfilled by introducing scalar and vector potentials.

The equations (83) describe the **‘creation’ of the fields from electric charges and currents**. As seen before these equations can be written in covariant form of 4-tensor structure. The components of the field strength appear in the field-strength tensor (76), i.e. we have to express the equations in terms of this tensor.

## 7. Maxwell equations

---

Indeed, we can construct two independent covariant equations with first order derivatives:

$$\sum_{\mu} \partial_{\mu} F^{\mu\nu} \quad \text{and} \quad \sum_{\mu} \partial_{\mu} (\dagger F)^{\mu\nu}, \quad (84)$$

where  $(\dagger F)_{\mu}$  is the **dual tensor to  $F_{\mu}$** . The first equation we can evaluate by writing the field-strength tensor in the form (84) via the potentials and take care of the Lorentz gauge :

$$\sum_{\mu} \partial_{\mu} F^{\mu\nu} = \sum_{\mu} \partial_{\mu} (\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) = \sum_{\mu} \partial_{\mu} \partial^{\mu} A^{\nu} = \mu_0 j^{\nu} \quad (85)$$

These equations apparently correspond to the **inhomogeneous** Eqs. (83). When rewriting the dual tensor in terms of the potentials we obtain:

$$(\dagger F)^{\mu\nu} = \frac{1}{2} \sum_{\rho, \sigma} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{2} \sum_{\rho, \sigma} \epsilon^{\mu\nu\rho\sigma} (\partial_{\rho} A_{\sigma} - \partial_{\sigma} A_{\rho}) = \sum_{\rho, \sigma} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma}. \quad (86)$$

By calculating the 4-divergence of (86) we find that due to the antisymmetry of the Levi-Civita tensor  $\epsilon^{\mu\nu\rho\sigma}$  and the exchange properties of the derivatives we get:

$$\partial_{\mu} (\dagger F)^{\mu\nu} = 0 \quad (87)$$

## 7. Maxwell equations

---

The advantage of the **Maxwell equations** (85) and (87) is that they are **gauge independent**. Indeed, a change in the gauge for the potentials implies

$$\tilde{A}_\mu = A_\mu + \partial_\mu \chi \quad (88)$$

where  $\chi(x)$  is an **arbitrary scalar field**. In fact, the field-strength tensor does not depend on such gauge transformations as seen from

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \quad (89)$$

This implies that the 4-potentials  $A_\mu$  and  $\tilde{A}_\mu$  describe the same physical fields. The invariance of (85) and (87) with respect to gauge transformations (88) is the starting point of the *standard model of elementary particle physics*.

## 8. Coulomb field

---

The **field of a charge  $q$  at rest** in the frame  $\Sigma'$  is:

$$\mathbf{E}'(\mathbf{r}') = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}'}{r'^3}; \quad \mathbf{B}'(\mathbf{r}') = 0 \quad (90)$$

In a frame moving with velocity  $\mathbf{v} = (v, 0, 0)$  relative to  $\Sigma'$  we obtain:

$$\begin{aligned} E_x &= E'_x = \frac{q}{4\pi\epsilon_0} \frac{\gamma(x - vt)}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}}, \\ E_y &= \gamma E'_y = \frac{q}{4\pi\epsilon_0} \frac{\gamma y}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}}, \\ E_z &= \gamma E'_z = \frac{q}{4\pi\epsilon_0} \frac{\gamma z}{(\gamma^2(x - vt)^2 + y^2 + z^2)^{3/2}}. \end{aligned} \quad (91)$$

Here  $x', y', z'$  has been written explicitly as a function of  $x, y, z$  using the Lorentz transformation of the coordinates. The field appears as in  $\Sigma'$  as a central field but is no longer isotropic. The factor  $\gamma^2$  in the square root differentiates the  $x$ -direction relative the  $y$ - and  $z$ -direction.

## 8. Coulomb Field

---

An observer measures also the **magnetic field**:

$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}), \quad (92)$$

since the charge  $q$  is moving and thus generating a current. For illustration we consider the case  $\gamma \gg 1$ :

i) Close to the  $x$ -axis ( $y, z \approx 0$ ) we get

$$E_x \approx \frac{1}{\gamma^2} \frac{q}{4\pi\epsilon_0} \frac{1}{(x - vt)^2}, \quad E_y = E_z = 0, \quad (93)$$

which implies a **reduction of the field strength** by a factor  $\gamma^{-2}$ .

ii) In the plane parallel to the  $y - z$ -plane through  $q$  we get:

$$E_x = 0, \quad E_y = \frac{\gamma y}{(y^2 + z^2)^{3/2}}, \quad E_z = \frac{\gamma z}{(y^2 + z^2)^{3/2}}, \quad (94)$$

which implies an **enhancement of the transverse field strength** by a factor of  $\gamma$ .

➔ The radial field lines are thus thinner in the direction of motion whereas they are enhanced in transverse direction:

# Covariant electrodynamics

---

## Summary:

**The basic equations of electrodynamics are covariant with respect to Lorentz transformations and have the same form in all inertial systems thus following the Einstein principle of relativity.**