

ON THE GENESIS OF THE CONCEPT OF COVARIANT

DIFFERENTIATION

Luca DELL'AGLIO (*)

ABSTRACT. — The purpose of this paper is to reconsider the genesis of the concept of covariant differentiation, which is interpreted as arising out of two traditions running through 19th-century research work. While the first tradition, of an algebraic nature, was responsible for the “algorithmic” emergence of the concept, the second, analytical in character, was essentially concerned with the import of covariant differentiation as a broader kind of differentiation. The methodological contrast that these two traditions exhibit, concerning the use of algebraic and variational methods, was mainly evidenced in Ricci-Curbastro’s work, and was a significant factor in the genesis of tensor analysis. The emergence of the notion of covariant differentiation in his research work may, indeed, be interpreted as the resolution of that methodological contrast into the definitive form of a conceptual synthesis.

RÉSUMÉ. — SUR L’ORIGINE DU CONCEPT DE DÉRIVATION COVARIANTE. Cet article se propose d’interpréter l’origine du concept de dérivation covariante comme conséquence de deux traditions de recherche au XIX^e siècle. Alors que la première tradition, de nature algébrique, est à l’origine de l’émergence «*algorithmique*» du concept, la seconde, de caractère analytique, se rapporte essentiellement à la signification de la dérivation covariante comme extension ou généralisation de la dérivation usuelle. L’opposition méthodologique que manifestent ces deux traditions, à propos de l’utilisation de méthodes algébriques ou variationnelles, apparaît principalement dans l’œuvre de Ricci-Curbastro, et fut un facteur fondamental dans la genèse de l’analyse tensorielle. L’émergence de la notion de dérivation covariante dans son travail de recherche peut, de fait, être interprétée comme la résolution de cette opposition méthodologique sous la forme décisive d’une synthèse conceptuelle.

1. INTRODUCTION

Emerging at the end of the 19th century with the work of the Italian

(*) Texte reçu le 3 février 1995, révisé le 24 juillet 1996.

Luca DELL’AGLIO, Università degli studi della Calabria, Dipartimento di matematica, 87036 Arcavacata di Rende (Cs), Italia.

mathematician G. Ricci-Curbastro, absolute differential calculus (and subsequently tensor analysis) appears historically as one of the most important links between Riemann's concept of space and the relativistic theory of gravitation. As an extension of the usual calculus to general geometrical contexts, this theory indeed represents one of the most important developments of Riemann's geometrical conceptions in the latter part of the 19th century. On the other hand, with reference to such notions as that of covariant differentiation and of the tensor, the theory set out by the Italian mathematician pointed to the possibility of an invariant formulation of analytical problems, possibly of a physical nature: a technical possibility which was to play a leading role in the mathematical expression of Einstein's ideas, some decades later.

Bearing that in mind, the aim of this paper is to provide a reconstruction of the emergence of the first fundamental concept of absolute differential calculus — that of covariant differentiation — as marking the convergence of various research traditions in mathematical thought, prevailing in the 19th century. More specifically, this reconstruction is based on a number of historiographical tenets, concerning different instances of the impact of the idea of invariance, which I shall now detail.

First of all, one may hold that there was an “algorithmic” genesis of the concept of covariant differentiation, arising out of a purely algebraic research tradition. As already suggested by other authors,¹ prior to Ricci-Curbastro's work, this concept had originated in Christoffel's approach, as the result of a research tradition, consisting in the application of the methods of the theory of algebraic invariants to analytical matters. In this context, the algorithm of covariant differentiation was used by Christoffel as a well-defined technique in a particular field of research, that of differential quadratic forms: in particular, it had the specific function of allowing a general programme to be carried out, that of the “reduction” of the theory of differential invariants to that of algebraic forms. As we shall see, this research tradition had clearly exerted an influence on Ricci-Curbastro in a period before his work was directly concerned with the creation of the absolute differential calculus. Such a methodological influence is no chance feature: as we shall see, an embryonic form of the algebraic research

¹ In particular, see the recent book [Reich 1994].

tradition on differential invariants was already at work in the mathematical community of post-Unification Italy, with the geometrical work of Casorati.

On the other hand, despite the fundamental significance of the algebraic tradition with respect to differential invariants, it is tenable that the conceptual origin of covariant differentiation — as a generalisation of the usual differentiation — was independent of that tradition. The appearance of a Riemannian differentiation, indeed, finds its true justification only when one takes into account the emergence of a second research tradition, which to some extent ran counter to the former from a methodological point of view. More specifically, this second tradition was concerned with a close investigation of “differential parameters”, as arising out of the work of the French mathematician G. Lamé and developed mainly through the research work of E. Beltrami. This new tradition made its presence felt in the process leading to the emergence of absolute differential calculus, most recognisably when Christoffel’s research programme was extended by Ricci-Curbastro to the study of differential parameters.

The point that needs to be emphasised is the contrast inherent in such a switch in topics of investigation. At that time, indeed, the research tradition concerned with differential parameters was grounded, methodologically speaking, on the use of the calculus of variations and only partly on algebraic methods. This was no chance feature, since this second tradition was closely connected to the thrust of classical mathematical physics, and hence to the study of partial differential equations. As we shall see, Ricci-Curbastro effected the introduction of the concept of covariant differentiation precisely for the purposes of furthering the study of differential equations, his aim being to arrive at an invariant expression of these equations in order to simplify their investigation. It is this very cross-over of the contexts of interpretation and methods — i.e., to use modern terminology, the analytical interpretation of an algebraic technique introduced to tackle some analytical problems — that warranted the emergence of the concept of covariant differentiation.

Thus, the emergence and the very genesis of the concept of covariant differentiation appears as a specific synthesis of many research traditions concerning the idea of invariance, running through the 19th century: differential invariants, differential parameters and algebraic invariants. This

fact — which, indeed, means that absolute differential calculus, together with Klein's "Erlangen programme", represented one of the most significant products of the idea of invariance in 19th-century mathematical thought — was especially significant with regard to the physical aspects of invariance that were to emerge with general relativity.

And, as a final point, reconstruction of the genesis of the concept of covariant differentiation makes it possible, *post factum*, to examine the specific features of Ricci-Curbastro's scientific work and, more generally, of the Italian mathematical community's contribution as specific contexts for the appearance of absolute differential calculus. In effect, from a strictly historical point of view, one may view the present paper as a comparative study of some aspects of Ricci-Curbastro's work in differential geometry.

2. RESEARCH TRENDS IN THE 19TH-CENTURY THEORY OF DIFFERENTIAL INVARIANTS

As is well known, the context of research in which Ricci-Curbastro's analytical methods originated was provided by the theory of differential invariants, i.e. the study of differential quantities that are invariant with respect to any particular transformation of coordinates.² In this manner, the Italian mathematician's work may be considered as an aspect of a more general phenomenon — the pervasiveness of the idea of invariance — which was a characteristic feature of a large part of mathematics throughout the 19th century [Bell 1945, chap. 20].

In this general context — where concepts of geometrical and algebraic invariance were coming to the fore — the study of differential invariants reflected various analytical requirements associated with the idea of invariance. Indeed, the modern theory of differential invariants reached its unified form only at the beginning of the 20th century,³ as the outcome of many research traditions at work in the course of the 19th century.

² According to M. Kline, tensor analysis "*is actually no more than a variation on an old theme, namely, the study of differential invariants associated primarily with a Riemannian geometry*" [Kline 1972, p. 1122]. On this subject see also [Reich 1994, 4.1.2.1], [Tonolo 1954, pp. 2–6].

³ This may be considered to be a result of Klein's thought. See [Veblen 1927, p. 15].

Apart from the approach of G. Halphen [1878] and S. Lie [1884] — which emerged much later in the century — there are essentially two theoretical thrusts which were of major importance in this field.⁴

The first direction — which will be referred to here as the “restricted [or special] theory of differential invariants” — arose from the context of 19th-century differential geometry. In effect, this was a direct sequel of Gauss’s geometrical opinions: according to this tradition, a differential invariant is the analytical reflection of the intrinsic properties of surfaces (such as the line element, curvature, and the angle between two directions on a surface).

At the same time, more general invariants — the so-called “differential parameters” — were being studied by another line of research, arising out of the work of Lamé on the equations of classical mathematical physics. In this context, differential parameters are quantities — such as the Laplacian of a function — by means of which it is possible to show the invariance of specific differential equations, in a well-defined geometrical situation.

For quite some time, these research traditions developed, to a large extent, independently. They pursued similar aims but in different fields of research: intrinsic geometry, on the one hand, and the theory of partial differential equations, on the other. They actually converged only in the post-Riemannian period.

Although exhibiting different concerns and activities, the two thrusts of research into differential invariants shared one common methodological element. Both traditions, indeed, were characterised by the implementation of two distinct technical methodologies: the theory of algebraic forms, on the one hand, and the calculus of variations, on the other. The function of these theoretical methods was operational, involved as they both were in the demonstration of the invariance (with respect to particular transformations of coordinates) of known differential quantities and the search for new, analogous, quantities.

From an operational standpoint, this methodological duality was of no particular significance for the development of the theory of differential invariants: as we shall see later, apart from some particular cases, the

⁴ On this subject, see [Reich 1973], [Struik 1933], [Veblen 1927], [Vincensini 1972], [Weitzenböck 1921].

two methodological approaches coexisted without any significant problems. On the other hand, the use of well-defined theoretical methods in a new domain of research may not be considered in an abstract fashion, irrespective of their contexts of origin. In other words, the problem to be analysed is to what extent the use of certain theoretical tools implied an actual carrying over of their original conceptual features into the new research context.

Specifically, in the case of the history of the theory of differential invariants, the two theoretical methods (the algebraic and the variational) clearly exhibited different conceptual backgrounds.

2.1. The programme of “algebraic reduction” of the theory of differential invariants

Owing to the analogy with the properties of algebraic invariants, the use of algebraic instruments was of particular significance for the history of the restricted theory of differential invariants. More generally, this approach was a natural development in 19th-century mathematical thought, and found a ready place, on the basis of the leading role devolving to the theory of algebraic forms during this century and of the consequent tendency, evinced by quite a few mathematicians of the time, to extol the centrality of algebraic methods.

Casorati and “elimination theory”

A highly significant example of the tendency to use algebraic methods in the theory of differential invariants in pre-Riemannian times is to be found in the work of F. Casorati, one of the most important figures of post-Unification Italian mathematics. Indeed, although they are known essentially for their analytical concerns, his works included a long paper of a geometrical nature about differential invariants [Casorati 1860–1861],⁵ which may be viewed as one of the first systematic studies on the topic. This in particular may be argued from the fact that, in this context, there was an explicit and general definition of differential invariants. More specifically, a differential invariant — *funzione inalterabile* — was introduced by Casorati [*Ibid.*, p. 136] as any function f such that

$$(1) \quad f(a_{11}, a_{12}, a_{22}, \partial a_{11}/\partial x_1, \partial a_{11}/\partial x_2, \text{etc.})$$

⁵ About this paper see [Bertini 1892, pp. 1221–1222], [Vivanti 1935, p. 135].

$$= f(A_{11}, A_{12}, A_{22}, \partial A_{11}/\partial X_1, \partial A_{11}/\partial X_2, \text{etc.})$$

where the a_{rs} and A_{rs} are the coefficients of the differential quadratic forms representing the metric element of a surface for the two different coordinate systems, x_r and X_r .

In his pursuit of a systematic investigation, Casorati insisted on two methodological elements whose significance was to be realised later.

On the one hand, from a structural point of view, his study was based on a preliminary classification of differential invariants by *ordini*, where the order of this invariant was defined in terms of that of the derivatives of the coefficients a_{rs} included in it. Consequently, from a methodological point of view, Casorati's analysis of differential invariants exhibits a "vertical" approach in which the study of the first orders is given pride of place in a natural way.

On the other hand, as we have mentioned, Casorati's investigation of differential invariants was essentially algebraic in character, since it was based solely on the "elimination theory". This second element — induced by the development of algebraic studies in Italy in the period around Unification [Bottazzini 1980] — may be considered as stemming from a definite methodological choice:

*"The purpose of this short paper is to show a method of finding the fundamental equations for the investigation of absolute properties; among such equations, the main one is that which expresses Gauss's famous theorem on the measure of curvature It is easy to point out that the way chosen by Gauss and other distinguished geometers to prove this theorem . . . , does not make apparent as fully as may be wished the principal source of the significance of that theorem, that is the fact of its being the simplest [equation] of such a class; nor [does it make apparent] those that actually follow it in order of simplicity."*⁶

⁶ "Scopo di questo breve lavoro è l'esposizione di un modo di trovare le equazioni fondamentali per lo studio delle proprietà assolute; fra le quali equazioni la principale è quella esprimente il celebre teorema di Gauss sulla misura della curvatura È facile rilevare che la via tenuta da Gauss nel dimostrare il detto teorema, e quelle seguite da altri insigni Geometri . . . , non mettono in tutta l'evidenza desiderabile nè ciò donde viene principalmente la importanza del medesimo, cioè di essere il più semplice possibile di siffatta categoria; nè quali indubbiamente sieno quelli che tengangli dietro per ordine di semplicità" [Casorati 1860, p. 134].

This critical attitude of Casorati towards earlier investigations of differential invariants arose out of his opinions concerning the significance of algebraic methods for the treatment of differential questions:

*“On these grounds especially, I believe the method to be worthy of some attention, that I am propounding here, which, consisting in a process of elimination without exceptions, necessarily leads us to find the equations sought for one after the other and exactly in the order of their importance.”*⁷

From a technical point of view, Casorati's method was based on the elimination of the X_h -derivatives of the functions x_r from the laws of transformation for the coefficients a_{rs}

$$(2) \quad \sum_r \sum_s a_{rs} \frac{\partial x_r}{\partial X_h} \frac{\partial x_s}{\partial X_k} = A_{hk} \quad (h, k = 1, 2),$$

and from their successive X_r -derivatives. This, indeed, makes it possible to obtain certain expressions that are independent of the coordinate system chosen, as containing only the quantities a_{rs} and their derivatives.

Through this algebraic method of resolution, Casorati was able to determine a large number of differential invariants of surfaces. In particular, he discussed the search for differential invariants up to the fourth order, first proving the non-existence of such invariants for the first order; going on to construct an invariant for the second (Gaussian curvature) and third orders and, finally, providing a well-defined procedure to obtain those of the fourth order.

It must be pointed out that this simple method of elimination was to be a characteristic instrument in much of the subsequent research on this subject, where algebraic methods were brought to bear on differential questions.

The methodological contrast between Lipschitz and Christoffel

As is well known, in the 19th century, the main impulse leading to the development of the theory of differential invariants came from Riemann's conceptions. More specifically, the lead came from the publication of the

⁷ *“È specialmente per questo riguardo ch'io credo possa meritare qualche attenzione il metodo che espongo, il quale, consistendo in un processo di eliminazione non soggetto ad eccezioni, conduce per necessità a trovare le equazioni in discorso l'una dopo l'altra precisamente in quell'ordine con cui si succedono nella importanza”* [Casorati 1860, p. 134].

German mathematician's *Habilitationsschrift* of 1868 [Riemann 1854]. The main technical topic of this paper — namely, the equivalence of differential quadratic forms (*Aequivalenzproblem*) — was in fact taken up by E.B. Christoffel and R. Lipschitz in papers published in the same issue of the “Journal de Crelle” one year later [Christoffel 1869a,b; Lipschitz 1869].⁸

Although bearing on the same technical argument,⁹ Christoffel's and Lipschitz's papers were methodologically quite different. Indeed, while Christoffel followed a purely algebraic approach to the *Aequivalenzproblem*, Lipschitz solved it by means of a “mixed” approach, where, in addition to algebraic methods, the calculus of variations played a significant role. This is actually the first occurrence of a methodological difference that was to characterise the later developments of the general theory of differential invariants.

More specifically, his constant reference to the methods of the theory of algebraic forms notwithstanding, it was essentially by means of the calculus of variations that Lipschitz arrived at the conditions for the transformability of a differential quadratic form into another with constant coefficients. As is well known, such conditions are given by the vanishing of the following 4-index symbol, already considered independently by Riemann

$$(3) \quad (ghki) = \frac{\partial}{\partial x_i} \Gamma_{gh,k} - \frac{\partial}{\partial x_h} \Gamma_{gi,k} + \sum_p (\Gamma_{gi}^p \Gamma_{hk,p} - \Gamma_{gh}^p \Gamma_{ik,p}),$$

where $\Gamma_{gh,k}$ and Γ_{gi}^p are the Christoffel symbols of the first and second kind.¹⁰ In addition, after 1869, Lipschitz extensively used the calculus of variations in a number of papers directly concerned with the search

⁸ Christoffel's works are discussed from different points of view in [Ehlers 1981], [Leichtweiss 1981], [Pinl 1981] and [Reich 1994].

⁹ As is well known, the topics considered by these papers were actually very similar: while Christoffel faced up to the problem in the general case, Lipschitz, in a way analogous to Riemann's [1861], but quite independently, investigated the conditions for the transformability of one differential quadratic form into another with constant coefficients.

¹⁰ The symbol (3) was introduced by Riemann in his *Commentatio* [1861] and independently by Christoffel and Lipschitz: see on this matter [Farwell, Knee 1990]. The expression given here is that of Christoffel, as it is relevant to the subsequent discussion of his work on differential quadratic forms.

for the differential invariants of a differential quadratic form [Lipschitz 1870, 1871]. In these papers, the original problem would be reduced to the theoretical domain of the calculus of variations, as a rule by means of a mechanical interpretation.¹¹ For instance, in order to determine the differential invariants of a differential quadratic form, Lipschitz would reduce the problem to that of obtaining the maximum and minimum values of the pressure exerted by a mechanical point moving on a surface defined by m equations in an n -dimensional space and having for its line element the differential quadratic form under consideration. This procedure involved an algebraic equation of degree $(n - m)$, of the form $D(\omega) = 0$, the coefficients from which may be used to construct the differential invariants.

It is worthy of note that, despite his intensive use of the calculus of variations, Lipschitz attached more importance to algebraic methods from a heuristic point of view. This emerges from the clear-cut distinction he made between the algebraic and the variational methods, where the former were denoted as “direct”:

*“The point of view from which the forms of n differentials are here being considered makes the bilinear form, which points to the conditions of integrability, and the quadrilinear form, that denotes the measure of curvature, appear as the first elements of a chain; there remains the need to continue this sequence of direct methods, in order to solve the given problem. In what follows, this problem will, on the contrary, be generally dealt with through an indirect method, whereby, for any forms, an associated problem of the calculus of variations shall be considered as solved.”*¹²

This methodological duality — which is similar to what, at the same time, was happening in the theory of differential parameters, as we shall see — was totally absent from Christoffel’s work. His investigation of the equivalence of differential quadratic forms was based solely on algebraic

¹¹ For the link between mechanics and geometry in Lipschitz’s work, see [Lützen 1995, pp. 34–45].

¹² *“Der Gesichtspunkt, von welchem aus die Formen von n Differentialen hier betrachtet sind, lässt die bilineare Form, welche auf die Bedingungen der Integrabilität hinweist, und die quadrilineare Form, welche auf das Krümmungsmass deutet, als die ersten Glieder einer Kette erscheinen; es bleibt das Bedürfniss, diese Reihe directer Methoden zur Lösung der gestellten Aufgabe fortzusetzen. Diese Aufgabe wird dagegen im Folgenden durch eine indirecte Methode, bei der für jede Form ein entsprechendes Problem der Variationsrechnung als gelöst vorausgesetzt ist, allgemein erledigt werden”* [Lipschitz 1869, p. 74].

arguments; a fact which is hardly surprising, from a historical point of view, since the German mathematician approached the *Aequivalenzproblem* from the angle of earlier studies on algebraic invariants [Christoffel 1868a,b].

The algebraic character of Christoffel's work on differential quadratic forms stems essentially from one theorem — the so-called *Reduktionssatz* [Klein 1927, p. 198] — which established a close link between the study of differential forms and the theory of algebraic invariants. The German mathematician arrived at this theorem by way of technical steps of fundamental significance for the emergence of tensor analysis, since they contained the first consideration of the technical expression of covariant differentiation: this justifies recounting these steps in outline.

After reducing the original problem — that of the transformability of one given quadratic differential form $F = \sum_{ik} \omega_{ik} dx_i dx_k$ into another $F' = \sum_{ik} \omega'_{ik} dx'_i dx'_k$, involving a locally reversible transformation: $x_i = x_i(x'_1, \dots, x'_n)$ — to the investigation of the following system of partial differential equations

$$(4) \quad \frac{\partial^2 x_\lambda}{\partial x'_\alpha \partial x'_{\alpha_s}} = \sum_r (\Gamma_{\alpha\alpha_s}^r)' u_r^\lambda - \sum_{i_s} \Gamma_{i_s}^\lambda u_{i_s}^i u_\alpha^i u_{\alpha_s}^{i_s},$$

where $u_\alpha^i = \partial x_i / \partial x'_\alpha$, Christoffel showed that the conditions of integrability of this Pfaffian system may be put in the form

$$(5) \quad (\alpha\beta\gamma\delta)' = \sum_{gkhi} (gkhi) u_\alpha^g u_\beta^h u_\gamma^k u_\delta^i,$$

where $(gkhi)$ is the Riemann 4-index symbol (3).

In this analytical context, Christoffel interpreted the expressions (5) algebraically, considering them as the *Transformationsrelationen*¹³ of a quadrilinear form G_4 having for coefficients the symbols $(gkhi)$; i.e., as the conditions required of the coefficients for the transformability of this multilinear form into another, G'_4 . It was this very view of expressions (5) that led Christoffel to his wholly algebraic treatment of the *Aequivalenzproblem*. That view, indeed, corresponded to his switching his attention

¹³ This term was taken up by Christoffel in the context of the theory of algebraic forms and, specifically, from Aronhold's work: [1863, p. 283]. This notion is historically significant, in that it constitutes an "algorithmic" definition of the tensor as given by Christoffel: see [Reich 1994, pp. 59–60].

to multilinear forms with the construction, starting from G_4 , of a particular sequence of multilinear differential forms $G_5, G_6, \dots$: we shall see in the following section how this construction was effected.

More specifically, Christoffel's solution of the *Aequivalenzproblem* was closely related to the consideration of a well-defined property of the sequence $G_4, G_5, G_6, \dots$ and of the sequence $G'_4, G'_5, G'_6, \dots$, obtained in like manner with G'_4 as starting point; i.e., this was linked to the realisation of the fact that equation $G_\mu = G'_\mu$ entails the consequent one, $G_{\mu+1} = G'_{\mu+1}$. Consequently, the *Reduktionssatz* asserts that the equivalence between the quadratic differential forms F and F' depends only on the algebraic compatibility of the system of equations: $F = F', G_4 = G'_4, G_5 = G'_5, \dots$.¹⁴

Thus, the differential forms $F, G_4, G_5, G_6, \dots$ may be considered as purely algebraic forms of their differentials and the *Aequivalenzproblem* may thus be reduced to a problem of equivalence of algebraic forms; and, ultimately — as a result of the theory of algebraic invariants — this was reduced by Christoffel to the coincidence between the sets of simultaneous invariants of the forms $F, G_4, G_5, G_6, \dots$ and those of forms $F', G'_4, G'_5, G'_6, \dots$.¹⁵

To sum up, it may be said that Christoffel's investigation of the Riemannian *Aequivalenzproblem* exhibited a programme of "reduction" of the theory of differential forms to the theory of algebraic forms. From an operational point of view, this approach was the opposite of that adopted by Lipschitz, which was characterised by its essential reliance on variational methods. In analogous fashion, following from this, the search for differential invariants brings out the fact that different approaches were used by the two German mathematicians. Indeed, Christoffel's method also implicitly entailed an algebraic reduction of the search for differential invariants: the common algebraic invariants of the forms $F, G_4, G_5, G_6, \dots$ may also be considered as differential invariants, even though this result was not explicitly stated by the German mathematician for the general case. Such a viewpoint is similar to that of Casorati, but is technically more powerful owing to the use of stronger algebraic methods.

¹⁴ See [Christoffel 1869a, p. 369].

¹⁵ For a modern version of this theorem on algebraic invariants, see [Weitzenböck 1923, pp. 199–203].

The “algorithmic” genesis of covariant differentiation

There are a number of reasons accounting for the central significance, as regards the emergence of tensor analysis, of the methodological contrast we have just outlined, between algebraic and variational methods. As we shall see, indeed, the same contrast — on a different level — was to reappear in Ricci-Curbastro’s work and characterises the genesis of that theory.

There is, however, one specific aspect in Christoffel’s solution of the *Aequivalenzproblem* that is directly connected to the rise of absolute differential calculus. As was already mentioned, in effect, the German mathematician’s algebraic treatment of the *Aequivalenzproblem* contained the first occurrence of what was later to be referred to as covariant differentiation by Ricci-Curbastro.¹⁶ It is necessary, however, to understand how it is possible to make this claim.

The introduction by Christoffel of the algorithms of “covariant differentiation” occurred in the middle of his 1869 paper, as the method to construct the sequence of differential multilinear forms $G_4, G_5, G_6, \dots$. It is interesting to observe how, bringing out a clear methodological analogy, Christoffel’s process made use of the “elimination theory” in a fashion similar to that which had characterised Casorati’s geometrical work.

Given a μ -linear form G_μ with coefficients $(i_1 i_2 \dots i_\mu)$, the x'_α -differentiation of their *Transformationsrelationen*

$$(6) \quad (\alpha_1 \alpha_2 \dots \alpha_\mu)' = \sum_{i_1 i_2 \dots i_\mu} (i_1 i_2 \dots i_\mu) u_{\alpha_1}^{i_1} \dots u_{\alpha_\mu}^{i_\mu}$$

leads to the equations

$$(7) \quad \begin{aligned} & \frac{\partial(\alpha_1 \alpha_2 \dots \alpha_\mu)'}{\partial x'_\alpha} \\ &= \sum_{ii_1 i_2 \dots i_\mu} \frac{\partial(i_1 i_2 \dots i_\mu)}{\partial x'_i} u_\alpha^i u_{\alpha_1}^{i_1} \dots u_{\alpha_\mu}^{i_\mu} \\ &+ \sum_{\lambda i_2 \dots i_\mu} \frac{\partial^2 x_\lambda}{\partial x'_\alpha \partial x'_{\alpha_1}} (\lambda i_2 \dots i_\mu) u_{\alpha_2}^{i_2} \dots u_{\alpha_\mu}^{i_\mu} \\ &+ \sum_{i_1 \lambda \dots i_\mu} \frac{\partial^2 x_\lambda}{\partial x'_\alpha \partial x'_{\alpha_2}} (i_1 \lambda \dots i_\mu) u_{\alpha_1}^{i_1} \dots u_{\alpha_\mu}^{i_\mu} + \dots \end{aligned}$$

¹⁶ The algorithms of covariant differentiation are also present in Lipschitz’s work, but only as a consequence of their use by Christoffel: see [Lipschitz 1871, p. 17].

Like Casorati, Christoffel used the expressions (4) to replace the second derivatives in equations (7), thus obtaining the following

$$(8) \quad (\alpha\alpha_1 \dots \alpha_\mu)' = \sum_{ii_1 \dots i_\mu} (ii_1 \dots i_\mu) u_\alpha^i u_{\alpha_1}^{i_1} \dots u_{\alpha_\mu}^{i_\mu},$$

where

$$(9) \quad (ii_1 \dots i_\mu) = \frac{\partial(i_1 i_2 \dots i_\mu)}{\partial x'_i} - \sum_\lambda [\Gamma_{ii_1}^\lambda (\lambda i_2 \dots i_\mu) + \Gamma_{ii_2}^\lambda (i_1 \lambda \dots i_\mu) + \dots].$$

As usual, the expressions (8) were considered by Christoffel as the *Transformationsrelationen* of a $(\mu+1)$ -linear form $G_{\mu+1}$ whose coefficients are given by the quantities (9). Thus, the previously-considered process yields the definition, taking G_μ as starting point, of a multilinear differential form $G_{\mu+1}$, for which the *Transformationsrelationen* (8) still hold; i.e., in such a way that the equation $G_\mu = G'_\mu$ entails as its consequent $G_{\mu+1} = G'_{\mu+1}$, as required.

One may, therefore, claim that the introduction of the algorithms (9) of covariant differentiation constituted the corner-stone of the entire algebraic project followed by Christoffel. Evidence for such a role is that, in the German mathematician's work, the expressions (9) were never considered from an analytical point of view but only from an algebraic one; actually, they were merely algorithms for the iterative production of differential forms. This may be underscored by pointing out that, in Christoffel's work, the expressions (9) were only special techniques introduced to reduce the original problem to another context, which was presumed to have a broader heuristic power. One may refer to such a functional device as a *linking technique*.¹⁷

2.2. The research tradition of the theory of differential parameters

As we shall see, the actual origin of the concept of covariant differentiation — understood as an extension of ordinary differentiation —

¹⁷ This notion may be seen to be closely connected with that of “transplantation”, as proposed by E. Koppelman [1975, p. 459]. More specifically, one may view a *linking technique* as a well-defined technique in a field of research which — when interpreted in another field of research — allows the application of the latter to the former. The reason for introducing this notion stems from the requirement to make a clear-cut difference between a technical and a conceptual stage in the history of the idea of covariant differentiation.

may be viewed as the result of a particular interpretation given by Ricci-Curbastro of Christoffel's algorithms (9). Analytical in nature, this interpretation arose in connection with a context of research — the theory of differential parameters — different from that in which the expressions (9) first emerged — the theory of differential forms. This fact tends to give the theory of differential parameters great significance for the history of absolute differential calculus.

The research programme of the early theory of differential parameters

Like the investigations of algebraic and differential invariants, the theory of differential parameters presented an embryonic stage of development characterised by the study of the invariant properties of certain well-defined quantities. On the other hand, unlike the investigations into the other types of invariants, which mostly arose in geometrical contexts, the study of differential parameters came from a typically analytical field, i.e., from classical French mathematical physics. It was indeed while working on elasticity theory in 1834 that Lamé demonstrated the invariance of the following expressions

$$(10) \quad \Delta_1 f = \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2,$$

$$(11) \quad \Delta_2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

which he respectively called “*paramètres différentiels du premier et du second ordre*” [Lamé 1834, p. 215]. After initially considering rectangular Cartesian coordinates, he generalised these results to the case of orthogonal curvilinear coordinates two decades later [Lamé 1859].

Lamé's attention to differential parameters was neither due to chance nor did it lack significance. Actually, it was connected with a general research programme in the field of partial differential equations. As is well known [Kline 1972, 28.6], this programme was based on the theory of systems of triply orthogonal families of surfaces [Reich 1973, VI.5], since it consisted in the search for a particular system of curvilinear coordinates, to reduce differential equations to a form resolvable by means of a separation of variables. This involved consideration of how one might express the differential equations of a given problem in a general form, i.e. with respect to a general system of curvilinear coordinates. This technical

programme, in particular, was already present in Lamé's paper of 1834, in which the differential equations for the propagation of light in the ether were expressed with respect to the system of curvilinear coordinates ρ, ρ_1, ρ_2 , with the only condition that ρ be the parameter "*des surfaces d'égale densité de l'éther*". More specifically, using some properties of orthogonal surfaces studied in the second part of his paper and essentially basing himself on consideration of the differential parameters of the first and second order, Lamé put these equations in the following form

$$(12) \quad \left\{ \begin{array}{l} P_1 = hh_2 \frac{\partial \varphi}{\partial \rho_2}, \quad P_2 = -hh_1 \frac{\partial \varphi}{\partial \rho_1}, \\ h^2 \frac{\partial}{\partial \rho} \left[h_1^2 \frac{\partial}{\partial \rho_1} (\rho^3 \varphi) \right] = h_1^2 \frac{\partial F}{\partial \rho_1}, \\ h^2 \frac{\partial}{\partial \rho} \left[h_2^2 \frac{\partial}{\partial \rho_2} (\rho^3 \varphi) \right] = h_2^2 \frac{\partial F}{\partial \rho_2}, \\ \rho \frac{h_1 h_2}{h} \left[\frac{\partial}{\partial \rho_1} \left(\frac{hh_1}{h_2} \frac{\partial \varphi}{\partial \rho_1} \right) + \frac{\partial}{\partial \rho_2} \left(\frac{hh_2}{h_1} \frac{\partial \varphi}{\partial \rho_2} \right) \right] \\ \quad \quad \quad + \rho \frac{\partial}{\partial \rho} \left(\frac{F}{\rho^3} \right) = \frac{1}{A} \frac{\partial^2 \varphi}{\partial t^2}, \end{array} \right.$$

where h, h_1, h_2 are the differential parameters of the first order of ρ, ρ_1, ρ_2 with respect to the rectilinear variables x, y, z ; f and F are functions of ρ, ρ_1, ρ_2, t ; and A is a numerical coefficient. In this fashion, in the final part of the paper, Lamé was able to solve the foregoing system of equations for the particular case of a single spherical and homogeneous particle acting "*sur l'éther environnant*".

In this technical programme, which was also the fundamental matter of Lamé's main work — his *Leçons sur les coordonnées curvilignes* [1859] —, differential parameters played an essential role; indeed, they represented the invariant quantities through which it was possible to express the given equations in an invariant form:

"*Cette constance de forme et de valeur explique, en quelque sorte, comment il se fait que presque toutes les équations aux différences partielles, qui concentrent les lois des phénomènes physiques, peuvent s'exprimer à l'aide de certaines fonctions-de-point et de leurs paramètres différentiels du second ordre, sans qu'il soit nécessaire de spécifier le système de coordonnées que l'on adopte*" [Lamé 1859, p. 24].

As a result, differential parameters — along with curvilinear coordinates — were to be extensively taken up after Lamé's research work.¹⁸ It is worthy of note that this reception was marked by the deep involvement of the Italian mathematical community, from the middle of the 19th century on. This development — which may be explained in terms of the rise of differential geometry, already manifest in pre-Unification Italy¹⁹ — is evidenced first and foremost by the work of Beltrami.

Beltrami and the “mixed” approach to the theory of differential parameters

The study of differential parameters was indeed one of the main topics of Beltrami's scientific production. It constituted, in fact, the principal common technical element linking his earlier research work in geometry and his later work on mathematical physics.²⁰

On the one hand, in effect, Beltrami extended the quantities defined by Lamé to the case of surfaces and manifolds in his “*Ricerche di analisi applicata alla geometria*” [1864–1865].²¹ It is worth noting that this extension was concomitant with a direct generalisation of Casorati's notion of *funzione inalterabile*; in particular, this was tantamount to a first general definition of differential parameters as quantities analytically depending on a set of arbitrary functions:

“But the idea of these functions, which may be termed absolute like the geometric properties which they represent, is susceptible of a useful extension. With the functions E, F, G , let us consider the other functions $\varphi, \psi, \dots$ of u, v , and suppose that the same change of variables that transforms expression (34) into (34')²² also transforms $\varphi, \psi, \dots$ into $\varphi', \psi', \dots$. An expression formed with $E, F, G, \varphi, \psi, \dots$ and their partial

¹⁸ Among the principal studies concerning differential parameters around 1850–1860, one may mention: [Brioschi 1854], [Chelini 1853], [Codazzi 1868], [Jacobi 1847], [Neumann 1860, 1867], [Somov 1865].

¹⁹ “*In my opinion Liouville was among the mathematicians who best understood the idea of intrinsic geometry around 1850, only rivaled perhaps by a few Italians such as Brioschi and Chelini*” [Lützen 1989, p. 86].

²⁰ For this central aspect of Beltrami's work, see: [Loria 1901], [Pascal 1901, pp. 71, 73, 77], [Struik 1981, p. 600], [Tazzioli 1993, pp. 3–4].

²¹ Differential parameters are also discussed in [Beltrami 1867a,b].

²² E, F, G and E', F', G' are the coefficients of the differential quadratic form in the two systems u, v and u', v' and equations (34) and (34') are the expressions of that form with respect to the two systems of coordinates.

*derivatives with respect to u , v will be termed invariable if ... it changes into another expression analogously formed with E' , F' , G' , φ' , ψ' , ... and their partial derivatives with respect to u' , v' .”*²³

On the other hand, Beltrami extensively employed differential parameters in his later work in mathematical physics, concerning the geometrical nature of physical space. In particular, he used such parameters to search for a general expression of the equations of classical mathematical physics — and, above all, in potential theory — in non-Euclidean contexts: a thrust of research which would lead to extensive developments in the German and Italian contexts of research shortly thereafter.²⁴ In addition to a clear Riemannian influence, these studies of Beltrami showed a renewal of Lamé's scientific orientation, now taken up in more general geometrical contexts.²⁵ In this manner, emphasis was again placed on the major significance of the mathematical tool of curvilinear coordinates (oblique, as opposed to the orthogonal ones considered by Lamé) in analytical research:

*“Now as regards the general usefulness of oblique curvilinear coordinates in physico-mathematical matters ... , it is not out of place to note that, considering the frequency of the cases in which the very nature of the problem suggests a priori a certain system of surfaces as an essential instrument for resolution ... , it is a natural assumption that discussion may often be made easier by use of formulae not bound to the hypothesis of threefold orthogonality.”*²⁶

²³ “Ma il concetto di queste funzioni, che si possono chiamare assolute come le proprietà geometriche che rappresentano, è suscettibile di un'utile estensione. Consideriamo, oltre le E , F , G , altre funzioni φ , ψ , ... di u , v , e supponiamo che quello stesso cambiamento di variabili il quale trasforma l'espressione (34) nella (34') trasformi parimente le φ , ψ , ... nelle φ' , ψ' , Un'espressione formata colle E , F , G , φ , ψ , ... e colle loro derivate parziali rispetto alle u , v si dirà invariabile quando ... essa si trasformerà in un'espressione formata analogamente colle E' , F' , G' , φ' , ψ' , ... e colle loro derivate parziali rispetto alle u' , v' ” [Beltrami 1864, p. 142].

²⁴ [Lipschitz 1870], [Schering 1870], [Lipschitz 1872], [Schering 1873], [Tonelli 1882], [Killing 1885]. On this matter, see [Tazzioli 1993, pp. 5–7].

²⁵ Unlike the French physical mathematician, however, Beltrami would appear to have followed more typically physical aims, in particular in connection with Maxwell's electromagnetic theory.

²⁶ “Quanto poi all'utilità di massima che può avere l'uso delle coordinate curvilinee oblique nelle questioni di fisica matematica ... non è fuor di luogo il notare che stante la frequenza dei casi in cui la natura stessa del problema suggerisce a priori un certo

Besides playing a major part in much of his scientific work, the topic of differential parameters was also developed by Beltrami in the guise of a purely analytical study. His paper, “Sulla teorica generale dei parametri differenziali” [Beltrami 1868b], may, indeed, be considered as the first autonomous analysis of such quantities. In particular, in this context, he proved the Riemannian invariance of Lamé’s quantities (10), (11), now expressed as follows

$$(13) \quad \Delta_1 U = \sum_{rs} A_{rs} \frac{\partial U}{\partial x_r} \frac{\partial U}{\partial x_s},$$

$$(14) \quad \Delta_2 U = \frac{1}{\sqrt{a}} \sum_r \frac{\partial}{\partial x_r} \left(\sqrt{a} \sum_s A_{rs} \frac{\partial U}{\partial x_s} \right),$$

where a is the determinant of the fundamental differential form and the A_{rs} are the coefficients of its reciprocal form. Moreover, he also demonstrated the invariance of the following expression — the so-called *parametro differenziale misto*

$$(15) \quad \Delta_1(UV) = \sum_{rs} A_{rs} \frac{\partial U}{\partial x_r} \frac{\partial V}{\partial x_s}.$$

This last work of Beltrami is of the greatest importance for a historical reconstruction of the genesis of tensor analysis. Indeed, it clearly showed the state of the theory of differential parameters a short time before Ricci-Curbastro’s work on the matter. This remark is of particular significance from a methodological point of view.

Like the studies of Lipschitz on the special theory of differential invariants, Beltrami’s research methodology was characterised by its “mixed” nature, as it was based on both algebraic and variational methods. More specifically, the invariance of the differential parameters (13), (15) was established only on the basis of certain properties of algebraic forms, for which the essentials of the theory were extensively discussed in the first part of Beltrami’s paper. Conversely, in his treatment of the expressions (14), Beltrami made use of the variation of an integral, with explicit reference to a paper by Jacobi on potential theory [Jacobi 1847]. The actual form of the differential parameters (14) derived from the application of

sistema di superficie come strumento essenziale di soluzione . . . è naturale presumere che la trattazione possa essere non di rado agevolata dall’uso di formole non vincolate all’ipotesi della triplice ortogonalità” [Beltrami 1884, p. 138].

the calculus of variations to the subject topic of differential invariants. Indeed, by extending the procedure employed by Jacobi, Beltrami arrived at these expressions by applying variational methods to the equation

$$(16) \quad \int^{(n)} \Delta_1 U \cdot \sqrt{a} \cdot dx_1 \cdots dx_n = \int^{(n)} \Delta_1 U \cdot \sqrt{b} \cdot dy_1 \cdots dy_n,$$

where b is the determinant of the fundamental form with regard to the variables y_i .²⁷ Thus, using the results of the calculus of variations for n -integrals yields

$$(17) \quad \int^{(n)} \delta U \sum_r \frac{\partial}{\partial x_r} (U_r \sqrt{a}) dx_1 \cdots dx_n = \int^{(n)} \delta U \sum_r \frac{\partial}{\partial y_r} (U'_r \sqrt{b}) dy_1 \cdots dy_n,$$

where

$$(18) \quad U_r = \frac{1}{2} \frac{\partial}{\partial \frac{\partial U}{\partial x_r}} (\Delta_1 U) = \sum_s A_{rs} \frac{\partial U}{\partial x_s};$$

and by way of (17), Beltrami was able to demonstrate the invariance of (14).

It is worth noting, however, that, notwithstanding the analogy of their methodological duality, Lipschitz's and Beltrami's opinions on the respective significance of the algebraic and variational methods did not coincide. Unlike that of the German mathematician, indeed, Beltrami's treatment of differential parameters did not explicitly indicate a definitely greater significance for algebraic methods over and above variational ones. In fact, the latter were often designated by Beltrami as more direct methods of research, thus evincing a different stance from that of the German mathematician. Thus, in the introduction of his essay on differential parameters, he wrote:

*"I hope that the simplicity of the method considered, which essentially does not differ from that of Jacobi . . . , may be conducive to considering it as the most natural and direct way to achieve the aim".*²⁸

²⁷ In particular, $b = ap^2$, where p is the Jacobian of the functions $x_i = x_i(y_1, \dots, y_n)$.

²⁸ *"Spero che la semplicità del metodo usato, il quale nei suoi principali lineamenti*

To sum up, it may be said that, around 1870, the theory of differential parameters exhibited a “mixed” character with algebraic methods coexisting with variational ones. Moreover, this “mixed” character appears more marked than what was happening at the same time in the special theory of differential invariants, as a result of the impact, in this latter context of research, of Christoffel’s work.

3. THE RISE OF TENSOR ANALYSIS IN THE CONTEXT OF THE WORK OF RICCI-CURBASTRO

The work of Ricci-Curbastro on the introduction of absolute differential calculus goes back to the final two decades of the 19th century, including gradual definitions of the main concepts and the first systematic accounts of the theory. As previously mentioned, the theoretical context from which the basic concepts of absolute differential calculus — and in particular that of covariant differentiation — emerged was provided by the theory of differential invariants. Starting from 1884, indeed, a major part of the Italian mathematician’s research work was directed to this field of research, being concerned with the topics of differential quadratic forms [Ricci-Curbastro 1884] and differential parameters [Ricci-Curbastro 1886a]. This part of Ricci-Curbastro’s work may be seen as taking place in a precise historical and methodological context.

On the one hand, it represented the stage of initial maturity of Ricci-Curbastro’s production. His previous contributions, indeed, may all be seen as arising out of his scientific background, which — apart from a year of specialisation in Munich, where he attended Klein’s and von Brill’s lectures — had unfolded entirely at the Scuola Normale Superiore of Pisa. Virtually all this early work of the Italian mathematician

non differisce da quello di Jacobi, . . . , induca la persuasione che la via da esso aperta è la più naturale e la più diretta per giungere allo scopo” [Beltrami 1868b, p. 75].

This attitude of Beltrami was altogether consistent with the extensive use of the calculus of variations in his research work on the non-Euclidean expression of the equations of classical mathematical physics. At first glance, then, one may view the divergence between Beltrami and Lipschitz as to the use of the term “direct” as resulting from the fact that this use occurred in connection with different fields of research: on the one hand, differential quadratic forms, with a potential connection with algebraic forms, and, on the other, differential parameters, in connection with the use of the calculus of variations in investigations in the field of mathematical physics.

was concerned with physico-mathematical matters, especially electromagnetism, as a result of the influence exerted on him by E. Betti [Ricci-Curbastro 1877b,c].²⁹ Not surprisingly, in 1880 Ricci-Curbastro started teaching mathematical physics at the University of Padua. It is in this context of research that his interest for differential invariants arose. In particular, in his only work to be published between 1880 and 1883 [Ricci-Curbastro 1882], his treatment of galvanic currents intersected with the topic of differential parameters.

On the other hand, from a methodological point of view, Ricci-Curbastro's papers on differential invariants exhibit a common element, namely, one and the same research programme. More specifically, these papers evidence the same systematic reduction of the study of differential invariants to that of algebraic invariants, which, as we have seen, had characterised Casorati's and, above all, Christoffel's approach. It is this fact, essentially, that enabled Ricci-Curbastro to bring a new twist to Christoffel's algorithms (9).

3.1. The first occurrence of the programme of “algebraic reduction” in the work of Ricci-Curbastro

There are various points in the course of Ricci-Curbastro's research work where the continuing impact may be recognised, of Casorati's and Christoffel's algebraic approach to the theory of differential invariants. The first instance was the previously-mentioned paper [Ricci-Curbastro 1884] on the theory of differential quadratic forms.

The first clue is proffered by the theoretical assumptions of the paper, showing clearly the influence of Casorati's work. Indeed, Ricci-Curbastro's investigation features a “vertical” aspect, being based on an initial classification of differential quadratic forms into *classi*, closely recalling Casorati's analysis of differential invariants into *ordini*. More specifically, a differential quadratic form $\varphi = \sum_{rs} a_{rs} dx_r dx_s$ was termed by Ricci-Curbastro of *classe* h , if h be the smallest positive number for which it is possible to express the form as follows

$$(19) \quad ds^2 = \sum_{r=1}^{n+h} dy_r^2.$$

²⁹ In Pisa, U. Dini also exerted an influence on Ricci-Curbastro, this being specifically vouched for by the paper [Ricci-Curbastro 1877a].

Wholly taken up by the study of forms of classes 0 and 1, the paper included, in its final part, a short analysis of the search for invariants of quadratic differential forms of the first class. And it is in this very context that the tradition may be recognised at work, of the algebraic approach to the theory of differential invariants. Let us see how this occurs.

After setting out the usual laws of transformation for the coefficients a_{rs} of a differential quadratic form (of class 1)

$$(20) \quad a_{rs} = \sum_{pq} b_{pq} \frac{\partial u_p}{\partial x_r} \frac{\partial u_q}{\partial x_s},$$

Ricci-Curbastro considered the quantities (pq) , defined as follows

$$(21) \quad (pq) = \frac{1}{\sqrt{a}} \begin{vmatrix} \frac{\partial^2 y_1}{\partial x_p \partial x_q} & \frac{\partial^2 y_2}{\partial x_p \partial x_q} & \cdots & \frac{\partial^2 y_{n+1}}{\partial x_p \partial x_q} \\ \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_{n+1}}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_{n+1}}{\partial x_n} \end{vmatrix},$$

where a is the determinant of the a_{rs} and the y_r are the variables appearing in (19). On this basis, one can see that the quantities (pq) vary in similar fashion to the coefficients a_{rs} of the fundamental form, i.e.

$$(22) \quad (rs) = \sum_{pq} (pq)' \frac{\partial u_p}{\partial x_r} \frac{\partial u_q}{\partial x_s}.$$

As in Christoffel's research work, this remark implied a wholly algebraic view of the topic of differential invariants by Ricci-Curbastro. Indeed, the differential quadratic form having the quantities (pq) as coefficients — and which the Italian mathematician called *forma derivata di φ* — shows itself algebraically similar to the fundamental one, due to the simultaneous validity of (20) and (22). In analogous fashion to Christoffel, this fact led Ricci-Curbastro to consider the two differential forms from a purely algebraic standpoint, considering their simultaneous algebraic invariants. However, owing to the special nature of the *forma derivata di φ* , these algebraic invariants will also be differential invariants. In this way, Ricci-Curbastro arrived at the following result — where Casorati's notion of *funzione inalterabile* reappears as *invariante differenziale di ordine m* of φ , i.e., as an expression depending only on the a_{rs} and their derivatives to the m -th order:

“For $n > 2$, every differential quadratic form of class 1 with n variables has n differential invariants of the 2nd order, which are obtained through construction, by the methods [already] known, of the set of absolute algebraic invariants that are common to φ and its derivative form.”³⁰

It is obvious that Ricci-Curbastro’s method of seeking differential invariants — unlike Lipschitz’s use of *indirecte Methoden* — is analogous to Christoffel’s solution of the Riemannian *Aequivalenzproblem*; in both instances, indeed, the method of resolution is based on the introduction of differential forms which are *covarianti* with respect to the fundamental form and, ultimately, on the search for their simultaneous algebraic invariants. In this respect, it is no mere chance that Ricci-Curbastro should directly refer to Christoffel’s work of 1869 at this point and, in particular, to the final part where the German mathematician had solved the three-dimensional case of the *Aequivalenzproblem* by proving the existence of three differential invariants of two differential quadratic forms.

Consequently, one may claim that there was a clear influence, of a purely methodological nature, of Christoffel’s work on Ricci-Curbastro’s. In particular, it is important to lay stress on the analogy between the German mathematician’s algorithms (9) and Ricci-Curbastro’s *forma derivata di φ* , whose coefficients are given by the expressions (21). Both constructs, in fact, carried out the essential role of making it feasible to establish a connection between problems involving differential invariants and an algebraic mode of resolution. In other words, Ricci-Curbastro’s *forma derivata di φ* had the same function, i.e. that of a *linking technique*, as the algorithms (9) in Christoffel’s work.³¹

3.2. Ricci-Curbastro’s algebraic approach to the theory of differential parameters

The second occurrence of the programme of “algebraic reduction” in

³⁰ “Ogni forma differenziale quadratica φ di 1^a classe ad n variabili, per $n > 2$, ammette n invarianti differenziali di 2° ordine, i quali si ottengono costruendo coi metodi noti, il sistema di invarianti algebrici assoluti comuni a φ ed alla sua forma derivata” [Ricci-Curbastro 1884, p. 170].

³¹ In effect, the expressions (21) — that in modern terms are the second fundamental coefficients of the hypersurface of line element φ in the $(n + 1)$ -dimensional Euclidean space — are implicitly connected to the algorithms (9) by Gauss equations [Ricci-Curbastro 1884, p. 159, (14)]. This however does not represent a real consideration of such algorithms.

Ricci-Curbastro's work concerns the theory of differential parameters. The significance of this development appears principally as a reflection on the "mixed" nature which, as we have already noted, characterised that theory at the time. Thus, Ricci-Curbastro's paper of 1886 on differential parameters may also be viewed as a breaking point as well as a breakthrough in the history of this theory.

Ricci-Curbastro's approach to the theory of differential parameters involved essentially two innovations, which mirrored precisely Casorati's approach to the special theory of differential invariants.

The first new departure was an element of systematicity. This merely implied the possibility of a wider theory, i.e., of an investigation no longer restricted to considering only special quantities of an intrinsic nature. This orientation by Ricci-Curbastro entailed producing an explicit expression of Beltrami's general definition of differential parameters. Moreover, in this context, Casorati's concept of the "order" of a differential invariant again appeared, in the guise of the following definition of differential parameters by Ricci-Curbastro:

*"We shall term differential parameters of the form itself [of the fundamental form] all expressions that include the coefficients of φ , one or more arbitrary functions and the derivatives of all these quantities, and that do not change their form when, for the variables $x_1, x_2, \dots, x_n$, new ones are substituted, $u_1, u_2, \dots, u_n$. Their order [of these parameters] is given by that of the derivatives of highest order included in them."*³²

The second element of innovation concerns the methodological heterogeneity of the earlier treatments of differential parameters. Like Casorati, Ricci-Curbastro viewed as "indirect" some of the methods initially implemented for these treatments:

"From this remark there naturally arises the doubt that, even restricting oneself to expressions of the second order, not solely those generally known ... be worthy of that name; and the more so since, to date, that very property has been demonstrated in their case by artificial and indirect methods."

³² *"Chiameremo parametri differenziali della forma stessa [della forma fondamentale], tutte le espressioni, che contengono i coefficienti di φ , una o più funzioni arbitrarie e le derivate di tutte queste quantità, e non cambiano forma quando alle variabili $x_1, x_2, \dots, x_n$ se ne sostituiscono delle nuove $u_1, u_2, \dots, u_n$. Il loro ordine si desume da quello delle derivate più alte in essi contenute"* [Ricci-Curbastro 1886a, p. 180].

*As a direct method for these investigations a natural candidate is that proposed by Prof. Casorati . . . , which consists in eliminating, as between two systems of quantities corresponding to two different systems of variables, the derivatives of the former with respect to the latter, in order to arrive at the equations that express precisely the essential property under consideration.”*³³

The true object of Ricci-Curbastro's criticism was the use of the calculus of variations in the theory of differential parameters. Although Lipschitz was not mentioned at this juncture, the foregoing passage was clearly referring to his qualitative distinction between *directe* and *indirecte Methoden*. Unlike the German mathematician, however, Ricci-Curbastro was not content with a theoretical distinction, subject to specific practical requirements. On the contrary, Ricci-Curbastro's approach called for the actual exclusion of the *indirecte Methoden* from the theory of differential parameters, which may only be studied in algebraic terms. One way of accounting for this stance of the Italian mathematician, again showing Christoffel's influence, is to claim that it was connected with his general, systematic aims, as in the case of the search for invariants of differential quadratic forms.

In any event, one might inquire why Ricci-Curbastro — with his strong background in mathematical physics — set himself against one of the most important technical instruments of this scientific tradition, and at a time when the calculus of variations was undergoing considerable developments.

To answer this historical question, one particular aspect of Ricci-Curbastro's scientific background must be examined. This aspect is specifically linked to the role played by von Brill, whose course of lectures the Italian mathematician had followed in Munich in 1877, as mentioned

³³ “Da questa osservazione emana naturalmente il dubbio che, anche limitandosi alle espressioni di 2° ordine, non soltanto quelle comunemente conosciute . . . meritino tal nome; e ciò tanto più che per essi la proprietà medesima è stata fino ad ora dimostrata con metodi indiretti ed artificiosi.

Come metodo diretto per tali indagini si offre naturalmente quello seguito dal prof. Casorati . . . che consiste nell'eliminare tra due sistemi di quantità corrispondenti a due diversi sistemi di variabili, le derivate delle une rispetto alle altre per giungere alle equazioni esprimenti appunto la ricordata proprietà essenziale” [*Ibid.*, p. 177]. The property of the differential parameters which Ricci-Curbastro was referring to here is naturally their intrinsic nature.

above. Indeed, Ricci-Curbastro's stance, laying greater store by algebraic methods, may be linked to Brill's approach to the theory of algebraic functions; his approach, indeed, had been characterised by a purely algebraic treatment of the subject, as opposed to the use of transcendental methods that had previously prevailed [Pogrebyssky 1981].

Thus, here is another possible influence, acting on Ricci-Curbastro from a purely methodological point of view, originating in the German research tradition on algebraic functions, as propounded by Clebsch, Gordan and thereafter by M. Noether and Brill. It is this further influence, in particular — together with that of Casorati —, which may well have been the main reason for Ricci-Curbastro's embracing Christoffel's algebraic approach to the theory of differential invariants. Such an influence, moreover, is hardly surprising from a historiographic point of view: in this respect, we need only recall the impact of that self-same research tradition on another segment of the Italian mathematical community at the time, i.e. the famous school of geometry founded by Segre and Enriques.

3.3. The emergence of the concept of covariant differentiation

Ricci-Curbastro's radical approach to the theory of differential parameters tended to emphasise the contrast between algebraic and variational methods which, as we have seen, characterised — to varying degrees — many lines of research into differential invariants. Indeed, one may claim that the Italian mathematician's opposition to the use of variational methods turned this divergence from a coexistence of methods showing differences in emphasis into an actual methodological conflict.

It is precisely this changed context that may be viewed as one of the principal reasons for the emergence of the idea of covariant differentiation. One consequence, indeed, was a clear-cut separation of various aspects that had gone into the research traditions on differential invariants. Specifically, this entailed the emergence of a break between the algebraic and analytical aspects of the question, to use these terms in the modern sense.

Broadly speaking, the algebraic aspect is linked to Christoffel's approach to the theory of differential quadratic forms and points to the tensor substratum of the notion of covariant differentiation — i.e., the validity of (8) — with its connection with the theory of multilinear forms. Conversely, the analytical aspect concerns the essential nature of differential parameters viewed as objects of the theory of partial differential

equations. As we shall see, this had already involved the emergence of the requirement for a broader concept of differentiation in the work of Lamé and of Beltrami.

These different aspects are clearly reflected in the process of genesis of the concept of covariant differentiation in Ricci-Curbastro's work, which may be divided into three basic stages. The first stage — characterised by the resurgence of Christoffel's algorithms (9) in the context of the theory of differential parameters — was essentially algebraic in nature, on the lines of the German mathematician's program of "algebraic reduction". Conversely, the second stage was strictly analytical in character, as it involved interpreting the expressions (9) as differential operators, in accordance with the research tradition on differential parameters. Finally, the last stage was characterised by the true autonomy of the concept of covariant differentiation — as evidenced by the total independence of this notion from a linguistic point of view. Of course, this separation into stages should not be considered in absolute terms, but only by way of a reconstruction of a specific rational process.

The algorithmic emergence of covariant differentiation in the work of Ricci-Curbastro

The particular approach adopted by Ricci-Curbastro for the theory of differential parameters — calling for a systematic treatment of the subject using a methodologically coherent procedure — was what led to his reappraisal of Christoffel's algorithms (9). Indeed, as we have already mentioned, the claim of the centrality of the *directe Methoden* in the theory of differential parameters coincided with a second instance of the implementation of Christoffel's programme of algebraic reduction in the Italian mathematician's work.

Indeed, Ricci-Curbastro's analysis of differential parameters [1886a] was grounded entirely on algebraic arguments. The investigation of differential parameters of order 0 and 1 actually hinged, in the first case (in order to prove the non-existence of such parameters), on the absence of absolute algebraic invariants of the fundamental form; and, in the second case, on certain points made by Beltrami about algebraic forms.³⁴

³⁴ These are the arguments which Beltrami had adduced to show the invariance of differential parameters (13), (15).

However, the algebraic character of Ricci-Curbastro's approach was peculiarly well suited for the investigation of the differential parameters of the higher orders. This investigation, indeed, was cast once again to the self-same methodological plan that had characterised Christoffel's work on differential quadratic forms and which the Italian mathematician had previously implemented to search for differential invariants of forms of class 1, as we have seen. In other words, in order to arrive at the differential parameters, Ricci-Curbastro brought in a differential form whose coefficients change in like manner to those of the fundamental form — i.e., a form *covariante a φ* . Thus, he could apply the results of the theory of algebraic invariants.³⁵ This is precisely why Ricci-Curbastro looked again to Christoffel's algorithms (9): it is interesting to see in more technical terms how this was achieved.

First, Ricci-Curbastro considered the laws of transformation for the coefficients a_{rs} of the fundamental form and for their derivatives $a_{rs}^{(g)}$ under a given change of coordinates

$$(23) \quad (a_{pq}) = \sum_{rs} a_{rs} x_r^{(p)} x_s^{(q)},$$

$$(24) \quad (a_{pq}^{(i)}) = \sum_{rs} a_{rs}^{(g)} x_g^{(i)} x_r^{(p)} x_s^{(q)} + \sum_{rs} a_{rs} [x_r^{(pi)} x_s^{(q)} + x_s^{(p)} x_r^{(qi)}],$$

where the $x_r^{(p)}$ have the same meaning as Christoffel's u_r^p . After introducing the Christoffel symbols of the first kind $a_{rs,i}$, and using the laws of transformation for them

$$(25) \quad (a_{rs,i}) = \sum_g x_g^{(i)} \left[\sum_{hk} a_{hk,g} x_h^{(r)} x_k^{(s)} + \sum_h a_{hg} x_h^{(rs)} \right],$$

Ricci-Curbastro arrived at the following expressions for the $x_h^{(rs)}$

$$(26) \quad x_h^{(rs)} = \sum_{pq} (c_{pq})(a_{rs,q}) x_h^{(p)} - \sum_{pqt} c_{hp} a_{qt,p} x_q^{(r)} x_t^{(s)},$$

³⁵ As a result of this paper of Ricci-Curbastro's, the method of studying differential parameters by means of algebraic covariant forms widely characterized the subsequent development of the theory of differential parameters [Somigliana 1890; Frobenius 1892; Knoblauch 1893, 1895]. On the other hand, while he noted the significance of Ricci-Curbastro's method, E. Padova — one of his colleagues at the University of Padua — insisted on using the calculus of variations [Padova 1887]. Further, Ricci-Curbastro's method was incorporated by Levi-Civita into his general treatment of differential invariants [Levi-Civita 1893-1894], which was also based on Lie's approach to the topic.

where the c_{rs} are the coefficients of the form reciprocal to the fundamental one.

Thus far, everything had proceeded along the lines of Christoffel's argument. Now, however, Ricci-Curbastro, in order to examine differential parameters, went on to consider an arbitrary function U and the laws of transformation for its derivatives $U^{(h)}$ and $U^{(hk)}$

$$(27) \quad (U^{(r)}) = \sum_h U^{(h)} x_h^{(r)},$$

$$(28) \quad (U^{(rs)}) = \sum_{hk} U^{(hk)} x_h^{(r)} x_k^{(s)} + \sum_h U^{(h)} x_h^{(rs)}.$$

With respect to U , Ricci-Curbastro also brought in expression (18) as envisaged by Beltrami, and now written as follows

$$(29) \quad U_r = \sum_s c_{rs} U^{(s)}.$$

Unlike Beltrami, who had used these quantities in a variational context, Ricci-Curbastro introduced them as the basis of his algebraic treatment of the search for differential parameters. In particular, by substituting (26), the expressions (28) may be written as follows

$$(30) \quad (U_{rs}) = \sum_{hk} U_{hk} x_h^{(r)} x_k^{(s)},$$

where the quantities U_{hk} are obtained from the U_r in the following manner

$$(31) \quad U_{hk} = U^{(hk)} - \sum_i a_{hk,i} U_i.$$

These last expressions may be seen as the coefficients of a differential quadratic form that is "covariant" with respect to the fundamental form, by virtue of (30); i.e., such that the coefficients of both forms change in like manner. As ever, this view of the matter led Ricci-Curbastro to reduce the search for differential parameters to that for certain algebraic invariants:

"If one constructs the system of absolute algebraic invariants, common to the form φ and to other forms having for coefficients respectively U_{rs} , V_{rs} , W_{rs} , ... all formed, like the U_{rs} , with the coefficients of the arbitrary

functions $U, V, W, \dots$, this yields differential parameters of the second order with any number of arbitrary functions.”³⁶

In the 1886 paper itself, this method of systematic generation of differential parameters was extended by Ricci-Curbastro to the higher-order cases. In particular, when outlining the analysis for the differential parameters of the third order, he also considered the following quantities

$$(32) \quad U_{h k j} = U^{(h k j)} - \sum_{p q} c_{p q} [a_{h k, p} U_{q j} + a_{j k, p} U_{q h} + a_{h j, p} U_{q k}] \\ - \sum_g [a_{k h, g}^{(j)} - \sum_{p q} c_{p q} a_{h k, q} a_{g j, q}] U_g$$

having the same function as the terms $U_{h k}$. In hindsight, these quantities, together with (31), may be seen as the first occurrences in Ricci-Curbastro’s work of what he was to call subsequently covariant differentiation. A minor formal difference concerning the application to the “contravariant” quantities (29) notwithstanding, they were wholly analogous to the expressions (9) of Christoffel. This analogy essentially concerns the function devolving to quantities (31), (32), for Ricci-Curbastro, and quantities (9) for Christoffel, in their programmes of “algebraic reduction” of the theory of differential invariants. That is to say, the expressions (31), (32) were once again cast in the self-same role, i.e. as *linking techniques*, as Christoffel’s quantities (9).

Thus, the resurrection of Christoffel’s algorithms (9) by Ricci-Curbastro does not appear surprising. In effect, the Italian mathematician’s introduction of quantities (31), (32) was no more than a variation, to suit the case of differential parameters, of the suggestion made by the German mathematician for the expressions (9): the basic proof of (30), indeed, is entirely analogous to that of Christoffel for (8), since in both instances the expressions were obtained through substitution, respectively in equation (27) — or its equivalent (7). Actually, this method had originated in elimination theory, once more manifesting the close connection between Ricci-Curbastro’s work and that of Casorati.

³⁶ “Se si costruisce il sistema di invarianti algebrici assoluti comuni alla forma φ ed a più forme rispettivamente di coefficienti $U_{rs}, V_{rs}, W_{rs}, \dots$ formati tutti analogamente alle U_{rs} coi coefficienti delle funzioni arbitrarie $U, V, W, \dots$, si ottengono dei parametri differenziali di $\mathcal{P}$ ordine con un numero qualsivoglia di funzioni arbitrarie” [Ricci-Curbastro 1886a, p. 183].

To sum up: it can be claimed that Ricci-Curbastro was heir to Christoffel with regard to his “algorithmic” genesis of the concept of covariant differentiation, since this emergence was closely connected to the programme of “algebraic reduction” that had characterised the German mathematician’s approach to the topic of differential invariants.³⁷

Differential parameters and the emergence of a broader concept of differentiation

In spite of their methodological similarities, the lines of research pursued by Ricci-Curbastro and Christoffel already diverged quite notably at this initial stage of the Italian mathematician’s work. In this context, indeed, the quantities (31), (32) were seen by Ricci-Curbastro as more general expressions of the customary second and third derivatives of function U .

This fundamental shift in emphasis occurred in singular fashion, in the introduction of Ricci-Curbastro’s paper on differential parameters; it was effected *en passant*, independently of the technical aims pursued in the paper. While pointing to a possible extension of the results arrived at, Ricci-Curbastro wrote:

*“Just as ... the second derivatives of U were expressed by the U_{rs} , for the third derivatives of U one substitutes, by way of analogous methods and results, the coefficients of a cubic form covariant with respect to φ , and, in general, for all the derivatives of U of the m -th order one is to substitute the coefficients of a form of degree m covariant with respect to the one under consideration.”*³⁸

There is an obvious difference between this passage and what had been involved in Christoffel’s work; there, indeed, as we have already pointed out, the expressions (9) were only given an algebraic meaning, as specific

³⁷ Oddly enough, Ricci-Curbastro [1886a] made no direct reference to Christoffel’s work. Nevertheless, acknowledgement of the decisive significance of German mathematician’s research work is to be found in many other places in Ricci-Curbastro’s research. For example: “*L’algorithme du Calcul différentiel absolu ... se trouve entier dans une remarque due à M. Christoffel*” [Ricci-Curbastro, Levi-Civita 1901, p. 127].

³⁸ “*Come ... le derivate seconde di U sono state espresse per le U_{rs} , così alle derivate terze di U , si sostituiscono con metodi e risultati analoghi i coefficienti di una forma cubica covariante a φ , e in generale alle derivate dell’ordine m -esimo di U si sostituiscono i coefficienti di una forma di grado m covariante alla proposta*” [Ricci-Curbastro 1886a, p. 179].

techniques for the progressive generation of differential forms.

How is one to account, then, for the conceptual step taken by Ricci-Curbastro?

One reason, of course, seems to be predominant in this context. In the research work pursued by Christoffel and Ricci-Curbastro one finds a clear shift in topics, from the Riemannian *Aequivalenzproblem* to the systematic investigation of differential parameters. From a technical point of view, this shift entailed a change in the way Christoffel's algorithms (9) were to be viewed: they now could be considered by Ricci-Curbastro in close connection with the analytical character of the arbitrary function U . The aforementioned minor formal difference between Christoffel's expressions (9) and Ricci-Curbastro's (31), (32) represented an obvious technical manifestation of this changed situation.

However, there is another, and more important, reason why this shift in subject matter, from the investigation of differential quadratic forms to that of differential parameters, was of fundamental significance as far as the emergence of a broader concept of differentiation was concerned. In effect, one can show that this idea had already been at work in the research on the theory of differential parameters.

An embryonic model of a generalisation of the usual differentiation had actually already been present in Lamé's work. In that context, the generalisation had focused essentially on the differential parameters of the second order, in order to emphasise their symbolic significance for analytical research. Thus, after setting out the principal equations of mathematical physics in terms of these parameters, Lamé claimed:

“En résumé, lorsqu’une classe de phénomènes physiques dépend des variations d’une certaine fonction-de-point, c’est presque uniquement par son paramètre différentiel du second ordre que cette fonction intervient. Comme si ce paramètre était une dérivée naturelle, plus essentielle, plus simple, et en même temps plus complète, que toutes les dérivées partielles, choisies plus ou moins arbitrairement, que l’on a l’habitude de considérer” [Lamé 1859, p. 25].

Lamé's call for something *plus essentiel* and *plus simple* than the usual differentiation had clearly stemmed from the intrinsic nature of the differential parameters. On the other hand, the notion of “intrinsic” properties characterised the entire tradition of the theory of differential parameters,

as indeed of other forms involved in the investigation of invariance; moreover, it had been closely connected with the development of mathematical, and, in particular, geometrical thought in the 19th century. Thus, it is not surprising that Beltrami — who was working in a Riemannian context — should have taken up and expanded Lamé's idea of a possible extension of the usual differentiation. Once again, attention was being focused on differential parameters, although now also including those of the first order. This involved a very abstract view of such quantities, as expounded by Beltrami, which may be recognised as an embryonic kind of the modern differential operators:

“The functions Δ_1, Δ_2 relative to a certain system of curves drawn on this surface can take an infinity of different values. The system of curves $\varphi = \text{const.}$, in fact, is not geometrically different from the system $f(\varphi) = \text{const.}$; but the parameters Δ_1 and Δ_2 are different according to whether one uses one form rather than the other. Indeed, there is no difficulty in establishing that

$$\Delta_1 f(\varphi) = f'(\varphi)\Delta_1\varphi, \quad \Delta_2 f(\varphi) = f'(\varphi)\Delta_2\varphi + f''(\varphi)(\Delta_1\varphi)^2,$$

*formulae which bear indeed a strong analogy with those which, in the calculus, are used for the differentiation of composite functions.”*³⁹

It is worth noting that this passage ends on a direct quotation of the words of Lamé we have just seen:

*“This fact, which is most noteworthy for function Δ_2 , accounts to some extent for the reason why this function is spontaneously introduced in many studies ‘as if, as M. Lamé says . . . , it were a natural derivative, that is more essential, more simple, and also more complete than all the partial derivatives that are usually considered and which are more or less arbitrarily chosen’.”*⁴⁰

³⁹ “Le funzioni Δ_1, Δ_2 relative ad un certo sistema di curve tracciate sulla superficie possono avere infiniti valori differenti. Infatti il sistema di curve $\varphi = \text{cost.}$ non differisce, geometricamente, dal sistema $f(\varphi) = \text{cost.}$; ma i parametri Δ_1 e Δ_2 sono differenti secondo che si adotta l’una o l’altra forma. Ed invero si trova facilmente

$$\Delta_1 f(\varphi) = f'(\varphi)\Delta_1\varphi, \quad \Delta_2 f(\varphi) = f'(\varphi)\Delta_2\varphi + f''(\varphi)(\Delta_1\varphi)^2,$$

formole che presentano una grande analogia con quelle che nel calcolo differenziale, servono alla differenziazione delle funzioni composte” [Beltrami 1864, pp. 151–152].

⁴⁰ “Questa circostanza, che è massimamente notevole per la funzione Δ_2 , rende in

In this context, while taking up this quest for a broader idea of differentiation, Ricci-Curbastro's research work involved the essential innovation of transferring attention from differential parameters to Christoffel's algorithms (9). This development was clearly signposted by Ricci-Curbastro's direct mention of Lamé, while laying the same emphasis on Christoffel's algorithms that the French mathematician had put on differential parameters of the second order:

*"When the line element has the form $\sum_r dx_r^2$, they coincide with the derivatives of U of the m -th order and it is possible to consider them, rather than differential parameters perhaps, as Lamé said of the latter, as something more essential, more simple, and also more complete than all partial derivatives."*⁴¹

The recurrence of this quotation clearly shows the emergence of a concept over the evolution of a research tradition, in connection with its specific operational circumstances.

Partial differential equations and the concept of covariant differentiation

The point, however, that must decisively corroborate the historical significance of the work of Lamé and Beltrami for the emergence of tensor analysis has to do with the reasons put forward by Ricci-Curbastro for considering the expressions (31), (32) as derivatives of a more general kind. These reasons, indeed, showed a direct connection with Lamé's research programme into the theory of partial differential equations. To wit, while switching attention to the quantities (31), (32), Ricci-Curbastro put forward the same requirements as for the theory of differential parameters. Once again, indeed, emphasis was placed on the usefulness of choosing a particular system of curvilinear coordinates in the study of differential equations:

certo modo ragione del perché questa funzione si introduca spontaneamente in un gran numero di ricerche 'come se essa, dice il sig. Lamé . . . , fosse una derivata naturale, più essenziale, più semplice ed in pari tempo più completa di tutte le derivate parziali che si sogliono considerare e che si scelgono più o meno arbitrariamente'" [Ibid., p. 152]. A similar purpose, of extending the usual meaning of differentiation is also present elsewhere in Beltrami's work [1867b].

⁴¹ *"Nel caso che l'elemento lineare abbia la forma $\sum_r dx_r^2$, essi coincidono colle derivate di ordine m di U e ad esse forse meglio che ai parametri differenziali, si addice il considerarle, come disse di questi il Lamé, come qualche cosa plus essentielle, plus simple et en même temps plus complète que toutes les dérivées partielles"* [Ricci-Curbastro 1886a, p. 179] (in French in the original text).

*“It seems to me that this very substitution should often turn out to be useful in analytical studies since, in the change of variables, the coefficients themselves explicitly introduce nothing but the first derivatives of the old [variables] with respect to the new ones and, depending on the form of the line element, they will indicate naturally, and in every case, the coordinates that are to be preferred, to give the greatest possible simplicity to the equations of the problem itself.”*⁴²

The fundamental import of these remarks lay in the fact that they effectively switched attention from an algebraic view of the quantities (31), (32) to an analytical one. In particular, this was concomitant with a shift in the significance of the implementation of Christoffel’s algorithms: i.e., a shift from their use as *linking techniques* within the framework of the theory of differential parameters to their employment as techniques for the investigation of partial differential equations. If the initial characterisation, as a functional device, of the expressions (31), (32) had brought about the “algorithmic” emergence of the concept of covariant differentiation, it was only their use for the study of partial differential equations that truly underpinned the introduction of this concept, as an extension of the usual differentiation.

The close link between the introduction, by Ricci-Curbastro, of the concept of covariant differentiation and Lamé’s research programme was further demonstrated in the Italian mathematician’s subsequent work. From this point of view, it is most important to note how the concept of covariant differentiation emerged. While the paper of 1886 on differential parameters had already made clear the essential import of covariant differentiation, it actually carried only general considerations, rather than the actual introduction of this concept, which, however, made its appearance in a paper of 1887 [Ricci-Curbastro 1887_a]. It is highly significant, at any rate, that this formal introduction came only after prior employment of the quantities (31), (32) in the context of the theory of partial differ-

⁴² *“Questa medesima sostituzione parmi debba spesso tornare utile nelle ricerche analitiche, perchè i coefficienti stessi non introducono nei cambiamenti di variabili, esplicitamente, se non le derivate prime delle antiche rispetto alle nuove e, dipendendo dalla forma dell’elemento lineare, indicheranno in ogni problema naturalmente le coordinate da preferire per dare alle equazioni del problema stesso la maggiore possibile semplicità”* [Ibid].

ential equations, along the lines of the research tradition on differential parameters.

This first form of the analytical use of covariant differentiation arose in the treatment of a particular extension of Lamé's research, namely the extension of the problem of systems of triply orthogonal families of surfaces — as developed by Darboux in his early research work [Darboux 1878] — to the case of n variables. Ricci-Curbastro discussed this matter in generalised form in a paper [Ricci-Curbastro 1886b] that immediately followed the one on differential parameters.⁴³ From an analytical point of view, the problem addressed by the Italian mathematician corresponded to the search for the existence conditions of $(n - 1)$ integrals $\rho_1, \rho_2, \dots, \rho_{n-1}$ of the equation

$$(33) \quad \sum_r Y_r \frac{\partial \varphi}{\partial x_r} = 0,$$

such as to be mutually orthogonal on a manifold with metric $ds^2 = \sum_{rs} a_{rs} dx_r dx_s$.⁴⁴ Ricci-Curbastro reduced the resolution of this problem to the study of a particular system of partial differential equations. In the course of this investigation, at one point and under well-defined conditions, Ricci-Curbastro introduced the following quantities, which are totally analogous to (29), (31) and (32)

$$(34) \quad \rho_r = \sum_s c_{rs} \frac{\partial \rho}{\partial x_s},$$

$$(35) \quad \rho_{rs} = \frac{\partial^2 \rho}{\partial x_r \partial x_s} - \sum_i a_{rs,i} \rho_i,$$

$$(36) \quad \rho_{rsq} = \frac{\partial^3 \rho}{\partial x_r \partial x_s \partial x_q} - \sum_{hk} c_{hk} [a_{rq,k} \rho_{hs} + a_{sq,k} \rho_{hr} + a_{sr,k} \rho_{hq}] \\ - \sum_k \left[\frac{\partial}{\partial x_q} a_{rs,k} - \sum_{hi} c_{hi} a_{kq,h} a_{rs,i} \right] \rho_k,$$

obtaining the equations

$$(37) \quad \begin{cases} \sum_{rsq} \rho_{rsq} \rho_q H_{rh} H_{sk} = 2 \sum_{pqrs} c_{pq} \rho_{pr} \rho_{qs} H_{rh} H_{sk}, \\ \sum_{rsq} \rho_{rsq} H_{qi} H_{rh} H_{sk} = 0. \end{cases}$$

⁴³ A technical analysis of this work — from a modern point of view — is to be found in [Tonolo 1961].

⁴⁴ I.e., such that $\sum_{rs} c_{rs} (\partial \rho_h / \partial x_r) (\partial \rho_k / \partial x_s) = 0$, with $h \neq k$.

In this process, the expressions (34), (35), (36) played the same functional role as the differential parameters in the work of Lamé and Beltrami, i.e. that of a method to produce the invariant expression of a given equation; the difference, however, lay in the fact that the quantities $\rho_r, \rho_{rs}, \rho_{rsq}$, as against differential parameters, were true generalisations of the usual derivatives. Thus, the equations (37) recast the initial problem in a generalised and invariant form and may indeed be considered as the first “tensor” formulation of a specific analytical problem. Moreover, they were very similar to the general — and, specifically, the non-Euclidean — expressions of the laws of mathematical physics that were being considered in the research work on differential parameters. Compared to the latter, however, the equations (37) were now involved in a programme which was more strictly mathematical in character. In Ricci-Curbastro’s context of research, indeed, the “tensor” formulation of the problem of systems of orthogonal families of surfaces played a fundamental technical role essentially in that it enabled him to obtain more information about the resolution of the given equations. The equations (37) indeed were of remarkable heuristic value for Ricci-Curbastro since they showed:

*“how the degree of difficulty as to the existence of the orthogonal systems in a manifold does not depend solely on the number of dimensions, but also on the nature of the manifold. Given the number n , the difficulty is smallest when the manifold is flat or Euclidean.”*⁴⁵

The same purposes were to be expressed by Ricci-Curbastro more clearly in an extended version of his paper on systems of orthogonal surfaces, published the following year:

*“I believe that the problem as stated and the results arrived at have their own intrinsic value as manifesting a new aspect of the theory of linear and homogeneous partial differential equations of the first order and as constituting, in highly general cases, a notable reduction of the problem of their integration.”*⁴⁶

⁴⁵ “come il grado di difficoltà per la esistenza dei sistemi ortogonali in una varietà non dipenda soltanto dal numero delle dimensioni, ma anche dalla natura della varietà stessa. Fermo il numero n , la difficoltà è minima se la varietà è piana od euclidea” [Ricci-Curbastro 1886b, p. 197].

⁴⁶ “Parmi che il problema enunciato e i risultati ottenuti abbiano un interesse loro proprio come quelli, che mettono in evidenza un nuovo aspetto della teoria delle equazioni lineari ed omogenee a derivate parziali di 1° ordine e costituiscono in casi

It is not surprising, therefore, that similar remarks were also to be found in the note of 1887 which included the formal introduction of covariant differentiation. Once again, the purposes of that introduction reflected the operational goals concerning the expression of an analytical problem — again in conformity with the research tradition of the theory of differential parameters:

*“The usefulness of this substitution is manifest, in particular in the investigations that are essentially independent of the nature of the manifold or of the choice of coordinates in any given manifold. Thus, for instance, those expressions . . . necessarily yield a simpler and clearer form for all expressions endowed with the characteristic property of differential parameters; and they have enabled me to put the equations, which the parameter of a family of $(n - 1)$ -dimensional loci in any n -dimensional manifold must satisfy . . . , in a form as simple as that given by Darboux for the case of a flat or Euclidean manifold with orthogonal Cartesian coordinates.”*⁴⁷

The stage of autonomy of the concept of covariant differentiation

In this overall context of continuity, the formal introduction of the notion of covariant differentiation was essentially conceptual in nature. Indeed, this introduction occurred in the briefest of forms, in an account solely concerned with the new kind of differentiation. Moreover, the introduction was presented in a highly schematic and technical fashion, being based on a complete quotation from the central part of Christoffel’s *Reduktionssatz*.

More specifically, after introducing the usual quantities

$$(38) \quad U_{r_1 r_2 \dots r_p r_{p+1}} = \frac{\partial U_{r_1 r_2 \dots r_p}}{\partial x_{r_{p+1}}} - \sum_{qs} c_{qs} \sum_h a_{r_h r_{p+1}, s} U_{r_1 r_2 \dots r_{h-1} q r_{h+1} \dots r_p},$$

molto generali una riduzione notevole del problema della loro integrazione” [Ricci-Curbastro 1887b, p. 205].

⁴⁷ “La utilità della sostituzione stessa in ispecie nelle ricerche, che sono per loro essenza indipendenti dalla natura della varietà o dalla scelta delle coordinate in una varietà data, è evidente. Così, per esempio, queste espressioni . . . danno necessariamente forma più semplice e perspicua a tutte le espressioni, che godono della proprietà caratteristica dei parametri differenziali, e mi hanno permesso di dare alle equazioni, cui deve soddisfare il parametro di una famiglia di luoghi ad $(n - 1)$ dimensioni in una varietà qual si voglia ad n dimensioni . . . , una forma tanto semplice quanto quella data dal Darboux nel caso in cui la varietà proposta sia piana od euclidea e le coordinate siano cartesiane ortogonali” [Ricci-Curbastro 1887a, p. 199].

Ricci-Curbastro reported Christoffel's statement in the following terms:

*"If the expressions $U_{r_1 r_2 \dots r_p}$ are the coefficients with p indices of a form which is covariant with respect to φ^2 , the $U_{r_1 r_2 \dots r_p r_{p+1}}$ given by (4) [i.e. (38)] are the coefficients with $(p+1)$ indices of a form which is also covariant with respect to φ^2 ."*⁴⁸

The explicit definition of covariant differentiation was no more than a consequence of this statement, from which, of course, the term "covariant" derived:

*"By means of the theorem demonstrated above, we can thus construct successively expressions with $2, 3, \dots, p$ indices, such that those with p indices are the coefficients of forms covariant with respect to φ^2 and include the derivatives of U up to the order p . Further it will be easily seen that these [expressions] will all be linear relative to the derivatives themselves and that each one will include just one derivative of order p . We shall call them covariant derivatives of order p in the manifold which is intrinsically defined by the expression φ^2 of the square of its line element."*⁴⁹

In spite of the algebraic character of its name, the new concept was highly analytical in nature, and was to be viewed as a differential operator. This is clearly vouched for by the fact that, in connection with Beltrami's work concerning differential parameters, a large part of Ricci-Curbastro's presentation [1887a] is devoted to a particular functional property of covariant differentiation, namely its partial commutativity.

Further, the essentially analytical nature of covariant differentiation was attested by a significant peculiarity of Ricci-Curbastro's foregoing formal definition. This definition, indeed, as in the case of the usual scalar differentiation, only concerns the single-function case. That is the meaning

⁴⁸ "Se le espressioni $U_{r_1 r_2 \dots r_p}$ sono coefficienti a p indici di una forma covariante a φ^2 , le $U_{r_1 r_2 \dots r_p r_{p+1}}$ date dalle (4) [(38)] sono coefficienti a $p+1$ indici di una forma pure covariante a φ^2 " [Ibid., p. 201].

⁴⁹ "Mediante il teorema sopra dimostrato possiamo dunque costruire successivamente delle espressioni con $2, 3, \dots, p$ indici, per guisa che quelle con p indici siano coefficienti di forme covarianti a φ^2 e contengano le derivate di U fino all'ordine p . Si vede di più facilmente che esse saranno tutte lineari rispetto alle derivate stesse, e che contengono ciascuna una sola derivata di ordine p . Noi le chiameremo derivate covarianti di ordine p nella varietà, che è definita in sè dalla espressione φ^2 del quadrato del suo elemento lineare" [Ibid., p. 202].

of the expression *derivazione di ordine p* used by the Italian mathematician. In other words, in this context, covariant differentiation was considered as a differentiation iteratively defined on function U . Of course, this also entailed an operation of differentiation on the quantities $U_{r_1 r_2 \dots r_p}$ — even though this consideration had not as yet been explicitly stated by Ricci-Curbastro. In other words, in this initial context, he had no intention of introducing a differentiation on “tensor fields”, i.e., on specific systems of functions.

The simplest way of accounting for this fact is to claim that, notwithstanding the extensive use of the algebraic substrate provided by the theory of multilinear forms, an autonomous concept of the “tensor” had not yet emerged. In effect, Ricci-Curbastro’s sole aim here was to define an analytical tool generalising customary differentiation and, in so doing, he disregarded the nature of what that tool was intended to be applied to. Hence, one may claim that, in Ricci-Curbastro’s work, the introduction of the notion of covariant differentiation conceptually anticipated and implied that of the “tensor field”.

Consequently, a complete technical formulation of the concept of covariant differentiation had to await the formalisation of the notion of tensor, i.e., until 1888. From then on, in every systematic treatment of his methods, Ricci-Curbastro would always term as covariant differentiation a differentiation defined on a particular system of functions [Ricci-Curbastro 1889, 1892; Ricci-Curbastro, Levi-Civita 1901]. For instance, in his 1888 essay “Delle derivazioni covarianti e controvarianti e del loro uso nella analisi applicata”, Ricci-Curbastro introduced covariant (and contravariant) differentiation after calling the expressions $U_{r_1 r_2 \dots r_m}$ *sistema m -plo covariante*:

*“The operation by which, in accordance with (8) [i.e., (38)], one goes over from the m -ply system $U_{r_1 r_2 \dots r_m}$ to the $(m+1)$ -ply system $U_{r_1 r_2 \dots r_{m+1}}$ is what I call differentiation covariant with respect to the differential form φ^2 of the latter system from the former one.”*⁵⁰

Thus, the objects of covariant differentiation were the systems of functions $U_{r_1 r_2 \dots r_m}$, the single-function case merely being a particular case.

⁵⁰ “La operazione, per cui secondo le (8) [(38)] si passa dal sistema $m^{\text{plo}} U_{r_1 r_2 \dots r_m}$ al sistema $(m+1)^{\text{plo}} U_{r_1 r_2 \dots r_{m+1}}$ è quella che chiamo derivazione covariante alla forma differenziale φ^2 del secondo sistema dal primo” [Ricci-Curbastro 1888, p. 251].

Hence, after also introducing the contravariant case, Ricci-Curbastro claimed:

*“I may thus state that, by means of the repeated application of covariant or contravariant differentiation, from one m -ply covariant or contravariant primitive system, others of the same nature can be obtained in indefinite numbers, i.e. one $(m + 1)$ -ply system, one $(m + 2)$ -ply system, etc. A single function U may be regarded as the most elementary [instance] of both the covariant and the contravariant systems.”*⁵¹

From a conceptual point of view, this may be seen as the starting point of the thrust towards a logical reconstruction of the theory, in which the concept of tensor would assume the dominant role.

4. CONCLUSION: WHY WAS TENSOR ANALYSIS BORN IN ITALY?

The foregoing considerations on the genesis of the concept of covariant differentiation also tell us something about the historical reason why tensor analysis was born in Italy, and indeed why it should have sprung from the work of Ricci-Curbastro. As a matter of fact, the situation of Italian differential geometry in the second half of the 19th century, as well as Ricci-Curbastro's position in this context appear highly specific. Let us spell this out, and adduce some reasons.

The crucial element of our subject is the relationship between Ricci-Curbastro's work and the mathematical idea of invariance: a relationship which appears more developed than in other fields of research in the post-Riemannian period. In particular, there was a difference between Ricci-Curbastro and the other mathematicians who continued Riemann's work on differential geometry. The works of these latter, indeed, show the presence of certain aspects of the mathematical idea of invariance — and of their respective research programmes — in the 19th century, but never all of these aspects actually together.

Indeed, Christoffel's work on differential quadratic forms developed on lines that were essentially concerned with two types of occurrence of the

⁵¹ “Potrà così dire che mediante la applicazione ripetuta della derivazione covariante o controvariante da un sistema m -plo primitivo covariante o controvariante, se ne possono ottenere altri della stessa natura in numero indefinito e cioè uno $(m + 1)^{plo}$ uno $(m + 2)^{plo}$ ecc. Una funzione unica U può riguardarsi come il più elementare tanto tra i sistemi covarianti quanto tra i controvarianti” [*Ibid.*, p. 252].

idea of mathematical invariance. On the basis of the *Reduktionssatz*, his procedure, in fact, consisted in the global application of the theory of algebraic invariants to the theory of Riemannian differential invariants, as we have seen. On the other hand, Lipschitz's work on differential invariants and that of Beltrami on differential parameters showed no such exclusive reliance on algebraic methods: their analyses, indeed, were essentially grounded on the calculus of variations, an instrument that remained alien to Christoffel's methodology. This was no mere chance event. The absence of the calculus of variations from Christoffel's work on differential quadratic forms may be seen as a specific reflection of the way he stood apart from the tradition of research on differential parameters, which, on the contrary, characterised Beltrami's (and, to some extent, Lipschitz's) work. Partly linked to this fact, the latter's work on differential invariants manifested an element of an analytical nature that was not to be found in Christoffel's work. That element was the considerable significance attributed to certain differential quantities (the differential parameters), owing to the central role they played in much analytical research, an importance which allowed Beltrami — as, before him, Lamé — to view them as expressions generalising the operation of customary differentiation. In this context the role played by the calculus of variations — which had no part in Lamé's work — was essentially that of amplifying the analytical import of the topic of differential invariants in Beltrami's research work.

Thus, the position occupied by Ricci-Curbastro is highly distinctive. His work, in effect, represented the intersection point of various influences, showing the simultaneous presence of elements associated with partly divergent research traditions.

Actually, the investigation that led Ricci-Curbastro to bring in the algorithm of covariant differentiation, as we have seen, was characterised by the global application of the methods of the theory of algebraic invariants to the theories of differential invariants and differential parameters. In this process, the natural premise and starting point was represented by Christoffel's work, consolidated by the algebraic nature of Casorati's geometrical research. This character of Ricci-Curbastro's work represented a true break with the earlier methods implemented in the investigation of differential invariants and parameters, essentially based as they were

on the calculus of variations. On the other hand, his analytical interpretation of the algorithm of covariant differentiation as a more general form of differentiation occurred as a result of the reappraisal of certain topics emanating from the research tradition on differential parameters. Thus, the original algorithm of covariant differentiation, introduced as a *linking technique* to reduce the investigation of an analytical problem (the equivalence of differential quadratic forms, the systematic search for differential parameters) to an algebraic perspective, was reconsidered by Ricci-Curbastro from an analytical point of view, on an independent level of interpretation.

Thus, compared with Christoffel, Ricci-Curbastro had recourse to a methodologically similar reliance on the theory of algebraic invariants for analytical matters, while generalising the German mathematician's interpretative viewpoint. At the same time, Ricci-Curbastro eschewed Beltrami's methodological approach to differential parameters, while taking up the conceptual import of the Italian mathematician's research work.

One may claim, therefore, that Ricci-Curbastro's work involved a peculiar "mix" of divergent research traditions. Thus, the genesis of the concept of covariant differentiation appears as the point of convergence of different contexts of development of the idea of invariance: algebraic invariants, differential invariants and parameters. And this is a fact that truly characterises the emergence of tensor analysis as one of the most important syntheses of the idea of invariance to have occurred in the second half of the 19th century.⁵²

On the other hand, one should also note that the conflict between research traditions, manifest in such an explicit form in Ricci-Curbastro's work, equally characterised the development of Italian differential geometry at large. It is essentially this element that may be of help in understanding the historical reason why tensor analysis should have emerged in that mathematical community. One may claim, indeed, that this conflict was also inherent in the virtual opposition between Casorati's algebraic approach to geometrical research and that of Beltrami, which was of a

⁵² One should note that, in addition, Klein also exerted an influence on Ricci-Curbastro, who had attended his courses at the University of Munich in 1877. Although it is not possible to consider this as a truly direct influence [Levi-Civita 1925, p. 393], it is clear that Ricci-Curbastro inherited from the German mathematician the awareness of the centrality of invariance in geometrical research.

physico-mathematical nature. Despite its virtual character, rooted as it was in a kind of operational coexistence,⁵³ this contrast in fact exhibited an intrinsic potentiality: the possibility of entertaining simultaneously different points of view concerning the use of the idea of invariance in contexts of research concerned with the investigation of differential matters.

BIBLIOGRAPHY

ARONHOLD (S.)

- [1863] Ueber eine fundamentale Begründung der Invariantentheorie, *Journal für die reine und angewandte Mathematik*, 62 (1863), pp. 281–345.

BELL (E.T.)

- [1945] *The development of mathematics*, New York: McGraw-Hill, 1945.

BELTRAMI (E.)

- [Opere] *Opere matematiche*, 4 vols., Milano: Ulrico Hoepli, 1902–1920.
 [1864–1865] Ricerche di analisi applicata alla geometria, *Giornale di matematiche*, 2 (1864), pp. 267–282, 297–306, 331–339, 355–375, and 3 (1865), pp. 15–22, 33–41, 82–91, 228–240, 311–314; *Opere* 1, pp. 107–198.
 [1867a] Intorno ad una trasformazione di variabili, *Ibid.*, 5 (1867), pp. 24–27; *Opere* 1, pp. 306–309.
 [1867b] Delle variabili complesse sopra una superficie qualunque, *Annali di matematica pura ed applicata*, (II) 1 (1867), pp. 329–366; *Opere* 1, pp. 318–353.
 [1868a] Sulle proprietà generali delle superficie d’area minima, *Memorie dell’Accademia delle scienze dell’Istituto di Bologna*, (II) 7 (1868), pp. 411–461; *Opere* 2, pp. 1–54.
 [1868b] Sulla teorica generale dei parametri differenziali, *Ibid.*, (II) 8 (1868), pp. 551–590; *Opere* 2, pp. 74–118.
 [1880–1882] Sulle equazioni generali dell’elasticità, *Ann. mat. pura appl.*, (II) 10 (1880–1882), pp. 188–211; *Opere* 3, pp. 383–407.
 [1884] Sull’uso delle coordinate curvilinee nelle teorie del potenziale e dell’elasticità, *Mem. Accad. sci. Ist. Bologna*, (IV) 6 (1884), pp. 401–448; *Opere* 4, pp. 136–179.

BERTINI (E.)

- [1892] Commemorazione del Prof. Francesco Casorati, *Rendiconti dell’Istituto lombardo di scienze e lettere*, (II) 25 (1892), pp. 1206–1236.

BOTTAZZINI (U.)

- [1980] Algebraische Untersuchungen in Italien, 1850–1863, *Historia mathematica*, 7 (1980), pp. 24–37.

BRIOSCHI (F.)

- [1854] *Teorica dei determinanti*, Pavia, 1854.

CASORATI (F.)

- [1860–1861] Ricerca fondamentale per lo studio di una certa classe di proprietà delle superficie curve, *Ann. mat. pura appl.*, (I) 3 (1860), pp. 363–379, and (I) 4 (1861), pp. 177–185; *Opere*, vol. 2, Roma: Cremonese, 1952, pp. 132–163.

⁵³ Indeed, Beltrami never criticized Casorati’s methodology and, moreover, he was make extensive use of algebraic methods in his treatment of differential parameters.

CHELINI (D.)

- [1853] Sulle formole fondamentali riguardanti la curvatura delle superficie, e delle linee, *Annali di scienze, matematiche e fisiche*, 4 (1853), pp. 337–394.

CHRISTOFFEL (E.B.)

- [GMA] *Gesammelte mathematische Abhandlungen*, 2 vols., Leipzig-Berlin: Teubner, 1910.
- [1868a] Beweis des Fundamentalsatzes der Invariantentheorie, *J. reine angew. Math.*, 68 (1868), pp. 246–252; *GMA* 1, pp. 269–276.
- [1868b] Theorie der bilinearen Functionen, *Ibid.*, 68 (1868), pp. 253–272; *GMA* 1, pp. 277–296.
- [1869a] Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades, *Ibid.*, 70 (1869), pp. 46–70; *GMA* 1, pp. 352–377.
- [1869b] Ueber ein die Transformation homogener Differentialausdrücke zweiten Grades betreffendes Theorem, *Ibid.*, 70 (1869), pp. 241–245; *GMA* 1, pp. 378–382.

CODAZZI (D.)

- [1868] Sulle coordinate curvilinee d'una superficie e dello spazio, *Ann. mat. pura appl.*, (II) 1 (1868), pp. 293–316.

DARBOUX (G.)

- [1878] Mémoire sur la théorie des coordonnées curvilignes et des systèmes orthogonaux, *Annales scientifiques de l'École normale supérieure*, (II) 7, pp. 101–150, 227–260, 275–348.

EHLERS (J.)

- [1981] Christoffel's work on the equivalence problem for Riemannian spaces and its importance for modern field theories of physics, in P.L. Butzer, F. Fehér (eds.), *E.B. Christoffel. The influence of his work on mathematics and the physical sciences*, Basel: Birkhäuser, 1981, pp. 526–542.

FARWELL (R.), KNEE (C.)

- [1990] The missing link: Riemann's 'Commentatio', differential geometry and tensor analysis, *Hist. math.*, 17 (1990), pp. 223–255.

FROBENIUS (G.)

- [1892] Ueber die in der Theorie der Flächen auftretenden Differentialparameter, *J. reine angew. Math.*, 110 (1892), pp. 1–36.

GAUSS (C.F.)

- [Werke] *Werke*, 12 vols., Königlichen Gesellschaft der Wissenschaften zu Göttingen ed., Leipzig-Berlin, 1863–1933; repr. Olms, 1981.

HALPHEN (G.)

- [1878] *Sur les invariants différentiels*, Thèse, Paris: Gauthier-Villars, 1878.

JACOBI (G.)

- [1847] Über eine particuläre Lösung der partiellen Differentialgleichung $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$, *J. reine angew. Math.*, 36 (1847), pp. 113–134; *Gesammelte Werke*, vol. II, Berlin, 1882, p. 191–216 (repr. Chelsea, 1969).

KILLING (W.)

- [1885] Die Mechanik in den Nicht-Euklidischen Raumformen, *J. reine angew. Math.*, 98 (1885), pp. 1–48.

KLEIN (F.)

- [1927] *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, vol. II, Berlin: Springer, 1927; repr. Chelsea, 1950.

KLINE (M.)

- [1972] *Mathematical thought from ancient to modern times*, Oxford: Oxford University Press, 1972.

KNOBLAUCH (J.)

- [1893] Zur Theorie der Differentialparameter, *J. reine angew. Math.*, 111 (1893), pp. 329–343.
 [1895] Zur simultanen Transformation quadratischer Differentialformen, *Ibid.*, 115 (1895), pp. 185–200.

KOPPELMAN (E.)

- [1975] Progress in mathematics, *Hist. math.*, 2 (1975), pp. 457–463.

LAMÉ (G.)

- [1834] Mémoire sur les lois de l'équilibre du fluide éthéré, *Journal de l'École polytechnique*, XIV, 23^e cahier (1834), pp. 191–288.
 [1859] *Leçons sur les coordonnées curvilignes et leurs diverses applications*, Paris, 1859.

LEICHTWEISS (K.)

- [1981] E.B. Christoffels Einfluss auf die Geometrie, in D. Ferus, W. Kühnel *et al.* (eds.), *Global differential geometry and global analysis*, New York-Berlin: Springer (Lecture Notes in Mathematics, 838), pp. 1–11.

LEVI-CIVITA (T.)

- [Opere] *Opere matematiche*, 6 vols., Bologna: Zanichelli, 1954–1973.
 [1893–1894] Sugli invarianti assoluti, *Atti dell'Istituto veneto di scienze, lettere ed arti*, (VII) 5 (1893–1894), pp. 1447–1523; *Opere* 1, pp. 41–100.
 [1925] Commemorazione del socio nazionale Prof. G. Ricci-Curbastro, *Atti dell'Accademia dei Lincei. Memorie*, (VI) 1 (1925), pp. 555–564; *Opere* 4, pp. 391–402.

LIE (S.)

- [1884] Über Differentialinvarianten, *Mathematische Annalen*, 24 (1884), pp. 537–578.

LIPSCHITZ (R.)

- [1869] Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen, *J. reine angew. Math.*, 70 (1869), pp. 71–102.
 [1870] Entwicklung einiger Eigenschaften der quadratischen Formen von n Differentialen, *Ibid.*, 71 (1870), pp. 274–287, 288–295.
 [1871] Fortgesetzte Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen, *Ibid.*, 72 (1871), pp. 1–56.
 [1872] Untersuchung eines Problemes der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist, *Ibid.*, 74 (1872), pp. 116–149.

LORIA (G.)

- [1901] Eugenio Beltrami e le sue opere matematiche, *Bibliotheca mathematica*, (III) 2 (1901), pp. 392–440.

LÜTZEN (J.)

- [1989] The geometrization of analytical mechanics. A pioneering contribution by Joseph Liouville (ca. 1850), in J. Mc Cleary, D.E. Rowe (eds.), *The history of modern mathematics*, vol. II, Boston: Academic Press, pp. 76–97.
 [1995] Interactions between mechanics and differential geometry in the 19th century, *Archive for History of Exact Sciences*, 47 (1995), pp. 1–72.

NEUMANN (C.)

- [1860] Zur Theorie der Elasticität, *J. reine angew. Math.*, 57 (1860), pp. 281–318.

- [1867] Kurzer Abriss einer Theorie der Kugelfunctionen und Ultrakugelfunctionen, *Zeitschrift für Mathematik und Physik*, 12 (1867), pp. 97–122.
- PADOVA (E.)
- [1887] Sulle espressioni invariabili, *Atti Accad. Lincei Mem.*, (IV) 4, pp. 4–17.
- PASCAL (E.)
- [1901] Commemorazione di Eugenio Beltrami, *Rc. Ist. lomb. sci. lett.*, (II) 34 (1901), pp. 57–108.
- PINL (M.)
- [1981] E.B. Christoffels Weg zum absoluten Differentialkalkül und sein Beitrag zur Theorie des Krümmungstensors, in P.L. Butzer, F. Fehér (eds.), *E.B. Christoffel, The influence of his work on mathematics and the physical sciences*, Basel: Birkhäuser, 1981, pp. 474–489.
- POGREBYSSKY (J.B.)
- [1981] Brill, Alexander Wilhelm von, in *Dictionary of scientific biography* (C. Gillispie ed.), vol. 1, 1981, p. 465.
- REICH (K.)
- [1973] Die Geschichte der Differentialgeometrie von Gauss bis Riemann (1828–1868), *Arch. Hist. Exact Sci.*, 11 (1973), pp. 273–382.
- [1994] *Die Entwicklung des Tensorkalküls*, Basel: Birkhäuser (Science Networks, Historical Studies, vol. 11), 1994.
- RICCI-CURBASTRO (G.)
- [Opere] *Opere*, 2 vols., Roma: Cremonese, 1956–1957.
- [1877a] Sopra un sistema di due equazioni differenziali lineari ... , *G. mat.*, 15 (1877), pp. 135–153; *Opere I*, pp. 17–39.
- [1877b] ‘Sopra la deduzione di una nuova legge fondamentale di elettrodinamica ...’ — ‘Sopra il modo di agire delle forze pondero-ed elettromotrici fra due conduttori filiformi ...’ —, per B. Clausius, *Nuovo cimento*, (III) 1 (1877), pp. 58–72, 89–106; *Opere I*, pp. 40–68.
- [1877c] Sulla teoria elettrodinamica di Maxwell, *Ibid.*, (III) 2 (1877), pp. 5–27, 93–116; *Opere I*, pp. 69–109.
- [1882] Sulla funzione potenziale di conduttori di correnti galvaniche costanti, *Atti Ist. veneto sci.*, (V) 12 (1882), pp. 1025–1048; *Opere I*, pp. 110–129.
- [1884] Principi di una teoria delle forme differenziali quadratiche, *Ann. mat. pura appl.*, (II) 12 (1884), pp. 135–167; *Opere I*, pp. 138–171.
- [1886a] Sui parametri e gli invarianti delle forme quadratiche differenziali, *Ibid.*, (II) 14 (1886), pp. 1–11; *Opere I*, pp. 177–188.
- [1886b] Sui sistemi di integrali indipendenti di una equazione lineare ed omogenea a derivate parziali del 1° ordine, *Atti dell’Accademia dei Lincei. Rendiconti*, (IV) 2 (1886), pp. 119–122, 190–194; *Opere I*, pp. 189–198.
- [1887a] Sulla derivazione covariante ad una forma quadratica differenziale, *Ibid.*, (IV) 3 (1887), pp. 15–18; *Opere I*, pp. 199–203.
- [1887b] Sui sistemi di integrali indipendenti di una equazione lineare ed omogenea a derivate parziali del 1° ordine, *Ann. mat. pura appl.*, (II) 15 (1887), pp. 127–159; *Opere I*, pp. 204–238.
- [1888] Delle derivazioni covarianti e controvarianti e del loro uso nella analisi applicata, in *Studi editi dall’Università di Padova a commemorare l’ottavo centenario della origine della Università di Bologna*, vol. III, Padova, 1888, pp. 3–23; *Opere I*, pp. 245–267.
- [1889] Sopra certi sistemi di funzioni, *Atti Accad. Lincei Rc.*, (IV) 5 (1889), pp. 112–118; *Opere I*, p. 268–275.

- [1892] Résumé de quelques travaux sur les systèmes variables de fonctions associés à une forme différentielle quadratique, *Bulletin des sciences mathématiques*, (II) 16 (1892), pp. 167–189; *Opere* I, pp. 288–310.
 - [1893] Di alcune applicazioni del calcolo differenziale assoluto alla teoria delle forme differenziali quadratiche binarie e dei sistemi a due variabili, *Atti Ist. veneto sci.*, (VII) 4 (1893), pp. 1336–1364; *Opere*, I, pp. 311–335.
- RICCI-CURBASTRO (G.), LEVI-CIVITA (T.)
- [1901] Méthodes de calcul différentiel absolu et leurs applications, *Math. Ann.*, 54 (1901), pp. 125–201; Ricci-Curbastro, *Opere* II, pp. 185–271; Levi-Civita *Opere* 1, pp. 479–559.
- RIEMANN (B.)
- [Werke] *Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, (H. Weber ed.), Leipzig: Teubner, 1876; 2nd ed. 1892; repr. New York: Springer, 1991.
 - [1854] Ueber die Hypothesen, welche der Geometrie zu Grunde liegen, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 13 (1868), pp. 133–152; *Werke*, pp. 272–287.
 - [1861] Commentatio mathematica, qua respondere tentatur quaestioni ab Ill^{ma} Academia Parisiensi propositae . . . , *Werke*, pp. 391–404. English transl. in [Farwell, Knee 1990, pp. 240–253].
- SCHERING (E.)
- [Werke] *Gesammelte mathematische Werke von Ernst Schering*, R. Haussner and K. Schering (eds.), 2 vols., Berlin: Mayer & Müller, 1902–1909.
 - [1870] Die Schwerkraft im Gaussischen Raume, *Nachrichten von der Gessellschaft der Wissenschaften zu Göttingen*, (1870), pp. 311–321; *Werke* 1, pp. 155–162.
 - [1873] Die Schwerkraft in mehrfach ausgedehnten Gaussischen und Riemannschen Räumen, *Ibid.* (1873), pp. 149–159; *Werke* 1, pp. 177–184.
- SOMIGLIANA (C.)
- [1890] Sulla trasformazione delle equazioni lineari, omogenee, a derivate parziali, con coefficienti costanti, *Ann. mat. pura appl.*, (II) 18 (1890), pp. 265–299.
- SOMOV (J.)
- [1865] Moyen d'exprimer directement en coordonnées curvilignes quelconques, orthogonales et obliques, les paramètres différentiels du premier et du second ordres et la courbure d'une surface, *Mémoires de l'Académie des sciences de St-Pétersbourg*, (VII) 8 (1865), pp. 1–45.
- SPEZIALI (P.)
- [1981] Ricci-Curbastro, Gregorio, in *Dictionary of scientific biography*, vol. 11, 1981, pp. 406–411.
- STRUİK (D.J.)
- [1933] Outline of a history of differential geometry, *Isis*, 20 (1933), pp. 161–191.
 - [1981] Beltrami, Eugenio, in *Dictionary of scientific biography*, vol. 1, 1981, pp. 599–600.
- TAZZIOLI (R.)
- [1993] Ether and theory of elasticity in Beltrami's work, *Arch. Hist. Exact Sci.*, 45 (1993), pp. 1–37.
- TONELLI (A.)
- [1882] Sopra la funzione potenziale in uno spazio di n dimensioni, *Ann. mat. pura appl.*, 10 (1882), pp. 291–321.

TONOLO (A.)

- [1954] Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita, *Rendiconti del Seminario matematico dell'Università di Padova*, 23 (1954), pp. 1–24.

- [1961] Sulle origini del calcolo di Ricci, *Ann. mat. pura appl.*, (IV) 53 (1961), pp. 189–207.

VEBLEN (O.)

- [1927] *Invariants of quadratic differential forms*, London: Cambridge University Press, 1927.

VINCENSINI (P.)

- [1972] La géométrie différentielle au XIX^e siècle, *Scientia*, 107 (1972), pp. 616–660.

VIVANTI (G.)

- [1935] Felice Casorati nel centenario della nascita, *Rendiconti del Seminario matematico e fisico di Milano*, 9 (1935), pp. 127–138.

WEITZENBÖCK (R.)

- [1921] Neuere Arbeiten der algebraischen Invariantentheorie. Differentialinvarianten, in W.F. Meyer., H. Mohrmann (eds.), *Encyklopädie der mathematischen Wissenschaften*, Leipzig: Teubner, vol. III D, 1902–1927, art. 10, pp. 1–71.

- [1923] *Invariantentheorie*, Groningen: P. Noordhoff, 1923.