

Tensors in a Nutshell

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1. Introduction

Tensors are used to represent quantities that are related to the directions in space. In these notes three-dimensional space is represented by a set of Cartesian axes numbered 1,2,3, corresponding to the coordinates x, y, z . A vector, which is a first rank tensor, is a quantity that has both magnitude and direction, and is specified by its components along each of the three axes. For example, an electric field $\mathbf{E}$ is specified by its components E_1, E_2, E_3 .

To see how higher rank tensors arise, consider a molecule (or other object) placed in an electric field. A dipole moment $\boldsymbol{\mu}$ (another vector) is induced in the molecule by the field. Each component μ_1, μ_2, μ_3 of the dipole moment depends on all three components of the field because, in general, the charge displacement caused by any one field component does not occur in a direction parallel to that component. So we write

$$\mu_1 = \alpha_{11}E_1 + \alpha_{12}E_2 + \alpha_{13}E_3 \quad (1)$$

$$\mu_2 = \alpha_{21}E_1 + \alpha_{22}E_2 + \alpha_{23}E_3 \quad (2)$$

$$\mu_3 = \alpha_{31}E_1 + \alpha_{32}E_2 + \alpha_{33}E_3 \quad (3)$$

The nine α coefficients represent the polarizability of the molecule. The polarizability is an example of a second rank tensor. Each component refers to two directions in space in such a tensor. The boldface symbols $\mathbf{E}$, $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$ refer to the full set of components of the corresponding vector or tensor.

These notes summarize some of the important mathematical properties of tensors in three-dimensional Cartesian space. The references in the bibliography may be consulted for more detailed expositions [1, 2, 3].

2. Definitions

A tensor of rank n (also called *order* n) is a set of 3^n quantities which obey certain rules of transformation when the coordinate axes are rotated. The transformation will be described in more detail below. A scalar is a tensor of rank 0 and a vector is a tensor of rank 1.

A tensor of rank 1 or greater is commonly denoted either by a boldface symbol such as $\mathbf{A}$ or by component notation such as $A_{\alpha\beta\gamma\dots}$, where the Greek subscripts may take the values 1,2,3 corresponding to the coordinate axes. In component notation the number of Greek subscripts is the rank of the tensor.

Tensors of rank 1 and 2 can be represented by familiar matrix notation. Then the components of $\mathbf{E}$ and $\boldsymbol{\alpha}$, mentioned above, can be written in the matrix arrays

$$\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \quad (4)$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad (5)$$

In the second rank tensor $\alpha_{\beta\gamma}$ is thus the element of row β and column γ of the matrix form of the tensor. In principle one could extend this idea to higher rank tensors by writing tensors as matrices of three or more dimensions. Of course, this presents serious typographical difficulties. But there are alternative matrix methods, to be described below, that are practical for tensors of any rank.

3. Tensor products

Let $\mathbf{p}$ and $\mathbf{q}$ be two first rank tensors (vectors). The direct product, or dyadic product, $\mathbf{pq}$ is the array of quantities obtained by multiplying each component p_α by each component q_β . The dyadic product thus has nine components and comprises a tensor of rank two. Let this tensor be called $\mathbf{T}$. Then we can write $\mathbf{T}$ in either boldface or component form as follows:

$$\mathbf{T} = \mathbf{pq} \quad (6)$$

$$T_{\alpha\beta} = p_\alpha q_\beta \quad (7)$$

Note that we can also write the dyadic product of two vectors in matrix form:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} p_1 q_1 & p_1 q_2 & p_1 q_3 \\ p_2 q_1 & p_2 q_2 & p_2 q_3 \\ p_3 q_1 & p_3 q_2 & p_3 q_3 \end{pmatrix} \quad (8)$$

In general one forms the direct product of an n -th rank tensor $A_{\alpha_1 \dots \alpha_n}$ and an m -th rank tensor $B_{\beta_1 \dots \beta_m}$ by multiplying each component of $\mathbf{A}$ with each component of $\mathbf{B}$, giving a tensor $\mathbf{C}$ of rank $n + m$. This is written in either boldface or component notation as follows.

$$\mathbf{C} = \mathbf{A} \mathbf{B} \quad (9)$$

$$C_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m} = A_{\alpha_1 \dots \alpha_n} B_{\beta_1 \dots \beta_m} \quad (10)$$

Sometimes the term *polyadic* product is used for the direct product of more than two vectors. For example, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are vectors, then we can form the third rank tensor $\mathbf{D}$ as the polyadic product

$$D_{\alpha\beta\gamma} = a_\alpha b_\beta c_\gamma \quad (11)$$

Likewise, we could take the polyadic product of $\mathbf{a}$ by itself N times, denoted $\mathbf{a}^N$, to obtain a tensor of rank N .

4. Tensor contractions

A contraction of a tensor is the result of setting two component indices equal to each other and summing over the three values of those indices. A common example is the dot product, or scalar product, of two vectors. Let S be the contraction of tensor $\mathbf{T}$ in eq 6.

$$S = T_{11} + T_{22} + T_{33} \quad (12)$$

$$= p_1 q_1 + p_2 q_2 + p_3 q_3 \quad (13)$$

Thus the contraction of $\mathbf{pq}$ is just the scalar product, normally written

$$S = \mathbf{p} \cdot \mathbf{q} \quad (14)$$

A widely accepted convention in writing tensor contractions is the summation convention of Einstein [4]: *when any two subscripts in a tensor expression are given the same symbol, it is implied that the contraction is formed*. Thus the symbol $T_{\alpha\alpha}$ means the same thing as S .

The convention for the dot product of higher rank tensors is illustrated by the following example, where $\mathbf{T}$ and $\mathbf{Q}$ are both second rank tensors.

$$(\mathbf{T} \cdot \mathbf{Q})_{\alpha\beta} = T_{\alpha\nu} Q_{\nu\beta} \quad (15)$$

That is, the contraction represented by the dot is a contraction of the rightmost index of the first factor with the leftmost index of the second factor.

Dot products between tensors of rank 1 or 2 are equivalent to conventional matrix products. For example, eqs 1–3 can be written

$$\boldsymbol{\mu} = \boldsymbol{\alpha} \cdot \mathbf{E} \quad (16)$$

which is equivalent to the product of matrices of eqs 4 and 5.

Multiple contractions of higher rank tensors are possible. For example, a double contraction of $\mathbf{T}$ and $\mathbf{Q}$ is expressed by

$$\mathbf{T} : \mathbf{Q} = T_{\alpha\beta} Q_{\beta\alpha} \quad (17)$$

where the summation convention is implied for index pairs $\alpha\alpha$ and $\beta\beta$.

An n -fold contraction is denoted by $\cdot n \cdot$. For example, if $\mathbf{A}$ and $\mathbf{B}$ are both n -th rank tensors, we can write

$$\mathbf{A} \cdot n \cdot \mathbf{B} = A_{\alpha_1 \dots \alpha_n} B_{\alpha_n \dots \alpha_1} \quad (18)$$

Note that the contraction indicated here produces a scalar quantity. In general each contraction of the direct product $\mathbf{AB}$ reduces the tensor rank by 2. An n -fold contraction thus reduces the rank by $2n$, which leaves rank 0 in this case.

The nature of a multiple contraction can be seen more clearly if we write out the case of a double contraction of a pair of second rank tensors:

$$\begin{aligned} \mathbf{A} : \mathbf{B} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ &\quad + A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \\ &\quad + A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33} \end{aligned} \quad (19)$$

We see that this can be written as the following matrix product:

$$\mathbf{A} : \mathbf{B} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{21} & A_{22} & A_{23} & A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \\ B_{31} \\ B_{12} \\ B_{22} \\ B_{32} \\ B_{13} \\ B_{23} \\ B_{33} \end{pmatrix} \quad (20)$$

Thus it is always possible to represent a higher order tensor by a matrix in which each row and/or column is indexed by one of the sets of component indices $\alpha_1 \dots \alpha_n$, letting these sets be arranged in some systematic manner as illustrated above. Then it is possible to write any higher order contraction as a matrix product.

5. Transformation of tensors under rotations

Consider a physical system that is fixed in space. Let S and S' be two sets of Cartesian axes that are rotated with respect to each other. A vector that characterizes some aspect of the physical system will be fixed in space, and must therefore have different components in S and S' . Call the vector $\mathbf{v}$ and $\mathbf{v}'$ in the two representations, respectively. The transformation of $\mathbf{v}$ under rotation of the coordinate axes is given by

$$\begin{pmatrix} v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (21)$$

where $\lambda_{\alpha\beta}$ is the direction cosine of axis α in S' with respect to axis β in S . The proof of eq 21 rests on the calculation of the projection of each component v_β on each axis α , and summing the projections on each axis α . We can abbreviate eq 21 by using the summation convention:

$$v'_\alpha = \lambda_{\alpha\beta} v_\beta \quad (22)$$

Consider now the polyadic product $\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_n$ formed from the n vectors $\mathbf{v}_1, \mathbf{v}_2$, etc. This product is transformed according to

$$v'_{1,\alpha_1} v'_{2,\alpha_2} \dots v'_{n,\alpha_n} = (\lambda_{\alpha_1\beta_1} v_{1,\beta_1}) (\lambda_{\alpha_2\beta_2} v_{2,\beta_2}) \dots (\lambda_{\alpha_n\beta_n} v_{n,\beta_n}) \quad (23)$$

Likewise any n -th rank tensor $\mathbf{A}$ is transformed under this rotation according to

$$A'_{\alpha_1 \dots \alpha_n} = \lambda_{\alpha_1\beta_1} \dots \lambda_{\alpha_n\beta_n} A_{\beta_1 \dots \beta_n} \quad (24)$$

This leads us to the formal definition of an n -th rank Cartesian tensor as *a set of 3^n quantities which transform under rotation of axes in the same manner as an n -th order polyadic*. Note that the quantities $\lambda_{\alpha\beta}$ do not constitute a tensor by this definition.

6. Isotropic tensors

An isotropic tensor is a tensor whose components do not change under any rotation of the coordinate system. A trivial case is any zero rank tensor, which is always invariant under rotation of the coordinate system. There is no isotropic first rank tensor.

The identity tensor $\mathbf{I}$, represented by the matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (25)$$

is a second rank isotropic tensor. The components of $\mathbf{I}$ are often denoted by the Kronecker delta, $\delta_{\alpha\beta}$, defined by

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \quad (26)$$

Any second rank isotropic tensor can be represented as the product of a scalar times $\mathbf{I}$; for example

$$a\mathbf{I} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad (27)$$

is such a tensor.

The permutation tensor ϵ , defined by

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} 1 & \text{if } \alpha\beta\gamma \text{ is any cyclic permutation of } 123 \\ -1 & \text{if } \alpha\beta\gamma \text{ is any cyclic permutation of } 321 \\ 0 & \text{if any two indices are equal} \end{cases} \quad (28)$$

is a third rank isotropic tensor. Any third rank isotropic tensor can be represented as the product of a scalar times ϵ .

The following are some useful relationships involving multiple contractions with isotropic tensors. Here $\mathbf{S}$ is a second rank tensor.

$$\mathbf{I} : \mathbf{I} = 3 \quad (29)$$

$$\mathbf{S} : \mathbf{I} = S_{11} + S_{22} + S_{33} \quad (30)$$

$$\epsilon \cdot 3 \cdot \epsilon = -6 \quad (31)$$

7. Isotropic averages of tensors

The isotropic average of a tensor $\mathbf{A}$, denoted by $\langle \mathbf{A} \rangle$, is the average over all orientations of the coordinate axes. It is equivalent to the average over a large number of identical physical systems oriented randomly in a single coordinate system. $\langle \mathbf{A} \rangle$ is necessarily an isotropic tensor, since it is invariant under any rotation of the coordinate axes.

Let $\mathbf{S}$ be a second rank tensor. To calculate its isotropic average, we make use of the fact that the sum of diagonal elements given by eq 30, called the *trace* of $\mathbf{S}$, is invariant under any rotation of the coordinate axes. We know further that the average must be of the form

$$\langle \mathbf{S} \rangle = a\mathbf{I} \quad (32)$$

where a is a scalar. Hence the problem is to find a . From the invariance of the trace we can write

$$\langle \mathbf{S} \rangle : \mathbf{I} = \langle \mathbf{S} : \mathbf{I} \rangle = \text{Tr} \mathbf{S} \quad (33)$$

where Tr denotes the trace. From eq 29 we have

$$\langle \mathbf{S} \rangle : \mathbf{I} = a\mathbf{I} : \mathbf{I} = 3a \quad (34)$$

Hence, from eqs 32–34,

$$\langle \mathbf{S} \rangle = \frac{1}{3}(\text{Tr} \mathbf{S})\mathbf{I} \quad (35)$$

Let $\mathbf{T}$ be a third rank tensor. The average must be of the form

$$\langle \mathbf{T} \rangle = b\epsilon \quad (36)$$

where b is a scalar. We make a triple contraction of both sides with ϵ and use eq 31:

$$\langle \mathbf{T} \rangle \cdot 3 \cdot \epsilon = b\epsilon \cdot 3 \cdot \epsilon = -6b \quad (37)$$

The left side of eq 37 is a contraction of the average tensor, which must be the same as the average of the contraction; i.e.

$$\langle \mathbf{T} \rangle \cdot 3 \cdot \epsilon = \langle \mathbf{T} \cdot 3 \cdot \epsilon \rangle \quad (38)$$

The right side of eq 38 is the average of a scalar, which is just the invariant value of the scalar. Hence, from eqs 36–38,

$$\langle \mathbf{T} \rangle = -\frac{1}{6}(\mathbf{T} \cdot 3 \cdot \epsilon)\epsilon \quad (39)$$

Exercises¹

1. Verify eq 29.
2. Verify eq 30.
3. Verify eq 31.
4. Prove that the identity tensor $\mathbf{I}$ is isotropic. (HINT: Show that the tensor is unchanged by the transformation in Section 5. Make use of the fact that $(\lambda_{\alpha 1}, \lambda_{\alpha 2}, \lambda_{\alpha 3})$ in eq 21 is the unit vector of axis α of system S' in the coordinate system S .)
5. Prove that the permutation tensor ϵ is isotropic.
6. Show that $\epsilon : \epsilon = -2\mathbf{I}$.
7. Show that $\epsilon : \mathbf{ab} = \mathbf{b} \times \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors.
8. Show that $\epsilon \cdot 3 \cdot \mathbf{abc} = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$, where $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are vectors.
9. If $\mathbf{S}$ is a second rank tensor, show that

$$\mathbf{S} : \epsilon = \begin{pmatrix} S_{32} - S_{23} \\ S_{13} - S_{31} \\ S_{21} - S_{12} \end{pmatrix}$$

References

- [1] H. Jeffreys. *Cartesian Tensors*. Cambridge University Press, Cambridge, 1974.
- [2] A. I. Borisenko and I. E. Tarapov. *Vector and Tensor Analysis with Applications*. Dover, New York, 1979.
- [3] J. Heading. *Matrix Theory for Physicists*. Wiley, New York, 1963.
- [4] A. Einstein. Die Grundlage der allgemeinen Relativitätstheorie. *Ann. Phys.*, 49:769, 1916.

¹Solutions are available from the author on request.