A study of 1/R expansions in toroidal coordinates PhL 5.22.10

I have done this now for spherical and cylindrical coordinates, so let's try to replicate things here for the toroidal case. From "Toroidal coordinates.doc" here are the main atomic combinations

$$\sqrt{ch\mu-cos\eta} \quad \text{times:} \quad \{P^{m}_{n-1/2}(ch\mu),\ Q^{m}_{n-1/2}(ch\mu)\} \quad e^{\pm inn} e^{\pm im\varphi} \qquad n,m = \text{integers}$$

As usual, I assume oscillatory in φ. Then η will be the other oscillatory, and μ the expo coordinate.

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Overview (added 10.10.10, 4 pages)

In **Section 0** part (a)**,** as my Smythian form candidate for 1/R, I make this choice of toroidal atoms:

$$f(\mathbf{r} \mid \mathbf{r}') = \sqrt{ch\mu-cos\eta}\ \Sigma_{nm}\ A_{nm}\ P^{m}_{n-1/2}(ch\mu_{<})\ Q^{m}_{n-1/2}(ch\mu_{>})\ e^{inn} e^{im\varphi} \qquad n,m = \text{all integers}$$

I then show that the above Smythian form exhibits the right behavior in the limits $\mu \to 0$ (huge donut radius) and $\mu \to \infty$ (donut shrinks to wire ring at focal distance).

In part (b) after some fiddling I am able to write $R = |\mathbf{r}-\mathbf{r}'|$ in the toroidal *coordinates* (not atoms) $\mathbf{r} = \mu,\eta,\varphi$ and $\mathbf{r}' = \mu',\eta',\varphi'$ as follows: (and I have verification from a certain Wilson pdf on this)

$$R^2 = 2a^2 \{ ch\gamma - cos(\eta-\eta')\} / [(ch\mu-cos\eta)(ch\mu'-cos\eta')]$$

$$1/R = (1/\sqrt{2}\ a) \sqrt{ch\mu-cos}\ \sqrt{ch\mu'-cos\eta'}\ /\sqrt{ch\gamma - cos(\eta-\eta')}$$

where $ch\gamma \equiv ch\mu\, ch\mu' - sh\mu\, sh\mu' \cos(\varphi-\varphi')$

In **Section 1** I use the pillbox method to evaluate A_{nm} . Note that φ and η are oscillatory, so the pillbox is in the μ direction, which is normal to a donut. I find, after doing two trig orthogonalities and after using the P,Q Wronskian and doing much general labor, that

$$A_{nm} = (q/\pi a) \sqrt{ch\mu'-cos\eta'}\ (-1)^{m} f(n-1/2,-m)\ e^{-inn'}\ e^{-im\varphi'}$$

and upon insertion into the above Smythian form we then find that (setting q = 1)

$$f(\mathbf{r} \mid \mathbf{r}') = 1/R = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{ch\mu' - cos\eta'}$$
$$\Sigma_{nm}\ (-1)^m f(n-1/2,-m)\, P^m_{n-1/2}(chu_<)\, Q^m_{n-1/2}(chu_>)\ e^{in(\eta-\eta')}\, e^{im(\varphi-\varphi')}$$

I then examined the symmetry under n and m reflection of pieces of the integrand and found that:

(a) The factor f(n-1/2,-m) → +f when n→ –n, and f(n-1/2,-m) → –f when m→ –m.
(b) Each Legendre function is symmetric under n → –n . For P this is obvious, and for Q it is only true since we have ν = n-1/2 and n = integer. Therefore, the entire summand is symmetric under n → -n.
(c) The entire summand is also symmetric under m → –m, this takes some work to show.

Since the summand is even under reflection of both n and m independently, we know that f is real. Here is a quick proof:

$$f = \Sigma_{nm}\ h_{nm}\, e^{in(\eta-\eta')}\, e^{im(\varphi-\varphi')} = \Sigma_{nm}\ h_{nm}\ [\cos(nx)+i\sin(nx)]\ [\cos(my)+i\sin(my)]$$

$$Im(f) = \Sigma_{nm}\ h_{nm} \cos(nx)\sin(my) + \Sigma_{nm}\ h_{nm} \sin(nx)\cos(my)$$

The first sum vanishes because h_{nm} is symmetric in m, while sin(my) is antisymmetric, so the summand as a whole is antisymmetric in m, and sum is over all integers m, positive and negative. The second term vanishes for the same reason with respect to index n.

Things had to come out this way, I just wanted to verify it. Once we know the above symmetries, we can rewrite our 1/R expansion in several other ways. With full n and m sums, we can replace either or both exponentials with cosines (I show this). Here are some rewrites: $f(n-1/2,-m) = \Gamma(n-m+1/2)/\Gamma(n+m+1/2)$

$$1/R = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{ch\mu' - cos\eta'}$$
$$\Sigma_{nm}\ (-1)^m f(n-1/2,-m) P^m_{n-1/2}(chu_<) Q^m_{n-1/2}(chu_>) \cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$$

$$1/R = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{ch\mu' - cos\eta'}$$
$$\Sigma_{n=0}^{\infty}\ \Sigma_{m=0}^{\infty}\ \varepsilon_n\,\varepsilon_m\ (-1)^m f(n-1/2,-m) P^m_{n-1/2}(chu_<) Q^m_{n-1/2}(chu_>) \cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$$

I find **verification** of this last 1/R form in MF, but they have the typo $(-i)^m$ or perhaps they have a different definition of the Q function which seems quite possible since MF are very "shy" of using the Bateman forms and tend to use their own notations. MF was 1953, the exact same year of Bateman. I then have a **second verification** from the Wilson 2005 PDF of this last result, and agreement is perfect.

In **Section 2** I first select the Green's charge location **r'** on the z axis. This means μ' = 0 since on infinite horizontal circle (donut), and then η' picks out a vertical circle (bowl) to give us some point on the z axis. For such a point **r'**, we find (using the simpler cylindrical coordinates with ρ' = 0) that

$$R^2 = \rho^2 + (z-z')^2 \qquad \rho = a\,sh\mu/(ch\mu - cos\eta)$$
$$z = a\,sin\eta/(ch\mu - cos\eta)$$
$$z' = a\,sin\eta'/(1 - cos\eta')$$

If we examine our last 1/R expansion above in the limit $\mu' \to 0$, I show that it becomes

$$1/R = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{1-cos\eta'}\;\; \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\eta')]$$

which then gives us this little hybrid [ie, cylindrical/toroidal] sum rule,

$$1/\sqrt{\rho^2+(z-z')^2} \;=\; (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{1-cos\eta'}\;\; \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\eta')]$$

We can run our point down the positive z axis to $z' = 0$ by taking $\eta' \to \pi$ where then

$$1/\sqrt{\rho^2+z^2} \;=\; (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{2}\;\; \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\pi)]$$

and if we express the LHS in toroidals and shuffle factors around, this becomes

$$(\pi/\sqrt{2})\,(1/\sqrt{ch\mu + cos\eta}) \;=\; \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\pi)] \qquad (*)$$
$$= \Sigma_{n=0}^{\infty}\varepsilon_n\,(-1)^n\, Q_{n-1/2}(ch\mu)\cos(n\eta)$$

This is essentially a specialized case of our toroidal 1/R expansion when **r'** = 0. Analogous equations from our cylindrical world are these:

$$1/\sqrt{\rho^2+z^2} \;=\; \int_0^{\infty} dk\, \exp(-k|z|)\, J_0(k\rho) \;=\; (2/\pi)\int_0^{\infty} dk\, \cos(kz)\, K_0(k\rho)$$

Notice that the toroidal world Smythian form has Σ_n whereas the cylindrical has $\int dk$. And notice that the non-trig atom in the toroidal world is the ring function $Q_{n-1/2}(ch\mu)$, whereas it is a Bessel function $Z_0(k\rho)$ in the cylindrical world.

If we replace $\eta \to \eta+\pi$ in (*) we get this simpler result

$$(\pi/\sqrt{2})\,(1/\sqrt{ch\mu - cos\eta}) \;=\; \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\mu)\cos(n\eta) \qquad (**)$$

which happens to appear on page 1304 of MF, but MF are missing the ε_n factor! (**errata**!)

$$\frac{1}{\sqrt{\cosh\mu - \cos\eta}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty} Q_{n-\frac{1}{2}}(\cosh\mu)\cos(n\eta) \qquad (10.3.79)$$

// should contain ε_n

If we set $a = \cosh\mu$, $b = 1$ and $\eta = x$, we can interpret the above (**) as the expansion direction of our little Fourier Cosine analysis (see doc just on this subject)

$$1/\sqrt{a - b\cos(x)} \;=\; (1/\pi)\sqrt{2/b}\;\; \Sigma_{n=0}^{\infty}\varepsilon_n\; Q_{n-1/2}(a/b)\cos(nx) \qquad \text{// expansion}$$
$$\int_0^{\pi} dx\, \cos(nx)/\sqrt{a - b\cos(x)} \;=\; \sqrt{2/b}\;\; Q_{n-1/2}(a/b) \qquad \text{// projection}$$

In **Section 3** I use specific "arbitrary" values for μ and η in the 1/R expansion (*) above,

$ch\mu = [\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho') \equiv B$

$\eta = \varphi - \varphi'$

and I quickly arrive at this result, where **r** and **r'** are *arbitrary* points expressed in cylindricals,

$$1/R = 1/(\pi\sqrt{\rho\rho'}) \ \Sigma_{n=0}^{\infty} \varepsilon_n \ Q_{n-1/2}(B) \cos[n(\varphi-\varphi')]$$

which is what I call the **Selvaggi/Snow 1/R expansion**. This thing is really expressed in cylindrical coordinates, but the toroidal atom $Q_{n-1/2}(B)$ appears in the sum, so it is a hybrid animal. By comparing this general 1/R expansion with those in my cylindrical doc, we conclude that

$$Q_{n-1/2}\{[\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')\} = \pi\sqrt{\rho\rho'} \int_0^{\infty} dk \exp(-k|z-z'|) J_n(k\rho) J_n(k\rho')$$

$$= 2\sqrt{\rho\rho'} \int_0^{\infty} dk \cos[k(z-z')] I_n(k\rho_<) K_n(k\rho_>)$$

If z = z', we get this interesting special case:

$$Q_{n-1/2}\{[\rho^2 + \rho'^2]/(2\rho\rho')\} = \pi\sqrt{\rho\rho'} \int_0^{\infty} dk \ J_n(k\rho) J_n(k\rho')$$

$$= 2\sqrt{\rho\rho'} \int_0^{\infty} dk \ I_n(k\rho_<) K_n(k\rho_>)$$

We could perhaps regard these as some "integral addition theorems" that live in the hybrid world of the Snow expansion.

In **Section 4** I first come up with a more general version of the toroidal sum rule which is this:

$$1 / \sqrt{ch\gamma - \cos(\eta-\eta')} = (\sqrt{2}/\pi) \ \Sigma_{n=0}^{\infty} \varepsilon_n \ Q_{n-1/2}(ch\gamma) \cos[n(\eta-\eta')] \qquad 4.5$$

where one can install any $ch\gamma$ one wants. Using $ch\gamma = B$ and other values shown in Section 3, this leads to the same Selvaggi 1/R expansion we got before. That is, $\eta' = \pi$, and so on.

Next, I develop this general 1/R toroidal expansion

$$1/R = (1/\pi a) \sqrt{ch\mu - \cos\eta} \sqrt{ch\mu' - \cos\eta'} \ \Sigma_{n=0}^{\infty} \varepsilon_n \ Q_{n-1/2}(ch\gamma) \cos[n(\eta-\eta')] \qquad 4.6$$

$$= (1/\sqrt{2}\, a) \sqrt{ch\mu - \cos\eta} \sqrt{ch\mu' - \cos\eta'} / \sqrt{ch\gamma - \cos(\eta-\eta')}$$

where $ch\gamma \equiv ch\mu \, ch\mu' - sh\mu \, sh\mu' \cos(\varphi-\varphi')$

This thing is somewhat analogous to the following spherical coordinates 1/R expansion

$$1/R = \Sigma_n \ (r_<)^n (r_>)^{-n-1} \ P_n(\cos\gamma) \qquad \text{// note m = 0}$$

where $\cos\gamma \equiv \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$

In cylindrical coordinates, the analogous items I think are these (these also have m = 0)

$$1/R = \int_0^\infty dk\, \exp(-k|z-z'|)\, J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,)$$

$$1/R = (2/\pi) \int_0^\infty dk\, \cos[k(z-z')]\, K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,)$$

where you have a single "atom" with a complicated argument.

I then compare my new 1/R toroidal expansion to the general Smythian one in Section 1 and this leads do the following **toroidal addition theorem**,

$$Q_{n-1/2}(ch\mu\, ch\mu' - sh\mu\, sh\mu'\cos(\varphi-\varphi')) = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')]\, (-1)^m\, f(n-1/2,-m) P^m_{n-1/2}(ch\mu_<) Q^m_{n-1/2}(ch\mu_>)$$

We may compare this with other addition theorems we have encountered:

$$P_n[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')]\, f(n,-m)\, P_n^m(\cos\theta)\, P_n^m(\cos\theta')]$$

$$J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')]\, J_m(k\rho)\, J_m(k\rho')$$

$$K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')]\, I_m(k\rho_<)\, K_m(k\rho_>)$$

All three coordinate systems have an azimuthal coordinate φ, and all three of these addition theorem types thus have a similar "look" to them.

I do not have direct verification for my toroidal addition theorem, but I think I have seen it somewhere! As with everything else, I expect it has a group theoretic interpretation.

In **Section 5** I conjecture briefly on what the 1/R expansion might look like had I taken the alternate Smythian form starting position where η is expo. The result would look something like this:

$$1/R = (1/\pi a)\sqrt{ch\mu - \cos\eta}\,\sqrt{ch\mu' - \cos\eta'} \qquad \text{// just a conjecture!}$$
$$\int_0^\infty d\tau\, \Sigma_{m=0}^{\infty} \varepsilon_m\, (-1)^m\, f(i\tau-1/2,-m) P^m_{i\tau-1/2}(chu_<) Q^m_{i\tau-1/2}(chu_>)\, e^{-\tau|\eta-\eta'|} \cos[m(\varphi-\varphi')]$$

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0. The 1/R expansion in toroidal coordinates: the Smythian Form

(a) Smythian Form. The potential of a point charge at the origin is q/R with R = r. If we move the point charge to some other origin **r'**, the potential is q/R where R = |**r** - **r'**|. In either case, the potential is a solution of the Laplace equation (away from the point charge) and can therefore be written in terms of the Laplace "atoms" of whatever coordinate system we want to use. For example, if we think of R as f(**r**) we know we can expand it this way in toroidals (the notation here means that **r'** is a parameter, **r** is the variable)

$$f(\mathbf{r} \mid \mathbf{r}') = \sqrt{ch\mu - cos\eta}\ \Sigma_{nm}\ A_{nm}\ P^{m}_{n-1/2}(ch\mu_<)\ Q^{m}_{n-1/2}(ch\mu_>)\ e^{in\eta}\ e^{im\varphi} \qquad n,m = \text{integers}$$

Our m is the usual azimuthal quantum number. As for n and cos(nη) stuff, we know that if we let η range from (0,2π), we create all the partial circles of the vertical circle family (bowls in toroidals) as shown in my Maple graphs of the above doc. I think these partial circles will "repeat" as you go (2π,4π) and so on, and that is what causes n to also be quantized to be an integer. Very similar to the reason for m to do this. And the atomic forms I have seen all say this same thing. An explanation of the choice for the P and Q appearance is given below after we review the coordinate parameterizations.

Here μ is the radial coordinate, and we know from our doc on toroidals that as you move away from the focus on the μ coordinate, $\mu \to 0$. See picture in that doc. So we really want $P^{m}_{n-1/2}(ch\mu)$ for this limit since P is happy at argument 1. Conversely, as we move the other way toward the focus with our horizontal circles, $\mu \to \infty$ and $ch\mu \to \infty$ and we need $Q^{m}_{n-1/2}(ch\mu)$ in this case. The Green's charge is at μ' somewhere. So we want $P^{m}_{n-1/2}(chu_<)\ Q^{m}_{n-1/2}(chu_>)$, and this is then how I made it appear in our Smythe atomic form above.

(b) expressing R in toroidal coordinates. Notice that we are parameterizing our two points **r** and **r'** this way

$$\mathbf{r} = a\,(sh\mu\cos\varphi,\ sh\mu\sin\varphi,\ \sin\eta)/(ch\mu - cos\eta)$$
$$\mathbf{r'} = a\,(sh\mu'\cos\varphi',\ sh\mu'\sin\varphi',\ \sin\eta')/(ch\mu' - cos\eta')$$

$$r^2 = a^2\,(sh^2\mu + \sin^2\eta)\,/\,(ch\mu - cos\eta)^2$$
$$r'^2 = a^2\,(sh^2\mu' + \sin^2\eta')\,/\,(ch\mu' - cos\eta')^2$$

$$\mathbf{r}\bullet\mathbf{r}' = a^2\,[\,sh\mu\,sh\mu'\cos(\varphi-\varphi') + \sin\eta\,\sin\eta'\,]\,/\,[(ch\mu - cos\eta)(\,ch\mu' - cos\eta')]$$

which is not all that pleasant a result. We then have

$$R^2 = |\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2\,\mathbf{r}\bullet\mathbf{r}' \quad = \text{a big mess.}$$

Here is what Maple says, if we let

$$\mu = \mu_1 \qquad \eta = \eta_1$$
$$\mu' = \mu_2 \qquad \eta' = \eta_2$$

$$R^2 = a^2 N/[(ch\mu - cos\eta)^2(\,ch\mu' - cos\eta')^2]$$

with numerator N given by :

$$-2\sinh(\mu 1)\sinh(\mu 2)\cos(\phi 1-\phi 2) - 2\sin(\eta 1)\sin(\eta 2) + 2\cosh(\mu 2)^2\cosh(\mu 1)^2 - 2\cosh(\mu 2)\cos(\eta 2)\cosh(\mu 1)^2$$
$$+ 2\cosh(\mu 2)\cos(\eta 2)\cos(\eta 1)^2 - 2\cos(\eta 2)^2\cos(\eta 1)^2 - 2\cosh(\mu 1)\cos(\eta 1)\cosh(\mu 2)^2$$
$$+ 2\cosh(\mu 1)\cos(\eta 1)\cos(\eta 2)^2$$

This does not seem very useful, does it! Another approach would be to start with cylindricals and write

$$R^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

then install

$$\rho = a\,\mathrm{sh}\mu\,/(\mathrm{ch}\mu-\cos\eta) \qquad z = a\sin\eta\,/(\mathrm{ch}\mu-\cos\eta)$$
$$\rho' = a\,\mathrm{sh}\mu'\,/(\mathrm{ch}\mu'-\cos\eta') \qquad z' = a\sin\eta'\,/(\mathrm{ch}\mu'-\cos\eta')$$

but this does not seem to buy much, it has to come out being the same mess. Looking at section (g1) of the toroidal notes, I just note in passing that we can invert the above two pairs of equations as follows:

$$\mathrm{th}\mu = 2a\rho\,/\,(a^2+\rho^2+z^2) \qquad \tan\eta = -\,2az/(a^2-\rho^2-z^2)$$
$$\mathrm{th}\mu' = 2a\rho'\,/\,(a^2+\rho'^2+z'^2) \qquad \tan\eta' = -\,2az'/(a^2-\rho'^2-z'^2)$$

Well, I have shown on scratch paper that the following is true

$$(R^2/2a^2)\,(\mathrm{ch}\mu-\cos\eta)\,(\mathrm{ch}\mu'-\cos\eta') = [\mathrm{ch}\mu\,\mathrm{ch}\mu' - \mathrm{sh}\mu\,\mathrm{sh}\mu'\cos(\varphi-\varphi')] - \cos(\eta-\eta')$$

where I must have done this using just the above R^2 written out in cylindrical coordinates and then using the replacements shown above for $\rho,\rho',z.z'$, so this does give an expression for R which is not TOO bad,

$$R^2 = 2a^2\,\{[\mathrm{ch}\mu\,\mathrm{ch}\mu' - \mathrm{sh}\mu\,\mathrm{sh}\mu'\cos(\varphi-\varphi')] - \cos(\eta-\eta')\}\,/\,[(\mathrm{ch}\mu-\cos\eta)\,(\mathrm{ch}\mu'-\cos\eta')]$$

We can see that the numerator [..] is at smallest $\mathrm{ch}\mu\,\mathrm{ch}\mu' - \mathrm{sh}\mu\,\mathrm{sh}\mu' = \mathrm{ch}(\mu-\mu')$ and so is ≥ 1, so we can represent it by another ch. So let's define, in analogy with $\cos\gamma$ from spherical coordinates,

$$\mathrm{ch}\gamma = \mathrm{ch}\mu\,\mathrm{ch}\mu' - \mathrm{sh}\mu\,\mathrm{sh}\mu'\cos(\varphi-\varphi')$$

Then we have (I add the first line just to have it here for comparison)

$$R^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

$$R^2 = 2a^2\,\{\,\mathrm{ch}\gamma - \cos(\eta-\eta')\}\,/\,[(\mathrm{ch}\mu-\cos\eta)\,(\mathrm{ch}\mu'-\cos\eta')]$$

$$1/R = (1/\sqrt{2}\,a)\,\sqrt{\mathrm{ch}\mu-\cos}\,\sqrt{\mathrm{ch}\mu'-\cos\eta'}\,/\sqrt{\mathrm{ch}\gamma - \cos(\eta-\eta')}$$

This result is <u>verified in the Wilson paper</u> which has this notation

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{(\cosh\xi-\cos\eta)^{1/2}(\cosh\xi'-\cos\eta')^{1/2}}{c[2\cosh\mu-2\cos(\eta-\eta')]^{1/2}}$$

$$\cosh\mu \triangleq \cosh\xi\cosh\xi' - \sinh\xi\sinh\xi'\cos(\phi-\phi') \geq 1$$

1. Use the pillbox condition to find the A_{mn} coefficients.

The general pillbox condition is this (Jackson units)

$$\partial_{q1} V_i - \partial_{q1} V_o = 4\pi q \, (h_1/h_2h_3) \, \delta(q_2-q_2') \, \delta(q_3-q_3')$$

where q_1 is the dimension perp to our partition of space into two regions at the point charge. For toroidals, we want $q_1 = \mu$, the radial coordinate. The Green's charge sits at some μ' with η' and φ'. So I will use the term "outer" to mean the smaller μ value, sort of inverted from the usual situation, see long paragraph just above. We then have (selecting where the prime goes on the P and Q)

$$V_i = \sqrt{ch\mu - cos\eta} \; \Sigma_{nm} \; A_{nm} \; P^m_{n-1/2}(chu') \; Q^m_{n-1/2}(chu) \; e^{in\eta} \, e^{im\varphi} \qquad \mu > \mu'$$
$$V_o = \sqrt{ch\mu - cos\eta} \; \Sigma_{nm} \; A_{nm} \; P^m_{n-1/2}(chu) \; Q^m_{n-1/2}(chu') \; e^{in\eta} \, e^{im\varphi} \qquad \mu < \mu'$$

So $q_1 = \mu$. We can then take $q_2 = \eta$ and $q_3 = \varphi$ and use these facts

$$h_1 = h_\mu = a/ [ch\mu - cos\eta]$$
$$h_2 = h_\eta = a/ [ch\mu - cos\eta]$$
$$h_3 = h_\varphi = a \, sh\mu/ [ch\mu - cos\eta]$$

which come from my toroidal doc, and which agrees with M&M page 191.

Our pillbox condition is then this: (note that $(h_1/h_2h_3) = 1/h_3$)

$$\partial_\mu V_i - \partial_\mu V_o = 4\pi q \, [ch\mu - cos\eta]/(a \, sh\mu) \, \delta(\eta-\eta') \, \delta(\varphi-\varphi')$$

We then need to compute the derivatives. Things are a little different now compared to previous coordinate systems because we have μ dependence in the leading factor. We need to know

$$\partial_\mu \sqrt{ch\mu - cos\eta} = (1/2) \, sh\mu \, / \sqrt{ch\mu - cos\eta}$$

We then have:

$$V_i = \sqrt{ch\mu - cos\eta} \; \Sigma_{nm} \; A_{nm} \; P^m_{n-1/2}(chu') \; Q^m_{n-1/2}(chu) \; e^{in\eta} \, e^{im\varphi} \qquad \mu > \mu'$$
$$V_o = \sqrt{ch\mu - cos\eta} \; \Sigma_{nm} \; A_{nm} \; P^m_{n-1/2}(chu) \; Q^m_{n-1/2}(chu') \; e^{in\eta} \, e^{im\varphi} \qquad \mu < \mu'$$

$$\partial_\mu V_i = \{(1/2) \, sh\mu \, / \sqrt{ch\mu - cos\eta} \, \} \; \Sigma_{nm} \; A_{nm} \; P^m_{n-1/2}(chu') \; Q^m_{n-1/2}(chu) \; e^{in\eta} \, e^{im\varphi}$$

$$+ \sqrt{ch\mu - cos\eta} \; \Sigma_{nm} \; A_{nm} \; P^m_{n-1/2}(chu') \; Q^m_{n-1/2}(chu)' \, sh\mu \; e^{in\eta} \, e^{im\varphi}$$

and to get $\partial_\mu V_o$ we just switch P ↔ Q. When we subtract the first two terms, they will contribute this amount to the difference:

$\{(1/2)\ sh\mu / \sqrt{ch\mu - cos\eta}\ \}\ \Sigma_{nm}\ A_{nm}\ e^{in\eta}\ e^{im\varphi}\ (\ P^m_{n-1/2}(chu')\ Q^m_{n-1/2}(chu) - Q^m_{n-1/2}(chu')\ P^m_{n-1/2}(chu)\}$

But when we evaluate at $\mu = \mu'$, this contribution will (thankfully) be 0 so we forget about it. The contribution of the other two terms is this:

$$\sqrt{ch\mu - cos\eta}\ \Sigma_{nm}\ A_{nm}\ \ sh\mu\ \ e^{in\eta}\ e^{im\varphi}\ W[P^m_{n-1/2}(chu), Q^m_{n-1/2}(chu)]$$

For "regular" (off the cut) P and Q functions, Leg prop doc tells us that

$$W[\ P_\nu^\mu(z), Q_\nu^\mu(z)]\ \ = e^{i\mu\pi}\ (1-z^2)^{-1}[\Gamma(1+\mu+\nu)/\ \Gamma(1-\mu+\nu)\ ]$$

or

$$W[\ P_\nu^m(z), Q_\nu^m(z)]\ \ = (-1)^m(1-z^2)^{-1}\ f(\nu,m)$$

or

$$W[\ P_\nu^m(chu), Q_\nu^m(chu)]\ \ = (-1)^m\ f(\nu,m)\ (1-ch^2\mu)^{-1}\ \ = (-1)^m\ f(\nu,m)\ 1/sh^2\mu$$

so at this point then we seem to have ($\nu = n-1/2$)

$$\partial_\mu V_i - \partial_\mu V_o\ =\ \sqrt{ch\mu - cos\eta}\ \ \Sigma_{nm}\ A_{nm}\ \ sh\mu\ \ e^{in\eta}\ e^{im\varphi}\ \ \{\ 1/sh^2\mu\ \}(-1)^m\ f(n-1/2,m)$$

$$=\ \sqrt{ch\mu - cos\eta}\ \ \Sigma_{nm}\ (-1)^m\ f(n-1/2,m)A_{nm}\ \ e^{in\eta}\ e^{im\varphi}\ /\ sh\mu$$

so our pillbox condition is then

$$4\pi q\ [ch\mu - cos\eta]/(a\ sh\mu)\ \delta(\eta-\eta')\ \delta(\varphi-\varphi')\ \ = \sqrt{ch\mu - cos\eta}\ \ \Sigma_{nm}\ (-1)^m\ f(n-1/2,m)A_{nm}\ \ e^{in\eta}\ e^{im\varphi}\ /\ sh\mu$$

and we get cancellation of the $sh\mu$ factor (this seems to happen in all coordinate systems) and we are left with

$$4\pi(q/a)\sqrt{ch\mu - cos\eta}\ \ \delta(\eta-\eta')\ \delta(\varphi-\varphi') = \Sigma_{nm}\ (-1)^m\ f(n-1/2,m)A_{nm}\ \ e^{in\eta}\ e^{im\varphi}$$

and it should now be a simple matter to extract the A_{nm} coefficients. For the moment, let's define

$$(-1)^m\ f(n-1/2,m)A_{nm} = B_{nm}$$

so we have then

$$4\pi(q/a)\sqrt{ch\mu - cos\eta}\ \ \delta(\eta-\eta')\ \delta(\varphi-\varphi') = \Sigma_{nm}\ B_{nm}\ \ e^{in\eta}\ e^{im\varphi}$$

First, apply $\int d\varphi\ e^{-im'\varphi}$ to both sides and use

$$(1/2\pi)\ \ \int_{-\pi}^{\pi}\ d\varphi\ e^{im\varphi}\ e^{-im'\varphi}\ \ =\ \delta_{mm'}\qquad\qquad\qquad\qquad //\ \text{orthogonality}$$

The RHS becomes

$$\Sigma_{nm}\ B_{nm}\ e^{inη} \int dφ\ e^{-im'φ}\ e^{imφ} = \Sigma_{nm}\ B_{nm}\ e^{inη}\ \{2πδ_{mm'}\}$$
$$= 2π\ \Sigma_n\ B_{nm}\ e^{inη}$$

and the LHS becomes $4π(q/a)\sqrt{chμ-cosη}\ \ δ(η-η')\ e^{-im'φ'}$ so we then have (replace m' with m)

$$4π(q/a)\sqrt{chμ-cosη}\ \ δ(η-η')\ e^{-imφ'} = 2π\ \Sigma_n\ B_{nm}\ e^{inη}$$

$$(2q/a)\sqrt{chμ-cosη}\ \ δ(η-η')\ e^{-imφ'} = \Sigma_n\ B_{nm}\ e^{inη}$$

Now apply $\int dη\ e^{-in'η}$ to both sides and use

$$(1/2π)\ \int_{-π}^{π}\ dη\ e^{inη}\ e^{-in'η} = δ_{nn'} \qquad \text{// orthogonality}$$

The RHS becomes

$$\Sigma_n\ B_{nm}\ \int dη\ e^{-in'η}\ e^{inη} = \Sigma_n\ B_{nm}\ \{2πδ_{nn'}\} = 2π\ BA_{n'm}$$

and the LHS becomes $(2q/a)\sqrt{chμ-cosη'}\ \ e^{-in'η'}\ e^{-imφ'}$ so we have (replace n' with n)

$$(2q/a)\sqrt{chμ-cosη'}\ \ e^{-inη'}\ e^{-imφ'} = 2π\ B_{nm}$$

$$B_{nm} = (q/πa)\ \sqrt{chμ'-cosη'}\ \ e^{-inη'}\ e^{-imφ'}$$

and thus

$$A_{nm} = B_{nm}\ (-1)^m\ f(n-1/2,-m)$$
$$= (q/πa)\ \sqrt{chμ'-cosη'}\ (-1)^m\ f(n-1/2,-m)\ e^{-inη'}\ e^{-imφ'}$$

So we now have our coefficients! I set μ = μ' because all this was done at the point charge. As expected, A_{nm} is a function of the coordinates of **r'** .

We can install these into our expansion above

$$V(\mathbf{r} \mid \mathbf{r'}) = q/R = \sqrt{chμ-cosη}\ \ \Sigma_{nm}\ A_{nm}\ P^m_{n-1/2}(chu_<)\ Q^m_{n-1/2}(chu_>)\ \ e^{inη}\ e^{imφ}$$

$$= (q/πa)\ \sqrt{chμ-cosη}\ \sqrt{chμ'-cosη'}\ \ \Sigma_{nm}\ (-1)^m\ f(n-1/2,-m) P^m_{n-1/2}(chu_<)\ Q^m_{n-1/2}(chu_>)\ \ e^{in(η-η')}\ e^{im(φ-φ')}$$

where we have

$$f(n-1/2,-m) = \Gamma(n-m+1/2)/\,\Gamma(n+m+1/2)$$

Before going another step, we want to look at the **symmetry of this factor** under inversion of n or m:

$$f(n-1/2,-m)P^{m}_{n-1/2}(chu_{<})\,Q^{m}_{n-1/2}(chu_{>}) \qquad (*)$$

(a) We know what the f factor does if m changes sign, it goes upside down. But what happens if n changes sign? We have to use the reflection formula which says $\Gamma(z)\,\Gamma(1-z) = \pi/\sin(\pi z)$. Setting $z = n-m+1/2$ we find that $1-z = 1-(n-m+1/2) = 1-n+m-1/2 = -n+m+1/2$, so

$$\Gamma(n-m+1/2)\,\Gamma(-n+m+1/2) = \pi/\sin(\pi[n-m+1/2]) = \pi/\cos\pi(n-m) = \pi\,(-1)^{n-m}$$

so that

$$\Gamma(n-m+1/2) = \pi\,(-1)^{n-m} / \Gamma(-n+m+1/2)$$

We then find that

$$f(n-1/2,-m) = \Gamma(n-m+1/2)/\,\Gamma(n+m+1/2)$$

$$= \{\,\pi\,(-1)^{n-m} / \Gamma(-n+m+1/2)\}/\,\{\,\pi\,(-1)^{n+m} / \Gamma(-n-m+1/2)\}$$

$$= \{\,1 / \Gamma(-n+m+1/2)\}/\,\{\,1 / \Gamma(-n-m+1/2)\}$$

$$= \Gamma(-n-m+1/2)/\,\Gamma(-n+m+1/2)$$

$$= f(-n-1/2,-m)$$

<u>So the f factor is simply symmetric when n → -n, and it flips over when m → -m.</u>

(b) In terms of the n reflection, our Leg prop doc shows that

$$P_{-\nu-1}{}^{\mu}(z) = P_{\nu}{}^{\mu}(z) \qquad \Rightarrow \qquad P_{-n-1/2}{}^{\mu}(z) = P_{n-1/2}{}^{\mu}(z)$$

The Q is less obvious. We start with:

$$Q_{-\nu-1}{}^{\mu}(z)\,\sin\pi(\nu-\mu) = Q_{\nu}{}^{\mu}(z)\,\sin\pi(\nu+\mu) - \pi\,e^{i\pi\mu}\cos\pi\nu\,P_{\nu}{}^{\mu}(z)$$

and set $\nu = n-1/2$ and $\mu = m$ to get

$$Q_{-n-1/2}{}^{m}(z)\,\sin\pi(n-1/2-m) = Q_{n-1/2}{}^{m}(z)\,\sin\pi(n-1/2+m) - \pi\,e^{inm}\cos\pi(n-1/2)\,P_{n-1/2}{}^{\mu}(z)$$

As we have seen elsewhere, the last term vanishes since $\cos\pi(n-1/2) = \pm\sin(\pi n) = 0$. We are left with

$$Q_{-n-1/2}^{m}(z)\cos\pi(n-m) = Q_{n-1/2}^{m}(z)\cos\pi(n+m)$$

But we know that $\cos\pi N = (-1)^N$ for so we then have

$$Q_{-n-1/2}^{m}(z)(-1)^{n-m} = Q_{n-1/2}^{m}(z)(-1)^{n+m}$$

$$Q_{-n-1/2}^{m}(z)(-1)^{-m} = Q_{n-1/2}^{m}(z)(-1)^{m}$$

$$Q_{-n-1/2}^{m}(z) = Q_{n-1/2}^{m}(z)$$

So both our P and Q functions are simply symmetric when $n \to -n$. This means that our entire factor quoted above in (*) is symmetric under n inversion.

(c) As for the m folding, from Leg prop doc we have

$$f(\nu,m)\ P_\nu^{-m}(z) = P_\nu^{m}(z)$$
$$f(\nu,m)\ Q_\nu^{-m}(z) = Q_\nu^{m}(z)$$

where $\nu = n-1/2$ as usual. Looking at our (*) factor we have [replace u with μ everwhere]

$$F(m) \equiv f(\nu,-m)P^{m}_{\nu}(\mathrm{chu}_<)\,Q^{m}_{\nu}(\mathrm{chu}_>)$$

$$F(-m) = f(\nu,m)P^{-m}_{\nu}(\mathrm{chu}_<)\,Q^{-m}_{\nu}(\mathrm{chu}_>) = f(\nu,m)\,P^{m}_{\nu}(\mathrm{chu}_<)\,Q^{m}_{\nu}(\mathrm{chu}_>)\,f(\nu,-m)\,f(\nu,-m)$$

$$= f(\nu,-m)\,P^{m}_{\nu}(\mathrm{chu}_<)\,Q^{m}_{\nu}(\mathrm{chu}_>) = F(m)$$

so our (*) factor is also simply symmetric under $m \to -m$.

These symmetries have to come out this way in order that the imaginary part of the double sum be zero! But we have explicitly shown this to be true in (a), (b), (c) here. So, we can now write our result in various ways. We start with the original way found above (n and m are over all integers)

$$1/R = (1/\pi a)\sqrt{\mathrm{ch}\mu-\cos\eta}\,\sqrt{\mathrm{ch}\mu'-\cos\eta'}$$
$$\Sigma_{nm}\ (-1)^m f(n-1/2,-m)P^{m}_{n-1/2}(\mathrm{chu}_<)Q^{m}_{n-1/2}(\mathrm{chu}_>)\,e^{in(\eta-\eta')}e^{im(\varphi-\varphi')}$$

If we look at the contribution to this sum from $\sin[n(\eta-\eta')]$, we know it is zero because the factor (*) noted above is symmetric under $n \to -n$. Thus we conclude that

$$1/R = (1/\pi a)\sqrt{\mathrm{ch}\mu-\cos\eta}\,\sqrt{\mathrm{ch}\mu'-\cos\eta'}$$
$$\Sigma_{nm}\ (-1)^m f(n-1/2,-m)P^{m}_{n-1/2}(\mathrm{chu}_<)Q^{m}_{n-1/2}(\mathrm{chu}_>)\cos[n(\eta-\eta')]e^{im(\varphi-\varphi')}$$

If we now ask about the contribution to this last sum from $\sin[m(\varphi-\varphi')]$, it is also zero since the factor is symmetric in $m \to -m$. Thus, we get this rewrite as well, and this form is then manifestly real:

$1/R = (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}$
$\Sigma_{nm}\ (-1)^m f(n-1/2,-m)P^m_{n-1/2}(chu_<)Q^m_{n-1/2}(chu_>)\cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$

Of course another way would be to start with the other factor, so we would get

$1/R = (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}$
$\Sigma_{nm}\ (-1)^m f(n-1/2,-m)P^m_{n-1/2}(chu_<)Q^m_{n-1/2}(chu_>)\ e^{in(\eta-\eta')}\cos[m(\varphi-\varphi')]$

Now due to our two symmetries just noted, we can reflect both the n and m sums (independently) and in the usual way we get this result:

$1/R = (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}$
$\Sigma_{n=0}^{\infty}\ \Sigma_{m=0}^{\infty}\ \varepsilon_n\ \varepsilon_m\ (-1)^m f(n-1/2,-m)P^m_{n-1/2}(chu_<)Q^m_{n-1/2}(chu_>)\cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$

where $f(n-1/2,-m) = \Gamma(n-m+1/2)/\Gamma(n+m+1/2)$

As noted in the toroidal doc, this result is actually stated in M&F, which I again quote: (a=1)

$$\frac{1}{R} = \frac{1}{\pi}\sqrt{(\cosh\mu-\cos\eta)(\cosh\mu_0-\cos\eta_0)}\sum_{m,n=0}^{\infty}\epsilon_m\epsilon_n(-i)^m\frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\cdot$$
$$\cdot\cos[m(\phi-\phi_0)]\cos[n(\eta-\eta_0)]\begin{cases}P^m_{n-\frac{1}{2}}(\cosh\mu)Q^m_{n-\frac{1}{2}}(\cosh\mu_0); & \mu_0>\mu\\ P^m_{n-\frac{1}{2}}(\cosh\mu_0)Q^m_{n-\frac{1}{2}}(\cosh\mu); & \mu_0<\mu\end{cases}$$
(10.3.81)

and my "suspicion" is now confirmed, MF have a typo: their $(-i)^m$ should be $(-1)^m$. And they are using a = 1 for the focal distance, so that my result shows the correct dimensions. [or they have a different phase definition of Q and maybe P and Q for all I know, recall CS and NCS stuff and $e^{im\mu}$ in Bateman Q tables.]

So I am very happy about getting this result!! It gives me a waypoint on the way to the next section.

Here is a second "verification" of the above result from a pdf paper "wilson toroid greens.doc" which appears in Electromagnetics in the ripe year 2005 (!) :

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{(\cosh\xi - \cos\eta)^{1/2}(\cosh\xi' - \cos\eta')^{1/2}}{c[2\cosh\mu - 2\cos(\eta-\eta')]^{1/2}}$$

$$= \frac{1}{c\pi}(\cosh\xi - \cos\eta)^{1/2}(\cosh\xi' - \cos\eta')^{1/2} \qquad (7)$$

$$\cdot \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \epsilon_n\epsilon_m(-1)^m \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)}$$

$$\cdot Q^m_{n-1/2}(\cosh\xi_>)\, P^m_{n-1/2}(\cosh\xi_<)\, \cos m(\phi-\phi')\, \cos n(\eta-\eta'),$$

where $$\cosh\mu \triangleq \cosh\xi\cosh\xi' - \sinh\xi\sinh\xi'\cos(\phi-\phi') \geq 1$$

These authors are using ξ in place of μ, but continue to use η for the bowl labels. The second equality here agrees exactly with my result above. The first equality is also correct and has been commented on in the previous section.

2. Development of some Toroidal Sum Rules

Now, what happens if we put our point charge on the z axis? This was our starting point in both the spherical and cylindrical systems. Then

$\mathbf{r'}$ = a (shμ' cosφ', shμ' sinφ', sinη')/(chμ'-cosη') = (0,0 ,z')

which clearly means μ' = 0 so you are on a "big" horizontal circle (fat toroid) that is coming down the z' axis and then your z' location is picked off by an η' circle (sphere). Certainly in this case μ' = $\mu_<$ and so we have

$P^m_{n-1/2}(\text{ch}u_<) = P^m_{n-1/2}(\text{ch}u') = P^m_{n-1/2}(1) = \delta_{m0}$

so the RHS of our equation

$1/R = (1/\pi a)\sqrt{\text{ch}\mu-\cos\eta}\,\sqrt{\text{ch}\mu'-\cos\eta'}$
$\Sigma_{n=0}^{\infty}\,\Sigma_{m=0}^{\infty}\,\varepsilon_n\,\varepsilon_m\,(-1)^m\, f(n-1/2,-m)P^m_{n-1/2}(\text{ch}\mu_<)Q^m_{n-1/2}(\text{ch}\mu_>)\cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$

becomes (azimuthal dependence is being removed)

$1/R = (1/\pi a)\sqrt{\text{ch}\mu-\cos\eta}\,\sqrt{1-\cos\eta'}\;\;\Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(\text{ch}\mu)\cos[n(\eta-\eta')]$

When I write out R on the LHS, however, it is still a big mess even with μ' = 0. However, I could write out this R in cylindrical coordinates which are a little less opaque. In general we have

$R^2 = |\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2\,\mathbf{r}\bullet\mathbf{r}' = \rho^2 + z^2 + \rho'^2 + z'^2 - 2\rho\rho'\cos(\varphi-\varphi') - 2\,zz'$

and for **r'** on the z axis this becomes $\rho' = 0$ so

$$R^2 = \rho^2 + (z-z')^2 \qquad \rho = a\,sh\mu/(ch\mu - cos\eta)$$
$$z = a\,sin\eta/(ch\mu - cos\eta)$$
$$z' = a\,sin\eta'/(1 - cos\eta')$$

so we are then saying that, for **r'** on the z axis, we have

$$1/R = 1/\sqrt{\rho^2 + (z-z')^2} = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{1 - cos\eta'}\;\; \Sigma_{n=0}^{\infty} \varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\eta')]$$

We might go further (as I did in the cylindrical 1/R doc) and set $z' = 0$. If we look carefully at our toroidal pictures, we see that this means $\eta' = \pi$ (not 0), so $cos\eta' = -1$ and $sin\eta' = 0$ and we then get

$$R^2 = \rho^2 + z^2 \qquad \rho = a\,sh\mu/(ch\mu - cos\eta)$$
$$z = a\,sin\eta/(ch\mu - cos\eta)$$

And then our 1/R equality becomes

$$1/R = 1/\sqrt{\rho^2 + z^2} = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{2}\;\; \Sigma_{n=0}^{\infty} \varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\pi)]$$

which is just our toroidal 1/R expansion where we put the Green's charge at the origin. We can of course then write

$$\sqrt{\rho^2 + z^2} = [\, a/(ch\mu - cos\eta)\,]\,\sqrt{sh^2\mu + sin^2\eta}$$

so

$$1/R = 1/\sqrt{\rho^2 + z^2} = [(ch\mu - cos\eta)/a]\; 1/\sqrt{sh^2\mu + sin^2\eta}$$

Our equality then becomes

$$[(ch\mu - cos\eta)/a]\; 1/\sqrt{sh^2\mu + sin^2\eta} = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{2}\;\; \Sigma_{n=0}^{\infty} \varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\pi)]$$

$$\pi\sqrt{ch\mu - cos\eta}\;/\sqrt{sh^2\mu + sin^2\eta} = \sqrt{2}\;\; \Sigma_{n=0}^{\infty} \varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\pi)]$$

$$(\pi/\sqrt{2})\,[\,(ch\mu - cos\eta)\,/\,(sh^2\mu + sin^2\eta)]^{1/2} = \Sigma_{n=0}^{\infty} \varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\pi)] \qquad (*)$$

Now process the LHS a bit:

$$\begin{aligned} LHS &= (\pi/\sqrt{2})\,[\,(ch\mu - cos\eta)\,/\,(ch^2\mu - 1 + sin^2\eta)]^{1/2} \\ &= (\pi/\sqrt{2})\,[\,(ch\mu - cos\eta)\,/\,(ch^2\mu - cos^2\eta)]^{1/2} \\ &= (\pi/\sqrt{2})\,[\,(ch\mu - cos\eta)\,/\,\{(ch\mu - cos\eta)\,(ch\mu + cos\eta)\}]^{1/2} \\ &= (\pi/\sqrt{2})\,[\,1/\,(ch\mu + cos\eta)]^{1/2} \end{aligned}$$

so we get this simpler result

$$(\pi/\sqrt{2})\,[\,1/(ch\mu + cos\eta)]^{1/2} = \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\pi)]$$

Every time we shift cos(x) by π, we change its sign, so this can be written as

$$(\pi/\sqrt{2})\,[\,1/(ch\mu + cos\eta)]^{1/2} = \Sigma_{n=0}^{\infty}\varepsilon_n\,(-1)^n\, Q_{n-1/2}(ch\mu)\cos(n\eta)$$

$$(\pi/\sqrt{2})\; 1/\sqrt{ch\mu + cos\eta} = \Sigma_{n=0}^{\infty}\varepsilon_n\,(-1)^n\, Q_{n-1/2}(ch\mu)\cos(n\eta) \qquad (*)$$

Once again, apart from an overall constant factor, this "toroidal sum rule" is just a special case of our general toroidal 1/R expansion where we have put the Green's charge at the origin, **r**' = 0.

NOTE: I continue my "addition theorem development" in Section 4 below.

3. Derivation of the Selvaggi/Snow 1/R Expansion

This "sum rule" (*) is valid for any values of μ and η we want, so let's emphasize this fact by rewriting the sum rule this way

$$(\pi/\sqrt{2})\,[\,1/(B + cos\alpha)]^{1/2} = \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(B)\cos[n(\alpha-\pi)]$$

where B is any positive real number and α is any angle. Let's make one further change, by defining

$$\beta \equiv \alpha - \pi \qquad \cos(\alpha) = \cos(\pi+\beta) = -\cos\beta$$

Our generic sum rule now becomes

$$(\pi/\sqrt{2})\,[\,1/(B - cos\beta)]^{1/2} = \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(B)\cos(n\beta)$$

or

$$1/D \equiv 1/\sqrt{B-cos\beta} = (\sqrt{2}/\pi)\,\Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(B)\cos(n\beta) \qquad (**)$$

which, by the way, we can interpret as just the Fourier Series expansion of $1/\sqrt{B-cos\beta}$, as I deal with in a ragged doc elsewhere. Now, *forgetting about* all earlier coordinates, let's consider two new arbitrary points **r** and **r**' in cylindrical coordinates. We know of course that

$$R^2 = \rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho'\cos(\varphi-\varphi')$$

Let's now apply our formula (**) above where we blindly select

$$B = [\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')$$
$$\beta = (\varphi-\varphi')$$

We then find that

$$D^2 = B - \cos\beta = [\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho') - \cos(\varphi-\varphi')$$

$$= [\rho^2 + \rho'^2 + (z-z')^2 - 2\rho\rho' \cos(\varphi-\varphi')]/ (2\rho\rho')$$

$$= R^2/2\rho\rho'$$

$$\Rightarrow \quad 1/D = \sqrt{2\rho\rho'}\ (1/R)$$

Then (**) becomes

$$\sqrt{2\rho\rho'}\ (1/R) = (\sqrt{2}/\pi) \Sigma_{n=0}^{\infty} \varepsilon_n\, Q_{n-1/2}(B) \cos(n(\varphi-\varphi'))$$

or

$$1/R = 1/(\pi\sqrt{\rho\rho'})\ \Sigma_{n=0}^{\infty} \varepsilon_n\, Q_{n-1/2}(B) \cos[n(\varphi-\varphi')] \qquad (***)$$

where $B = [\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')$

and this is precisely the "Selvaggi" 1/R expansion. I think there is a simple geometric interpretation here, but I cannot quite nail it down, so I will defer that to some future time. The question I cannot answer is this: what is the interpretation of B ? Ignoring this question, we can conclude that the Selvaggi expansion has been *derived from* the toroidal sum rule, and is some kind of geometric transformation or application of it. In the *sum rule*, the trig angle is really η which is *not* an azimuthal angle at all. But in the Selvaggi thing, we really have a difference of azimuths.

So what should we call this Selvaggi thing? It seems more of a cylindrical coordinates 1/R expansion, but it contains the toroidal Q function atom! The Selvaggi paper doesn't apply any name to this expansion.

Now one thing we can do at this point is compare our Selvaggi 1/R expansion to our *cylindrical* general 1/R expansion which is either of these forms:

$$1/R = \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')] \int_0^{\infty} dk \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')$$

$$1/R = (2/\pi) \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')] \int_0^{\infty} dk \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)$$

$$1/R = 1/(\pi\sqrt{\rho\rho'}) \Sigma_{m=0}^{\infty} \varepsilon_m \cos[m(\varphi-\varphi')]\, Q_{m-1/2}\{[\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')\} \qquad \text{// Selvaggi}$$

where we changed the summation variable to m in the Selvaggi form. We can now make a term by term match to conclude that the following equations must be true, with $B = [\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')$:

$$1/(\pi\sqrt{\rho\rho'})\ Q_{m-1/2}(B) = \int_0^{\infty} dk \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')$$

$$1/(\pi\sqrt{\rho\rho'})\ Q_{m-1/2}(B) = (2/\pi) \int_0^{\infty} dk \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)$$

Let's rewrite these as

$$Q_{m-1/2}\{[\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')\} = \pi\sqrt{\rho\rho'} \int_0^\infty dk\, \exp(-k|z-z'|)\, J_m(k\rho)\, J_m(k\rho')$$

$$Q_{m-1/2}\{[\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')\} = 2\sqrt{\rho\rho'} \int_0^\infty dk\, \cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>)$$

I guess you could call these things "addition theorems" where the sum is an integral over k. They can be compared to the cylindrical and spherical addition theorems which are these

$$J_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) = \Sigma_{m=0}^{\infty}\varepsilon_m \cos[m(\varphi-\varphi')]\, J_m(k\rho)\, J_m(k\rho')$$
$$K_0(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi')}\,) = \Sigma_{m=0}^{\infty}\varepsilon_m \cos[m(\varphi-\varphi')]\, I_m(k\rho_<)\, K_m(k\rho_>)$$

$$P_n[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')] = \Sigma_{m=0}^{\infty}\varepsilon_m \cos[m(\varphi-\varphi')]\, f(n,-m)\, P_n^m(\cos\theta)\, P_n^m(\cos\theta')]$$

Somehow the roles of m and k are swapped. In the "normal" addition theorems we are summing over m which is associated with the azimuth φ, and the quantity k (or n for spherical) is fixed. But in our Q addition theorems, we are instead summing over k, and m is fixed.

4. Development of a new toroidal 1/R expansion and the addition theorem.

My earlier attempt here led to a big mess which I have deleted and replaced it with what follows.

Let's now give it one more go using the "new" form I found for 1/R. Start again with

$$1/\sqrt{\rho^2 + (z-z')^2} = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{1-cos\eta'}\;\; \Sigma_{n=0}^{\infty}\varepsilon_n\, Q_{n-1/2}(ch\mu)\cos[n(\eta-\eta')]$$

$$\rho = a\, sh\mu/(ch\mu - cos\eta)$$
$$z = a\, sin\eta/(ch\mu - cos\eta)$$
$$z' = a\, sin\eta'/(1 - cos\eta')$$

where we have μ' = 0 since we are on the z axis. Our general 1/R formula is this:

$$1/R = (1/\sqrt{2}\, a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}\,/\sqrt{ch\gamma - cos(\eta-\eta')} \qquad 4.1$$

$$\text{where} \quad ch\gamma \equiv ch\mu\, ch\mu' - sh\mu\, sh\mu'\cos(\varphi-\varphi') \qquad 4.2$$

and we want to set μ' = 0 so chμ' = 1 and chγ = chμ, so that we have

$$1/R = (1/\sqrt{2}\, a)\sqrt{ch\mu-cos\eta}\,\sqrt{1-cos\eta'}\,/\sqrt{ch\mu - cos(\eta-\eta')} \qquad 4.3$$

We have then shown that

$$(1/\sqrt{2}\, a)\sqrt{ch\mu-cos\eta}\,\sqrt{1-cos\eta'}\,/\sqrt{ch\mu - cos(\eta-\eta')}$$

$$= (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{1-cos\eta'}\;\;\Sigma_{n=0}^{\infty}\varepsilon_n\,Q_{n-1/2}(ch\mu)\cos[n(\eta-\eta')]$$

We can cancel factors on both sides to find that

$$1/\sqrt{ch\mu\;-\cos(\eta-\eta')}\;\;=\;(\sqrt{2}/\pi)\;\Sigma_{n=0}^{\infty}\varepsilon_n\,Q_{n-1/2}(ch\mu)\cos[n(\eta-\eta')] \qquad 4.4$$

and we have then done pretty will in "exposing" chμ on the LHS. The sum rule shown here must be value for any chμ we like. So we could install the full chγ on both sides to get

$$1/\sqrt{ch\gamma\;-\cos(\eta-\eta')}\;\;=\;(\sqrt{2}/\pi)\;\Sigma_{n=0}^{\infty}\varepsilon_n\,Q_{n-1/2}(ch\gamma)\cos[n(\eta-\eta')] \qquad 4.5$$

Now multiply both sides by $(1/\sqrt{2}\,a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}$. The LHS becomes general 1/R and we have then shown that

$$1/R = (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}\;\;\Sigma_{n=0}^{\infty}\varepsilon_n\,Q_{n-1/2}(ch\gamma)\cos[n(\eta-\eta')] \qquad 4.6$$

$$= (1/\sqrt{2}\,a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}\,/\sqrt{ch\gamma\;-\cos(\eta-\eta')}$$

where $\quad ch\gamma \;\equiv\; ch\mu\,ch\mu' - sh\mu\,sh\mu'\cos(\varphi-\varphi')$

The first line above is the addition theorem I have been looking for. I now want to make sure earlier results in previous sections are still OK. Suppose in 4.5 se set η' = π. It then reads

$$1/\sqrt{ch\gamma\;+\cos\eta}\;\;=\;(\sqrt{2}/\pi)\;\Sigma_{n=0}^{\infty}\varepsilon_n\,(-1)^n Q_{n-1/2}(ch\gamma)\cos(n\eta) \qquad 4.7$$

and this agrees with the second (*) in Section 2 above.

Now what about Selvaggi? We can derive it exactly as before, so all is well, but I am wondering if I don't now have some "interpretation" of things. Is $B = [\rho^2 + \rho'^2 + (z-z')^2]/(2\rho\rho')$ some special case of the quantity chγ ? I know from above that

$$B = D^2 + \cos(\varphi-\varphi') \;=\; R^2/2\rho\rho' \;+ \cos(\varphi-\varphi')$$

$$ch\gamma \;\equiv\; ch\mu\,ch\mu' - sh\mu\,sh\mu'\cos(\varphi-\varphi')$$

The only way these two could be the same is if cos(φ-φ') = 0 in which case we would have

$$B = R^2/(2\rho\rho') \qquad\qquad R^2 \;= 2\,a^2\,\{ch\mu\,ch\mu' - \cos(\eta-\eta')\}\;/\;[(ch\mu-cos\eta)\,(ch\mu'-cos\eta')]$$
$$ch\gamma \;\equiv\; ch\mu\,ch\mu'$$

We can then write

$$B = R^2/(2\rho\rho') \;= a^2\,\{ch\mu\,ch\mu' - \cos(\eta-\eta')\}\;/\;[(ch\mu-cos\eta)\,(ch\mu'-cos\eta')(\rho\rho')]$$

But we know that

$$\rho = a\,sh\mu/(ch\mu-cos\eta) \quad \Rightarrow \quad (ch\mu-cos\eta)\,\rho = a\,sh\mu$$
$$\rho' = a\,sh\mu'/(ch\mu'-cos\eta') \quad \Rightarrow \quad (ch\mu'-cos\eta')\,\rho' = a\,sh\mu'$$

so we then have

$$B = a^2\,\{ch\mu\,ch\mu' - cos(\eta-\eta')\}\,/\,[\,a^2\,sh\mu\,sh\mu'] = \{ch\mu\,ch\mu' - cos(\eta-\eta')\}/\,[\,sh\mu\,sh\mu']$$

But this does not in general give B = chγ. I am just not finding any useful "interpretation" here.

What about addition theorems? We have shown the following 1/R expansion

$$1/R = (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}\;\Sigma_{n=0}^{\infty}\varepsilon_n\,Q_{n-1/2}(ch\gamma)\cos[n(\eta-\eta')] \qquad 4.6$$

$$= (1/\sqrt{2}\,a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}\,/\sqrt{ch\gamma - cos(\eta-\eta')}$$

where $ch\gamma \equiv ch\mu\,ch\mu' - sh\mu\,sh\mu'\cos(\varphi-\varphi')$

We usually set this equal to some other 1/R expansion and try to compare terms. Our toroidal form was this:

$$1/R = (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}$$
$$\Sigma_{n=0}^{\infty}\;\Sigma_{m=0}^{\infty}\,\varepsilon_n\,\varepsilon_m\,(-1)^m\,f(n-1/2,-m)P^m_{n-1/2}(chu_<)Q^m_{n-1/2}(chu_>)\cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$$

If we set these things equal we obtain:

$$(1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}\;\Sigma_{n=0}^{\infty}\varepsilon_n\,Q_{n-1/2}(ch\gamma)\cos[n(\eta-\eta')]$$

$$= (1/\pi a)\sqrt{ch\mu-cos\eta}\,\sqrt{ch\mu'-cos\eta'}$$
$$\Sigma_{n=0}^{\infty}\;\Sigma_{m=0}^{\infty}\,\varepsilon_n\,\varepsilon_m\,(-1)^m\,f(n-1/2,-m)P^m_{n-1/2}(chu_<)Q^m_{n-1/2}(chu_>)\cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$$

or

$$\Sigma_{n=0}^{\infty}\varepsilon_n\,Q_{n-1/2}(ch\gamma)\cos[n(\eta-\eta')] =$$

$$\Sigma_{n=0}^{\infty}\;\Sigma_{m=0}^{\infty}\,\varepsilon_n\,\varepsilon_m\,(-1)^m\,f(n-1/2,-m)P^m_{n-1/2}(chu_<)Q^m_{n-1/2}(chu_>)\cos[n(\eta-\eta')]\cos[m(\varphi-\varphi')]$$

or

$$Q_{n-1/2}(ch\gamma) = \Sigma_{m=0}^{\infty}\,\varepsilon_m\,(-1)^m\,f(n-1/2,-m)P^m_{n-1/2}(chu_<)Q^m_{n-1/2}(chu_>)\cos[m(\varphi-\varphi')]$$

where $ch\gamma \equiv ch\mu\,ch\mu' - sh\mu\,sh\mu'\cos(\varphi-\varphi')$

So THIS is what we would call "the toroidal addition theorem". We could for fun consider the special case of this where we put **r'** back on the z axis with μ' = 0 and we know

$$P^m_{n-1/2}(ch\mu_<) = P^m_{n-1/2}(ch\mu')P^m_{n-1/2}(1) = \delta_{m0} \qquad \text{and} \qquad ch\gamma = ch\mu$$

so we end up then with

$$Q_{n-1/2}(ch\mu) = \varepsilon_0 (-1)^0 f(n-1/2,0)\, Q^0_{n-1/2}(chu_>) \cos(0) = Q_{n-1/2}(ch\mu)$$

so things check out OK.

Next topic: is there some kind of transform to worry about? In our atomic approach here our two SL problems were the trivial ones involving φ and η. But if you were to use $e^{-n\eta}$ then you get those cone functions and they do have a SL problem, as noted elsewhere, called Mehler-Foch.

5. Something I did not do in this doc: the "other" Smythian form.

I could have taken the "other" Smythian form at the start, which might be something like this:

$$f(\mathbf{r} \mid \mathbf{r}') = \sqrt{ch\mu - cos\eta} \int_0^\infty d\tau \, \Sigma_m \, A_{\tau m} \, P^m_{i\tau-1/2}(ch\mu_<) \, Q^m_{i\tau-1/2}(ch\mu_>) \, e^{-\tau|\eta|} \, e^{im\varphi}$$

where now μ is oscillatory and η is expo, the reverse of our previous case. One would then do the pillbox in the η direction to deduce the $A_{\tau m}$ coefficients. More clarity is needed regarding the exponential, I am just doing a symbolic outline here. The pillbox would require Mehler-Fock orthogonality as well as the usual trig φ orthogonality. I have never done this, and I have never seen it done, so it would be "advanced" work that perhaps I don't need to do right now. Based on Green's symmetry, I would expect the result to have this *general form*:

$$1/R = (1/\pi a)\sqrt{ch\mu - cos\eta}\,\sqrt{ch\mu' - cos\eta'}$$
$$\int_0^\infty d\tau \, \Sigma_{m=0}^\infty \varepsilon_m \, (-1)^m f(i\tau-1/2,-m) P^m_{i\tau-1/2}(chu_<) Q^m_{i\tau-1/2}(chu_>) \, e^{-\tau|\eta-\eta'|} \cos[m(\varphi-\varphi')]$$

but I expect there to be extra factors not shown. Perhaps I could derive this using contour integration methods to get the sum into the τ integral, and then start with the already known result. I would then obtain perhaps a sum rule and an addition theorem and something like the Selvaggi/Snow expansion. I dimly recall maybe seeing one or two papers using this kind of stuff.