Various toroidal coordinates used by M&F PhL 12.30.13

1. What toroidal coordinates do I use in my Stackel paper?

Answer: I use $(\xi_1,\xi_2,\xi_3) = (\eta,\theta,\psi)$ where the latter are the M&S coordinates! These are scale factors:

$$h_1 = h_2 = a/[ch(\xi_1)-\cos(\xi_2)]$$
$$h_3 = a\, sh(\xi_1)/[ch(\xi_1)-\cos(\xi_2)]$$

I then compute a Stackel matrix using these coordinates:

$$\Phi = \begin{pmatrix} a(1) & -1 & -1/sh^2(\xi_1) \\ d(2) & 1 & 0 \\ g(3) & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1/sh^2(\xi_1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (7.7)$$

Later I get these same matrices doing the calculation a different way.

I get these three ODE's

$$L_1X_1 = \partial_1^2X_1 + \coth(\xi_1)\,\partial_1X_1\} + [(1/4) - k_2^2 - k_3^2/sh^2(\xi_1)]X_1 = 0$$
$$L_2X_2 = \partial_2^2X_2 + k_2^2X_2 = 0$$
$$L_3X_3 = \partial_3^2X_3 + k_3^2X_3 = 0 \quad .$$

and then I get the harmonics two different ways:

$$[P^q_{p-1/2}(ch\xi_1), Q^q_{p-1/2}(ch\xi_1)\ [\ \overset{osc}{\sin(p\xi_2), \cos(p\xi_2)}\ ]\ [\ \overset{osc}{\sin(q\xi_3), \cos(q\xi_3)}\ ] * [ch(\xi_1)-\cos(\xi_2)]^{1/2} .$$

$$[P^q_{i\tau-1/2}(ch\xi_1), Q^q_{i\tau-1/2}(ch\xi_1)\ [\ \overset{osc}{sh(\tau\xi_2), ch(\tau\xi_2)}\ ]\ [\ \overset{osc}{\sin(q\xi_3), \cos(q\xi_3)}\ ] * [ch(\xi_1)-\cos(\xi_2)]^{1/2}$$

2. What toroidal coordinates do I use in my Bowl paper?

The first thing I do is just state these two atomic forms:

$$\xi \text{ in } (0,\infty) \qquad u \text{ in } (0,2\pi) \qquad \varphi \text{ in } (0,2\pi)$$

$$(1) \qquad \sqrt{ch\xi - \cos u}\ \ [\overset{expo}{P_{n-1/2}^m(ch\xi)}, \overset{osc}{Q_{n-1/2}^m(ch\xi)}\ ]\ \ [\ \overset{osc}{\sin(nu),\cos(nu)}\ ]\ \ [\ \overset{osc}{\sin(m\varphi),\cos(m\varphi)}\ ]$$

$$(2) \qquad \sqrt{ch\xi - \cos u}\ \ [\overset{osc}{P_{i\tau-1/2}^m(ch\xi)}, \overset{expo}{Q_{i\tau-1/2}^m(ch\xi)}\ ]\ [\ \overset{expo}{\exp(\tau u), \exp(-\tau u)}\ ]\ \ [\ \overset{osc}{\sin(m\varphi),\cos(m\varphi)}\ ]$$

So here is the connection to the Stackel paper

Stackel paper $(\xi_1,\xi_2,\xi_3) = (\eta,\theta,\psi)$ of M&S
Bowl paper: $(\xi,u,\varphi) = (\xi_1,\xi_2,\xi_3) = (\eta,\theta,\psi)$ of M&S

So at least I am reasonably consistent in the two papers.

Now in another of my docs ("toroidal coordinates.doc") where I am using these same (ξ,u,φ) coordinates, I write these equations:

$x = a\cos\varphi\ \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$ $\rho = a\ \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$ $\rho^2 = x^2 + y^2$
$y = a\sin\varphi\ \mathrm{sh}\xi/(\mathrm{ch}\xi - \cos u)$
$z = a\sin u/(\mathrm{ch}\xi - \cos u)$

3. What toroidal coordinates do M&F use on their page 666? (Their only Volume 1 foray)

On this page, the coordinates have two different names:

set one: (ξ_1,ξ_2,ξ_3) set two: (μ,θ,φ)

connection between the two MF coordinates: (from below)

$\xi_1 = \mathrm{ch}(\mu)$
$\xi_2 = \cos(\theta)$
$\xi_3 = \cos(\varphi)$

I can then translate their various equations to the (μ,θ,φ) coordinates:

$x = a\cos\varphi\ \mathrm{sh}\mu\ /(\mathrm{ch}\mu - \cos\theta)$
$y = a\sin\varphi\ \mathrm{sh}\mu\ /(\mathrm{ch}\mu - \cos\theta)$
$z = a\sin\theta\ /(\mathrm{ch}\mu - \cos\theta)$

If I compare these to my own equations in Section 2 above, I find that

$(\mu,\theta,\varphi)_{\mathrm{MF666}} = (\xi,u,\varphi)_{\mathrm{PLbowl}}$

Now let's take a look at the scale factors given below by MFp666:

$h_1 = -\ a\ [\mathrm{sh}\mu]^{-1}/(\mathrm{ch}\mu - \cos\theta)$
$h_2 = a\ [\sin\theta]^{-1}/(\mathrm{ch}\mu - \cos\theta)$
$h_3 = a\ \mathrm{sh}\mu\ [\sin\varphi]^{-1}/(\mathrm{ch}\mu - \cos\theta)$

These do not match mine because THESE h_i are for the MF ξ_i coordinates, not for the MF (μ,θ,ε) coordinates. OK, I think I understand this now, and I don't really care about their ξ_i coordinates and thus I don't care about their Stackel matrix in those coordinates. I never comment on this anywhere, but perhaps I should.

Toroidal Coordinates

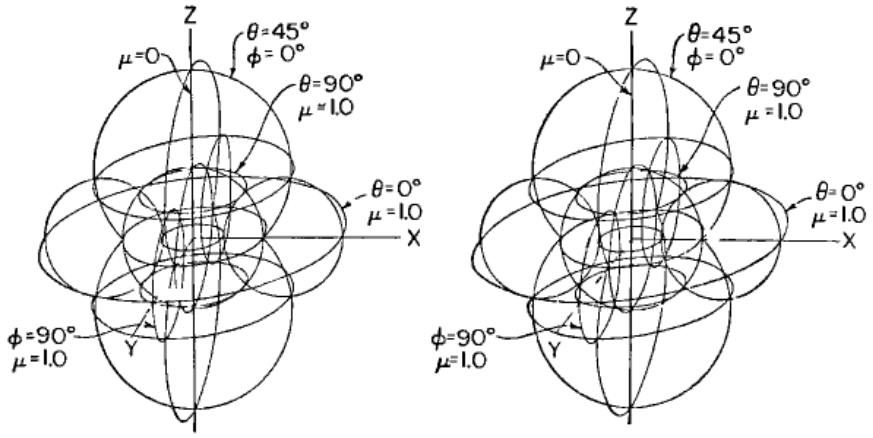

Fig. 5.26.

$$x = a\xi_3 \frac{\sqrt{\xi_1^2 - 1}}{\xi_1 - \xi_2}; \quad y = a \frac{\sqrt{(\xi_1^2 - 1)(1 - \xi_3^2)}}{\xi_1 - \xi_2}; \quad z = a \frac{\sqrt{1 - \xi_2^2}}{\xi_1 - \xi_2}$$

$$h_1 = \frac{a}{(\xi_1 - \xi_2)\sqrt{\xi_1^2 - 1}}; \quad h_2 = \frac{a}{(\xi_1 - \xi_2)\sqrt{1 - \xi_2^2}};$$

$$h_3 = \frac{a}{(\xi_1 - \xi_2)} \sqrt{\frac{\xi_1^2 - 1}{1 - \xi_3^2}}$$

$$f_1 = \xi_1^2 - 1; \quad f_2 = \sqrt{1 - \xi_2^2}; \quad f_3 = \sqrt{1 - \xi_3^2}$$

$$r = \sqrt{x^2 + y^2 + z^2} = a\sqrt{(\xi_1 + \xi_2)/(\xi_1 - \xi_2)}$$

$$R^2 = \frac{a}{\xi_1 - \xi_2}; \quad u = \frac{a^2}{(\xi_1 - \xi_2)^2}; \quad k_1^2 = \tfrac{1}{4}$$

$$\xi_1 = \cosh \mu; \quad \xi_2 = \cos \vartheta; \quad \xi_3 = \cos \varphi$$

$$S = \begin{vmatrix} \frac{1}{(\xi_1^2 - 1)} & \frac{-1}{(\xi_1^2 - 1)} & \frac{-1}{(\xi_1^2 - 1)^2} \\ 0 & \frac{1}{(1 - \xi_2^2)} & 0 \\ 0 & 0 & \frac{1}{1 - \xi_3^2} \end{vmatrix} = \frac{1}{(\xi_1^2 - 1)(1 - \xi_2^2)(1 - \xi_3^2)}$$

4. What toroidal coordinates do MF use in Volume II?

Here is what is on page 1301: (and following pages deal with toroidals)

$$z = \frac{a \sin \eta}{\cosh \mu - \cos \eta}; \quad x = \frac{a \sinh \mu \cos \phi}{\cosh \mu - \cos \eta}; \quad y = \frac{a \sinh \mu \sin \phi}{\cosh \mu - \cos \eta}$$

$$h_\mu = h_\eta = \frac{a}{\cosh \mu - \cos \eta}; \quad h_\phi = \frac{a \sinh \mu}{\cosh \mu - \cos \eta} \qquad (10.3.75)$$

$$\nabla^2 \psi = \frac{1}{h_\mu{}^3} \left[\frac{1}{\sinh \mu} \frac{\partial}{\partial \mu} \left(h_\mu \sinh \mu \frac{\partial \psi}{\partial \mu} \right) + \frac{\partial}{\partial \eta} \left(h_\eta \frac{\partial \psi}{\partial \eta} \right) + \frac{h_\mu}{\sinh^2 \mu} \frac{\partial^2 \psi}{\partial \phi^2} \right]$$

This is consistent with their Volume 1 stuff EXCEPT they have now renamed the second variable to be η instead of θ. So here they have (μ,η,φ) whereas in Volume 1 they had (μ,θ,φ). This is then the same as my coordinates (ξ,u,φ). The h_i here agree with mine.

So OK, there is no other stuff in Volume 2, so this completes my survey.

$(\mu,\eta,\varphi)_{\texttt{MFp1301}} = (\mu,\theta,\varphi)_{\texttt{MFp666}} = (\xi,u,\varphi)_{\texttt{PLbowl}} = (\xi_1,\xi_1,\xi_3)_{\texttt{PLstackel}} = (\eta,\theta,\psi)_{\texttt{MoonSp}}$

$(\xi_1,\xi_1,\xi_3)_{\texttt{MFp666}}$ = totally different coordinates