

THE MEHLER-FOCK TRANSFORM OF GENERAL ORDER AND ARBITRARY INDEX AND ITS INVERSION

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ABSTRACT. An integral transform involving the associated Legendre function of zero order, $P_{-\frac{1}{2}+i\tau}(x)$, $x \in [1, \infty)$, as the kernel (considered as a function of τ), is called Mehler-Fock transform. Some generalizations, involving the function $P_{-\frac{1}{2}+i\tau}^{\mu}(x)$, where the order μ is an arbitrary complex number, including the case when $\mu = 0, 1, 2, \dots$, have been known for some time. In this present note, we define a general Mehler-Fock transform involving, as the kernel, the Legendre function $P_{-\frac{1}{2}+t}^{\mu}(x)$, of general order μ and an arbitrary index $-\frac{1}{2} + t$, $t = \sigma + i\tau$, $-\infty < \tau < \infty$. Then we develop a symmetric inversion formulae for these transforms. Many well-known results are derived as special cases of this general form. These transforms are widely used for solving many axisymmetric potential problems.

KEY WORDS AND PHRASES. Mehler-Fock transform, Kontorovich-Lebedev transforms, Legendre functions of first and second kinds, Macdonald function, Bessel functions.

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1. INTRODUCTION.

To solve many axisymmetric potential problems, integral transforms involving associated Legendre functions as kernels, are used widely. When the associated Legendre function of the first kind, $P_{-\frac{1}{2}+i\tau}(x)$, $x \in [1, \infty)$, is used as the kernel, the transform is called Mehler-Fock transform of order zero, [1, p.175(8,9)], and index $-\frac{1}{2} + i\tau$, $-\infty < \tau < \infty$. Some generalizations of Mehler-Fock transformation involve more general associated Legendre function, namely, $P_{-\frac{1}{2}+i\tau}^m(x)$, m , a non-negative integer [2, p.390] and $P_{-\frac{1}{2}+i\tau}^{\mu}(x)$, where $\mu \neq 0$ [3], have been known for some time, [cf 4]. In all these generalizations, the emphasis has been to generalize the order μ of the Legendre function $P_{\nu}^{\mu}(x)$, but no attempt has been made to extend the results for an arbitrary index ν . In this note we define transforms, involving a more general form of the kernel function, the associated Legendre function of the first kind $P_{-\frac{1}{2}+t}^{\mu}(x)$, with index $t = \sigma + i\tau$, $-\infty < \tau < \beta$ and μ an arbitrary complex number. An inversion formula is developed, also, involving the Legendre function $P_{-\frac{1}{2}+t}^{\mu}(x)$ as

kernel, thus establishing a symmetric inversion theory for general Mehler-Fock transformation. Also, a Parseval type relation for these general transforms is produced. All the well-known results are derived as special cases.

2. THE PRELIMINARY RESULTS.

We note below, for future reference, some of the properties of the associated Legendre function of the first kind, $P_{\nu}^{\mu}(z)$ of order μ and index ν for unrestricted ν, μ and z [1, III]. The function is one-valued and regular in the complex plane supposed cut along the real axis from 1 to $-\infty$.

$$P_{\nu}^{\mu}(z) = P_{-\nu-1}^{\mu}(z) \quad (2.1)$$

$$P_{\nu}^m(z) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} P_{\nu}^{-m}(z), \quad m = 0, 1, 2, \dots \quad (2.2)$$

An important case is when $\mu = 0$ and ν is an non-negative integer. Then

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n,$$

the Legendre polynomial.

Now, consider the representation, [1, p.128 (28)],

$$P_{\nu}^{\mu}(z) = 2^{\mu} (z^2-1)^{-\frac{1}{2}\mu} \frac{(z+\sqrt{z^2-1})^{\nu+\mu}}{\Gamma(1-\mu)} {}_2F_1\left(-\nu-\mu, \frac{1}{2}-\mu; 1-2\mu; \frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right)$$

Re $z > 1$ and Re $\mu < 1$.

Also, [2, p.78(1)],

$${}_2F_1(a, b; c; z) = \frac{\Gamma^2(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \tau^{a-1} (1-\tau)^{c-a-1} d\tau \\ \times \int_0^1 \frac{t^{b-1} (1-t)^{c-b-1}}{(1-t\tau z)^c} dt,$$

Re $c > \text{Re } b > 0$.

Using the inequality

$$\frac{1}{(1-t\tau z)^c} \leq \frac{1}{(1-z)^c}, \quad c > 0,$$

it is clear that

$${}_2F_1(a, b; c; z) \leq \frac{1}{(1-z)^c}.$$

Thus

$$P_{\nu}^{\mu}(z) \leq \frac{2^{\mu} (z+\sqrt{z^2-1})^{\nu-\mu+1}}{\Gamma(1-\mu) (z^2-1)^{\frac{1}{2}\mu} (z-\sqrt{z^2-1})^{1-2\mu}}, \quad \text{Re } \mu < \frac{1}{2},$$

whence,

$$P_{\nu}^{\mu}(x) \sim x^{\nu} \quad \text{as } x \rightarrow \infty \quad (2.3)$$

and

$$P_{\nu}^{\mu}(x) \sim (x^2-1)^{-\frac{1}{2}\mu} \quad \text{as } x \rightarrow 1+ . \quad (2.4)$$

Also, that

$${}_2F_1(\sigma+i\tau, b; c; z) = O(1), \quad \text{as } |\tau| \rightarrow \infty,$$

whence

$$P_{\sigma+i\tau}^{\mu}(z) = O(1) \quad \text{as } |\tau| \rightarrow \infty, \quad (2.5)$$

provided Re $\mu < \frac{1}{2}$.

We shall also need the following result, [5, p.75(76)].

Lemma 1.

$$\begin{aligned} \text{Let } f(v) &= O(e^{-v}), & \text{as } v \rightarrow \infty \\ &= O(v^{\frac{1}{2}}), & \text{as } v \rightarrow 0. \end{aligned}$$

If

$$f(x) = \frac{1}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} t K_t(x) g(t) dt, \quad \alpha > -\frac{1}{2}, \quad (2.6)$$

$$\text{then } g(t) = \int_0^\infty \frac{f(v)}{v} I_t(v) dv, \quad (2.7)$$

where $K_t(v)$ and $I_t(v)$ are the Macdonald function and the Bessel function respectively of order t , a complex number. Next let us consider the contour integral

$$I = \int_C z K_z(u) g(z) dz,$$

where C is the closed contour in the z -plane as shown. Now $K_z(u)$ is an entire function of z and it is clear that $g(z)$ is regular inside and on C from (2.7) above.

Thus, by Cauchy's theorem of residues,

$$I = 0,$$

or

$$\begin{aligned} 0 &= i \int_{-\tau}^{\tau} (\sigma + iy) K_{\sigma+iy}(u) g(\sigma+iy) dy \\ &+ \int_0^{-\sigma} (x + i\tau) K_{x+i\tau}(u) g(x+i\tau) dx \\ &+ i \int_{\tau}^{-\tau} (-\sigma + iy) K_{-\sigma+iy}(u) g(-\sigma+iy) dy \\ &+ \int_{-\sigma}^{\sigma} (x - i\tau) K_{x-i\tau}(u) g(x-i\tau) dx \\ &= I_1 + I_2 + I_3 + I_4, \text{ say,} \end{aligned}$$

Now the existence of the integral in (2.6), implies that

$$|(x+i\tau) K_{x+i\tau}(u) g(x+i\tau)| \rightarrow 0 \quad \text{as } |\tau| \rightarrow \infty,$$

thus

$$|I_2| \text{ and } |I_4| \text{ both vanish as } |\tau| \rightarrow \infty.$$

And we now have,

$$I_1 = -I_3 \text{ or } |\tau| \rightarrow \infty$$

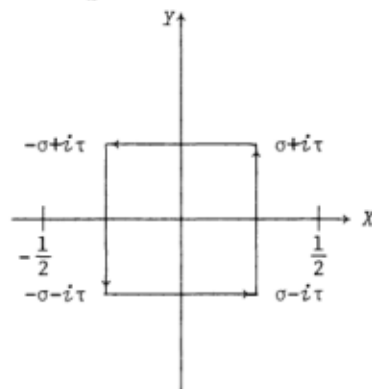
or,

$$i \int_{-\infty}^{\infty} (\sigma + iy) K_{\sigma+iy}(u) g(\sigma+iy) dy = -i \int_{-\infty}^{\infty} (-\sigma + iy) K_{-\sigma+iy}(u) g(-\sigma+iy) dy$$

on simplifying, we obtain

$$\begin{aligned} \int_{\sigma-i\infty}^{\sigma+i\infty} z K_z(u) g(z) dz &= - \int_{\sigma-i\infty}^{\sigma+i\infty} z K_z(u) g(-z) dz \\ &= if(u), \end{aligned}$$

using (2.6) and the fact that $K_{-z}(u) = K_z(u)$.



Thus,

$$f(u) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z K_z(u) [g(z) - g(-z)] dz \quad (2.8)$$

From the definition of the function g given by (2.7),

$$g(z) - g(-z) = \int_0^\infty \frac{f(v)}{v} [I_z(v) - I_{-z}(v)] dv ,$$

or

$$-\frac{\pi}{2} \frac{g(z) - g(-z)}{\sin(\pi z)} = \int_0^\infty \frac{f(v)}{v} K_z(v) dv , \quad (2.9)$$

If we let

$$-\frac{\pi}{2} \frac{g(z) - g(-z)}{\sin(\pi z)} = F(z), \text{ say,}$$

then (2.8) and (2.9) reduce to

$$f(u) = \frac{i}{\pi^2} \int_{\sigma-i\infty}^{\sigma+i\infty} z \sin(\pi z) K_z(u) F(z) dz$$

and

$$F(z) = \int_0^\infty \frac{f(v)}{v} K_z(v) dv ,$$

respectively.

Now we have a generalized Kontorovich-Lebedev transforms in a symmetrical form, (cf. 6]. Thus, we have shown:

Lemma 3. Let $f(v) = O(e^{-v})$, as $v \rightarrow \infty$
 $= O(v^{\frac{1}{2}})$, as $v \rightarrow 0$.

If

$$f(v) = \frac{i}{\pi^2} \int_{\sigma-i\infty}^{\sigma+i\infty} z \sin(\pi z) K_z(v) F(z) dz , \quad (2.10)$$

then

$$F(z) = \int_0^\infty \frac{f(v)}{v} K_z(v) dv , \quad (2.11)$$

where $|\sigma| < \frac{1}{2}$.

If we set $\sigma = 0$, the above pair reduces to

$$f(v) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) K_{i\tau}(v) F(i\tau) d\tau$$

where

$$F(i\tau) = \int_0^\infty \frac{f(v)}{v} K_{i\tau}(v) dv ,$$

giving us the usual Kontorovich-Lebedev transformation [2, p.361].

Lemma 4. If $\psi(t) \in L(\sigma-i\infty, \sigma+i\infty)$, $|\sigma| < \frac{1}{2}$, and $\text{Re } \mu < 1$,

then

$$\int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) P_{-\frac{1}{2}+t}^\mu(y) dt = \sqrt{\frac{2}{\pi}} x^{\mu-\frac{1}{2}} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) K_t(x) dt .$$

Proof. Consider the double integral

$$\left| \int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) P_{-\frac{1}{2}+t}^\mu(y) dt \right|$$

$$\begin{aligned}
& \leq \int_1^\infty \int_{\sigma-i\infty}^{\sigma+i\infty} \left| \frac{e^{-xy} \psi(t)}{(y^2-1)^{\mu/2}} P_{-\frac{1}{2}+t}^\mu(y) \right| dt dy \\
& = \int_1^\infty \int_{-\infty}^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} |\psi(\sigma+i\tau) P_{-\frac{1}{2}+\sigma+i\tau}^\mu(y)| d\tau dy \\
& = \int_1^\delta \int_{-\infty}^\infty |\cdot| d\tau dy + \int_\delta^\infty \int_{-\infty}^\infty |\cdot| d\tau dy, \quad \delta > 1. \\
& \leq \int_1^\delta \int_{-\infty}^\infty \left| \frac{e^{-xy}}{(y^2-1)^{\mu/2}} \psi(\sigma+i\tau) (y^2-1)^{\mu/2} \right| d\tau dy + \int_\delta^\infty \int_{-\infty}^\infty \left| \frac{e^{-xy}}{(y^2-1)^{\mu/2}} \psi(\sigma+i\tau) \right| d\tau dy \\
& = \int_1^\delta \frac{e^{-xy}}{(y^2-1)^\mu} dy \int_{-\infty}^\infty |\psi(\sigma+i\tau)| d\tau + \int_\delta^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \times \\
& \quad \int_{-\infty}^\infty |\psi(\sigma+i\tau)| d\tau < \infty, \text{ due to the hypotheses.}
\end{aligned}$$

Hence

$$\int_1^\infty \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} \psi(t) P_{-\frac{1}{2}+t}^\mu(y) dt dy$$

is absolutely convergent, and therefore the change of order of integration is possible.

And,

$$\begin{aligned}
& \int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) P_{-\frac{1}{2}+t}^\mu(y) dt \\
& = \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) dt \int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} P_{-\frac{1}{2}+t}^\mu(y) dy \\
& = \sqrt{\frac{2}{\pi}} x^{\mu-\frac{1}{2}} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) K_t(x) dt, \quad \operatorname{Re} \mu < 1.
\end{aligned}$$

by evaluating the y -integral, [6, p.179(1)].

3. The main results and special cases.

Theorem 1.

$$\begin{aligned}
\text{Let } f(y) &= O(e^{-y}), & \text{as } y \rightarrow \infty \\
&= O(1), & \text{as } y \rightarrow 1+
\end{aligned}$$

If

$$f(y) = \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) P_{-\frac{1}{2}+t}^\mu(y) F(t) dt \quad (3.1)$$

then

$$F(t) = \int_1^\infty f(y) P_{-\frac{1}{2}+t}^\mu(y) dy \quad (3.2)$$

$$\text{where } t = \sigma + i\tau, \quad -\infty < \tau < \infty, \quad |\sigma| < \frac{1}{2} - \mu, \quad \operatorname{Re} \mu < 1.$$

Proof. Using the estimates (2.3) and (2.4) of the function $P_{-\frac{1}{2}+t}^\mu(y)$, along with the conditions imposed on f , the integral in (3.2) defining the function g , exists. Note that in order to ensure the existence of the integral in (3.1), representing the function f , one must have $t^{1-2\mu} F(t) \in L(\sigma-i\infty, \sigma+i\infty)$ at least.

Now let,

$$\phi(t) \equiv \frac{\pi}{2} \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) F(t) \quad (3.3)$$

then from (3.1)

$$f(y) = \frac{i}{\pi^2} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \phi(t) P_{-\frac{1}{2}+t}^{\mu}(y) dt. \quad (3.4)$$

Next, we define an integral operator.

$$L_f \equiv L_{\mu}[f(y); y \rightarrow x] \equiv \int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} f(y) dy, \quad \operatorname{Re} \mu < 1.$$

Note that this is a linear, self adjoint operator, a sort of generalized incomplete Laplace operator. Now applying this operator to both sides of the equation (3.4) above, we obtain

$$\begin{aligned} \int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} f(y) dy &= \frac{i}{\pi^2} \int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \phi(t) \cdot P_{-\frac{1}{2}+t}^{\mu}(y) dt \\ &= \frac{i}{\pi^2} \sqrt{\frac{2}{\pi}} x^{\mu-\frac{1}{2}} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \phi(t) K_t(x) dt, \end{aligned}$$

due to lemma 4.

Or,

$$\psi(x) \equiv \sqrt{\frac{\pi}{2}} x^{\frac{1}{2}-\mu} \int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} f(y) dy = \frac{i}{\pi^2} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \phi(t) K_t(x) dt,$$

whence, according to lemma 3, we have the inversion,

$$\phi(t) = \int_0^{\infty} \frac{\psi(v)}{v} K_t(v) dv,$$

provided the integral exists. Also,

$$\begin{aligned} \phi(t) &= \sqrt{\frac{\pi}{2}} \int_0^{\infty} K_t(v) v^{-\frac{1}{2}-\mu} dv \int_1^{\infty} \frac{e^{-vy}}{(y^2-1)^{\mu/2}} f(y) dy \\ &= \sqrt{\frac{\pi}{2}} \int_1^{\infty} \frac{f(y)}{(y^2-1)^{\mu/2}} dy \int_0^{\infty} e^{-vy} v^{-\frac{1}{2}-\mu} K_t(v) dv. \end{aligned}$$

The change of order of integration can be justified due to absolute convergence, by making use of the estimates of the Macdonald's function $K_t(v)$ along with the conditions imposed on the function f . Now the v -integral can be evaluated, [6,p.198(27)], to give,

$$\phi(t) = \frac{\pi}{2} \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) \int_1^{\infty} f(y) P_{-\frac{1}{2}+t}^{\mu}(y) dy.$$

And from (3.3), we have finally,

$$F(t) = \int_1^{\infty} f(y) P_{-\frac{1}{2}+t}^{\mu}(y) dy,$$

as required.

Thus we have a symmetric transformation, in that, if

$$f(y) = \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) P_{-\frac{1}{2}+t}^{\mu}(y) F(t) dt,$$

then

$$F(t) = \int_1^{\infty} f(y) P_{-\frac{1}{2}+t}^{\mu}(y) dy,$$

defining a more general form of the Mehler-Fock transform of order μ and arbitrary index $-\frac{1}{2} + t$, where $t = \sigma + i\tau$, $-\infty < \tau < \infty$.

Corollary: If the function f , satisfies the conditions of the above theorem, then

$$f(y) = \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_1^{\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) P_{-\frac{1}{2}+t}^{\mu}(y) P_{-\frac{1}{2}+t}^{\mu}(u) f(u) du dt. \quad (3.5)$$

Note that from the definition of g , in (3.3), we have

$$g(-t) = g(t) \quad (3.6)$$

Now we shall look at some of the special cases.

Let $\sigma = 0$. Then (3.1) reduces to

$$\begin{aligned} f(y) &= \frac{i}{2\pi} \int_{-i\infty}^{i\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) P_{-\frac{1}{2}+t}^{\mu}(y) F(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau \sinh(\pi \tau) \Gamma\left(\frac{1}{2} - \mu - i\tau\right) \Gamma\left(\frac{1}{2} - \mu + i\tau\right) P_{-\frac{1}{2}+i\tau}^{\mu}(y) F(i\tau) d\tau. \end{aligned}$$

Hence

$$f(y) = \frac{1}{\pi} \int_0^{\infty} \tau \sinh(\pi \tau) \Gamma\left(\frac{1}{2} - \mu - i\tau\right) \Gamma\left(\frac{1}{2} - \mu + i\tau\right) P_{-\frac{1}{2}+i\tau}^{\mu}(y) F(i\tau) d\tau,$$

using the properties (2.1) and (3.6),

$$\text{where } F(i\tau) = \int_1^{\infty} f(y) P_{-\frac{1}{2}+i\tau}^{\mu}(y) dy,$$

giving us a generalized Mehler-Fock transformation [3].

Further if we put $\mu = 0$, then the above pair is reduced to

$$f(y) = \int_0^{\infty} \tau \coth(\pi \tau) P_{-\frac{1}{2}+i\tau}(y) F(i\tau) d\tau$$

and

$$F(i\tau) = \int_1^{\infty} f(y) P_{-\frac{1}{2}+i\tau}(y) dy;$$

the usual Mehler-Fock transformation of zero order [2, p.389]. Next, we shall derive, formally, a Parseval type relation. Let us define the function $F(t)$ and $G(t)$ to be the generalized Mehler-Fock transforms of the function $f(y)$ and $g(y)$ respectively, so that

$$F(t) = \int_1^{\infty} f(y) P_{-\frac{1}{2}+t}^{\mu}(y) dy$$

and

$$G(t) = \int_1^{\infty} g(y) P_{-\frac{1}{2}+t}^{\mu}(y) dy,$$

where $t = \sigma + i\tau$, $|\sigma| < \frac{1}{2}$, $\text{Re } \mu < 1$ and $-\infty < \tau < \infty$. And of course, as proved above

$$f(y) = \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) P_{-\frac{1}{2}+t}^{\mu}(y) F(t) dt,$$

and a similar formula for $g(y)$.

Consider the integral,

$$\begin{aligned} &\frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) F(t) G(t) dt \\ &\equiv \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) F(t) G(t) dt, \text{ say} \end{aligned}$$

$$\begin{aligned}
&= \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) F(t) i^t \int_1^\infty g(y) P_{-\frac{1}{2}+t}^\mu(y) dy \\
&= \frac{i}{2\pi} \int_1^\infty g(y) dy \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t) F(t) P_{-\frac{1}{2}+t}^\mu(y) dy \\
&= \int_1^\infty g(y) f(y) dy .
\end{aligned}$$

Thus we have,

Theorem 2.

Let F and G be the generalized Mehler-Fock transforms of f and g . Then,

$$\begin{aligned}
&\frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - \mu - t\right) \Gamma\left(\frac{1}{2} - \mu + t\right) F(t) G(t) dt \\
&= \int_1^\infty f(y) g(y) dy
\end{aligned} \tag{3.7}$$

This is a formal derivation, but the analysis can be justified by absolute convergence, using suitable conditions on the functions involved. If we set $\sigma = 0$ in (3.7), we can easily deduce that

$$\begin{aligned}
&\frac{1}{\pi} \int_0^\infty \tau \sinh(\pi\tau) \Gamma\left(\frac{1}{2} - \mu - i\tau\right) \Gamma\left(\frac{1}{2} - \mu + i\tau\right) F(i\tau) G(i\tau) d\tau \\
&= \int_1^\infty f(y) g(y) dy ,
\end{aligned} \tag{3.8}$$

a known result [3]. If, further, we put $\mu = 0$, then (3.8) reduces to [2, p.394],

$$\frac{1}{\pi i} \int_0^\infty \tau \tanh(\pi\tau) F(i\tau) G(i\tau) d\tau = \int_1^\infty f(y) g(y) dy ,$$

where,

$$F(i\tau) = \int_1^\infty f(y) P_{-\frac{1}{2}+i\tau}^\mu(y) dy$$

and a similar representation for $G(i\tau)$.

Although the functions F and f satisfying (3.1) and (3.2) are defined for restricted values of μ , one can formally extend the results as follows:

If

$$f(y) = \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t) \Gamma\left(\frac{1}{2} - m - t\right) \Gamma\left(\frac{1}{2} - m + t\right) P_{-\frac{1}{2}+t}^m(y) F(t) dt, \tag{3.9}$$

then

$$F(t) = \int_1^\infty f(y) P_{-\frac{1}{2}+t}^m(y) dy \tag{3.10}$$

where $t = \sigma + i\tau$, $|\sigma| < \frac{1}{2}$, $-\infty < \tau < \infty$ and $m = 0, 1, 2, \dots$. And all the results derived above hold. For instance, if $\sigma = 0$, then we have from above,

$$f(y) = (-1)^m \int_0^\infty \tau \tanh(\pi\tau) \frac{\Gamma\left(\frac{1}{2} - m + i\tau\right)}{\Gamma\left(\frac{1}{2} + m + i\tau\right)} P_{-\frac{1}{2}+i\tau}^m(y) F(i\tau) d\tau$$

where

$$F(i\tau) = \int_1^\infty f(y) P_{-\frac{1}{2}+i\tau}^m(y) dy .$$

If we let

$$\tilde{f}_m^s(\tau) \equiv \frac{\Gamma(\frac{1}{2} - m + i\tau)}{\Gamma(\frac{1}{2} + m + i\tau)} F(i\tau),$$

then

$$\tilde{f}_m^s(\tau) = \int_1^\infty f(y) \frac{\Gamma(\frac{1}{2} - m + i\tau)}{\Gamma(\frac{1}{2} + m + i\tau)} P_{-\frac{1}{2}+i\tau}^m(y) dy,$$

or

$$\tilde{f}_m^s(\tau) = \int_0^\infty f(y) P_{-\frac{1}{2}+i\tau}^{-m}(y) dy$$

using the property (2.2), where

$$f(y) = (-1)^m \int_0^\infty \tau \tanh(\pi\tau) P_{-\frac{1}{2}+i\tau}^m(y) \tilde{f}_m^s(\tau) d\tau$$

giving us a pair of Mehler-Fock transform of order m , [2, p.416]. It may be pointed out here that one can prove Lemma 4, using the result of Theorem 1. In other words, assume (3.1) and (3.2) hold one can show that if

$$f(v) = \frac{i}{\pi z} \int_{\sigma-i\infty}^{\sigma+i\infty} z \sin(\pi z) K_z(v) F(z) dz$$

then

$$F(z) = \int_0^\infty \frac{f(v)}{v} K_z(v) dv, \quad |\sigma| < \frac{1}{2}$$

the so called generalized Kontrovich-Lebedev transform. Consequently one can say that there is an equivalence between the generalized Mehler-Fock and generalized Kontrovich-Lebedev transform.

A slightly different form of the Mehler-Fock transformation and its inversion, can easily be established, by making use of the pair of Kontrovich-Lebedev transforms [3, p.75(76)],

$$f(x) = \frac{1}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} t K_t(x) g(t) dt \quad (3.11)$$

and

$$g(t) = \int_0^\infty \frac{f(v)}{v} I_t(v) dv \quad (3.12)$$

That is,

Theorem 3.

If

$$f(y) = \frac{1}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} z g(z) P_{-\frac{1}{2}+z}^\mu(y) dz, \quad y \in [1, \infty) \quad (3.13)$$

then

$$g(z) = e^{i\mu\pi} \int_1^\infty f(v) Q_{-\frac{1}{2}+z}^{-\mu}(v) dv, \quad \operatorname{Re} \mu < 1, \quad |\sigma| < \frac{1}{2} \quad (3.14)$$

where P_v^μ and Q_v^μ are the associated Legendre functions of first and second kind respectively.

Alternatively,

$$f(y) = \frac{e^{i\mu\pi}}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_1^\infty z P_{-\frac{1}{2}+z}^\mu(y) Q_{-\frac{1}{2}+z}^{-\mu}(v) dv dz$$

The proof of the above theorem is on the same lines as the proof of Theorem 1, the linear operator used in the proof will now be

$$L[f(y); y \rightarrow x] \equiv \int_0^{\infty} e^{-xy} y^{\mu-\frac{1}{2}} f(y) dy ,$$

which on applying to the equation (3.13), will reduce it to the form (3.11). Then using the inversion (3.10), will produce equation (3.14), as desired.

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