Comments on the Jacobian 6.3.09

Where is my proof that the Jacobian is the correct "volume transformation factor" when you change coordinates? The answer is that my proof is strewn around in various documents and is not really collected in any one place, so I will try to do that here.

1. In my notes "M&M chapter 4_5.doc" I review M&M's treatment of distance, area and volume in their discussion of curvilinear coordinates. In that discussion, the new coordinates are $x'_i = q_i$ and the metric tensor is $g_{ij} = G_{ij}$.

(a) The **distance** situation is this:

$$ds^2 = d\mathbf{r}\bullet d\mathbf{r} = [\mathbf{e}_{(i)} dq^i] \bullet [\mathbf{e}^{(j)} dq_j] = \delta_{ij}\, dq^i\, dq_j = dq^i\, dq_i = dq^i\, (G_{ij}\, dq^j) = G_{ij}\, dq^i\, dq^j$$

where the main point is this:

$$ds^2 = g_{ij}\, dq^i\, dq^j$$

(b) The **area** situation is this

$$d\mathbf{S}_1 = d\mathbf{s}_2 \times d\mathbf{s}_3 \qquad \text{or} \qquad d\mathbf{S}_i = \tfrac{1}{2}\, \varepsilon_{ijk}\, d\mathbf{s}_j \times d\mathbf{s}_k \qquad \text{// sums implied}$$

$$dS_1 = \sqrt{g_{22}g_{33} - g_{23}^2}\; dq^2\, dq^3 \qquad \text{// 5-6a}$$

and in general

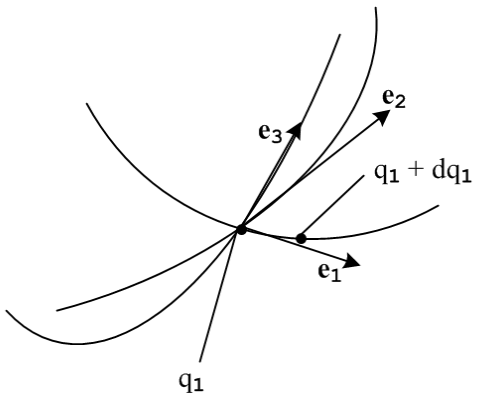
$$dS_i = \tfrac{1}{2}\, \varepsilon_{ijk}\, \sqrt{g_{jj}g_{kk} - g_{jk}^2}\; dq^j\, dq^k \qquad \text{// sums implied}$$

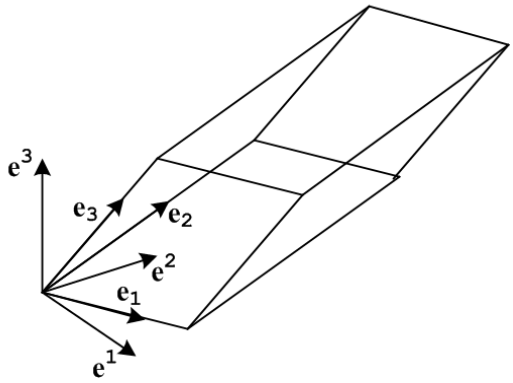
(c) The **volume** situation is this:

$$dV = d\mathbf{s}_1 \bullet d\mathbf{S}_1 = d\mathbf{s}_1 \bullet d\mathbf{s}_2 \times d\mathbf{s}_3 = \mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)}\; dq^1\, dq^2\, dq^3 = v\, dq^1\, dq^2\, dq^3$$

where v is the volume of the parallelepiped $v = [\mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)}]$.

We start in Cartesian coordinates where $dV = dx^1 dx^2 dx^3$ and we go to our curvilinear coordinates (which means any new coordinate system!) and "v" is the volume of the little volume parallelepiped with unit vector edges. We then scale each edge by its dq^i and we get $dV = v\, dq^1\, dq^2\, dq^3$, which I think is extremely clear so there is no confusion about what "v" means. I show a picture of this parallelepiped at the start of "MM Chap 4_5 integration.doc" which I replicate here:





2. Now, the *claim* is made that "v", which is the volume scaling factor, is given by this simple formula:

$v = \sqrt{\det(g)} \equiv \sqrt{g}$ // both sides are rotational scalars

where we now use symbol "g" both to mean matrix g_{ij} and to mean the determinant of this matrix!

I give a proof of this fact in a dedicated section of "M&M chapter 4_5.doc". M&M give their own proof as well. Here is a rough outline of my proof that $g = v^2$. We first write

$$g = \det(g) = \varepsilon_{ijk}\, g_{1i} g_{2j} g_{3k} = \varepsilon_{ijk}\, (\mathbf{e}_{(1)} \bullet \mathbf{e}_{(i)})\, (\mathbf{e}_{(2)} \bullet \mathbf{e}_{(j)})\, (\mathbf{e}_{(3)} \bullet \mathbf{e}_{(k)})$$

and then we write

$$v^2 = [\mathbf{e}_{(1)} \bullet \mathbf{e}_{(2)} \times \mathbf{e}_{(3)}]^2$$

where now everything is in terms of those lower-case covariant unit vectors. I then compact down on the notation and write our desired theorem $g = v^2$ in this way { $\mathbf{e}_{(1)} = \mathbf{1}$ and $\mathbf{e}_{(j)} = \mathbf{j}$, etc. }

$$\varepsilon_{ijk}\, (\mathbf{1} \bullet \mathbf{i})\, (\mathbf{2} \bullet \mathbf{j})\, (\mathbf{3} \bullet \mathbf{k}) = [\mathbf{1} \bullet \mathbf{2} \times \mathbf{3}]^2$$

which can be written out "in components" as

$$\varepsilon_{ijk}\, (\mathbf{1}_m \mathbf{i}_m)\, (\mathbf{2}_n \mathbf{j}_n)\, (\mathbf{3}_s \mathbf{k}_s) = [\mathbf{1}_m \varepsilon_{mns} \mathbf{2}_n \mathbf{3}_s]\ [\mathbf{1}_a \varepsilon_{abc} \mathbf{2}_b \mathbf{3}_c]$$

I then show that these two sides are equal using an argument based on totally antisymmetric tensors, see "M&M chapter 4_5.doc" for further details.

3. But how does this all relate to the Jacobian? So far we have said nothing about the Jacobian which we know is the determinant of the transformation matrix!

$$J \equiv \det(T) = \left| \frac{\partial x^i}{\partial q^k} \right| = | T_k{}^i |$$

If we "start in Cartesian coordinates" and move to non-Cartesians, we know that (various places)

$$g_{kp} = T_k^i\, T_p^i = \frac{\partial x^i}{\partial q^k}\frac{\partial x^i}{\partial q^p}$$

so at least we have some connection between J and $v^2 = \det(g)$. The problem is to now prove that this is true:

$J = \sqrt{\det(g)} = \sqrt{g}$ // since we know from (2) above that $\sqrt{g} = v$

I prove this in " M&M chap5 support.doc " and here is a rough outline of my proof that $J^2 = g$. We start by writing these expressions for the items of interest,

$J = \varepsilon_{stu} T_1^s T_2^t T_3^u$ = jac1
$J = \varepsilon_{ijk} T_i^1 T_j^2 T_k^3$ = jac2

$J^2 = \text{jac2} * \text{jac1} = (\varepsilon_{ijk}\, T_i^1\, T_j^2\, T_k^3)(\varepsilon_{stu}\, T_1^s\, T_2^t\, T_3^u)$

On the other hand, we write, making use of the above fact that $g_{kp} = T_k^i\, T_p^i$,

$g = \varepsilon_{ijk}\, g_{1i} g_{2j} g_{3k} = \varepsilon_{ijk}\, T_1^s\, T_i^s\, T_2^t\, T_j^t\, T_3^u\, T_k^u$ //6 sums and 6 T's

You can see that it is at least feasible that $J^2 = g$ since both sides involve sums of products of six T factors. I show that the equality is in fact true using an antisymmetric tensor argument which is similar to that used in the previous proof.

4. So now we gather up all the above information and state our conclusions:

$J = \sqrt{\det(g)} = \sqrt{g} = v$ // our Big Double Theorem !

where

$J \equiv |\det(T)| = \left|\frac{\partial x^i}{\partial q^k}\right| = |\, T_k^i\,| = \varepsilon_{ijk} T_i^1 T_j^2 T_k^3$ // Jacobian

$dV = v\, dq^1\, dq^2\, dq^3 = dx^1 dx^2 dx^3$ // volume element

$g_{kp} = T_k^i\, T_p^i = \frac{\partial x^i}{\partial q^k}\frac{\partial x^i}{\partial q^p}$ // metric tensor

so our volume transformation can be written:

$dx^1 dx^2 dx^3 = \left|\frac{\partial x^i}{\partial q^k}\right| dq^1\, dq^2\, dq^3$ // = dV

which is certainly easy to remember since it looks like this single variable formula

$$dx = \left(\frac{dx}{dq}\right) dq$$

Here are more standard notations for the Jacobian:

$$d\,y_1 \cdots d\,y_n = \left|\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}\right| d\,x_1 \cdots d\,x_n.$$

$$J = \left|\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}\right|.$$

$$\mathbf{J}(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}.$$

which in my notation would be

$$dx^1 dx^2 dx^3 = \left|\frac{\partial x^i}{\partial q^k}\right| dq^1\, dq^2\, dq^3 = J(q^1, q^2, q^3)\, dq^1\, dq^2\, dq^3$$

5. I have expressed most results here for the case n=3, but at least the volume-related results are valid in any number of dimensions, though M&M don't treat the general case. Eventually I will find a book that does all this stuff right. I think these would be some of the generalizations:

$$v^2 = [\mathbf{1}\bullet\ \mathbf{2} \times \mathbf{3}]^2 = \varepsilon_{ijk}\,(\mathbf{1}\bullet\ \mathbf{i})\,(\mathbf{2}\bullet\ \mathbf{j})\,(\mathbf{3}\bullet\ \mathbf{k}) \rightarrow \varepsilon_{ijk\ell\ldots}\,(\mathbf{1}\bullet\ \mathbf{i})\,(\mathbf{2}\bullet\ \mathbf{j})\,(\mathbf{3}\bullet\ \mathbf{k})\,(\mathbf{4}\bullet\boldsymbol{\ell}) \ldots$$

$$= \varepsilon_{i1i2i3..in}\,(\mathbf{e}_{(1)}\bullet\ \mathbf{e}_{(i1)})\,(\mathbf{e}_{(2)}\bullet\ \mathbf{e}_{(i2)}) \ldots\ldots (\mathbf{e}_{(n)}\bullet\ \mathbf{e}_{(in)})$$

$$J \equiv |\det(T)| = \left|\frac{\partial x^i}{\partial q^k}\right| = |\,T_k{}^i\,| = \varepsilon_{stuv\ldots} T_1{}^s T_2{}^t T_3{}^u T_4{}^v \ldots.$$

$$g_{kp} = T_k{}^i\, T_p{}^i = \frac{\partial x^i}{\partial q^k}\frac{\partial x^i}{\partial q^p} \qquad // \sum_{i=1}^{n} \text{ implied}$$

$$g = \det(g_{ij}) = \varepsilon_{ijk\ell\ldots}\, g_{1i} g_{2j} g_{3k} g_{3\ell} \ldots$$

$$dx^1 dx^2 dx^3 \ldots = \left|\frac{\partial x^i}{\partial q^k}\right| dq^1\, dq^2\, dq^3 \ldots \qquad // = dV$$

I would first show that $g = v^2$ as in item (2) above, then I would show that $J = \sqrt{g}$ as in (3) above.

6. Does the above formula for v work in the case n = 2? It had better. The generalized formula for v given above applied to the case n = 2 says this:

$$v^2 = \varepsilon_{ij}(\mathbf{1}\bullet \mathbf{i})\,(\mathbf{2}\bullet \mathbf{j}) \qquad ??$$

We know that $v = \sin\theta$ because $v^2 = (\mathbf{1} \times \mathbf{2})^2$. Ie, the area of a parallelogram with unit vector edges is just $\sin\theta$. But if we write out the above expression for v^2, we get

$$v^2 = \varepsilon_{ij}(\mathbf{1}\bullet \mathbf{i})\,(\mathbf{2}\bullet \mathbf{j}) = (\mathbf{1}\bullet \mathbf{1})\,(\mathbf{2}\bullet \mathbf{2}) - (\mathbf{1}\bullet\mathbf{2})^2 = 1 - \cos^2\theta = \sin^2\theta$$

QED. In the case n = 4, I don't offhand know a simple "geometric formula" for the volume of the 4D parallelepiped, so I then have to rely on the general formula.