

THE UNIVERSITY OF MELBOURNE
SEMESTER 1, 2006
DEPARTMENT OF MATHEMATICS AND STATISTICS
620-233 VECTOR ANALYSIS (ADVANCED)
EXAM

Exam duration — 3 hours

Reading time — 15 minutes

This paper consists of 8 pages.

Examination Papers with Common Content:

- This paper contains questions in common with those for the subject 620-231 Vector Analysis which is being held at the same time.

Instructions to Invigilators:

- Initially, students are to receive a 14 page script book.

Authorized Materials:

- No calculators or computers are permitted.
- No written or printed material may be brought into the examination room.

Instructions to Students:

- There are 10 questions on this examination paper.
- The number of marks allocated to a question are shown below the question.
- The total number of marks for the whole exam is 100.
- There are formulas of vector identities, curvilinear coordinates and geometry of surfaces on pages 6–8 that you may use in this examination.

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1. (a) Let $f(x, y) = (2x^2, 3y^2)$, $g(u, v) = u^2 - v^2$ and $h = g \circ f$. Calculate $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ at $(x, y) = (1, 1)$ using the chain rule. Verify your answer by direct substitution.

- (b) Find the limit as $(x, y) \rightarrow (0, 0)$ or show that no limit exists for the following functions:

$$\text{i) } \frac{x^3 y^6}{x^2 + y^4}, \quad \text{ii) } \frac{\log(1 + x + y)}{\sqrt{x^2 + y^2}}.$$

- (c) Let

$$g(x, y) = \begin{cases} \frac{2x^3 + y}{x^2 + |y|} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that $g(x, y)$ is not continuous at $(0, 0)$.

- (d) Show that $\frac{\partial g}{\partial x}$ exists at $(0, 0)$ and calculate its value.

10 marks

2. Using the method of Lagrange multipliers, find the minimum of the distance squared function $d(x, y, z) = x^2 + y^2 + z^2$ for the points in common with the cone $z^2 = 4(x^2 + y^2)$ and the plane $2y + 4z = 5$.

10 marks

3. Let $\mathbf{c}$ be the curve parametrized by

$$\mathbf{c}(t) = (\sin t - t \cos t, 2, \cos t + t \sin t), \quad t > 0.$$

- (a) Calculate the arc length of $\mathbf{c}$ between $t = 0$ and $t = 2\pi$.
(b) Calculate the curvature of the curve.

10 marks

4. (a) Let $f(x, y, z)$ and $g(x, y, z)$ be any C^1 scalar functions. Show that

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \nabla^2 g$$

- (b) Let $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. Using vector identities or otherwise, calculate the following quantities where they are defined

$$\text{i) } \nabla^2(3r^2) \quad \text{ii) } \nabla \times \left(\frac{\mathbf{r}}{r^2} \right)$$

10 marks

5. (a) Evaluate the integral

$$\iint_D \exp(x+y) dx dy$$

where D is the domain bounded by the lines $y = 0$, $y = x$ and $y = -x + 2$.

- (b) Consider the integral

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \exp(x^2 + y^2) dy dx$$

- i. Sketch the domain of integration, carefully labelling all the boundary curves.
- ii. Let $x = u \cos v$ and $y = u \sin v$ be a change of variables. Calculate the Jacobian for this change of variables and hence evaluate the integral.

10 marks

6. Let R be the solid region bounded by the plane $z = 0$ and the cone $z = 3 - 3\sqrt{x^2 + y^2}$. The z coordinate of the centre of mass is given by

$$z_c = \frac{\iiint_R z \mu(x, y, z) dV}{\iiint_R \mu(x, y, z) dV}$$

where $\mu(x, y, z)$ is the mass density. If $\mu(x, y, z)$ is constant, calculate z_c .

10 marks

7. Let S be a surface parameterised by

$$x = u, \quad y = v, \quad z = u^2 + v^2, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1$$

- (a) Describe the curves of constant u .
- (b) Find tangent vectors to the curves of constant u and constant v and hence find a normal vector to the surface S .
- (c) Find the mass of a thin metal sheet in the shape of the surface S if the mass density per unit area, $m(x, y) = \sqrt{4x^2 + 4y^2 + 1}$.

10 marks

8. Green's theorem can be written as

$$\int_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy$$

- (a) Verify Green's theorem for the vector field $\mathbf{F} = (x^2, x^2 - y^2)$ and the domain D in common with the two regions $y \leq 1$ and $y \geq x^2$.
- (b) If the vector field is changed to $\mathbf{F} = \left(\frac{1}{x}, \frac{1}{y-1} \right)$ then Green's theorem is no longer valid for the domain D . Why?

10 marks

9. Let $f : D \subset \mathbb{R} \rightarrow \mathbb{R}^+$ be a positive C^2 function. A surface of revolution is obtained by rotating the graph of f around the x -axis. A surface of revolution has the parametrisation

$$\Phi(u, v) = (v, 0, 0) + f(v)(0, \cos u, \sin u),$$

for $v \in D$ and $u \in [0, 2\pi)$.

- (a) Compute the coefficients of the first fundamental form of a surface of revolution.
- (b) Compute the coefficients of the second fundamental form of a surface of revolution.

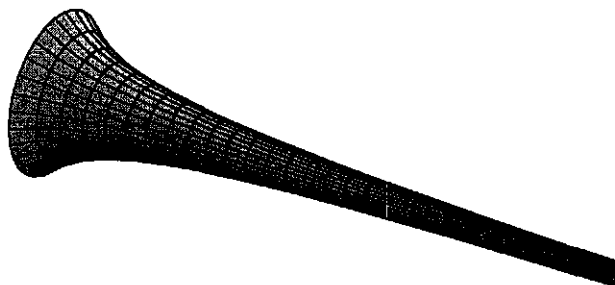


Figure 1: Gabriel's horn

Gabriel's horn is the surface of revolution obtained by rotating the function $f : x \mapsto 1/x$, $x \in [1, \infty)$, around the x -axis.

- (c) Show that the Gaussian curvature of Gabriel's horn is negative everywhere.
- (d) Compute the volume of Gabriel's horn.
- (e) Compute the area of Gabriel's horn. Can Gabriel's horn store enough polish to make its divine surface shine everywhere?

10 marks

10. In this question we will use Maxwell's equation for the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the magnetic vector potential. We will also use the cylindrical coordinates ρ , ϕ and z , and write the unit vectors in this coordinate system as $\hat{\rho}$, $\hat{\phi}$ and $\hat{z}$.

Consider the infinitely long cylinder $x^2 + y^2 = a^2$, $-\infty < z < \infty$. Let $\mathbf{A} = \mathbf{A}_{\text{in}}$ for $0 \leq \rho \leq a$ and $\mathbf{A} = \mathbf{A}_{\text{out}}$ for $\rho \geq a$, where $\mathbf{A}_{\text{in}}$ and $\mathbf{A}_{\text{out}}$ are C^2 vector fields.

- (a) Let $\mathbf{A}_{\text{in}} = \frac{1}{2}B_0\rho\hat{\phi}$. Compute the magnetic field $\mathbf{B}_{\text{in}}$ inside the cylinder from Maxwell's equation using cylindrical coordinates.
- (b) We define the magnetic field outside the cylinder to be zero, $\mathbf{B}_{\text{out}} = \mathbf{0}$. Compute the magnetic flux $\phi = \iint_D \mathbf{B} \cdot d\mathbf{S}$ where D is the disk $x^2 + y^2 \leq R^2$, $z = 0$ and $R > a$. Take the up-pointing normal.
- (c) Use Stokes' theorem to show that outside the cylinder $\mathbf{A}_{\text{out}} \neq \mathbf{0}$, even though $\mathbf{B}_{\text{out}} = \mathbf{0}$.
- (d) Assuming that $\mathbf{A}_{\text{out}} = g(\rho)\hat{\phi}$, compute $g(\rho)$ using Maxwell's equation and Stokes' theorem.
- (e) Show that for any C^2 function f the vector potential $\mathbf{A}' = \mathbf{A} + \nabla f$ produces the same magnetic field.
- (f) Compute $\mathbf{A}_{\text{out}} + \nabla f$ with $f(\rho, \phi, z) = -\frac{1}{2}a^2B_0\phi$. How do you reconcile your result with (c) and (e)?

10 marks

BASIC IDENTITIES OF VECTOR ANALYSIS

Let $f(x, y, z)$ and $g(x, y, z)$ be scalar functions, $\mathbf{F}$ and $\mathbf{G}$ be vector fields in R^3 and β be any constant.

1. $\nabla(f + g) = \nabla f + \nabla g$
2. $\nabla(\beta f) = \beta \nabla f$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ provided $g \neq 0$.
5. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$
6. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$
7. $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$
8. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
9. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
10. $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$
11. $\nabla \times (\nabla f) = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
13. $\nabla \cdot (\nabla f \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

ORTHOGONAL CURVILINEAR COORDINATES

Let $u_1, \dots, u_n$ be orthogonal curvilinear coordinates with scale factors $h_1, \dots, h_n$ and unit tangent vectors $\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_n$. Let $f = f(u_1, \dots, u_n)$ and $\mathbf{F} = \sum_{i=1}^n F_i(u_1, \dots, u_n) \hat{\mathbf{u}}_i$. Then

$$\begin{aligned} dV &= h_1 \cdots h_n du_1 \cdots du_n \\ ds &= \sqrt{h_1^2 du_1^2 + \cdots + h_n^2 du_n^2} \\ \text{grad } f &= \sum_{i=1}^n \frac{1}{h_i} \frac{\partial f}{\partial u_i} \hat{\mathbf{u}}_i \\ \text{div } \mathbf{F} &= \frac{1}{h_1 \cdots h_n} \sum_{i=1}^n \frac{\partial}{\partial u_i} \left(\frac{h_1 \cdots h_n}{h_i} F_i \right) \\ \text{curl } \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \\ \nabla^2 f &= \frac{1}{h_1 \cdots h_n} \sum_{i=1}^n \frac{\partial}{\partial u_i} \left(\frac{h_1 \cdots h_n}{h_i^2} \frac{\partial f}{\partial u_i} \right) \end{aligned}$$

For polar coordinates (r, θ) : $h_1 = 1$, $h_2 = r$.

For cylindrical coordinates (ρ, ϕ, z) : $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

For spherical coordinates (r, θ, ϕ) : $h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$.

GEOMETRY OF SURFACES

Let $\Phi : U \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ be a regular parametrization, and let $\mathbf{n} : U \rightarrow \mathbb{R}^3$ be a unit normal vector field on S . Then:

- The first fundamental form of S is $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ where

$$E = \Phi_u \cdot \Phi_u, \quad F = \Phi_u \cdot \Phi_v, \quad G = \Phi_v \cdot \Phi_v.$$

- The area element is $dS = \|\Phi_u \times \Phi_v\| = \sqrt{EG - F^2}$.

- The second fundamental form has coefficients

$$e = \Phi_{uu} \cdot \mathbf{n}, \quad f = \Phi_{uv} \cdot \mathbf{n}, \quad g = \Phi_{vv} \cdot \mathbf{n}.$$

- The principal curvatures of S are

$$\kappa_1, \kappa_2 = H \pm \sqrt{H^2 - K},$$

where H and K are the mean and Gaussian curvature given by

$$H = \frac{Ge + Eg - 2Ff}{2(EG - F^2)}, \quad K = \frac{eg - f^2}{EG - F^2}.$$