A Quick Tour of Standard Orthogonal Coordinate Systems PhL 11.8.09

Following the ordering in M&M p 177. Pictures from wiki, one from Wolfram. I could make my own pictures with Maple.

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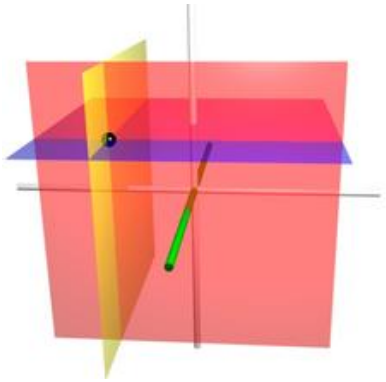
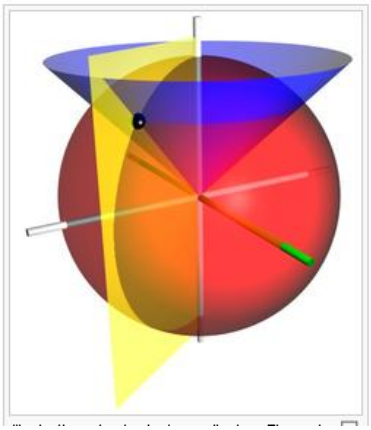
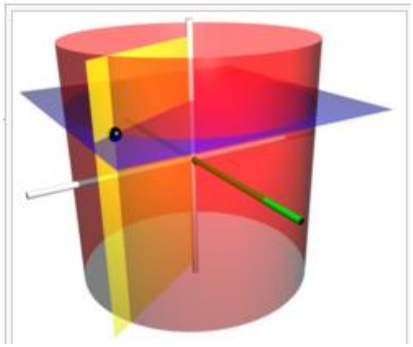
5.3 Cartesian



5.4 Spherical



5.5 Cylindrical (= extruded 2D polar)

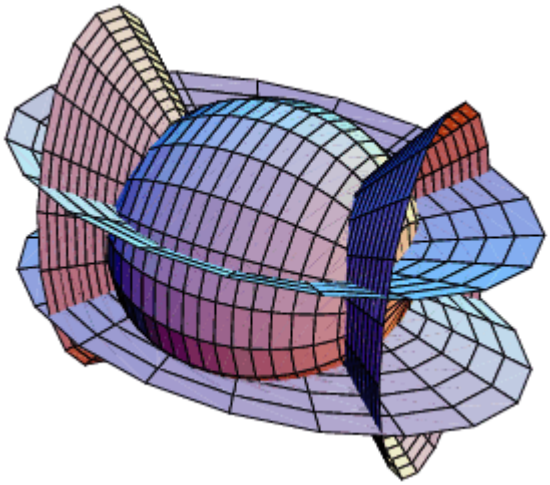


$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$

5.6 Confocal Ellipsoidal (aka Ellipsoidal)

The trick here is that ξ, η and ζ have different legal ranges relative to the constants, and this causes the three equations to have different signs for the terms and hence be different type surfaces. Only the ξ surfaces are ellipsoids, others are hyperboloids. Each (ξ, η, ζ) maps to 8 points (x,y,z) so 1 ↔ 8, see this just looking at the picture. This is the only coordinate system in this survey that has general ellipsoids as a constant variable surface. Very messy MF section, Laplace solutions are Lame functions.

$$\frac{x^2}{a^2+\xi}+\frac{y^2}{b^2+\xi}+\frac{z^2}{c^2+\xi}=1$$
$$\frac{x^2}{a^2+\eta}+\frac{y^2}{b^2+\eta}+\frac{z^2}{c^2+\eta}=1$$
$$\frac{x^2}{a^2+\zeta}+\frac{y^2}{b^2+\zeta}+\frac{z^2}{c^2+\zeta}=1,$$



where $-c^2<\xi<\infty$, $-b^2<\eta<-c^2$, and $-a^2$ constant ξ are confocal ellipsoids, surfaces of Cohn-Vossen 1999, pp. 22-23). For every $(x, y$ in octants.

Solving for x, y, and z gives

$$x^2=\frac{(a^2+\xi)(a^2+\eta)(a^2+\zeta)}{(b^2-a^2)(c^2-a^2)}$$
$$y^2=\frac{(b^2+\xi)(b^2+\eta)(b^2+\zeta)}{(a^2-b^2)(c^2-b^2)}$$
$$z^2=\frac{(c^2+\xi)(c^2+\eta)(c^2+\zeta)}{(a^2-c^2)(b^2-c^2)}.$$

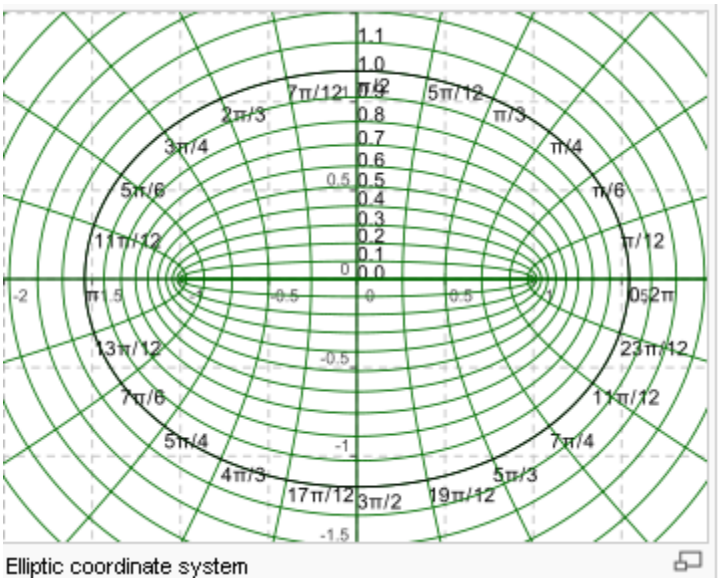
The Laplacian is

$$\nabla^2\Psi=(\eta-\zeta)f(\xi)\frac{\partial}{\partial\xi}\left[f(\xi)\frac{\partial\Psi}{\partial\xi}\right]+(\zeta-\xi)f(\eta)\frac{\partial}{\partial\eta}\left[f(\eta)\frac{\partial\Psi}{\partial\eta}\right]+(\xi-\eta)f(\zeta)\frac{\partial}{\partial\zeta}\left[f(\zeta)\frac{\partial\Psi}{\partial\zeta}\right],$$

where

$$f(x)\equiv\sqrt{(x+a^2)(x+b^2)(x+c^2)}.$$

5.x 2D Elliptical , as needed for the next *three* 3D coordinate systems



$$x = a\ \cosh\mu\ \cos\nu$$
$$y = a\ \sinh\mu\ \sin\nu$$

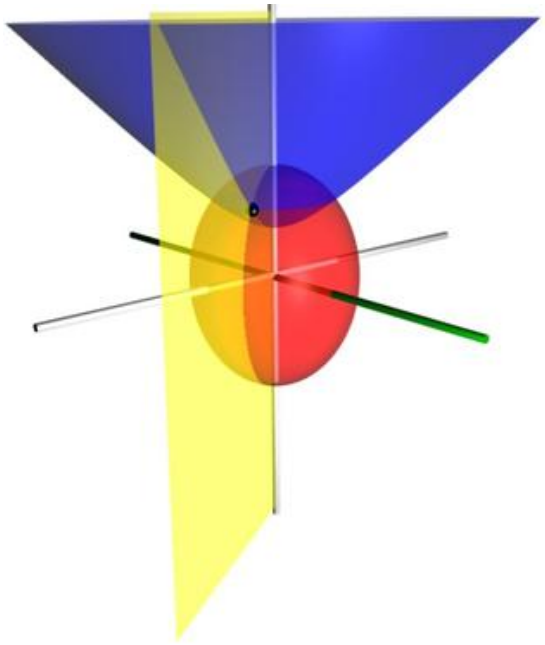
$$x = a\,\sigma\,\tau$$
$$y^2 = a^2\left(\sigma^2 - 1\right)\left(1 - \tau^2\right).$$

σ = coshμ and τ = cosν

This is the form appropriate for extension to *prolate* spheroidal below where I think σ = chu and τ = η. If we were doing this 2D for the oblate, we would have $\sigma^2 + 1$ above where σ = shu = ξ and τ = η.

5.7 Prolate Spheroidal

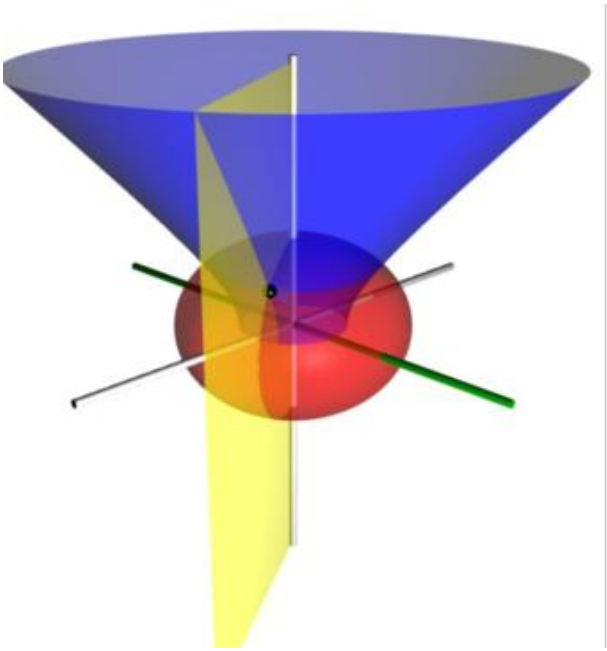
-- hyperbola in xz plane centered on the z axis rotated about the z axis



$$x = a\sqrt{\left(\sigma^2 - 1\right)\left(1 - \tau^2\right)}\cos\phi$$
$$y = a\sqrt{\left(\sigma^2 - 1\right)\left(1 - \tau^2\right)}\sin\phi$$
$$z = a\ \sigma\ \tau$$

5.8 Oblate Spheroidal

-- hyperbola in xz plane centered on x axis, rotated about the z axis

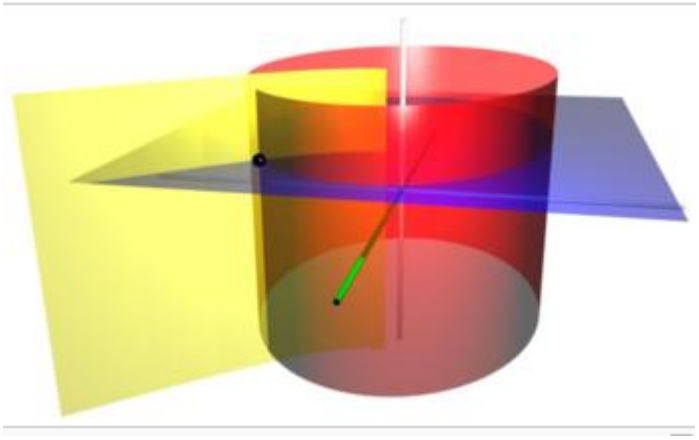


$$x = a\sqrt{(1+\zeta^2)(1-\xi^2)}\,\cos\phi$$
$$y = a\sqrt{(1+\zeta^2)(1-\xi^2)}\,\sin\phi$$
$$z = a\zeta\xi$$

In MF these are written

x = a $\sqrt{\xi^2+1}$ $\sqrt{1-\eta^2}$ cosφ = a $\sqrt{\xi^2+1}$ sinν cosφ
y = a $\sqrt{\xi^2+1}$ $\sqrt{1-\eta^2}$ sinφ = a $\sqrt{\xi^2+1}$ sinν sinφ
z = a ξ η = a ξ cosν

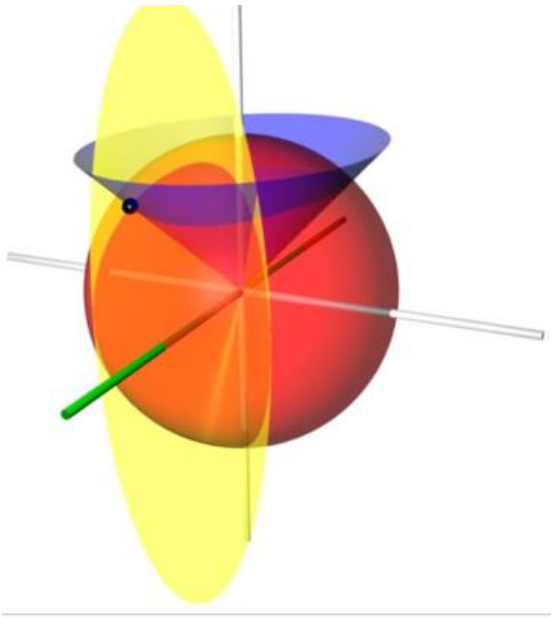
5.9 Elliptical Cylindrical (= extruded 2D elliptical)



$$x = a\ \cosh\mu\ \cos\nu$$
$$y = a\ \sinh\mu\ \sin\nu$$
$$z = z$$

Notice that the yellow here is not a plane, it is a piece of an extruded hyperbola.

5.10 Conical



$$x = \frac{r\mu\nu}{bc}$$

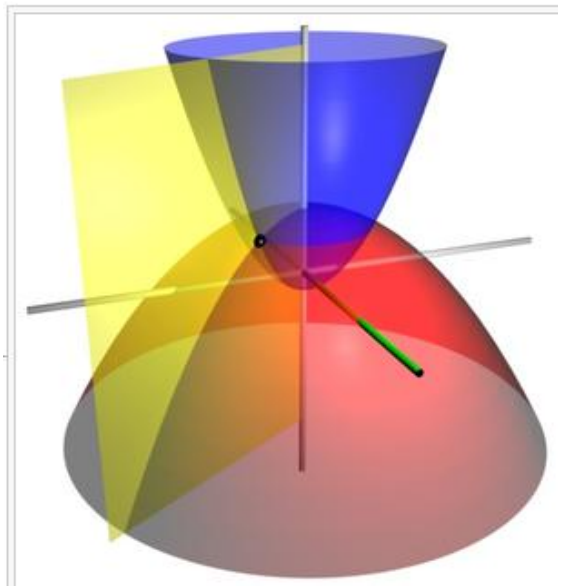
$$y = \frac{r}{b}\sqrt{\frac{(\mu^2 - b^2)(\nu^2 - b^2)}{(b^2 - c^2)}}$$

$$z = \frac{r}{c}\sqrt{\frac{(\mu^2 - c^2)(\nu^2 - c^2)}{(c^2 - b^2)}}$$

One should understand that the cones in this picture both, in general, have elliptical cross sections, so strictly speaking these are not really "cones", they are "elliptical cones".

5.11 Confocal Paraboloidal (Parabolic coordinates)

Here x and y are different, so blue and red are *not* surfaces of revolution, but the picture does not show it very well.



$$x^2 = \frac{(A - \lambda)(A - \mu)(A - \nu)}{B - A}$$

$$y^2 = \frac{(B - \lambda)(B - \mu)(B - \nu)}{A - B}$$

$$z = \frac{1}{2}(A + B - \lambda - \mu - \nu)$$

where the following limits apply to the coordinates

$\lambda < B < \mu < A < \nu$

Consequently, surfaces of constant λ are elliptic paraboloids

$$\frac{x^2}{\lambda - A} + \frac{y^2}{\lambda - B} = 2z + \lambda$$

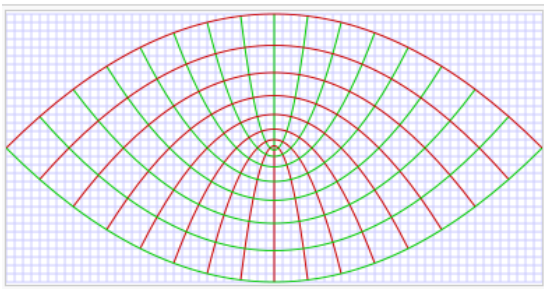
and surfaces of constant ν are likewise

$$\frac{x^2}{\nu - A} + \frac{y^2}{\nu - B} = 2z + \nu$$

whereas surfaces of constant μ are hyperbolic paraboloids

$$\frac{x^2}{\mu - A} + \frac{y^2}{\mu - B} = 2z + \mu$$

5.12 Parabolic Coordinates 2D and 3D, which is just a rotation of the 2D



Two-dimensional parabolic coordinates (σ,τ) are defined by the equations

$$x = \sigma\tau$$

$$y = \frac{1}{2}\left(\tau^2 - \sigma^2\right)$$

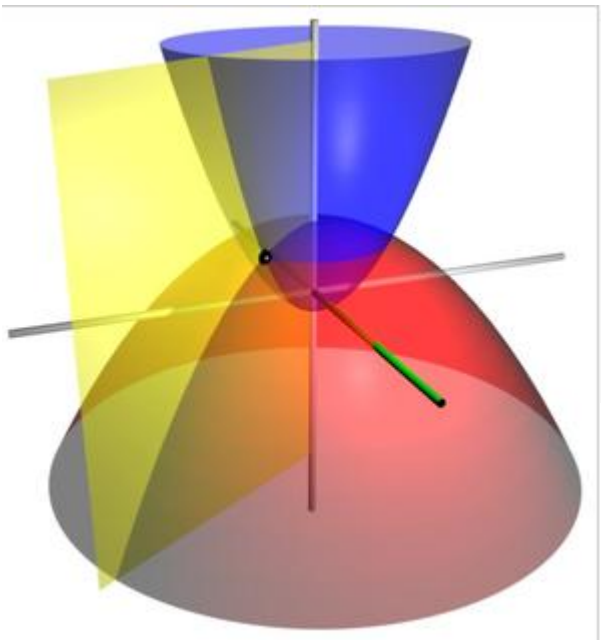
The curves of constant σ form confocal parabolae

$$2y = \frac{x^2}{\sigma^2} - \sigma^2$$

that open upwards (i.e., towards $+y$), whereas the curves of constant τ form confocal parabolae

$$2y = -\frac{x^2}{\tau^2} + \tau^2$$

that open downwards (i.e., towards $-y$). The foci of all these parabolae are located at the origin.



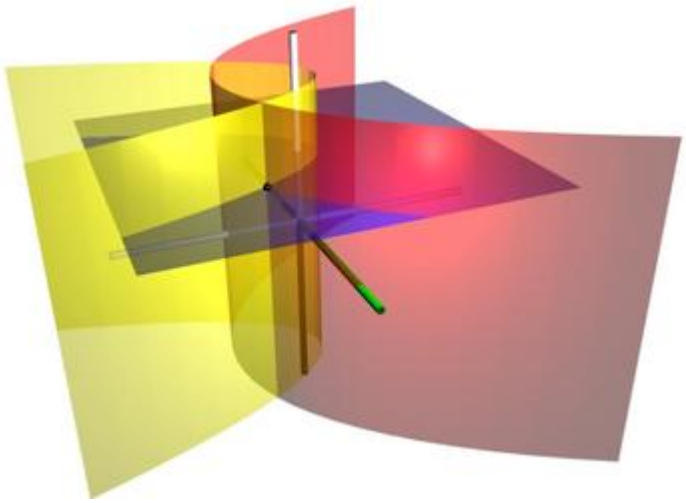
$$x = \sigma\tau\cos\varphi$$

$$y = \sigma\tau\sin\varphi$$

$$z = \frac{1}{2}\left(\tau^2 - \sigma^2\right)$$

5.13 Parabolic Cylindrical (= extruded 2D parabolic)

(a blue slice gives same 2D parabolic seen above)



The parabolic cylindrical coordinates (σ,τ,z) are (x,y,z) by:

$$x = \sigma\tau$$
$$y = \frac{1}{2}\left(\tau^2 - \sigma^2\right)$$
$$z = z$$

The surfaces of constant σ form confocal parabo

$$2y = \frac{x^2}{\sigma^2} - \sigma^2$$

that open towards $+y$, whereas the surfaces of c

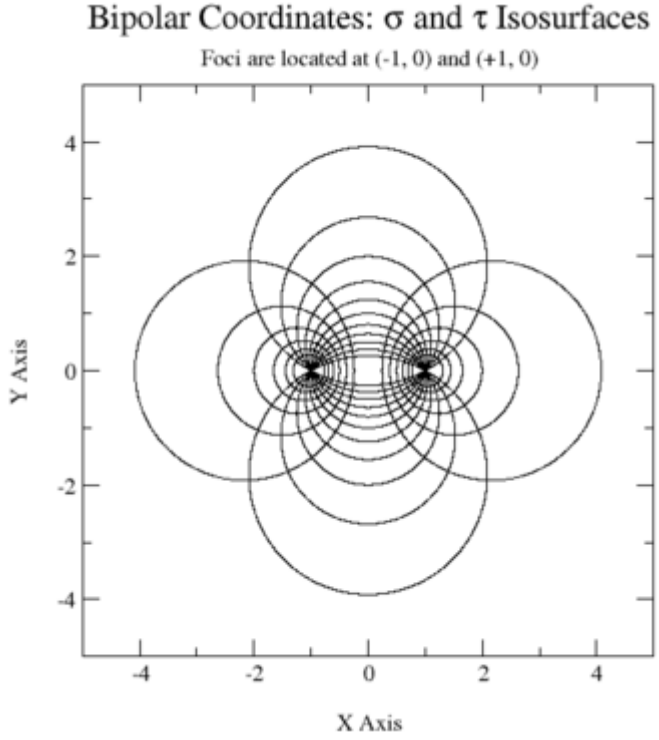
$$2y = -\frac{x^2}{\tau^2} + \tau^2$$

that open in the opposite direction, i.e., towards $-$ simple formula as well

$$r = \sqrt{x^2 + y^2} = \frac{1}{2}\left(\sigma^2 + \tau^2\right)$$

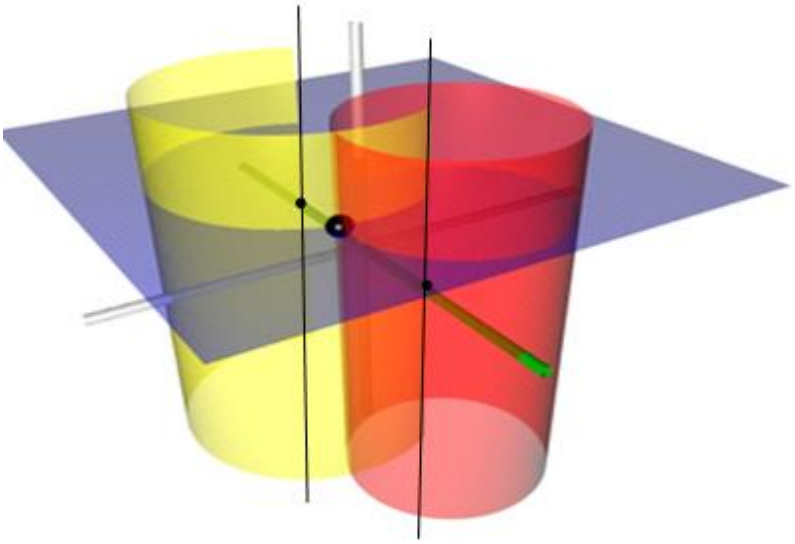
5.14 Bipolar cylindrical 2D and 3D -- (3D = extruded 2D bipolar)

Here is the 2D bipolar coordinates situation. This is Ahlfors's Steiner Circles of his page 85 and dust jacket. All curves are exact circles so when you rotate this below (about vertical axis) to get 3D toroidal, the toroid donut has a circular cross section. Perhaps use this to get the potential of a metal donut.



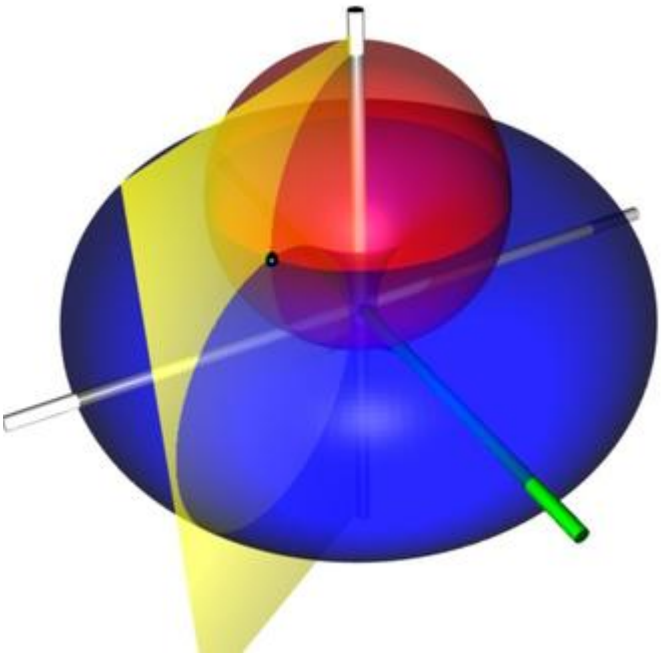
$$x = a\,\frac{\sinh\tau}{\cosh\tau - \cos\sigma}$$
$$y = a\,\frac{\sin\sigma}{\cosh\tau - \cos\sigma}$$

This below is the Bipolar Cylindrical system, just an extrusion of the above 2D bipolar. I have added lines showing the two bipolar axes, and this shows that the red is Apollonius encircling one pole, while the yellow touches both poles.



$$x = a\,\frac{\sinh\tau}{\cosh\tau - \cos\sigma}$$
$$y = a\,\frac{\sin\sigma}{\cosh\tau - \cos\sigma}$$
$$z = z$$

5.15 Toroidal (aka Bispherical) (= rotated 2D bipolar)



$$x = a\,\frac{\sinh\tau}{\cosh\tau - \cos\sigma}\cos\phi$$
$$y = a\,\frac{\sinh\tau}{\cosh\tau - \cos\sigma}\sin\phi$$
$$z = a\,\frac{\sin\sigma}{\cosh\tau - \cos\sigma}$$

There should be an upper red spheroid and a mirror image lower one.
