THESE ARE NOW JUST SCRATCH NOTES, BUT SAVE FOR POSSIBLE FUTURE USE
done when developing the curvilinear meta notes 11.20.09 or so.

(e) Assignment: How can all of the above be extended from N=3 to N ≥ 4 dimensions?

Let's try a more systematic notation for our two cases so far

$$(\mathbf{e}^{\mathbf{i}})_{\mathbf{a}} = (1/2!)\, \varepsilon_{\mathbf{i,n1,n2}}\, \varepsilon_{\mathbf{a,m1,m2}}\, (\mathbf{e}_{\mathbf{n1}})^{\mathbf{m1}} (\mathbf{e}_{\mathbf{n2}})^{\mathbf{m2}} / v \qquad N = 3$$

$$(\mathbf{e}^{\mathbf{i}})_{\mathbf{a}} = (1/3!)\, \varepsilon_{\mathbf{i,n1,n2,n3}}\, \varepsilon_{\mathbf{a,m1,m2,m3}}\, (\mathbf{e}_{\mathbf{n1}})^{\mathbf{m1}} (\mathbf{e}_{\mathbf{n2}})^{\mathbf{m2}} (\mathbf{e}_{\mathbf{n3}})^{\mathbf{m3}} / v \qquad N = 4$$

Our general case will be

$$(\mathbf{e}^{\mathbf{i}})_{\mathbf{a}} = (1/(N-1)!)\, \varepsilon_{\mathbf{i,n1,n2,n3..n(N-1)}}\, \varepsilon_{\mathbf{a,m1,m2,m3..n(N-1)}}\, [\, \Pi_{\mathbf{k=1}}^{\mathbf{N-1}} (\mathbf{e}_{\mathbf{nk}})^{\mathbf{mk}} \,]/v \qquad N = N$$

where n(N-1) means n_{N-1} . We might compact down the ε labels a bit in this way, where like {n} is the set $n_1, n_2 ..... n_{N-1}$ and similarly for {m}. Then we get

$$(\mathbf{e}^{\mathbf{i}})_{\mathbf{a}} = (1/(N-1)!)\, \varepsilon_{\mathbf{i,\{n\}}}\, \varepsilon_{\mathbf{a,\{m\}}}\, [\, \Pi_{\mathbf{k=1}}^{\mathbf{N-1}} (\mathbf{e}_{\mathbf{nk}})^{\mathbf{mk}} \,]/v \qquad N = N$$

Things are a little clearer if we rotate the indices on both ε factors to the left one place

$$(\mathbf{e}^{\mathbf{i}})_{\mathbf{j}} = (1/(N-1)!)\, \varepsilon_{\mathbf{\{n\},i}}\, \varepsilon_{\mathbf{\{m\},j}}\, [\, \Pi_{\mathbf{k=1}}^{\mathbf{N-1}} (\mathbf{e}_{\mathbf{nk}})^{\mathbf{mk}} \,]/v \qquad N = N$$

Now we put this notation to its first use. Consider

$$\mathbf{e}_{\mathbf{a}} \bullet \mathbf{e}^{\mathbf{i}} = (\mathbf{e}_{\mathbf{a}})^{\mathbf{j}} (\mathbf{e}^{\mathbf{i}})_{\mathbf{j}} = (\mathbf{e}_{\mathbf{a}})^{\mathbf{j}} \{ \quad (1/(N-1)!)\, \varepsilon_{\mathbf{\{n\},i}}\, \varepsilon_{\mathbf{\{m\},j}}\, [\, \Pi_{\mathbf{k=1}}^{\mathbf{N-1}} (\mathbf{e}_{\mathbf{nk}})^{\mathbf{mk}} \,]/v \quad \}$$

In order to make the leading factor $(\mathbf{e}_{\mathbf{a}})^{\mathbf{j}}$ *look* like the missing last term in the product, we set

j = m_N = a summation index a = n_N = a fixed value

Then

$$\mathbf{e}_{\mathbf{nN}} \bullet \mathbf{e}^{\mathbf{i}} = (\mathbf{e}_{\mathbf{a}})^{\mathbf{mN}} (\mathbf{e}^{\mathbf{i}})_{\mathbf{mN}} = (\mathbf{e}_{\mathbf{nN}})^{\mathbf{mN}} \{ \quad (1/(N-1)!)\, \varepsilon_{\mathbf{\{n\},i}}\, \varepsilon_{\mathbf{\{m\},mN}}\, [\, \Pi_{\mathbf{k=1}}^{\mathbf{N-1}} (\mathbf{e}_{\mathbf{nk}})^{\mathbf{mk}} \,]/v \quad \}$$

$$= (1/(N-1)!)\, \varepsilon_{\mathbf{\{n\},i}}\, \varepsilon_{\mathbf{\{m\},mN}}\, [\, \Pi_{\mathbf{k=1}}^{\mathbf{N}} (\mathbf{e}_{\mathbf{nk}})^{\mathbf{mk}} \,]/v$$

In any given non-zero term in the sum, the sets {n} and {m} contain all the integers between 1 and N-1, in some order. For example, a term for N= 5 might be (WRONG!)

$$\varepsilon_{\mathbf{1,3,2,4,i}}\, \varepsilon_{\mathbf{4,5,1,2,nM}}$$

The only way this factor cannot vanish is it i = nM = 5. Applying this to all the terms in the sum, we may conclude that $\mathbf{e}_{nN} \bullet \mathbf{e}^{i} = 0$ unless nN = i .

In the case nN = i , we have

$$\mathbf{e}_i \bullet \mathbf{e}^i = (1/(N-1)!)\ \varepsilon_{\{n\},i}\ \varepsilon_{\{m\},i}\ [\ \Pi_{k=1}^{N} (\mathbf{e}_{nk})^{mk}\ ]/v \qquad \text{// no sum on i}$$

From our Lemma below we know that

$$v = \det(\mathbf{e}_a{}^b) = (1/N!)\ \varepsilon_{\{n\},nN}\ \varepsilon_{\{m\},mN}\ [\ \Pi_{k=1}^{N} (\mathbf{e}_{nk})^{mk}\ ]$$

Now recall from above the expression for v as a determinant

$$v = \varepsilon_{ijk\ell m\ldots}\ (\mathbf{e}_1)^i\ (\mathbf{e}_2)^j\ (\mathbf{e}_3)^k (\mathbf{e}_4)^\ell (\mathbf{e}_5)^m \ldots.$$

$$= \varepsilon_{\{m\},mN}\ (\mathbf{e}_1)^{m1}\ (\mathbf{e}_2)^{m2}\ (\mathbf{e}_3)^{m3} (\mathbf{e}_4)^{m4} (\mathbf{e}_5)^{m5}\ \ldots.\ (\mathbf{e}_N)^{mN}$$

This result is unchanged if we replace the set of lower indices with some other partition of the integers 1 through N, as long as we compensate with a sign reflecting the number of "column swaps" we did. But that sign is going to be

This is of course true for any rearrangement of the set 1,2,3,4...

Comments on determinants

The formula for the determinant of a matrix can be written

$$\det(A) = \varepsilon_{m1,m2,m3\cdots mN}\ A_{1,m1}\ A_{2,m2}\ A_{3,m3}\ \ldots\ldots\ A_{N,mN}$$

If the *fixed* set of integers (n1,n2,n3..nN) is a rearrangement of (1,2,3..N), then if we replace the first indices of the A factors with this rearranged set, the determinant is the same except for a sign which is determined by the number of row swaps implied by the rearrangement. That sign is precisely $\varepsilon_{n1,n2,n3\ldots nN}$, so we can rewrite the above as (sums on the mi, but no sums on the ni)

$$\det(A) = \varepsilon_{n1,n2,n3\cdots nN}\ \varepsilon_{m1,m2,m3\ldots mN}\ A_{n1,m1}\ A_{n2,m2}\ A_{n3,m3}\ \ldots\ldots\ A_{nN,mN} \qquad n_i \text{ not summed}$$

$$= \varepsilon_{n1,n2,n3\cdots nN}\ \varepsilon_{m1,m2,m3\ldots mN}\ [\ \Pi_{k=1}^{N} A_{nk,mk}\ ]$$

Suppose now we pick a specific value for n_N, but we sum over all the remaining n_i indices. Then we get (N-1)! replications of det(A) in the sum, so we could say

$$\det(A) = (1/(N-1)!)\ \varepsilon_{n1,n2,n3\cdots nN}\ \varepsilon_{m1,m2,m3\ldots mN}\ [\ \Pi_{k=1}^{N} A_{nk,mk}\ ] \qquad \text{// no sum on } n_N$$

There are N! possible fixed rearrangements of the integers1 to N, so if we now allow summation over all the ni indices as well as the mi ones, we have

$$\det(A) = (1/N!)\ \varepsilon_{n1,n2,n3..nN}\ \varepsilon_{m1,m2,m3...mN}\ A_{n1,m1}\ A_{n2,m2}\ A_{n3,m3}\ .....\ A_{nN,mN}$$

If we adopt our notation above and replace A with e and put the second index up, we get

$$\det(\mathbf{e}_a{}^b) = (1/N!)\ \varepsilon_{\{n\},nN}\ \varepsilon_{\{m\},mN}\ [\ \Pi_{k=1}^{N}\ (\mathbf{e}_{nk})^{mk}\ ]$$

$$\det(A) = \varepsilon_{n1,n2,n3..nN}\ \varepsilon_{m1,m2,m3...mN}\ A_{n1,m1}\ A_{n2,m2}\ A_{n3,m3}\ .....\ A_{nN,mN} \qquad n_i \text{ not summed}$$

$$= \Sigma_{m1,m2...mN}\ \varepsilon_{m1,m2,m3...mN}\ \{\ [\ \Pi_{k=1}^{N}\ A_{nk,mk}\ ]\ \varepsilon_{n1,n2,n3..nN}\ \}$$

Now in each term in the multiple sum, suppose we choose nN = mN. Then we can write

$$\det(A) = \Sigma_{m1,m2...mN}\ \varepsilon_{m1,m2,m3...mN}\ \{\ [\ \Pi_{k=1}^{N}\ A_{nk,mk}\ ]\ \varepsilon_{n1,n2,n3..mN}\ \}$$

Now if we sum over all possible rearrangements of n1,n2,,,,,n(N-1), we will generate (N-1)! copies of det(A), so we can then say

$$\det(A) = (1/(N-1)!)\ \Sigma_{m1,m2...mN}\ \Sigma_{n1,n2...n(N-1)}\ \varepsilon_{m1,m2,m3...mN}\ \{\ [\ \Pi_{k=1}^{N}\ A_{nk,mk}\ ]\ \varepsilon_{n1,n2,n3..mN}\ \}$$

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Appendix on Matrix Forms

Let's make sure things are well-defined:

$$(T_{up})_{ab}\ \equiv\ T_a{}^b$$

$$[(T_{up})^T]_{ab}\ =\ (T_{up})_{ba}\ = T_b{}^a$$

$$[(T_{up})^{-1}]_{ab}\ =\ ???$$

Does up commute with inverse??? Consider this equation

$$(T_{up})\ (T_{up})^{-1} = 1$$

which nobody can deny. Then

$$(T_{up})_{ab}\ (T_{up})^{-1}{}_{bc} = \delta_{ac}$$

$$T_a{}^b\ (T_{up})^{-1}{}_{bc} = \delta_{ac} \qquad \text{we know that } (T_{up})^{-1}{}_{bc}\ = T^c{}_b \text{ from orthonormality}$$

Question: does the notation T^{-1} all by itself mean anything? Well, does T mean anything? T could mean *either* of two different matrices, so it is ambiguous, therefore T^{-1} is also ambiguous. But certainly the quantity $(T_{up})^{-1}$ is not ambiguous. Maybe we could make this *definition*:

$$(T^{-1})_{up} \equiv (T_{up})^{-1}$$

$$[(T^{-1})_{up}]_{ab} = [(T_{up})^{-1}]_{ab}$$

$$(T^{-1})_a{}^b = [(T_{up})^{-1}]_{ab} = T^b{}_a$$

Then we have

$$T_a{}^b \ (T^{-1})_b{}^c = \delta_a{}^c$$

So, T and T^{-1} solo are ambiguous, but T_{up} and $(T^{-1})_{up}$ are not ambiguous.